ON THE INVARIANCE OF CYCLIC ELEMENTS UNDER POINTWISE ALMOST PERIODIC TRANSFORMATION GROUPS

by

JEAN-MARIE PAUL PAGES

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
the Department of Mathematics in
the Graduate School of the
University of Alabama

UNIVERSITY, ALABAMA

1972

ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. William J. Gray for suggesting the topics in this paper and for his encouragement during the period of research and preparation.

CONTENTS

Chapter I. PRELIMINARIES........................ 1

Chapter II. ENVELOPING SUBCONTINUA..............12

Chapter III. THE INVARIANCE OF CYCLIC ELEMENTS.............................24

Bibliography 35

STANDING HYPOTHESIS

In this paper, all topological spaces are assumed to be Hausdorff.

Chapter I

PRELIMINARIES

A. INTRODUCTION

The study of invariance of cyclic elements of a Peano continuum may be said to have begun with the appearance of the Scherrer fixed point theorem in 1926, [9], which states that a dendrite has the fixed point property for homeomorphisms. A few years later in 1932, Borsuk, [2], used Whyburn's cyclic element theory and his own theory of retracts to show that a dendrite has the fixed point property. In 1939, Ayres, [1], proved that if $f:X \rightarrow X$ is a pointwise almost periodic homeomorphism and X is a semi-locally connected continuum, then the union of all cyclic elements of X which are invariant under f is what in our notation is a "cyclic envelope". In 1939, Kelley proved that an auto-homeomorphism of a continuum of X leaves invariant a subcontinuum of X which has no cutpoint of itself [8]. In 1949, Wallace proved that if (X,T) is a transformation group, where X is a continuum, and T is an abelian group, then T leaves invariant a subcontinuum of X which has no cutpoint of itself, and raised the question as to whether or not a similar result holds if "T is abelian" is replaced by "T is compact", [11]. This problem was solved by Gray in 1968, [6].

In this dissertation, the above studies are continued. The results presented here may be summarized as follows:

Let (X,T) be a transformation group, where X is a semi-locally connected continuum. X is T-irreducible if no proper non-empty subcontinuum of X is invariant. It is shown that if X is T-irreducible and contains a cutpoint, then X contains at least two endpoints, and (X,T) is almost periodic at each endpoint of X; furthermore, if (X,T) is pointwise almost periodic, then X is a minimal orbit closure. If A is a closed subset of X, E(A), the enveloping subcontinuum of A, is defined to be the intersection of all nodal sets containing A. E(A) is characterized in terms of the convex hull of A. A is a cyclic envelope if E(A) = A. It is shown that if either T is abelian and (X,T) is pointwise almost periodic, or if (X,T) is pointwise regularly almost periodic, then the union, I(T), of all cyclic elements of X is a cyclic envelope, and if C is a non-invariant cyclic element of X, then the enveloping subcontinuum of $\overline{TC}$ contains exactly one invariant cyclic element of X. These last results extend to the setting of transformation groups some theorems of Ayres and Whyburn on pointwise almost periodic homeomorphisms.

B. CONTINUA

Definition 1.1. A continuum is a compact connected Hausdorff space.

Definition 1.2. A point x of X is a cutpoint of X if $X \setminus x$ is not connected, and otherwise is a non-cutpoint.

The following theorem can be found in [7].

Theorem 1.3. A non-trivial Hausdorff continuum has at least two non-cutpoints.

Definition 1.4. Let X be a continuum. A closed subset A of X is a nodal set if Fr(A) contains at most one point.

Definition 1.5. A subcontinuum C of X is a universal subcontinuum (U.S.C.) of X if $C \cap A$ is a continuum for each subcontinuum A of X.

The proof of the following theorem is trivial.

Theorem 1.6. A nodal set is a U.S.C.

Theorem 1.7. The intersection of arbitrary many U.S.C. is a U.S.C.

Proof: Let $\{A_\alpha : \alpha \in \mathcal{A}\}$, be a collection of U.S.C., let $K = \bigcap \{A_\alpha : \alpha \in \mathcal{A}\}$, and let C be a subcontinuum of X. Let $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$. Since $A_{\alpha_1} \cap C$ is a continuum, $A_{\alpha_2} \cap A_{\alpha_1} \cap C$ is a continuum. Repeating this argument we find that $A_{\alpha_n} \cap A_{\alpha_{n-1}} \cap \dots \cap A_{\alpha_1} \cap C$ is a continuum. Thus the intersection over any finite subcollection of $\{A_\alpha \cap C : \alpha \in \mathcal{A}\}$ is a continuum. Therefore $K \cap C$ is a continuum, see [7].

Definition 1.8. A space is said to be semi-locally connected (s.l.c.) if each point of X has arbitrarily small neighborhoods whose complements have a finite number of components.

Theorem 1.9. A locally connected continuum is s.l.c.

Proof: Let U be a neighborhood of $x \in X$, and choose a neighborhood V of x such that $\overline{V} \subset U$. The components of $X \setminus \overline{V}$ are open sets and so finitely many of them $C_1, \dots, C_n$ cover the compact set $X \setminus U$. Let $W = X \setminus (\overline{C_1} \cup \dots \cup \overline{C_n})$. Then $V \subset W \subset U$ and thus W is a neighborhood of x whose complement has a finite number of components.

Notation: If $x \in X$, $X \setminus x = U | V$ means that $X \setminus x$ is the union of two disjoint non-empty open sets U and V.

Definition 1.10. Let $x, y, z \in X$. Then x is said to separate y and z if there is a separation $X \setminus x = U | V$ with $y \in U$ and $z \in V$. E(a,b) denotes the set of all points which separate a and b together with $a \cup b$.

Definition 1.11. Define an order on $E(a,b)$ as follows: Let a be the least and b the largest element of $E(a,b)$. If $x, y \in E(a,b)$, then $x \leq y$ if and only if $x \in E(a,y)$. $\leq$ is called the separation order on $E(a,b)$.

The order topology on $E(a,b)$ coincides with the subspace topology, see [7].

Definition 1.12. A point $e \in X$ is an endpoint of X if e has arbitrarily small neighborhoods whose closures are nodal sets.

Theorem 1.13. If A and B are U.S.C. which meet, then $A \cup B$ is a U.S.C.

Proof: Let C be a subcontinuum of X. Assume that $(A \cup B) \cap C$ is not connected. Then $A \cap C \neq \emptyset \neq B \cap C$. Also $A \cap B \cap C = \emptyset$ for otherwise $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ would be connected. Now $B \cup C$ is a continuum and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is a union of nonempty disjoint closed sets. This contradicts the fact that A is a U.S.C. Thus $(A \cup B) \cap C$ is a continuum.

Theorem 1.14. Let α be a collection of U.S.C. Then has the finite intersection property if and only if for every pair $A_1, A_2 \in \alpha$ we have $A_1 \cap A_2 \neq \emptyset$. Furthermore if α has the finite intersection property and C is a continuum, then $\alpha \cup \{C\}$ has the finite intersection property if and only if C meets each member of α.

Proof: Suppose that for some positive interger n, any n elements of α have a point in common. Choose $A_1, \ldots, A_{n+1} \in \alpha$. Then $[(A_1 \cap \ldots \cap A_{n-1}) \cup A_{n+1}] \cap A_n = (A_1 \cap \ldots \cap A_n) \cup (A_{n+1} \cap A_n)$ is a continuum which is the union of two closed non-empty sets $A_1 \cap \ldots \cap A_n$ and $A_{n+1} \cap A_n$. Thus $A_1 \cap \ldots \cap A_{n+1} \neq \emptyset$ and α has the finite intersection property. The converse is trivial.

The second statement is proved similarly. Assuming that any n elements of $\alpha \cup \{C\}$ have a nonempty intersection it suffices to show that $C \cap A_1 \cap \ldots \cap A_n \neq \emptyset$ where $A_i \in \alpha$. This is done by letting $C = A_{n+1}$ in the proof of the first statement.

Theorem 1.15. Let X be a s.l.c. continuum and a,b,x. X. Then if $x \notin E(a,b)$, a and b lie in a continuum $C \subset X \setminus x$.

Proof: Let α be the collection of all neighborhoods of x whose complements have a finite number of components. If $U \in \alpha$ and $a \notin U$, let R(a,U) be the component of $X \setminus U$ containing a and let $R(a,U) = \emptyset$ if $a \in U$. If $a \neq x$, there are $U \in \alpha$ for which $R(a,U) \neq \emptyset$. Define $R(a) = \bigcup \{R(a,U) : U \in \alpha\}$. If $U, V \in \alpha$ with $\overline{V} \subset U$, then since $X \setminus V$ has only finitely many components, the intersection with $X \setminus \overline{V}$ of each of these is open in $X \setminus \overline{V}$, hence in X. Thus R(a,U) is contained in an open set which is contained in $R(a,V)$. Thus $R(a,U)$ lies in the interior of $R(a)$. This means that $R(a)$ is open.

Now suppose $z \in X \setminus x$. If $R(z) \neq R(a)$ then for each U in α, $R(a,U)$ does not meet $R(z,U)$. Let $U,V,W \in \alpha$ with $W \subset U \cap V$. Then
$R(a,U) \cap R(z,V) \subset R(a,W) \cap R(z,W) = \emptyset$. This means that $R(z)$ does not meet $R(a)$.

Consequently, $R(a)$ is a component of $X \setminus x$ and the components of $X \setminus x$ are open. If $x \notin E(a,b)$, we must have $b \in R(a)$. Then $b \in R(a,U)$ for some $U \in \alpha$, and $R(a,U)$ is a continuum.

In the proof above we have also proved the following.

Theorem 1.16. If X is a s.l.c. continuum and $x \in X$, each component of $X \setminus x$ is open.

Theorem 1.17. Let X be a s.l.c. continuum. If x is a non-cutpoint of X, x has arbitrarily small neighborhoods whose complements are connected.

Proof: We use the notation of the proof of 1.15. Let $V \in \alpha$ and define $S(a,V) = \emptyset$ if $a \in \overline{V}$ and otherwise, $S(a,V) = R(a,V) \cap X \setminus \overline{V}$. If $W \in \alpha$, and $\overline{V} \subset W$, then $R(a,W) \subset S(a,V) = \emptyset$. Thus, since $X \setminus x$ is connected, $X \setminus x = R(a) = \bigcup \{S(a,V) : V \in \alpha\}$ for each $a \neq x$. Then $x = \bigcap \{X \setminus S(a,V) : V \in \alpha\}$. Let U be a neighborhood of x. Since $X \setminus S(a,V)$ is compact for all $V \in \alpha$, there exists a V such that $X \setminus S(a,V) \subset U$, see [7]. Then $X \setminus R(a,V)$ is the required neighborhood of x.

<u>Theorem 1.18</u>. Every U.S.C. of a s.l.c. continuum is s.l.c.

<u>Proof</u>: Let $A \subset X$ be a U.S.C. and X be s.l.c. Let $x \notin A$ and N be a neighborhood of x in A. Let U be a neighborhood of x in X such that $N = U \cap A$. Let $C_i, i = 1,\ldots,n$, be the components of $X \setminus U$. The statement follows from the fact that if C is a component of $A \setminus N$ then $C = C_i \cap A$ for some i.

<u>Theorem 1.19</u>. If C is a U.S.C. of a s.l.c. continuum X and $x,y,z \in C$ are such that z separates x from y in C, then z separates x from y in X.

<u>Proof</u>: Let $C \setminus z = U | V$ with $x \in U$, $y \in V$. If z does not separate x from y, then by theorem 1.15. there is a continuum $D \subset X \setminus z$ with $x \cup y \subset D$. Then $D \cap C$ is a continuum meeting U and V, and so must contain z, which is absurd.

<u>Theorem 1.20</u>. If $a,b \in X$, and X is s.l.c. continuum then $E(a,b)$ is closed.

<u>Proof</u>: Let $x \notin E(a,b)$. Then by theorem 1.15. there is a continuum C containing $a \cup b$ such that $C \subset X \setminus x$. But since $a \cup b \subset C$ we must have $E(a,b) \subset C$. Hence $\overline{E(a,b)} \subset C \subset X \setminus x$. Thus $x \notin \overline{E(a,b)}$. So $E(a,b)$ is closed.

C. TRANSFORMATION GROUPS

Definition 1.21. A topological transformation group consists of a topological space X, a topological group T and a continuous map π: $T \times X \to X$ satisfying

1) $\pi(e,x) = x$ for every $x \in X$
2) $\pi(s, \pi(t,x)) = \pi(st,x)$ for every $s,t \in T$ and $x \in X$.

In this case we use the notation (X,T) to indicate a transformation group. Also if $x \in X$ and $t \in T$, $\pi(t,x)$ is denoted by tx and if $A \subset X$ and $S \subset T$, $\pi(S \times A) = \{tx : x \in A \text{ and } t \in S\}$ is denoted by SA.

Definition 1.22. Let (X,T) be a transformation group and $A \subset X$. Then A is said to be T-invariant provided that $TA \subset A$.

Definition 1.23. Let (X,T) be a transformation group. Then $A \subset X$ is said to be a minimal orbit closure, provided that A is an orbit closure, i.e. $A = \overline{Tx}$ for some $x \in X$, and A does not properly contain an orbit closure.

Definition 1.24. A subset A of a topological group T is said to be syndetic in T provided that $T = KA$ for some compact subset K of T.

Definition 1.25. Let $x \in X$. The transformation group (X,T) is said to be almost periodic at x provided that if U is a neighborhood of x, then there exists a syndetic subset A of T such that

$Ax \subset U$. (X,T) is said to be pointwise almost periodic provided that it is almost periodic at every point of X.

Definition 1.26. (X,T) is said to be regularly almost periodic at x provided that if U is a neighborhood of x, then there exists a syndetic invariant subgroup A of T such that $Ax \subset U$.

Theorem 1.27. If X is regular and T is pointwise almost periodic, then every orbit closure is minimal.

Proof: Let $x \in X$. Assume that $\overline{Tx}$ is not minimal. Then there exists $y \in \overline{Tx}$ such that $x \notin \overline{Ty}$. Let U be a neighborhood of x for which $\overline{U} \cap \overline{Ty} = \emptyset$. Since T is almost periodic at x there exists a syndetic subset A of T such that $Ax \subset U$. Let K be a compact subset of T for which $T = KA$. Then $Tx \subset KU$. Since $K^{-1} y \cap \overline{U} = \emptyset$, $X \setminus \overline{U}$ is a neighborhood of $K^{-1} y$. Since $\{y\}$ and K^{-1} are compact, there exists a neighborhood V of y such that $K^{-1} V \subset X \setminus \overline{U}$. Then $K^{-1} V \cap \overline{U} = \emptyset$ and $V \cap K\overline{U} = \emptyset$. However, $y \in \overline{Tx}$, so that $Tx \cap V \neq \emptyset$. Since also $Tx \subset KU$, we have $V \cap KU \neq \emptyset$. This is a contradiction.

The proof of the following theorem may be found in [4].

Theorem 1.28. Let X be compact. Then there exists a point of X which is discretely almost periodic under T.

Definition 1.29. Let (X,T) be a transformation group where X is a continuum. X is said to be T-irreducible if no proper nonempty subcontinuum of X is T-invariant.

The following theorem is due to A. D. Wallace [11].

Theorem 1.30. Let (X,T) be a topological transformation group where X is a Hausdorff continuum. If X is T-irreducible and T is abelian, then X has no cutpoints.

W. J. Gray [6] proved the following extension to Wallace's theorem.

Theorem 1.31. Let (X,T) be a topological transformation group where X is a Hausdorff continuum and one of the following conditions is satisfied:

(i) T is compact

(ii) X is s.l.c. and T is pointwise regularly almost periodic.

Then T leaves invariant a nonempty subcontinuum of X which contains no cutpoints of itself.

Chapter II

ENVELOPING SUBCONTINUA

In this chapter we will introduce the concept of the enveloping subcontinuum and give a structure theorem for it. Throughout this chapter X will be assumed to be a semi-locally connected continuum.

Definition 2.1. A true cyclic element of a continuum X is a non-trivial subcontinuum of X which is maximal with respect to having no cutpoints of itself.

Thus any connected subset of X which contains no cutpoint of itself is contained in a true cyclic element of X.

Definition 2.2. Let X be a continuum. If a and b are in X then a is said to be conjugate to b, $a \sim b$, if $E(a,b) = a \cup b$, i.e. if a and b are separated by no point of X.

Definition 2.3. The conjugancy class C(a) of $a \in X$ is defined by $C(a) = \{x \in X: a \sim x\}$.

Theorem 2.4. C(a) is the intersection of all nodal sets containing a in their interior. Hence C(a) is a U.S.C. of X.

Proof: Let be the collection of all nodal sets A such that a Int A. Then $C(a) \subset \bigcap \{A : A \in \alpha\}$. On the other hand if $x \notin C(a)$, there exists $y \in X$ such that $X \setminus y = U | V$ with $a \in U$ and $x \in V$. Then $x \notin U \cup y \in \alpha$. Thus $x \notin \bigcap \{A : A \in \alpha\}$.

The second statement follows from theorems 1.6 and 1.7.

Theorem 2.5. If x is a non-cutpoint of a s.l.c. continuum X and C(x) = x, then x is an endpoint of X.

Proof: Let U be a neighborhood of x. Then, by theorem 1.17, U contains a neighborhood V of x with $X \setminus V$ connected. Since C(x) = x, x is the intersection of all nodal sets of X containing x in their interior. Since nodal sets are compact there exist finitely many such nodal sets $A_i, i = 1, \ldots, n$, such that the intersection of those A_i is contained in V and thus does not meet the continuum $X \setminus V$. Therefore, if α is the collection of all nodal sets containing x in their interior, the collection $\alpha \cup \{X \setminus V\}$ does not have the finite intersection property. Then by theorem 1.14, $X \setminus V$ fails to meet a nodal set A of α. Then $A \subset U$, and hence x is an endpoint.

Theorem 2.6. If $x \in X$ and $C(x)$ is non-degenerate, then $C(x)$ has no cutpoint of itself except possibly x.

Proof: Suppose $C(x)$ has a cutpoint $y \neq x$. Write $C(x) \setminus y = U \mid V$ with $x \in U$. $C(x)$ is a U.S.C. and so Theorem 1.19. implies that y separates x from any point of V in X. But this is impossible since $V \subset C(x)$.

Theorem 2.7. Let X be a s.l.c. continuum. If x X and $C(x)$ is non-degenerate, then $C(x)$ is the union of all true cyclic elements of X containing x. If R is a component of $C(x) \setminus x$, then $\overline{R}$ is a true cyclic element.

Proof: The first statement follows from the second since $C(x)$ is the union of the closures of the components of $C(x) \setminus x$. To prove the second statement let R be a component of $C(x) \setminus x$. Now x is not a cutpoint of $\overline{R}$, and if $y \neq x$ were a cutpoint of R then since $\overline{R}$ is a U.S.C. of $C(x)$, by theorem 1.19, y would be a cutpoint of $C(x)$, which is impossible. Then $\overline{R}$ has no cutpoint of itself and hence is contained in a true cyclic element $C \subset C(x)$. If $y \in C \setminus \overline{R}$, then x must separate y from each point of $\overline{R}$ in $C(x)$, hence in X (1.19.) contradicting the fact that C is a true cyclic element. Therefore $C = \overline{R}$.

Corollary 2.8. Each true cyclic element of a s.l.c. continuum X is a U.S.C.

Proof: Let C be a true cyclic element of X and let $x \in C$. Then by the theorem $C \subset C(x)$. Since C(x) is the union of the closures of the components of $C(x) \setminus x$, and since C meets exactly one such component R, we must have $C \subset \overline{R}$. But $\overline{R}$ is itself a true cyclic element. Therefore $C = \overline{R}$. So C is a U.S.C. of C(x) and hence of X.

The proof of the following corollary is trivial.

Corollary 2.9. If x is a non-cutpoint of the s.l.c. continuum X and C(x) is non-degenerate then C(x) is a true cyclic element.

It is known that if X is a metric space, each true cyclic element contains a non-cutpoint. The following example shows that this is not so in general.

Example 2.10. Let S^1 denote the unit circle. For each $z \in S^1$, let $I_z = [0,1]$. Let $Y = \prod \{I_z : z \in S^1\}$, and let $S^1 \times Y$ have the product topology. Consider the subspace $X = \{(z,f) \in S^1 \times Y : f(w) = 0 \text{ if } w \in S^1 \setminus z\}$. If $(z,f) \notin X$, then $f(w) \neq 0$ for some $w \in S^1 \setminus z$. Let $p_w : Y \to I_w$ be the w-th projection, and let N be a neighborhood of z not containing w. Then $N \times p_w^{-1}((0,1])$ is a neighborhood of (z,f) which does not meet X. Thus X is closed in $S^1 \times Y$.

Now if $a,b \in [0,1]$, and $z \in S'$, let $[a,b]_z =$ $\{(z,f) \in X : a \leq f(z) \leq b\}$. If $\theta = \prod\{0 \in I_z : z \in S'\}$, the set $(S' \times \theta) \cup [0,1]_z$ is connected for each $z \in S'$. Hence X, being the union of all such sets, is connected.

We next show that each point $(z,f) \in X$ for which $f(z) < 1$ is a cutpoint. First note that each point $(z, \theta) \in X$ separates the sets

$$(0,1]_z = [S' \times p_z^{-1}((0,1])] \cap X$$

and $X \setminus [0,1]_z = [(S' \setminus z) \times Y] \cap X$

i.e. every (z, θ) is a cutpoint of X. Also each point $(z,f) \in (0,1)_z$ separates the sets

$$(f(z),1]_z = [S' \times p_z^{-1}((f(z),1])] \cap X$$

and

$$X \setminus [f(z),1]_z = [S' \times p_z^{-1}([0,f(z))) \cup (S' \setminus z) \times Y] \cap X,$$

and thus every $(z,f) \in (0,1)_z$ is a cutpoint of X.

Finally we prove that X is locally connected. If $(z,f) \in (0,1]_z$, choose t for which $0 < t < f(z)$. Then $(t,1]_z$ is connected and open in X, so X is locally connected at (z,f). If $z \in S'$, let

$$R = (U \times [p_{z_1}^{-1}(U_1) \cap \ldots \quad p_{zn}^{-1}(U_n)]) \cap X$$

be a basic neighborhood of (z,θ), where U is open in S' and U_i is open in I_{z_i}. Since $0 \in U_i$ for each i, there is an $\varepsilon > 0$ such that $[0, \varepsilon) \subset U_i$ for each i. Let V be a connected open neighborhood of z with $V \subset U$. Let

$$W = V \times [p_{z_1}^{-1}([0, \varepsilon)) \cap \ldots \cap p_{z_n}^{-1}([0, \varepsilon))] \cap X.$$

Then $W \subset R$, and W is the union of all sets of the form

$$\{(V \times \theta) \cup [0,1]_w : w \in V \setminus \{z_1, \ldots, z_n\}\}$$

and

$$\{(V \times \theta) \cup [0, \varepsilon)_w : w \in \{z_1, \ldots, z_n\} \cap V\}$$

Since each of these is connected, W is connected. Therefore X is locally connected.

The only true cyclic element of X is $S^1 \times \theta$. X consists only of endpoints and cutpoints, however no pair of points of $S^1 \times \theta$ can be separated in $S^1 \times \theta$ and thus X is not a tree.

Definition 2.11. A cyclic element of a s.l.c. continuum X is an endpoint, a cutpoint, or a true cyclic element.

Theorem 2.12. If x and y are distinct points of a s.l.c. continuum X and C(x) and C(y) are non-degenerate then $C(x) \cap C(y)$ is either empty, a cutpoint of X, or a true cyclic element.

Proof: Suppose $C(x) \cap C(y) \setminus z = U | V$ with $r \in U$, $t \in V$. Then by theorem 1.19 since $C(x) \cap C(y)$ is a U.S.C. z separates r and t in X, hence separates r and t in C(x) and C(y). Then by theorem 2.6, $x = z$ and $y = z$. Therefore since $x \neq y$, $C(x) \cap C(y)$ contains no cutpoint of itself.

If $C(x) \cap C(y)$ is non-degenerate, it is contained in a true cyclic element C. Let

$a, b \in C(x) \cap C(y)$

and $z \in C$. Then $x \sim b$, $b \sim a$, $a \sim z$. Since b does not separate x and a, and a does not separate b and z we must have $x \sim z$. Hence $C \subset C(x)$. Similarly $C \subset C(y)$. Therefore $C = C(x) \cap C(y)$.

If $C(x) \cap C(y)$ is degenerate but not empty, let $z = C(x) \cap C(y)$. Then $x \cup z$ and $y \cup z$ are contained in true cyclic elements $E \subset C(x)$ and $F \subset C(y)$. Then $E \cap F = z$, and z is a cutpoint of the U.S.C. $E \cup F$. Hence, by theorem 1.19, z is a cutpoint of X.

<u>Corollary 2-13</u>. Let C be a true cyclic element and $x \notin C$. Then x can be conjugate to at most one point of C.

<u>Proof</u>: Let $y, z \in C$. Then y is conjugate to z so that $C(y) \cap C(z)$ is non-degenerate and hence is a true cyclic element which contains y and z. Then $C = C(y) \cap C(z)$, so that if $x \notin C$, x cannot be conjugate to both y and z.

<u>Theorem 2.14</u>. If C is a true cyclic element of a s.l.c. continuum X and $x \notin C$, some point of C separates a point of C from x.

<u>Proof</u>: Suppose the thoerem is false. Choose $y \in C$ for which $y \notin C(x)$. Then $E(y,x)$ contains a point w for which $y < w < x$, where $\leq$ is the separation order on $E(y,x)$. Choose $z \in C \setminus y$. Then, by

assumption, y cannot separate z from x, and hence by theorem 1.15, $z \cup x$ is contained in a subcontinuum $D \subset X \setminus y$. Then $E(y,x) \setminus y \subset D$, hence $y \notin \overline{E(y,x) \setminus y}$. Since $E(y,x)$ is a compact ordered space, there is a first element $w \in E(y,x)$ following y.

Let R be the component of $X \setminus C$ containing x. Since $\overline{R} \cup C$ is a continuum containing $x \cup y$, we have $E(y,x) \subset \overline{R} \cup C$. Since $w \notin C$, $w \in R$.

Now if $y \sim w$, we cannot have $z \sim w$ because $w \notin C$. But since $z \sim y$ and $w \sim y$, we must have $y \in E(z,w)$, i.e. $X \setminus y = U \mid V$, where U is the component of $X \setminus y$ containing z, and $w \in V$. Then $R \subset V$ and $y \in E(z,x)$ which is impossible. If $y \not\sim w$, then $E(y,w) \setminus (y \cup w) \neq \emptyset$, and this contradicts the choice of w. This completes the proof.

<u>Theorem 2.15</u>. Let C be a true cyclic element of the s.l.c. continuum X. The components of $X \setminus C$ are open sets with one-point boundaries.

<u>Proof</u>: Let R be a component of $X \setminus C$ and choose $x \in R$. By the theorem above we can find $y \in C$ and a separation $X \setminus y = U \mid V$ where U is a component of $X \setminus y$ with $C \cap U \neq \emptyset$ and $x \in V$. Then $C \subset U \cup y$ and $R \subset V$. Thus R is a component of V, hence of $X \setminus y$. Thus y is the boundary of R and by theorem 1.16 R is open.

Definition 2.16. Let X be a s.l.c. continuum and $A \subset X$. The convex hull of A, H(A), is the union of A and all points which separate a pair of points of A.

H(A) contains A and contains E(x,y) whenever $x,y \in H(A)$, and is the smallest subset of X with these properties.

Theorem 2.17. Let F be a closed subset of a s.l.c. continuum X. If a point x of X is not in the convex hull of F, then F lies in a subcontinuum of X not containing x.

Proof: Choose a neighborhood U of x with $U \cap F = \emptyset$. Since X is s.l.c., there exists a neighborhood V of x with $V \subset U$ for which $X \setminus V$ has finitely many components. Let $C, \ldots, C_n$ be those components of $X \setminus V$ which meet F. Since x separates no pair of points of F, then, by theorem 1.15, there are subcontinua

$$K_1, \ldots, K_n \subset X \setminus x$$

for which K_i meets C_i and C_{i+1} for $i = 1, \ldots, n$.
Then

$$C_1 \cup K_1 \cup C_2 \cup K_2 \cup \ldots \cup C_{n-1} \cup K_{n-1} \cup C_n$$

is the desired continuum.

Corollary 2.18. The convex hull of a closed subset of X is closed.

Proof: Let F be a closed subset of X and suppose that x is not in the convex hull of F. Then let C be a continuum such that $F \subset C \subset X \setminus x$. Now $H(F) \subset C$ so that $\overline{H(F)} \subset C$ and so $x \notin \overline{H(F)}$. Therefore $H(F)$ is closed.

Theorem 2.19. Let F be a closed non-degenerate subset of a s.l.c. continuum X. Suppose $x \notin H(F)$ and x is conjugate to at most one point of H(F). Then there is a nodal set A with $F \subset A$ and $x \notin A$.

Proof: Suppose x lies in a true cyclic element C of X. Since all points of C are conjugate to each other, and $F \subset H(F)$, C contains at most one point of F. Then F meets a component R of $X \setminus C$. By theorem 2.15 R has a one-point boundary $y = \overline{R} \cap C$. Let S be the union of all components of $X \setminus y$ which lie in $X \setminus C$ and meet F. If F meets a component R_1 of $X \setminus C$ with $\overline{R_1} \cap C = z \neq y$, then $y \cup z \subset H(F)$. But since $x, y, z \in C$, x is conjugate to y and z, which is impossible. Therefore $F \subset S \cup C$.

Suppose $y \neq x$. Then C cannot contain a member of F other than y, for otherwise y would be in H(F). Thus $S \cup y$ is a nodal set containing F but not containing x.

Now if $y = x$, then S is a component of $X \setminus x$, for otherwise x would separate two points of F which is impossible since $x \notin H(F)$. We also have $F \subset S$. If x lies in a true cyclic element C_1 of $\overline{S}$, we repeat the above argument with S replacing X and find a point

y_1 and the union S_1 of the components $\overline{S} \setminus y_1$, which meet F and lie in $\overline{S} \setminus C_1$. This time $x \neq y_1$, since S is connected, and $S_1 \cup y_1$ is the required nodal set. If x lies in no true cyclic element of $\overline{S}$, replace X by $\overline{S}$ in the following argument.

If x lies in no true cyclic element of X, then $C(x) = x$. By theorem 2.17 F lies in a continuum C of X not containing x. By theorem 2.4, x is the intersection of all nodal sets containing x in their interior. Therefore, by theorem 1.14, C does not meet some nodal set A with $x \in \text{Int } A$. Then $X \setminus \text{Int } A$ is a nodal set containing F and not containing x. This completes the proof.

Definition 2.20. Let X be a s.l.c. continuum and $A \subset X$. The enveloping subcontinuum of A, E(A), is the intersection of all nodal sets containing A.

Theorem 2.21. If A is a closed subset of a s.l.c. continuum X, the enveloping subcontinuum of A is the union of H(A) and all true cyclic elements of X which meet H(A) in at least two points.

Proof: If B is a nodal set containing A then B contains H(A). Let C be a true cyclic element of X meeting H(A) in two distinct points a and b, and let $x \in C \setminus a \cup b$. Suppose $x \notin B$. Then since B contains a and b, the boundary point of B separates x from either a or b, which is impossible. Therefore B also contains any true cyclic element of X meeting H(A) in at least two points.

Now suppose $x \in H(A)$ and x is not in any true cyclic element meeting H(A) in two points. Then x can be conjugate to at most one point of H(A): if $x \sim y$ and $x \sim z$ with $y, z \in H(A)$ and $y \neq z$, then $x \cup y$ $x \cup z$ lie in true cyclic elements E and F of X. If E = F, x is in a true cyclic element meeting H(A) in two points, which is excluded. If $E \neq F$ then their intersection consists of a single point, so $x = E \cap F$ and x separates two points of H(A) in the U.S.C. $E \cup F$, and hence $x \in H(A)$, which is impossible. Then, by theorem 2.19, there is a nodal set B with $A \subset B$ and $x \notin B$. Therefore $x \notin E(A)$ and the proof is complete.

<u>Theorem 2.22</u> If A is a closed nonempty subset of a s.l.c. continuum X, then E(A) is a union of cyclic elements of X.

<u>Proof</u>: We show that if $x \in E(A)$ then there is a cyclic element C of X with $x \in C \subset E(A)$. If x is an endpoint or a cutpoint, the result is obvious. Suppose now that x is a non-cutpoint of X other than an endpoint. Let B be a nodal set of X containing A. Then $x \in \text{Int } B$, so that, by theorem 2.4, $C(x) \subset B$. Therefore $C(x) \subset E(A)$. Now since x is neither a cutpoint nor an endpoint C(x) is non-degenerate and hence, by theorems 2.6 and 2.7, C(x) is a true cyclic element.

CHAPTER III

THE INVARIANCE OF CYCLIC ELEMENTS

A. INTRODUCTION

In this chapter we will apply the results we have obtained thus far and obtain some information about a transformation group (X,T) where X is a s.l.c. continuum which is T-irreducible and contains a cutpoint. It will be shown that in this case, X has at least two endpoints, and (X,T) is almost periodic at each endpoint of X; X contains exactly one minimal orbit closure, and consequently if (X,T) is pointwise almost periodic, then X is a minimal orbit closure.

Next we turn our attention to transformation groups (X,T), where X is a s.l.c. continuum and (X,T) is pointwise almost periodic. It is shown here that the union, I(T) of all invariant cyclic elements of X is a cyclic envelope. If C is a non-invariant cyclic element, the enveloping subcontinuum of $\overline{TC}$ contains exactly one invariant cyclic element. Other results of this type are obtained. These results are all extensions of theorems proved by Ayres [1].

B. THE INVARIANCE OF CYCLIC ELEMENTS

We first make the following observation: Let (X,T) be a transformation group. If X is T-irreducible, then no proper U.S.C. of X can contain a non-empty invariant set I. For otherwise, the intersection, D, of all U.S.C. which contain I is a proper subcontinuum of X containing I. Now D is invariant and thus X is not T-irreducible.

Theorem 3.1. Let (X,T) be a transformation group, where X is a s.l.c. continuum which is T-irreducible and contains a cutpoint. Then X contains at least two endpoints.

Proof: Let z be any non-cutpoint. We need only show that X contains an endpoint e different from z.

We first note that if $X \setminus x = U \mid V$, then by the observation above, neither $U \cup x$ nor $V \cup x$ can contain the orbit Tx.

Let E be the set of cutpoints of X. For each $x \in E$ write

$$X \setminus x = R(x) \mid S(x)$$

where $R(x)$ is the component of $X \setminus x$ containing z. Now suppose $y \in S(x) \cap E$. Then since $\overline{R(x)}$ is connected, does not contain y, and meets the component $R(y)$, we must have $\overline{R(x)} \subset R(y)$. Then $\overline{S(y)}$ is connected, does not contain x, and meets $S(x)$, so that $\overline{S(y)} \subset S(x)$. Therefore $y \in S(x) \cap E$ implies $\overline{S(y)} \subset S(x)$.

Order the collection $\{\overline{S(x)} : x \in E\}$ by inclusion and extract a subcollection $\{\overline{S(y)} : y \in Y\}$ which is maximal with respect to being totally ordered by inclusion.

Let

$$S = \bigcap \{S(y) : y \in Y\}.$$

We prove that S contains no cutpoint of X. Suppose $x \in S$ is a cutpoint of X. For each $y \in Y \setminus x$, we have $x \in S(y)$ and hence $\overline{S(x)} \subset S(y)$. Now S(x) contains a cutpoint p of X and $\overline{S(p)} \subset S(x)$, so that $\overline{S(p)}$ is properly contained in $\overline{S(y)}$ for each $y \in Y$. Then $\{\overline{S(y)} : y \in Y\}$ is not maximal, which is a contradiction.

Now S is an intersection of nodal sets, and so is a U.S.C. which contains no cutpoints of X. Then, by theorem 1.19, S has no cutpoints of itself, and by construction, S is maximal with respect to this property. Thus S is either an endpoint or a true cyclic element. Since S contains no cutpoint of X, we have, by theorem 2.14, that S cannot be a true cyclic element. Therefore S is an endpoint. Clearly $S \neq z$ and the proof is complete.

<u>Theorem 3.2</u>. Let (X,T) be a transformation group, where X is a s.l.c. continuum which is T-irreducible and contains a cutpoint. If e is an endpoint of X, then $e \in \overline{Tx}$ for each $x \in X$.

Proof: Suppose e is an endpoint of X, and $e \notin \overline{Tx}$ for some $x \in X$. Then $X \setminus \overline{Tx}$ is a neighborhood of e, and there exists a neighborhood U of e whose closure is a nodal set contained in $X \setminus \overline{Tx}$. Then $\overline{Tx} \subset X \setminus U$ which contradicts the fact that X is T-irreducible.

Theorem 3.3. Let (X,T) be a transformation group, where X is s.l.c. continuum which is T-irreducible and contains a cutpoint. Then (X,T) is almost periodic at each endpoint of X, and X contains exactly one minimal orbit closure.

Proof: By theorem 1.28, T is almost periodic at some point $x \in X$. Then by [4,4.09], (X,T) is almost periodic at each point of $\overline{Tx}$. Therefore, by theorem 3.2, (X,T) is almost periodic at each endpoint of X.

Now, by theorem 1.27, X contains at least one minimal orbit closure. By [4,2.12], distinct minimal orbit closures are disjoint. Then theorems 3.1 and 3.2 imply that there is only one such minimal orbit closure.

Theorem 3.4. Let (X,T) be a transformation group, where X is a s.l.c. continuum which is T-irreducible and contains a cutpoint. If (X,T) is pointwise almost periodic, then X is a minimal orbit closure. In addition, no cyclic element of X has an interior point.

Proof: The first statement follows from theorems 1.27 and 3.3. Now if C is a true cyclic element of X then Int C contains no endpoints. Since $\overline{Te} = X$ for any endpoints e of X, Int C = $\emptyset$.

We now extend some results of Ayres, [1], on pointwise almost periodic homeomorphisms to transformation groups.

Theorem 3.5. Let (X,T) be a pointwise almost periodic transformation group where X is a s.l.c. continuum for which $X = A \cup B$ where A and B are nodal sets with $A \cap B = x$. If tA meets A and tA meets B for each $t \in T$, then x is fixed under T.

Proof: For each $t \in T$, tA and tB are U.S.C., so that, by theorem 1.14, the collections $\{tA:t \in T\}$ and $\{tB:t \in T\}$ have the finite intersection property. Let

$$C = \bigcap \{tA:t \in T\}$$

and

$$D = \bigcap \{tB:t \in T\}$$

Then C and D are nonempty T-invariant subcontinua of X.

Assume x is not fixed. Choose $t \in T$ for which $tx \neq x$. Suppose $tx \in B$. Since tA is connected and meets both A and B, we must have $x \in t(A \setminus x)$, so that $x \notin tB$. Since tB is connected, meets B and does not contain x we must have $tB \subset B$. Thus $x \notin D$. Since $tx \neq x$ and $tx \in B$ we must have $tx \notin A$, so that $x \notin t^{-1}A = A$.

Thus $x \notin C$. Then x separates each point of C from each point of D since $C \subset A$, $D \subset B$ and neither C nor D contain the cutpoint x of X. Let

$$H = \bigcap \{E(y,z) : y \in C,\ z \in D\} .$$

Since C and D are T-invariant and homeomorphic images of cutpoints are cutpoints, H is T-invariant. Since $x \in H$, $Tx \subset H$. By theorem 1.20, each E(y,z) is closed, so that H is closed. Hence $\overline{Tx} \subset H$. Since the separation order is a total order, any E(y,z) induces a total order on H, and T acts as a group of order isomorphisms on H. Then since $\overline{Tx}$ is a compact ordered space, it contains a largest element w. Now, if $s \in T$, then for any $y \in \overline{Tx}$, $sy \leq w$. Therefore $sw \leq w$ and $s^{-1}w \leq w$ for every $s \in T$. Then $sw = w$ for every $s \in T$, and so $Tw = w$ and $\overline{Tw} = w$. Now, by theorem 1.27, $\overline{Tx}$ is a minimal orbit closure. Then, by [4,2.12], $w = \overline{Tw} = \overline{Tx}$. Then since $x \in \overline{Tx}$ we must have $w = x$, and x is fixed, a contradiction. This completes the proof.

<u>Theorem 3.6</u>. Let (X,T) be a pointwise almost periodic transformation group, where X is a non-trivial s.l.c.continuum. If T leaves an endpoint e of X fixed, then T leaves infinitely many cutpoints of X fixed.

<u>Proof</u>: Choose a point $x \neq e$ and let U be a neighborhood of x with $e \notin \overline{U}$. Let A be a syndetic

subset of T for which $Ax \subset U$, and let K be a compact subset of T for which KA = T. Then $\overline{Ax} \subset \overline{U}$, so that $e \notin \overline{Ax}$. Since e is fixed under T, we have $e \notin K\overline{Ax} = \overline{Tx}$. Choose a nodal set B with $e \in \text{Int } B$ and $B \subset X \setminus \overline{Tx}$. Let y be the boundary of B. Since $e \in B$ and $\overline{Tx} \subset X \setminus B$, tB meets B and $t(X \setminus B)$ meets $X \setminus B$ for each $t \in T$, by theorem 3.5, y must be fixed. Since e is an endpoint, E(e,y) is an infinite set and again theorem 3.5 implies that each point of E(e,y) is fixed under T.

Definition 3.7. Let X be a s.l.c. continuum and $A \subset X$. A is a cyclic envelope if E(A) = A.

Theorem 3.8. Let (X,T) be a pointwise almost periodic transformation group where X is a s.l.c. continuum. Let I(T) be the union of all invariant cyclic elements of X. If either T is abelian or (X,T) is pointwise regularly almost periodic, then I(T) is a nonempty cyclic envelope.

Proof: Theorem 1.31 implies that T leaves invariant a nonempty subcontinuum which contains no cutpoints of itself. Then it follows that $I(T) \neq \emptyset$. If I(T) is a point, it is either an endpoint or a cutpoint, and we are through. Thus assume I(T) is non-degenerate. We first show that if $x \in X$ separates a pair of points of I(T), then x is fixed under T. Suppose $X \setminus x = U \mid V$ where both U and V meet $\overline{I(T)}$. Than U and V are open, hence both meet I(T), so that $U \cup x$ and $V \cup x$ both contain an invariant cyclic element. Therefore by theorem 3.5, x is fixed under T.

Next we will show that if $x \in \overline{I(T)}$, then x lies in an invariant cyclic element. This will prove that $\overline{I(T)} = I(T)$, i.e. I(T) is closed.

Suppose $x \sim y$ with $y \in I(T)$. Let C be a true cyclic element containing $x \cup y$. Assume that x lies in no invariant cyclic element of X. If $I(T) \subset C$, then since I(T) is non-trivial, C is a true cyclic element for which $tC \cap C$ contains at least two points for all $t \in T$. Hence $tC = C$ for all $t \in T$, so that C is invariant. But this is contrary to our assumption. Thus I(T) meets a component R of $X \setminus C$. Let $z = \overline{R} \cap C$. Then z separates a pair of points of I(T), and thus as we have seen above, z must be fixed under T. Let S be the union of all components of $X \setminus C$ whose closures meet C in the point z and write $X \setminus z = S \mid U$. Then $x \in U$, so I(T) meets U. Hence $\overline{U}$ contains an invariant cyclic element A. Now suppose $A \subset C$. Then since $z \in C$ is fixed, $tC \cap C$ contains at least two points for all $t \in T$, and since C is a true cyclic element, we must have $tC = C$ for all $t \in T$, which again is contrary to our assumption. Thus A cannot be contained in C. Then A meets a component Q of $X \setminus C$, and $\overline{Q}$ contains an invariant cyclic element. Let $w = \overline{Q} \cap C$, then, as in the case above, w is fixed under T. Then C contains two fixed points, hence is invariant. This contradiction shows that x must lie in some invariant cyclic element of x.

Now suppose $x \not\sim y$ for all $y \in I(T)$. Choose any $y \in I(T)$. Then $E(x,y)$ contains a point $w \notin x \cup y$. Any such w is fixed since it separates two points of $\overline{I(T)}$. If $E(x,y)$ contains a first element w following x in its separation order, then we must have $x \sim w$, which is impossible since w is fixed and hence is contained in I(T). Since the order topology on $E(x,y)$ coincides with the subspace topology, x is a limit point of fixed points, hence is fixed. If x is an endpoint or cutpoint we are through. If x is neither an endpoint nor a cutpoint, then by corollary 2.9, C(x) is a true cyclic element, hence is invariant. Therefore in all cases x lies in an invariant cyclic element, and I(T) is closed.

We now prove that $E(I(T)) = I(T)$ using theorem 2.21. If x separates a pair of points of I(T) then x must be fixed. Thus any cyclic element which is contained in H(I(T)) is invariant.

Now let A be a true cyclic element of X meeting H(I(T)) in at least two points, say x and y. Then $x \in B$ and $y \in C$ where B and C are invariant cyclic elements of X. If B = C, then A and B are cyclic elements which meet in at least two points, so that A = B = C, and A is an invariant cyclic element. If $B \neq C$, assume $B \cap C \neq \emptyset$. Then A.B.C are U.S.C. such that $A \cap B, B \cap C, B \cap C$ are all nonempty. Then, by theorem 1.14, $A \cap B \cap C \neq \emptyset$, so $A \cap B = A \cap C = A \cap B \cap C$ is a point, which is impossible since $x \neq y$. Therefore if $B \neq C$, $B \cap C = \emptyset$. If B is non-degenerate, $A \cap B$ separates each point of B from each point of $A \setminus B$, and since both B and $A \setminus B$ meet I(T), both must contain an invariant cyclic element, and thus $A \cap B$ must be fixed. On the other hand, if B is degenerate, $A \cap B$ is fixed since B is invariant. Similarly $A \cap C$ is fixed. Then A is a true cyclic element containing two fixed points, and hence must be invariant.

We have shown that $E(I(T) \subset I(T))$. The other inclusion is trivial, and thus E(I(T)) = I(T).

<u>Theorem 3.9</u>. Let (X,T) be a pointwise almost periodic transformation group, where X is s.l.c. and either T is abelian or (X,T) is pointwise regularly almost periodic. If C is a non-invariant cyclic element, the enveloping subcontinuum of $\overline{TC}$ contains exactly one invariant cyclic element of X.

<u>Proof</u>: $E(\overline{TC})$ is an invariant non-trivial subcontinuum. Then, by theorem 1.31, some cyclic element D of $E(\overline{TC})$ is invariant. If D is an endpoint of $E(\overline{TC})$, theorem 3.6 implies that some cutpoint x of

$E(\overline{TC})$ is invariant, and since $E(\overline{TC})$ is a U.S.C., x is a cutpoint of X. If D is a cutpoint of $E(\overline{TC})$, then D is also a cutpoint of X. If D is a true cyclic element of $E(\overline{TC})$, then by theorem 2.22, D is also a true cyclic element of X. Thus $E(\overline{TC})$ contains an invariant cyclic element of X.

If $E(\overline{TC})$ contains two invariant cyclic elements of X, then there exists a cutpoint x which separates two points of $\overline{TC}$, and, by theorem 3.5, x must be fixed. Let $\{R_\alpha\}$ be the collection of all components of $X \setminus x$ which meet TC. Since each R_α is open, each R_α meets TC, and since C is a cyclic element, each $\overline{R_\alpha}$ contains tC for some $t \in T$. If $R_\alpha \neq R_\beta$, let $tC \subset \overline{R_\alpha}$ and $sC \subset \overline{R_\beta}$. Then $st^{-1}R_\alpha = R_\beta$. Since $E(\overline{TC}) \subset \bigcup \{R_\alpha\} \cup x$, $E(\overline{TC})$ contains no invariant cyclic element other than x. This completes the proof.

As a final note, observe that there exist dendrites which are minimal orbit closures under their total homeomorphism group. One such example was constructed by Doyle and Hocking [3]. Thus pointwise almost periodicity of a transformation group with a s.l.c. phase space is not sufficient to insure the existence of an invariant cyclic element.

BIBLIOGRAPHY

1. Ayres, W. L., On Transformations having Periodic Properties, Fund. Math., Vol. 33, 95-103, 1939.

2. Borsuk, K., Sur les retractes, Fund. Math., Vol. 31 152-170, 1931.

3. Doyle, P. H. and Hocking, J. G., Invertible Spaces, Amer. Math. Monthly, Vol. 68, 959-965, 1961.

4. Gottschalk, W. H. and Hedlund, G. A., Topological Dynamics, Amer. Math. Soc. Colloq. Publ. XXXVI, Providence, 1955.

5. Gray, W. J., A Fixed Point Theorem for Commuting Monotone Functions, Canad. J. Math. Vol. 21, 502-504, 1969.

6. ____________, A Note on Group Invariant Continua, J. Austral. Math. Soc., Vol. 8, 310-312, 1968.

7. Hocking, J. G. and Young, G. S., Topology, Reading, Mass., 1961.

8. Kelley, J. L., Fixed Sets under Homeomorphisms, Duke J. Math., Vol. 5, 535-537, 1939.

9. Scherrer, W., Uber Ungeschlossene Stetige Kurven, Math. Zeitschr., Vol. 24, 125-130, 1926.

10. Wallace, A. D., Monotone transformations, Duke J. Math., Vol. 9, 487-506, 1942.

11. ____________, Group Invariant Continua, Fund. Math., Vol. 36, 119-124, 1949.

12. Whyburn, G. T., Analytic Topology, Amer. Math. Soc. Colloq. Publ. XXVIII, New York, 1942.