

Particle Classification and Dynamics in $GL(2,C)$ Gravity

A. Stern – University of Alabama

Deposited 05/02/2019

Citation of published version:

Stern, A. (2009): Particle Classification and Dynamics in $GL(2,C)$ Gravity. *Physical Review D*, 79(10). DOI: <https://doi.org/10.1103/PhysRevD.79.105017>

Particle classification and dynamics in $GL(2, C)$ gravity

A. Stern

Department of Physics and Astronomy, University of Alabama, Tuscaloosa, Alabama 35487, USA

(Received 13 March 2009; published 20 May 2009)

A relatively simple approach to noncommutative gravity utilizes the gauge theory formulation of general relativity and involves replacing the Lorentz gauge group by a larger group. This results in additional field degrees of freedom which either must be constrained to vanish in a nontrivial way or require physical interpretation. With the latter in mind, we examine the coupling of the additional fields to point particles. Nonstandard particle degrees of freedom should be introduced in order to write down the most general coupling. The example we study is the $GL(2, C)$ central extension of gravity given by Chamseddine, which contains two $U(1)$ gauge fields and a complex vierbein matrix, along with the usual spin connections. For the general coupling one should then attach two $U(1)$ charges and a complex momentum vector to the particle, along with the spin. The momenta span orbits in a four-dimensional complex vector space and are classified by $GL(2, C)$ invariants and by their little groups. In addition to orbits associated with standard massive and massless particles, a number of novel orbits can be identified. We write down a general action principle for particles associated with any nontrivial orbit and show that it leads to corrections to geodesic motion. We also examine the classical and quantum theory of the particle in flat space-time.

DOI: 10.1103/PhysRevD.79.105017

PACS numbers: 11.10.Nx, 02.40.Gh, 04.20.-q

I. INTRODUCTION

The standard gauge theory formalism for gravity [1–4] is based on the Lorentz group, or equivalently, on its $SL(2, C)$ covering group. A central extension to the $GL(2, C)$ gauge group has been proposed by Chamseddine and some properties of the resulting theory have been investigated [5]. The $GL(2, C)$ gauge theory has the advantage over the standard gauge theory formulation in that it allows for a straightforward generalization to the noncommutative version of the theory. Unlike in the more sophisticated treatment of noncommutative gravity given by Aschieri *et al.* [6], the diffeomorphism symmetry of the commutative theory is not preserved in this approach. However, its technical simplicity makes it much more amenable for practical applications. These applications include the computation of noncommutative corrections to the known solutions of general relativity. Such computations have been of recent interest [7–16].

Although the noncommutative generalization of $GL(2, C)$ gauge theory is straightforward, its physical interpretation as a gravity theory is not—due to the presence of additional field degrees of freedom. Two different interpretations of the noncommutative $GL(2, C)$ gauge theory are possible, which we now mention:

- (1) One approach is to eliminate the additional degrees of freedom by expressing the noncommutative $GL(2, C)$ gauge fields in terms of the commutative $SL(2, C)$ gauge fields using the Seiberg-Witten map [17]. The standard metric tensor, Lorentz curvature, and torsion of the commutative theory can be utilized to determine the physical consequences of the noncommutative dynamics. The disadvantage of

this approach is that the Seiberg-Witten map leads to complicated nonlinear and nonlocal constraints which must then be imposed on the noncommutative fields.¹ Solving the field equations with these constraints would be a formidable task. (Actually, just writing down the field equations is nontrivial in this case, because the noncommutative action should be varied with respect to the independent fields of the commutative theory.)

- (2) One drops these complicated constraints in the second approach and instead treats all the $GL(2, C)$ gauge fields as independent degrees of freedom. This makes the field equations easier to solve, but has the disadvantage of introducing fields in the gravitational theory which have no analog in the standard gauge theory formulation. The degrees of freedom now include two $U(1)$ gauge fields, a set of complex vierbein fields, and the usual spin connections. A physical interpretation of the extra degrees of freedom is then required in this approach and this can already be addressed in the commutative theory.

Motivated by the second approach, we shall examine the physical content of the $GL(2, C)$ central extension of the standard gauge theory formulation of gravity by coupling to test particles. We require that the particle interaction be invariant under $GL(2, C)$ gauge transformations, as well as

¹In a very recent article [18], conditions involving just the charge conjugation operator are imposed on the noncommutative fields which eliminate the additional degrees of freedom in the noncommutative limit. However, they do not eliminate the additional degrees of freedom from the noncommutative theory without also using the Seiberg-Witten map.

general coordinate transformations and reparametrizations of the evolution parameter. In this regard, it is easy to find a $GL(2, C)$ invariant metric tensor, from which the usual point particle Lagrangian can be constructed. Geodesic motion with respect to the $GL(2, C)$ invariant metric tensor results. However, this particle Lagrangian is not general because it does not take into account all possible particle degrees of freedom. These include the particle spin and two $U(1)$ charges, the latter of which can couple to the two $U(1)$ gauge fields in the $GL(2, C)$ gauge theory.

More curious is the fact that a complex momentum vector, or equivalently two real momentum vectors, should be attached to the particle in order to couple to the complex vierbein fields mentioned above. Under the action of $GL(2, C)$, these momentum vectors span orbits in a four-dimensional complex vector space. In analogy with the usual classification of orbits in \mathbb{R}^4 for relativistic particles, here particles are classified by orbits in a \mathbb{C}^4 (or equivalently, \mathbb{R}^8). The latter are labeled by $GL(2, C)$ invariants, or by their little groups. One of the $GL(2, C)$ invariants is quadratic and can be associated with the “mass,” while another is quartic and does not have a familiar interpretation. One additional invariant can also be found. Up to now the discussion has not taken into account the particle spin (or charges). Three more invariants can easily be constructed (from the analog of Pauli-Lubanski vectors) when spin is present. Because of the large number of invariants, a large variety of different classes of orbits are possible. One such class of orbits can be identified with standard massive particles, while several other disconnected orbits can be used to describe massless particles. Some nonstandard orbits can be identified as well.

An action principle can be formulated which is applicable to all of the nontrivial orbits, and it is a generalization of the action for a relativistic spinning particle [19]. (See also [20].) The particle action is constructed from the real invariant bilinears for $GL(2, C)$. In addition to being invariant under $GL(2, C)$ gauge transformations, it is also invariant under general coordinate transformations, reparametrizations in the evolution parameter, and transformations generated by the orbit’s little group. Coupling of the spin and the two $U(1)$ charges is achieved with the use of a Wess-Zumino type term, and the corresponding equations of motion are a generalization of the Mathisson-Papapetrou equations [21,22] to the $GL(2, C)$ gauge theory. They contain the Lorentz forces associated with the two $U(1)$ fields. A general class of solutions to the equations of motion can be found. We show that they lead to deviations from geodesic motion even in the case of zero spin and charge. The usual dynamics for relativistic particles is recovered upon specializing to flat space-time, although for one class of orbits studied here, the particle can contain additional degrees of freedom.

This article is organized as follows: In Sec. II we review the standard gauge theory formulation of gravity based on

the $SL(2, C)$ gauge group. The extension to $GL(2, C)$ gauge group is given in Sec. III. There we write down the $GL(2, C)$ invariant metric tensor, as well as other invariants of the theory. The standard particle Lagrangian obtained from that metric tensor is discussed. A classification of particles based on orbits in the four-dimensional complex momentum space is given in Sec. IV. We write down a $GL(2, C)$ invariant action for arbitrary orbits in Sec. V and obtain the Euler-Lagrange equations of motion along with the general solutions. With the inclusion of the $GL(2, C)$ invariant Wess-Zumino term, the action is generalized to include interactions with the particle spin and two $U(1)$ charges in Sec. VI. We specialize to flat space-time in Sec. VII where the $GL(2, C)$ gauge symmetry is broken. In the quantum theory, the particle carries representations of a 16-dimensional algebra, containing the Poincaré algebra. We write down the algebra in Sec. VIII and construct the Hilbert space using the method of induced representations. Concluding remarks are given in Sec. IX.

II. STANDARD GAUGE THEORY FORMULATION OF GRAVITY

In the standard gauge theory formulation of gravity [1–4], one introduces spin connection and vierbeins, $\omega_\mu^{ab} = -\omega_\mu^{ba}$ and e_μ^a , respectively. $a, b, \dots = 0, 1, 2, 3$ are Lorentz indices which are raised and lowered using the flat metric tensor $[\eta_{ab}] = \text{diag}(-1, 1, 1, 1)$, while $\mu, \nu \dots$ denote space-time indices. The space-time metric is

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad (2.1)$$

and it is invariant under local Lorentz transformations. Infinitesimal Lorentz variations δ_λ of ω_μ^{ab} and e_μ^a are of the form

$$\begin{aligned} \delta_\lambda \omega_\mu^{ab} &= \partial_\mu \lambda^{ab} + \omega_\mu^{ac} \lambda_c^b - \omega_\mu^{bc} \lambda_c^a, \\ \delta_\lambda e_\mu^a &= e_\mu^b \lambda_b^a, \end{aligned} \quad (2.2)$$

where $\lambda^{ab} = -\lambda^{ba}$ are infinitesimal gauge parameters. The Lorentz curvature and torsion are defined by

$$R_{\mu\nu}^{ab} = \partial_{[\mu} \omega_{\nu]}^{ab} + \omega_{[\mu}^{ac} \omega_{\nu]c}^b, \quad (2.3)$$

$$\tau_{\mu\nu}^a = \partial_{[\mu} e_{\nu]}^a + \omega_{[\mu}^{ab} e_{\nu]b}, \quad (2.4)$$

respectively, and satisfy the Bianchi identities

$$\begin{aligned} \partial_{[\mu} R_{\nu\rho]}^{ab} - R^a_{c[\mu\nu} \omega_{\rho]}^{cb} + \omega^a_{c[\mu} R_{\nu\rho]}^{cb} &= 0, \\ \partial_{[\mu} \tau_{\nu\rho]}^a - R^a_{b[\mu\nu} e_{\rho]}^b + \omega^a_{b[\mu} \tau_{\nu\rho]}^b &= 0. \end{aligned} \quad (2.5)$$

The Lagrangian density for pure gravity is proportional to the Lorentz invariant

$$\epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd}. \quad (2.6)$$

It shall be convenient for us to introduce γ matrices and utilize Dirac spinor notation. [23] We define

$$\omega_\mu = \frac{1}{2}\omega_\mu^{ab}\sigma_{ab}, \quad e_\mu = e_\mu^a\gamma_a, \quad (2.7)$$

with $\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{1}$ and $\sigma_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$. $\mathbb{1}$ denotes the 4×4 unit matrix and $[\]$ the matrix commutator. Then, for example, (2.1) can be written as

$$g_{\mu\nu} = \frac{1}{4}\text{tr } e_\mu e_\nu, \quad (2.8)$$

using $\text{tr } \gamma_a \gamma_b = 4\eta_{ab}$, while (2.2) becomes

$$\delta_\lambda \omega_\mu = \partial_\mu \lambda + i[\omega_\mu, \lambda], \quad \delta_\lambda e_\mu = i[e_\mu, \lambda], \quad (2.9)$$

where $\lambda = \frac{1}{2}\lambda^{ab}\sigma_{ab}$.

III. EXTENSION TO $GL(2, C)$ GAUGE THEORY

A. Motivation

The Lorentz [or $SL(2, C)$] algebra no longer closes upon going to the noncommutative version of the standard gauge theory formulation. In the canonical approach to noncommutative field theories, one replaces the pointwise product between functions by the star product, more specifically, the Groenewold-Moyal star product

$$\star = \exp\left\{\frac{i}{2}\Theta^{\mu\nu}\tilde{\partial}_\mu\tilde{\partial}_\nu\right\}. \quad (3.1)$$

Here $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$ are constant matrix elements corresponding to the noncommutativity parameters and $\tilde{\partial}_\mu$ and $\tilde{\partial}_\nu$ are, respectively, left and right derivatives with respect to some coordinates x^μ of a smooth manifold. The commutator $[A, B]$ between any two matrix-valued functions A and B in the commutative theory is then replaced by the star commutator, $[A, B]_\star = A \star B - B \star A$ in the noncommutative theory. As a consequence, the commutators $[\omega_\mu, \lambda]$ and $[e_\mu, \lambda]$ appearing in the gauge variation (2.9) are replaced by

$$[\omega_\mu, \lambda]_\star = \frac{1}{8}\{\omega_\mu^{ab}, \lambda^{cd}\}_\star[\sigma_{ab}, \sigma_{cd}] + \frac{1}{8}[\omega_\mu^{ab}, \lambda^{cd}]_\star\{\sigma_{ab}, \sigma_{cd}\}, \quad (3.2)$$

$$[e_\mu, \lambda]_\star = \frac{1}{4}\{e_\mu^a, \lambda^{bc}\}_\star[\gamma_a, \sigma_{bc}] + \frac{1}{4}[e_\mu^a, \lambda^{bc}]_\star\{\gamma_a, \sigma_{bc}\}, \quad (3.3)$$

in the noncommutative theory. Here $\{, \}$ denotes the anticommutator, and $\{, \}_\star$ the star anticommutator, $\{a, b\}_\star = a \star b + b \star a$. For the Groenewald-Moyal star, $[a, b]_\star$ is imaginary for any two real-valued functions a and b , while $\{a, b\}_\star$ is real. The $SL(2, C)$ gauge algebra no longer closes due to the presence of the second term on the right-hand side of (3.2). Moreover, from the right-hand side of (3.3), noncommutative gauge transformations do not leave the space of vierbeins invariant. The anticommutator $\{\sigma_{ab}, \sigma_{cd}\}$ appearing in (3.2) is a linear combination of γ_5 and the unit matrix $\mathbb{1}$, while the anticommutator

$\{\sigma_{ab}, \gamma_c\}$ appearing in (3.3) is a linear combination of $\gamma_5\gamma_c$.

Following [5], closure of the gauge algebra is recovered upon enlarging the gauge group from $SL(2, C)$ to $GL(2, C)$. For this one introduces $GL(2, C)$ connections A_μ and gauge parameters Λ

$$A_\mu = \omega_\mu + a_\mu\mathbb{1} + ib_\mu\gamma_5, \quad \Lambda = \lambda + \alpha\mathbb{1} + i\beta\gamma_5, \quad (3.4)$$

where a_μ and b_μ are two $U(1)$ potentials and α and β are two infinitesimal functions on space-time. In addition, Ref. [5] replaces e_μ^a by a complex vierbein matrix $e_\mu^a + if_\mu^a$. Equivalently, upon writing

$$E_\mu = e_\mu + f_\mu, \quad f_\mu = f_\mu^a\gamma_5\gamma_a, \quad (3.5)$$

one can then write down a consistent set of noncommutative $GL(2, C)$ gauge variations δ_Λ

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda + i[A_\mu, \Lambda]_\star, \quad \delta_\Lambda E_\mu = i[E_\mu, \Lambda]_\star \quad (3.6)$$

This leads to rather involved variations for the component fields ω_μ^{ab} , a_μ , b_μ , e_μ^a , and f_μ^a [5].

As stated in the introduction, our interest is to study the physical content of the new degrees of freedom in this model; i.e., those not present in the $SL(2, C)$ gauge theory formulation of gravity. They are contained in the fields a_μ , b_μ , and f_μ^a . Thanks to the existence of the Seiberg-Witten map [17] between commutative and noncommutative gauge theories, these issues can be addressed at the commutative level, meaning $\Theta^{\mu\nu} = 0$. The $GL(2, C)$ gauge variations (3.6) reduce to

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda + i[A_\mu, \Lambda], \quad \delta_\Lambda E_\mu = i[E_\mu, \Lambda] \quad (3.7)$$

in this limit, and the resulting variations of the component fields are now easy to write down:

$$\delta_\Lambda \omega_\mu^{ab} = \partial_\mu \lambda^{ab} + \omega_\mu^{ac}\lambda_c^b - \omega_\mu^{bc}\lambda_c^a, \quad (3.8)$$

$$\delta_\Lambda a_\mu = \partial_\mu \alpha, \quad (3.9)$$

$$\delta_\Lambda b_\mu = \partial_\mu \beta, \quad (3.10)$$

$$\delta_\Lambda e_\mu^a = e_\mu^b\lambda_b^a + 2f_\mu^a\beta, \quad (3.11)$$

$$\delta_\Lambda f_\mu^a = f_\mu^b\lambda_b^a + 2e_\mu^a\beta. \quad (3.12)$$

The two sets of vierbeins are invariant under the action of one of the $U(1)$ subgroups of $GL(2, C)$, while they get mixed under the action of the other $U(1)$. A $GL(2, C)$ invariant field action was found in [5], which in the linear approximation yielded massive modes in addition to the massless graviton.

In what follows we shall introduce a test particle in the commutative $GL(2, C)$ gauge theory and examine possible

$GL(2, C)$ invariant interactions. We therefore need to construct $GL(2, C)$ invariants, one of which should be the metric tensor.

B. The metric tensor and other $GL(2, C)$ invariants

We need to identify the metric tensor in order to connect the $GL(2, C)$ gauge theory to a theory of space-time. We require that the metric tensor transform nontrivially only under general coordinate transformations. It should therefore be invariant under the action of the $GL(2, C)$ gauge group. We note in this regard that (2.1) is only invariant under the $SL(2, C)$ subgroup of $GL(2, C)$ and can no longer serve as the metric tensor. In order to recover the $SL(2, C)$ gauge theory when $f_\mu^a \rightarrow 0$, we need the metric tensor to reduce to (2.1) in this limit.

Two space-time dependent $GL(2, C)$ invariant bilinears can be constructed from the two sets of vierbeins in E_μ :

$$\mathfrak{g}_{\mu\nu} = \frac{1}{4} \text{tr} E_\mu E_\nu = e_\mu^a e_{a\nu} - f_\mu^a f_{a\nu}, \quad (3.13)$$

$$\mathbb{B}_{\mu\nu} = \frac{1}{4} \text{tr} \gamma_5 E_\mu E_\nu = f_\mu^a e_{a\nu} - e_\mu^a f_{a\nu}, \quad (3.14)$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and we used $\text{tr} \gamma_5\gamma_a\gamma_b = 0$. Higher order invariants can also be defined; e.g.,

$$\mathfrak{h}_{\mu\nu\rho\sigma} = \text{tr} E_\mu E_\nu E_\rho E_\sigma, \quad \mathfrak{k}_{\mu\nu\rho\sigma} = \text{tr} \gamma_5 E_\mu E_\nu E_\rho E_\sigma. \quad (3.15)$$

The quadratic invariant $\mathfrak{g}_{\mu\nu}$ given in (3.13) is symmetric in the space-time indices and it reduces to (2.1) when f_μ^a vanish. It can therefore be identified with the metric tensor in the $GL(2, C)$ gauge theory. $\mathbb{B}_{\mu\nu}$ and $\mathfrak{k}_{\mu\nu\rho\sigma}$ are antisymmetric in all space-time indices, while $\mathfrak{h}_{\mu\nu\rho\sigma}$ is symmetric under cyclic permutations. The volume integral of $\mathfrak{k}_{\mu\nu\rho\sigma}$ serves as a cosmological term in the gravity action [5]. $\mathbb{B}_{\mu\nu}$, as well as $\mathfrak{g}_{\mu\nu}$, can be used to write down $GL(2, C)$ invariant couplings to strings. Here, however, we shall only be concerned with point particles.

The flat space-time metric tensor is recovered for E_μ equal to

$$E_\mu^{\text{flat}} = c_1 \gamma_\mu + c_2 \gamma_5 \gamma_\mu, \quad (3.16)$$

where the constants c_1 and c_2 satisfy $c_1^2 - c_2^2 = 1$. The vacuum (3.16) breaks the $GL(2, C)$ gauge symmetry to a $U(1)$ gauge symmetry, the latter being associated with the variations (3.9), in addition to the global Lorentz symmetry. Massive and massless modes were shown to follow from a $GL(2, C)$ invariant field action upon expanding about the flat space-time metric (3.16)

$$E_\mu = E_\mu^{\text{flat}} + \bar{e}_\mu^a \gamma_a + \bar{f}_\mu^a \gamma_5 \gamma_a, \quad (3.17)$$

where \bar{e}_μ^a and \bar{f}_μ^a are small perturbations [5]. The massless modes were shown to have spin two and were thus identified with gravitons. They correspond to the linear combi-

nations $\rho_\mu^a = c_1 \bar{e}_\mu^a - c_2 \bar{f}_\mu^a$. The same linear combinations appear as small perturbations in the $GL(2, C)$ invariant metric tensor $\mathfrak{g}_{\mu\nu}$, since substituting (3.17) in (3.13) gives

$$\mathfrak{g}_{\mu\nu} = \eta_{\mu\nu} + \rho_{a\mu} \delta_\nu^a + \rho_{a\nu} \delta_\mu^a. \quad (3.18)$$

Thus, as in standard gravity theories, the metric tensor contains all the graviton modes. The massive modes of the theory are present in $\mathbb{B}_{\mu\nu}$ and the higher order invariants (3.15).

Finally, other invariants can be constructed from the curvature and torsion, which for the $GL(2, C)$ gauge theory are

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + i[A_\mu, A_\nu], \quad (3.19)$$

$$T_{\mu\nu} = \partial_{[\mu} E_{\nu]} + i[A_{[\mu}, E_{\nu]}],$$

respectively. The former contains the Lorentz curvature (2.3) and two $U(1)$ curvatures

$$F_{\mu\nu} = \frac{1}{2} R_{\mu\nu}^{ab} \sigma_{ab} + \partial_{[\mu} a_{\nu]} \mathbb{1} + i \partial_{[\mu} b_{\nu]} \gamma_5. \quad (3.20)$$

The latter can be decomposed according to

$$T_{\mu\nu} = t_{\mu\nu}^a \gamma_a + u_{\mu\nu}^a \gamma_5 \gamma_a, \quad (3.21)$$

with torsion tensors $t_{\mu\nu}^a$ and $u_{\mu\nu}^a$ defined by

$$\begin{aligned} t_{\mu\nu}^a &= \partial_{[\mu} e_{\nu]}^a + \omega^a{}_{b[\mu} e_{\nu]}^b + 2f_{[\mu}^a b_{\nu]} \\ u_{\mu\nu}^a &= \partial_{[\mu} f_{\nu]}^a + \omega^a{}_{b[\mu} f_{\nu]}^b + 2e_{[\mu}^a b_{\nu]} \end{aligned} \quad (3.22)$$

thus generalizing the Lorentz torsion (2.4). Now the Bianchi identities are

$$\partial_{[\mu} F_{\nu\rho]} + i[A_{[\mu}, F_{\nu\rho]}] = 0, \quad (3.23)$$

$$\partial_{[\mu} T_{\nu\rho]} + i[A_{[\mu}, T_{\nu\rho]}] + i[E_{[\mu}, F_{\nu\rho]}] = 0.$$

$GL(2, C)$ invariant field actions were constructed from the curvature (3.20) and vierbeins (3.5) in [5]. Here, however, we shall not be concerned with the dynamics of the fields, and rather treat them as external in the point particle action.

C. A simple particle action

The action for a point particle should possess the necessary symmetries, which here include invariance under $GL(2, C)$ gauge transformations, general coordinate transformations, and reparametrizations of the evolution parameter. It should also reduce to the usual coupling to gravity in the absence of the additional fields of $GL(2, C)$ gauge theory, i.e., a_μ , b_μ , and f_μ^a . For a point particle with mass $m \neq 0$, an obvious choice is

$$S_0 = \int d\tau \mathcal{L}_0, \quad \mathcal{L}_0 = m \sqrt{-\mathfrak{g}_{\mu\nu}(z) \dot{z}^\mu \dot{z}^\nu}, \quad (3.24)$$

where $\dot{z}^\mu = \frac{dz^\mu}{d\tau}$, $z^\mu(\tau)$ being the particle's space-time coordinates, and τ parametrizes its world line. It possesses all of the required symmetries, and reduces to the standard action for a massive particle when f_μ^a vanish. The Euler-

Lagrange equations of motion correspond to equations of parallel transport for the vector $\mathcal{L}_0^{-1}\dot{z}^\mu$

$$\frac{D}{D\tau}(\mathcal{L}_0^{-1}\dot{z}^\lambda) \equiv \frac{d}{d\tau}(\mathcal{L}_0^{-1}\dot{z}^\lambda) + \Gamma_{\mu\nu}^\lambda(\mathcal{L}_0^{-1}\dot{z}^\mu)\dot{z}^\nu = 0, \quad (3.25)$$

where $\Gamma_{\mu\nu}^\lambda$ are the Christoffel symbols constructed from the metric tensor $\mathfrak{g}_{\mu\nu}$, provided that $\mathfrak{g}_{\mu\nu}$ is nonsingular. As usual, we can perform a reparametrization such that the transformed \mathcal{L}_0 is a constant, thereby recovering the geodesic equations. This corresponds to transforming τ to the proper time, i.e., $d\tau^2 \rightarrow -\mathfrak{g}_{\mu\nu}(z)dz^\mu dz^\nu$.

Although it is reassuring that we recover geodesic motion, the $GL(2, C)$ gauge theory contains more degrees of freedom than is found in standard gravity theory, and so more particle interactions are possible. In addition to spin, the particle can have two $U(1)$ charges, say q and \tilde{q} , associated with the two $U(1)$ gauge fields. The interaction terms

$$\int d\tau(qa_\mu(z) + \tilde{q}b_\mu(z))\dot{z}^\mu \quad (3.26)$$

can then be considered. Moreover, in addition to $\mathfrak{g}_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu$, the total action can involve the higher order $GL(2, C)$ invariant $\mathfrak{h}_{\mu\nu\rho\sigma}(z)\dot{z}^\mu\dot{z}^\nu\dot{z}^\rho\dot{z}^\sigma$. In what follows we give a systematic approach to writing down particle dynamics in this theory and show that the general action contains such higher order invariant terms, as well as interaction terms (3.26).

IV. PARTICLE CLASSIFICATION

A. Orbits in momentum space

Particles are standardly classified by the orbits traced out in four-dimensional momentum space under the action of the Lorentz group. If p^a denotes the particle momenta, then the action is generated by the variations

$$\delta_\lambda(p^a\gamma_a) = i[p^a\gamma_a, \lambda], \quad (4.1)$$

where $\lambda = \frac{1}{2}\lambda^{ab}\sigma_{ab}$. Six distinct orbits can be identified, only two of which are physically relevant and they correspond to positive energy massive and massless particles. (See, for example, [24,25].) We now replace $p^a\gamma_a$ by some matrices P , the Lorentz group by $GL(2, C)$, and the variations (4.1) by

$$\delta_\Lambda P = i[P, \Lambda], \quad (4.2)$$

where Λ was defined in (3.4). For closure we need P to be a linear combination of both γ_a and $\gamma_5\gamma_a$ matrices. Thus momentum space must be enlarged to an eight-dimensional real vector space \mathbb{R}^8 spanned by real vectors, say p^a and \tilde{p}^a . (Alternatively, we can introduce the complex momentum vector $p^a + i\tilde{p}^a$.) Upon writing

$$P = p^a\gamma_a + \tilde{p}^a\gamma_5\gamma_a, \quad (4.3)$$

it follows that p^a and \tilde{p}^a transform under $GL(2, C)$ as the vierbeins e_μ^a and f_μ^a in (3.11) and (3.12), i.e.,

$$\delta_\Lambda p^a = p^b\lambda_b^a + 2\tilde{p}_\mu^a\beta, \quad \delta_\Lambda \tilde{p}^a = \tilde{p}^b\lambda_b^a + 2p^a\beta. \quad (4.4)$$

Many more distinct orbits are possible upon enlarging the momentum space to \mathbb{R}^8 . These orbits are generated by the adjoint action (4.2) and can be classified by their $GL(2, C)$ invariants. For this, it is convenient to reexpress p^a and \tilde{p}^a in terms of the following 2×2 Hermitian matrices \mathcal{P} and $\tilde{\mathcal{P}}$:

$$\begin{aligned} \mathcal{P} &= (p^0 + \tilde{p}^0)\mathbb{1} + (p^i + \tilde{p}^i)\sigma_i, \\ \tilde{\mathcal{P}} &= (p^0 - \tilde{p}^0)\mathbb{1} - (p^i - \tilde{p}^i)\sigma_i, \end{aligned} \quad (4.5)$$

σ_i being the Pauli matrices. \mathcal{P} and $\tilde{\mathcal{P}}$ transform under $GL(2, C)$ according to

$$\mathcal{P} \rightarrow \mathcal{P}' = M\mathcal{P}M^\dagger, \quad \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}' = M^{\dagger-1}\tilde{\mathcal{P}}M^{-1}, \quad (4.6)$$

where M is a $GL(2, C)$ matrix written in the defining representation. This agrees with (4.4) for infinitesimal transformations. The space of all \mathcal{P}' and $\tilde{\mathcal{P}}'$ generated from \mathcal{P} and $\tilde{\mathcal{P}}$ in (4.6) defines an orbit. Then

$$\mathcal{C}^{(2)} = p_a p^a - \tilde{p}_a \tilde{p}^a = -\frac{1}{2}\text{tr} \mathcal{P}\tilde{\mathcal{P}}, \quad (4.7)$$

$$\mathcal{C}^{(4)} = (p_a p^a + \tilde{p}_a \tilde{p}^a)^2 - 4(p_a \tilde{p}^a)^2 = \det \mathcal{P} \det \tilde{\mathcal{P}} \quad (4.8)$$

are quadratic and quartic invariants, respectively, and serve to label the orbit. The former defines the invariant norm of the momenta and reduces to minus the mass-squared when $\tilde{p}^a \rightarrow 0$. The latter can be reexpressed as $\mathcal{C}^{(4)} = \frac{1}{2} \times (\text{tr} \mathcal{P}\tilde{\mathcal{P}})^2 - \frac{1}{2} \text{tr}(\mathcal{P}\tilde{\mathcal{P}})^2$. Although neither

$$\det \mathcal{P} = -(p^a + \tilde{p}^a)(p_a + \tilde{p}_a) \quad (4.9)$$

nor

$$\det \tilde{\mathcal{P}} = -(p^a - \tilde{p}^a)(p_a - \tilde{p}_a) \quad (4.10)$$

are invariant under $GL(2, C)$ transformations, their signs are; i.e., $\det \mathcal{P}$ and $\det \tilde{\mathcal{P}}$ are either positive, negative, or zero for all points on any orbit. Thus they can also be used to label the orbits. Of course, for $\mathcal{C}^{(4)} \neq 0$ they are not independent. There thus exist at least three invariants which can be used to classify the orbits. (More will be obtained in Sec. VII B upon including the spin.)

For the action (3.24) considered previously, a reasonable choice for the ‘‘momenta’’ p^a and \tilde{p}^a is

$$p^a = \frac{mv^a}{\sqrt{\tilde{v}^a \tilde{v}_a - v^a v_a}}, \quad \tilde{p}^a = \frac{m\tilde{v}^a}{\sqrt{\tilde{v}^a \tilde{v}_a - v^a v_a}}, \quad (4.11)$$

with ‘‘velocities’’ v^a and \tilde{v}^a given by

$$v^a = e^a_\mu(z)\dot{z}^\mu, \quad \tilde{v}^a = f^a_\mu(z)\dot{z}^\mu. \quad (4.12)$$

They transform under the action of $GL(2, C)$ as p^a and \tilde{p}^a , respectively. Then $\tilde{v}^a \tilde{v}_a - v^a v_a$ is invariant and corresponds to $(\mathcal{L}_0/m)^2$. With the choice (4.11), it follows that the canonical momenta $\pi_\mu = \partial \mathcal{L}_0 / \partial \dot{z}^\mu$ are equal to the linear combinations

$$\pi_\mu = p_a e^a_\mu - \tilde{p}_a f^a_\mu. \quad (4.13)$$

For p^a and \tilde{p}^a defined this way, $\mathcal{C}^{(2)}$ is minus the mass-squared, while the invariant $\mathcal{C}^{(4)}$ is dynamically determined

$$\mathcal{C}^{(2)} = -m^2, \quad \mathcal{C}^{(4)} = m^4 \left(2 - \frac{h_{\mu\nu\rho\sigma}(z)\dot{z}^\mu \dot{z}^\nu \dot{z}^\rho \dot{z}^\sigma}{(2g_{\eta\xi}(z)\dot{z}^\eta \dot{z}^\xi)^2} \right). \quad (4.14)$$

In the special case where all the f^a_μ vierbeins can be gauge transformed away using a $GL(2, C)$ gauge transformation, it follows that,

$$h_{\mu\nu\rho\sigma}(z)\dot{z}^\mu \dot{z}^\nu \dot{z}^\rho \dot{z}^\sigma = (2g_{\eta\xi}(z)\dot{z}^\eta \dot{z}^\xi)^2, \quad (4.15)$$

and hence $\mathcal{C}^{(4)}$ reduces to m^4 .

B. General classification

We now drop the definitions of p^a and \tilde{p}^a as given in (4.11) and consider general orbits generated by (4.6) in \mathbb{R}^8 . These orbits can be classified using the invariants $\mathcal{C}^{(2)}$, $\mathcal{C}^{(4)}$, and $\det \mathcal{P}$ (and/or $\det \tilde{\mathcal{P}}$). As we shall see later, further quantities are needed to classify orbits with $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = \det \mathcal{P} = \det \tilde{\mathcal{P}} = 0$.

General orbits are defined by the set of all $\{\mathcal{P}\}$ and $\{\tilde{\mathcal{P}}\}$ with

$$\mathcal{P} = NKN^\dagger, \quad \tilde{\mathcal{P}} = N^{\dagger-1} \tilde{K}N^{-1}, \quad (4.16)$$

where N denote $GL(2, C)$ matrices written in the defining representation, while K and \tilde{K} are constant 2×2 Hermitian matrices

$$\begin{aligned} K &= (k^0 + \tilde{k}^0)\mathbb{1} + (k^i + \tilde{k}^i)\sigma_i, \\ \tilde{K} &= (k^0 - \tilde{k}^0)\mathbb{1} - (k^i - \tilde{k}^i)\sigma_i, \end{aligned} \quad (4.17)$$

which we can associate with a fiducial point (k, \tilde{k}) on the orbit. \mathcal{P} and $\tilde{\mathcal{P}}$ are invariant under

$$N \rightarrow N' = Ne^{i\alpha}, \quad (4.18)$$

corresponding to a $U(1)$ gauge symmetry. More generally, there is a gauge symmetry associated with the right action on N by the little group $G_{k, \tilde{k}} = \{n\}$ of both K and \tilde{K} :

$$N \rightarrow N' = Nn, \quad (4.19)$$

where

$$nKn^\dagger = K \quad \text{and} \quad n^{\dagger-1} \tilde{K}n^{-1} = \tilde{K}. \quad (4.20)$$

As is usual, the little groups are isomorphic for all points on

an orbit and can therefore be used to classify the orbits $\{\mathcal{P}\}$ and $\{\tilde{\mathcal{P}}\}$ in \mathbb{R}^8 .

Among the many possible orbits are those which have fiducial points $(k, 0)$. If we restrict to transformations by the $SL(2, C)$ subgroup of $GL(2, C)$, then provided $k \neq 0$, the familiar orbits for massive particles, massless particles, and tachyons are swept out in the four-dimensional subspace of \mathbb{R}^8 spanned by p , while only a point at the origin results in the \tilde{p} -subspace. For this reason, we shall identify $GL(2, C)$ orbits going through fiducial point $(k, 0)$ with massive particles, massless particles, and tachyons, which result for the different choices for $k \neq 0$.

ia) Massive particle: $k^a = m\delta_0^a$ and $\tilde{k} = 0$, $m \neq 0$. For this case, the invariants satisfy $\mathcal{C}^{(2)} = -m^2 < 0$, $\mathcal{C}^{(4)} = m^4$ and $\text{sign det } \mathcal{P} = \text{sign det } \tilde{\mathcal{P}} = +$. (4.14) reduces to this case when (4.15) holds. Here $K = \tilde{K} = m\mathbb{1}$, having identical little groups equal to $G_{(m, \tilde{0}, 0)} = U(2)$, and so \mathcal{P} and $\tilde{\mathcal{P}}$ both span $GL(2, C)/U(2)$. As in the case of orbits obtained under the action of just the Lorentz group, we can divide this case into two subcases with $m > 0$ and $m < 0$. This is since there is no $GL(2, C)$ transformation (4.6) that connects the two subcases. We examine dynamics in flat space-time in Sec. VII, and recover the usual massive particle system in this case. This is despite the presence of two momentum vectors p^a and \tilde{p}^a .

ib) Massless particle: $k = (\nu, 0, 0, \nu)$ and $\tilde{k} = 0$, $\nu \neq 0$. All of the invariants vanish in this case, $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = \det \mathcal{P} = \det \tilde{\mathcal{P}} = 0$. Now $K = \nu(\mathbb{1} + \sigma_3)$ and $\tilde{K} = \nu(\mathbb{1} - \sigma_3)$, which have a common little group $G_{(\nu, 0, 0, \nu), 0} = U(1) \times E(2)$. It is generated by $\mathbb{1}$, σ_3 , $\sigma_1 + i\sigma_2$, $\sigma_2 - i\sigma_1$. [Note from (4.20) that the little group acts differently on K and \tilde{K} .] Now \mathcal{P} and $\tilde{\mathcal{P}}$ both span $GL(2, C)/(U(1) \times E(2))$. As with i(a), this case can be subdivided into $\nu > 0$ and $\nu < 0$, since there is no $GL(2, C)$ transformation (4.6) that connects the two subcases.

ic) Tachyon: $k^a = \kappa\delta_3^a$ and $\tilde{k} = 0$, $\kappa \neq 0$. The invariants are $\mathcal{C}^{(2)} = \kappa^2 > 0$, $\mathcal{C}^{(4)} = \kappa^4$, and $\text{sign det } \mathcal{P} = \text{sign det } \tilde{\mathcal{P}} = -$. Here $K = -\tilde{K} = \kappa\sigma_3$, and so there is again a common little group $G_{(0, 0, 0, \kappa), 0} = U(1) \times SO(2, 1)$, generated by $\mathbb{1}$, σ_3 , $i\sigma_1$, $i\sigma_2$. Consequently, \mathcal{P} and $\tilde{\mathcal{P}}$ both span $GL(2, C)/(U(1) \times SO(2, 1))$.

Complementary to the previous cases, we can consider orbits having fiducial points $(0, \tilde{k})$. If we again restrict to transformations by the $SL(2, C)$ subgroup of $GL(2, C)$, then provided $\tilde{k} \neq 0$, the familiar orbits for massive particles, massless particles, and tachyons are swept out in the four-dimensional subspace of \mathbb{R}^8 spanned by \tilde{p} , while only a point at the origin results in p subspace. The various cases now result from different choices for \tilde{k} .

ii(a) $k^a = 0$, $\tilde{k}^a = \tilde{m}\delta_0^a$, $\tilde{m} \neq 0$. This is the complement of i(a). Now the invariants are $\mathcal{C}^{(2)} = \tilde{m}^2 > 0$, $\mathcal{C}^{(4)} = \tilde{m}^4$, $\text{sign det } \mathcal{P} = \text{sign det } \tilde{\mathcal{P}} = +$. Here the signs of $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(4)}$ are the same as i(c), but this case has opposite signs for $\det \mathcal{P}$ and $\det \tilde{\mathcal{P}}$. Therefore, i(c) and ii(a) define distinct

orbits. Now $K = -\bar{K} = \tilde{m}\mathbb{1}$ and, as in case i(a), both have the little group $G_{0,(\tilde{m},\tilde{0})} = U(2)$, and so \mathcal{P} and $\bar{\mathcal{P}}$ both span $GL(2, C)/U(2)$. $\tilde{m} > 0$ and $\tilde{m} < 0$ correspond to disconnected orbits. In Sec. VII we show that this case is unphysical.

ii(b) $k^a = 0$, $\tilde{k}^a = (\tilde{\nu}, 0, 0, \tilde{\nu})$, $\tilde{\nu} \neq 0$. This is the complement of i(b). $\tilde{\nu} > 0$ and $\tilde{\nu} < 0$ correspond to disconnected orbits. As with i(b), all invariants vanish: $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = \det\mathcal{P} = \det\bar{\mathcal{P}} = 0$. Furthermore, $K = \tilde{\nu}(\mathbb{1} + \sigma_3)$ and $\bar{K} = -\tilde{\nu}(\mathbb{1} - \sigma_3)$ have a common little group which is the same as the little group for i(b), i.e., $G_{0,(\tilde{\nu},0,0,\tilde{\nu})} = U(1) \times E(2)$. Despite having the same invariants and little group, i(b) and ii(b) define distinct orbits. This is because there is no $GL(2, C)$ transformation (4.12) from $K = \tilde{\nu}(\mathbb{1} + \sigma_3)$, $\bar{K} = -\tilde{\nu}(\mathbb{1} - \sigma_3)$ to $K = \tilde{\nu}(\mathbb{1} + \sigma_3)$, $\bar{K} = \tilde{\nu}(\mathbb{1} - \sigma_3)$, as such a transformation would have to be in the little group of K , but not in the little group of \bar{K} . Therefore, the invariants and little groups are not sufficient to distinguish all possible orbits.

ii(c) $k^a = 0$, $\tilde{k}^a = \tilde{\kappa}\delta_3^a$, $\tilde{\kappa} \neq 0$. The invariants are $\mathcal{C}^{(2)} = -\tilde{\kappa}^2 < 0$, $\mathcal{C}^{(4)} = \tilde{\kappa}^4$, and $\text{sign det}\mathcal{P} = \text{sign det}\bar{\mathcal{P}} = -$, so here the signs of $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(4)}$ are the same as for the massive particle orbits i(a), but with opposite signs for $\det\mathcal{P}$ and $\det\bar{\mathcal{P}}$. Now $K = \bar{K} = \tilde{\kappa}\sigma_3$, and the resulting little groups are $G_{0,(0,0,0,\tilde{\kappa})} = U(1) \times SO(2, 1)$, as was true for i(c). So as in that case, \mathcal{P} and $\bar{\mathcal{P}}$ both span $GL(2, C)/(U(1) \times SO(2, 1))$.

In all of the previous cases, K and \bar{K} had identical little groups and \mathcal{P} and $\bar{\mathcal{P}}$ spanned identical orbits. More gen-

erally, one can consider cases where K and \bar{K} each have different little groups and therefore \mathcal{P} and $\bar{\mathcal{P}}$ span different orbits. Two special examples correspond to either \mathcal{P} or $\bar{\mathcal{P}}$ vanishing, implying a trivial orbit for either \mathcal{P} or $\bar{\mathcal{P}}$. These orbits have $k^a = \pm\tilde{k}^a$ and $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = 0$.

iii) $k^a = \tilde{k}^a$. This implies $K = 2(k^0\mathbb{1} + k^i\sigma_i)$ and $\bar{K} = 0$, and hence $\det\mathcal{P} = -4p_a p^a$ and $\det\bar{\mathcal{P}} = 0$. Three separate subcases can then be considered: iii(a) $\det\mathcal{P} > 0$, iii(b) $\det\mathcal{P} = 0$, and iii(c) $\det\mathcal{P} < 0$. They have little groups $U(2)$, $U(1) \times E(2)$, and $U(1) \times SO(2, 1)$, respectively.

iv) $k^a = -\tilde{k}^a$. This implies $K = 0$ and $\bar{K} = 2(k^0\mathbb{1} - k^i\sigma_i)$, and hence $\det\mathcal{P} = 0$, along with $\det\bar{\mathcal{P}} = -4p_a p^a$. Again three separate subcases can then be considered: iv (a) $\det\bar{\mathcal{P}} > 0$, iv(b) $\det\bar{\mathcal{P}} = 0$, and iv(c) $\det\bar{\mathcal{P}} < 0$, having little groups $U(2)$, $U(1) \times E(2)$, and $U(1) \times SO(2, 1)$, respectively.

The invariants and little groups for iii(b) and iv(b) agree with cases i(b) and ii(b). However, all cases correspond to distinct orbits for \mathcal{P} and $\bar{\mathcal{P}}$, as no $GL(2, C)$ transformations (4.6) connect the different assignments for K and \bar{K} . In Sec. VII, we shall show that all four cases of orbits lead to the same dynamics in flat space-time, namely, that of a massless particle. Actually, all iii and iv orbits are associated with massless particles. More surprisingly, ii(c) also describes a massless particle, although it contains additional degrees of freedom.

We summarize the results for the various orbits in the table below.

Invariants and little groups for various orbits

	$\mathcal{C}^{(2)}$	$\mathcal{C}^{(4)}$	$\det\mathcal{P}$	$\det\bar{\mathcal{P}}$	$G_{k,\bar{k}}$
i(a)	$-m^2$	m^4	+	+	$U(2)$
i(b)	0	0	0	0	$U(1) \otimes E(2)$
i(c)	κ^2	κ^4	-	-	$U(1) \otimes SO(2, 1)$
ii(a)	\tilde{m}^2	\tilde{m}^4	+	+	$U(2)$
ii(b)	0	0	0	0	$U(1) \otimes E(2)$
ii(c)	$-\tilde{\kappa}^2$	$\tilde{\kappa}^4$	-	-	$U(1) \otimes SO(2, 1)$
iii(a)	0	0	+	0	$U(2)$
iii(b)	0	0	0	0	$U(1) \otimes E(2)$
iii(c)	0	0	-	0	$U(1) \otimes SO(2, 1)$
iv(a)	0	0	0	+	$U(2)$
iv(b)	0	0	0	0	$U(1) \otimes E(2)$
iv(c)	0	0	0	-	$U(1) \otimes SO(2, 1)$

V. PARTICLE DYNAMICS

Here we write down a $GL(2, C)$ invariant particle action which applies for all nontrivial orbits. The approach is along the lines of [19] which yielded general $SL(2, C)$ invariant particle actions. We also obtain the equations of motions and a general class of solutions.

A. $GL(2, C)$ invariant Lagrangians

The Lagrangian can be constructed from real invariant bilinears for $GL(2, C)$. There are two such bilinears, each of which have norm (4.7). To see this we introduce another set of 2×2 Hermitian matrices, denoted by \mathcal{V} and $\bar{\mathcal{V}}$, which are defined to transform, respectively, as \mathcal{P} and $\bar{\mathcal{P}}$ in (4.6). Then both $\text{tr}\mathcal{P}\bar{\mathcal{V}}$ and $\text{tr}\bar{\mathcal{V}}\mathcal{P}$ are invariant. Upon

writing

$$\begin{aligned}\mathcal{V} &= (v^0 + \tilde{v}^0)\mathbb{1} + (v^i + \tilde{v}^i)\sigma_i, \\ \tilde{\mathcal{V}} &= (v^0 - \tilde{v}^0)\mathbb{1} - (v^i - \tilde{v}^i)\sigma_i,\end{aligned}\quad (5.1)$$

along with (4.5), then two independent invariant bilinears can be expressed as

$$\text{tr}(\mathcal{P}\tilde{\mathcal{V}} + \mathcal{V}\tilde{\mathcal{P}}) = -4(p_a v^a - \tilde{p}_a \tilde{v}^a), \quad (5.2)$$

$$\text{tr}(\mathcal{P}\tilde{\mathcal{V}} - \mathcal{V}\tilde{\mathcal{P}}) = -4(\tilde{p}_a v^a - p_a \tilde{v}^a). \quad (5.3)$$

They are proportional to (4.7) when $\mathcal{P} = \mathcal{V}$ and $\tilde{\mathcal{P}} = \tilde{\mathcal{V}}$.

We shall use the definitions of \mathcal{P} and $\tilde{\mathcal{P}}$ as given in (4.16) in writing down the general particle Lagrangian. v^a and \tilde{v}^a will denote the ‘‘velocities’’ defined in (4.12). The matrices in (5.1) can then be expressed as

$$\mathcal{V} = \mathcal{E}_\mu(z)\dot{z}^\mu, \quad \tilde{\mathcal{V}} = \tilde{\mathcal{E}}_\mu(z)\dot{z}^\mu, \quad (5.4)$$

where $\mathcal{E}_\mu(x)$ and $\tilde{\mathcal{E}}_\mu(x)$ are the space-time dependent 2×2 Hermitian matrices

$$\begin{aligned}\mathcal{E}_\mu &= (e_\mu^0 + f_\mu^0)\mathbb{1} + (e_\mu^i + f_\mu^i)\sigma_i, \\ \tilde{\mathcal{E}}_\mu &= (e_\mu^0 - f_\mu^0)\mathbb{1} - (e_\mu^i - f_\mu^i)\sigma_i.\end{aligned}\quad (5.5)$$

The particle Lagrangian \mathcal{L}_K can be written down using the invariants (5.2) and (5.3). The particle degrees of freedom in this case are $z^\mu(\tau)$, $N(\tau)$, and $N^\dagger(\tau)$. A general expression for the Lagrangian is $\rho \text{tr} \mathcal{P}\tilde{\mathcal{V}} + \bar{\rho} \text{tr} \mathcal{V}\tilde{\mathcal{P}}$, where ρ and $\bar{\rho}$ are constants. These constants can be absorbed into the definitions of K and \bar{K} , respectively, and so without any loss generality we can define

$$\mathcal{L}_K = -\frac{1}{4}\text{tr}(NKN^\dagger\tilde{\mathcal{E}}_\mu(z) + N^{\dagger-1}\bar{K}N^{-1}\mathcal{E}_\mu(z))\dot{z}^\mu. \quad (5.6)$$

In the case of the orbits i(a) for a massive particle and i(b) for a massless particle, the Lagrangian (5.6) reduces to

$$\mathcal{L}_K^{\text{ia}} = -\frac{m}{4}\text{tr}(NN^\dagger\tilde{\mathcal{E}}_\mu(z) + (NN^\dagger)^{-1}\mathcal{E}_\mu(z))\dot{z}^\mu, \quad (5.7)$$

$$\begin{aligned}\mathcal{L}_K^{\text{ib}} &= -\frac{\nu}{4}\text{tr}(N(\mathbb{1} + \sigma_3)N^\dagger\tilde{\mathcal{E}}_\mu(z) \\ &\quad + N^{\dagger-1}(\mathbb{1} - \sigma_3)N^{-1}\mathcal{E}_\mu(z))\dot{z}^\mu,\end{aligned}\quad (5.8)$$

respectively. The corresponding particle action $\mathcal{S}_K = \int d\tau \mathcal{L}_K$ is invariant under reparametrizations $\tau \rightarrow \tau' = f(\tau)$ and transformations under the action of the little group (4.19). The $GL(2, C)$ gauge symmetry appears upon treating the fields dynamically, with the associated gauge transformations:

$$\begin{aligned}N(\tau) &\rightarrow N'(\tau) = MN(\tau), \\ \mathcal{E}_\mu(z) &\rightarrow \mathcal{E}'_\mu(z) = M\mathcal{E}_\mu(z)M^\dagger, \\ \tilde{\mathcal{E}}_\mu(z) &\rightarrow \tilde{\mathcal{E}}'_\mu(z) = M^{\dagger-1}\tilde{\mathcal{E}}_\mu(z)M^{-1},\end{aligned}\quad (5.9)$$

where $M = M[z(\tau)]$ is a $GL(2, C)$ matrix. The action

is then also invariant under general coordinate transformations.

B. Equations of motion

We next obtain the Euler-Lagrange equations which follow from variations of N , N^\dagger , and z^μ in the Lagrangian (5.6). General variations of N lead to

$$\mathcal{P}\tilde{\mathcal{V}} = \mathcal{V}\tilde{\mathcal{P}}, \quad (5.10)$$

while variations of N^\dagger lead to its Hermitian conjugate. Upon expanding these equations of motion in ‘‘velocity’’ and ‘‘momentum’’ components one gets

$$v^{[a}p^{b]} - \tilde{v}^{[a}\tilde{p}^{b]} = 0, \quad (5.11)$$

$$v^a\tilde{p}_a - \tilde{v}^a p_a = 0. \quad (5.12)$$

In general, these equations, along with (4.7) and (4.8), may not uniquely determine p^a and \tilde{p}^a in terms of v^a and \tilde{v}^a . The Euler-Lagrange equations that follow from variations of z^μ in (5.6) are

$$\text{tr}\left(\frac{d\mathcal{P}}{d\tau}\tilde{\mathcal{E}}_\mu + \frac{d\tilde{\mathcal{P}}}{d\tau}\mathcal{E}_\mu\right) = \text{tr}(\tilde{\mathcal{P}}\partial_{[\mu}\mathcal{E}_{\nu]} + \mathcal{P}\partial_{[\mu}\tilde{\mathcal{E}}_{\nu]})\dot{z}^\nu. \quad (5.13)$$

These equations can be reexpressed in a covariant form upon introducing the covariant derivatives

$$\begin{aligned}D_\tau\mathcal{P} &= \frac{d\mathcal{P}}{d\tau} + i(\mathcal{A}_\mu\mathcal{P} - \mathcal{P}\mathcal{A}_\mu^\dagger)\dot{z}^\mu, \\ D_\tau\tilde{\mathcal{P}} &= \frac{d\tilde{\mathcal{P}}}{d\tau} + i(\mathcal{A}_\mu^\dagger\tilde{\mathcal{P}} - \tilde{\mathcal{P}}\mathcal{A}_\mu)\dot{z}^\mu,\end{aligned}\quad (5.14)$$

where \mathcal{A}_μ is the $GL(2, C)$ connection, now expressed in the defining representation. It gauge transforms as

$$\mathcal{A}_\mu \rightarrow \mathcal{A}'_\mu = M\mathcal{A}_\mu M^{-1} + i\partial_\mu M M^{-1}, \quad (5.15)$$

where $M = M(x)$ is a $GL(2, C)$ matrix. The infinitesimal version of (5.15) was given in (3.7). In terms of component gauge potentials ω_μ^{ab} , b_μ , and a_μ , \mathcal{A}_μ is given by

$$\mathcal{A}_\mu = \frac{1}{4}(\epsilon_{ijk}\omega_\mu^{ij} + 2i\omega_\mu^{0k})\sigma_k + \frac{1}{2}(a_\mu + ib_\mu)\mathbb{1}. \quad (5.16)$$

Then (5.13) can be rewritten as

$$\text{tr}(D_\tau\mathcal{P}\tilde{\mathcal{E}}_\mu + D_\tau\tilde{\mathcal{P}}\mathcal{E}_\mu) = \text{tr}(\tilde{\mathcal{P}}\mathcal{T}_{\mu\nu} + \mathcal{P}\tilde{\mathcal{T}}_{\mu\nu})\dot{z}^\nu, \quad (5.17)$$

where we used equations of motion (5.10). $\mathcal{T}_{\mu\nu}$ and $\tilde{\mathcal{T}}_{\mu\nu}$ denote the $GL(2, C)$ generalization of the torsion, here written as 2×2 Hermitian matrices:

$$\begin{aligned}\mathcal{T}_{\mu\nu} &= \partial_{[\mu}\mathcal{E}_{\nu]} + i\mathcal{E}_{[\mu}\mathcal{A}_{\nu]}^\dagger - i\mathcal{A}_{[\nu}\mathcal{E}_{\mu]}, \\ \tilde{\mathcal{T}}_{\mu\nu} &= \partial_{[\mu}\tilde{\mathcal{E}}_{\nu]} + i\tilde{\mathcal{E}}_{[\mu}\mathcal{A}_{\nu]} - i\mathcal{A}_{[\nu}\tilde{\mathcal{E}}_{\mu]}.\end{aligned}\quad (5.18)$$

They transform as $\mathcal{E}_\mu(x)$ and $\tilde{\mathcal{E}}_\mu(x)$, respectively, and hence the left and right-hand sides of (5.17) are invariant

under $GL(2, C)$ gauge transformations. $\mathcal{T}_{\mu\nu}$ and $\bar{\mathcal{T}}_{\mu\nu}$ can also be expressed in terms of the component torsion fields $t_{\mu\nu}^a$ and $u_{\mu\nu}^a$ which were defined in (3.22),

$$\begin{aligned}\mathcal{T}_{\mu\nu} &= (t_{\mu\nu}^0 + u_{\mu\nu}^0)\mathbb{1} + (t_{\mu\nu}^i + u_{\mu\nu}^i)\sigma_i, \\ \bar{\mathcal{T}}_{\mu\nu} &= (t_{\mu\nu}^0 - u_{\mu\nu}^0)\mathbb{1} - (t_{\mu\nu}^i - u_{\mu\nu}^i)\sigma_i.\end{aligned}\quad (5.19)$$

The right-hand side of (5.17) vanishes for the case of zero torsion. In order for the covariant derivatives of \mathcal{P} and $\bar{\mathcal{P}}$ to then vanish we would further need the vierbein matrices e_{μ}^a and f_{μ}^a to be nonsingular and $e_{\mu}^a f_b^{\mu} = f_{\mu}^a e_b^{\mu} = 0$, where e_b^{μ} and f_b^{μ} are the inverses of e_{μ}^a and f_{μ}^a , respectively.

C. Solutions

Here we first obtain a general class of solutions to equations of motion (5.10) which are valid when \mathcal{V} and $\bar{\mathcal{V}}$ are nonsingular matrices. We can apply the results to the various orbits discussed in Sec. IV. For the choice i(a) associated with massive particles, we obtain an effective Lagrangian containing corrections to the naive Lagrangian (3.24), and thus yielding corrections to geodesic motion. We also find deviations from null curves for orbits i(b) associated with massless particles. Finally, we examine the case of singular matrices \mathcal{V} and $\bar{\mathcal{V}}$. We are unable to find any physically meaningful solutions in that case.

I. $\det \mathcal{V} \neq 0, \det \bar{\mathcal{V}} \neq 0$

We first note that $\bar{\mathcal{V}}^{-1}$ and \mathcal{V}^{-1} transform under the action of $GL(2, C)$ as \mathcal{P} and $\bar{\mathcal{P}}$, respectively. So here since both \mathcal{V} and $\bar{\mathcal{V}}$ are nonsingular matrices, we may write down the following solutions to (5.10):

$$\mathcal{P} = \varsigma \mathcal{V} + \varpi \bar{\mathcal{V}}^{-1}, \quad \bar{\mathcal{P}} = \varsigma \bar{\mathcal{V}} + \varpi \mathcal{V}^{-1}, \quad (5.20)$$

where ς and ϖ are real and invariant under $GL(2, C)$ transformations. For the special case where $\varpi = 0$ these solutions say that the ‘‘momenta’’ p_a and \tilde{p}_a are proportional to the ‘‘velocities’’ v_a and \tilde{v}_a , as in (4.11). More generally, ς and ϖ are constrained by (4.7) and (4.8). Substituting (5.20) into these constraints gives rather involved conditions for ς and ϖ

$$\begin{aligned}-2\mathcal{C}^{(2)} &= \left(\varsigma^2 + \frac{\varpi^2}{\det(\mathcal{V}\bar{\mathcal{V}})} \right) \text{tr}(\mathcal{V}\bar{\mathcal{V}}) + 4\varsigma\varpi, \\ \mathcal{C}^{(4)} &= \varsigma^4 \det(\mathcal{V}\bar{\mathcal{V}}) + \frac{\varpi^4}{\det(\mathcal{V}\bar{\mathcal{V}})} \\ &\quad + \varsigma^2 \varpi^2 \left(\frac{(\text{tr} \mathcal{V}\bar{\mathcal{V}})^2}{\det(\mathcal{V}\bar{\mathcal{V}})} - 6 \right) - 4\varsigma\varpi\mathcal{C}^{(2)},\end{aligned}\quad (5.21)$$

where we used the identity $2 \det(\mathcal{V}\bar{\mathcal{V}}) = (\text{tr} \mathcal{V}\bar{\mathcal{V}})^2 -$

$\text{tr}(\mathcal{V}\bar{\mathcal{V}})^2$. Solutions for ς and ϖ can then in principle be expressed as functions of the invariants $\mathcal{C}^{(2)}, \mathcal{C}^{(4)}$,

$$\text{tr}(\mathcal{V}\bar{\mathcal{V}}) = -2g_{\mu\nu}(z)\dot{z}^{\mu}\dot{z}^{\nu} \quad (5.22)$$

and

$$\det(\mathcal{V}\bar{\mathcal{V}}) = (2g_{\mu\nu}(z)g_{\rho\sigma}(z) - \frac{1}{4}h_{\mu\nu\rho\sigma}(z))\dot{z}^{\mu}\dot{z}^{\nu}\dot{z}^{\rho}\dot{z}^{\sigma}. \quad (5.23)$$

The solutions for ς and ϖ are highly nontrivial for arbitrary values of $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(4)}$. They simplify considerably upon specifying particular orbits. As an example, we now consider the orbits i(a) associated with massive particles. The calculations depend on the values $\mathcal{C}^{(2)} = -m^2$ and $\mathcal{C}^{(4)} = m^4$, but not on $\text{sign} \det \mathcal{P}$ and $\text{sign} \det \bar{\mathcal{P}}$. Therefore the results also apply for the orbits ii(c), which in Sec. VII A will be shown to correspond to *massless* particles. There are two real solutions for ς and ϖ in this case:

$$\begin{aligned}\varsigma &= \pm \frac{m}{\sqrt{\text{tr}(\mathcal{V}\bar{\mathcal{V}}) + 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}}, \\ \varpi &= \pm \frac{m(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}{\sqrt{\text{tr}(\mathcal{V}\bar{\mathcal{V}}) + 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}},\end{aligned}\quad (5.24)$$

which is valid for $\det(\mathcal{V}\bar{\mathcal{V}}) \geq 0, \text{tr}(\mathcal{V}\bar{\mathcal{V}}) + 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2} > 0$, and

$$\begin{aligned}\varsigma &= \pm \frac{m}{\sqrt{\text{tr}(\mathcal{V}\bar{\mathcal{V}}) - 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}}, \\ \varpi &= \mp \frac{m(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}{\sqrt{\text{tr}(\mathcal{V}\bar{\mathcal{V}}) - 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}},\end{aligned}\quad (5.25)$$

which is valid for $\det(\mathcal{V}\bar{\mathcal{V}}) \geq 0, \text{tr}(\mathcal{V}\bar{\mathcal{V}}) - 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2} > 0$. We can eliminate one of the solutions by demanding that they are well defined in the limit $f_{\mu}^a \rightarrow 0$, since we recover the standard gravity theory in this limit. This limit coincides with the condition (4.15), and $\text{tr}(\mathcal{V}\bar{\mathcal{V}}) + 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}$ vanishes as a result. The solutions (5.24) are singular in this limit and we reject them for that reason. From (5.20), we thus have

$$\begin{aligned}\mathcal{P} &= \pm m \frac{\mathcal{V} - (\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2} \bar{\mathcal{V}}^{-1}}{\sqrt{\text{tr}(\mathcal{V}\bar{\mathcal{V}}) - 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}}, \\ \bar{\mathcal{P}} &= \pm m \frac{\bar{\mathcal{V}} - (\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2} \mathcal{V}^{-1}}{\sqrt{\text{tr}(\mathcal{V}\bar{\mathcal{V}}) - 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}}}.\end{aligned}\quad (5.26)$$

Substituting this solution back into the Lagrangian (5.7) gives

$$\mathcal{L}_K^{ia} \propto \sqrt{\text{tr}(\mathcal{V}\bar{\mathcal{V}}) - 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2}} \propto \sqrt{-g_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu - \sqrt{(2g_{\mu\nu}(z)g_{\rho\sigma}(z) - \frac{1}{4}h_{\mu\nu\rho\sigma}(z))\dot{z}^\mu\dot{z}^\nu\dot{z}^\rho\dot{z}^\sigma}}. \quad (5.27)$$

It reduces to the naive Lagrangian (3.24) when $f_\mu^a \rightarrow 0$, as then the condition (4.15) applies. Geodesic motion is then recovered in this limit. More generally, however, (5.27) will give corrections to geodesic motion.

There are no solutions to (5.21) when $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = 0$ for arbitrary $\text{tr}(\mathcal{V}\bar{\mathcal{V}})$ and $\det(\mathcal{V}\bar{\mathcal{V}})$. This case corresponds to the orbits i(b) for massless particles, as well as for orbits ii(b). On the other hand, there can be consistent solutions when

$$\text{tr}(\mathcal{V}\bar{\mathcal{V}}) \pm 2(\det(\mathcal{V}\bar{\mathcal{V}}))^{1/2} = 0. \quad (5.28)$$

This condition coincides with null curves $g_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu \rightarrow 0$ in the limit that $f_\mu^a \rightarrow 0$, but yields corrections to null curves when $f_\mu^a \neq 0$. ς and ϖ are not completely determined in this case, but are instead constrained by $\varsigma + 2\varpi/\text{tr}(\mathcal{V}\bar{\mathcal{V}}) = 0$, and so

$$\begin{aligned} \mathcal{P} &= \varsigma(\mathcal{V} - \frac{1}{2}\text{tr}(\mathcal{V}\bar{\mathcal{V}})\bar{\mathcal{V}}^{-1}), \\ \bar{\mathcal{P}} &= \varsigma(\bar{\mathcal{V}} - \frac{1}{2}\text{tr}(\mathcal{V}\bar{\mathcal{V}})\mathcal{V}^{-1}). \end{aligned} \quad (5.29)$$

Substituting the solution back into the Lagrangian (5.7) this time gives zero. The solutions (5.29) do not apply for orbits iii and v where either \mathcal{P} or $\bar{\mathcal{P}}$ vanish. For orbits iii we would need that $\bar{\mathcal{V}} = \frac{1}{2}\text{tr}(\mathcal{V}\bar{\mathcal{V}})\mathcal{V}^{-1}$, while for iv we need $\mathcal{V} = \frac{1}{2}\text{tr}(\mathcal{V}\bar{\mathcal{V}})\bar{\mathcal{V}}^{-1}$. However, these are equivalent conditions when both \mathcal{V} and $\bar{\mathcal{V}}$ are nonsingular and hence (5.29) reduce to trivial solutions.

There are alternative solutions which are only valid for orbits iii and iv. We can set $\mathcal{P} = \varsigma\mathcal{V} + \varpi\bar{\mathcal{V}}^{-1}$ and $\bar{\mathcal{P}} = 0$. Solutions of (5.10) then require that $\varsigma\mathcal{V}\bar{\mathcal{V}} + \varpi\mathbb{1} = 0$. We get the same condition upon demanding that $\mathcal{P} = 0$ and $\bar{\mathcal{P}} = \varsigma\bar{\mathcal{V}} + \varpi\mathcal{V}^{-1}$ satisfies (5.10). Thus,

$$\mathcal{P} = \varsigma(\mathcal{V} - \frac{1}{2}\text{tr}(\mathcal{V}\bar{\mathcal{V}})\bar{\mathcal{V}}^{-1}), \quad \bar{\mathcal{P}} = 0 \quad (5.30)$$

and

$$\mathcal{P} = 0, \quad \bar{\mathcal{P}} = \varsigma(\bar{\mathcal{V}} - \frac{1}{2}\text{tr}(\mathcal{V}\bar{\mathcal{V}})\mathcal{V}^{-1}) \quad (5.31)$$

are solutions provided that

$$\mathcal{V}\bar{\mathcal{V}} = \frac{1}{2}\text{tr}(\mathcal{V}\bar{\mathcal{V}})\mathbb{1}. \quad (5.32)$$

The former applies for orbits iii and the latter for orbits iv.

2. $\det \mathcal{V} = 0$ or $\det \bar{\mathcal{V}} = 0$

This case only applies when $\mathcal{C}^{(4)} = 0$. Here we can in principle allow for $\mathcal{C}^{(2)} \neq 0$, although we did not consider these kinds of orbits in Sec. IV B. With the exception of (5.30) and (5.31), the solutions obtained above are invalid

for singular \mathcal{V} or $\bar{\mathcal{V}}$.² The solutions (5.26) are ill defined in the limit where either $\det \mathcal{V}$ or $\det \bar{\mathcal{V}}$ (or both) vanish. It follows that (5.23) must also vanish in this limit. Notice that this condition is different from (4.15), and unlike (4.15) it is not satisfied when $f_\mu^a \rightarrow 0$.

If only \mathcal{V} is singular we can write

$$\mathcal{P} = \varsigma\mathcal{V} + \varpi\bar{\mathcal{V}}^{-1}, \quad \bar{\mathcal{P}} = \bar{\varsigma}\bar{\mathcal{V}}, \quad (5.33)$$

while if only $\bar{\mathcal{V}}$ is singular we can have

$$\mathcal{P} = \varsigma\mathcal{V}, \quad \bar{\mathcal{P}} = \bar{\varsigma}\bar{\mathcal{V}} - \varpi\mathcal{V}^{-1}. \quad (5.34)$$

They are solutions to the equation of motion (5.10) provided $(\bar{\varsigma} - \varsigma)\mathcal{V}\bar{\mathcal{V}} = \varpi\mathbb{1}$. ς , $\bar{\varsigma}$ and ϖ are related by the quadratic invariant. One gets $\mathcal{C}^{(2)} = \bar{\varsigma}^2\varpi/(\varsigma - \bar{\varsigma})$ and $\mathcal{C}^{(2)} = \varsigma^2\varpi/(\varsigma - \bar{\varsigma})$, respectively, for the two cases.

If, on the other hand, both \mathcal{V} and $\bar{\mathcal{V}}$ are singular, then the ‘‘momenta’’ \mathcal{P} and $\bar{\mathcal{P}}$ are proportional to the ‘‘velocities’’ \mathcal{V} and $\bar{\mathcal{V}}$,

$$\mathcal{P} = \varsigma\mathcal{V}, \quad \bar{\mathcal{P}} = \bar{\varsigma}\bar{\mathcal{V}}, \quad (5.35)$$

where here we need either $\mathcal{V}\bar{\mathcal{V}} = 0$ or $\varsigma = \bar{\varsigma}$. In the former case $\mathcal{C}^{(2)} = 0$, and in the latter $\mathcal{C}^{(2)} = -\frac{1}{2}\varsigma^2\text{tr}(\mathcal{V}\bar{\mathcal{V}})$.

Other special solutions to the equations of motion (5.23) are

$$\tilde{p}^a = \pm p^a, \quad \tilde{v}^a = \pm v^a. \quad (5.36)$$

They correspond to either $\bar{\mathcal{P}} = \bar{\mathcal{V}} = 0$ or $\mathcal{P} = \mathcal{V} = 0$, and thus $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = 0$. These solutions are relevant for the orbits iii and iv. However, (5.36) implies that $(e_\mu^a(z) \mp f_\mu^a(z))\dot{z}^\mu = 0$, which then also means $g_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu = 0$. It follows that $g_{\mu\nu}$ is a singular metric tensor and hence these solutions can only occur in a singular space-time.

VI. WESS-ZUMINO TERM

The dynamics discussed in the previous section only applies for particles with zero spin and zero charge. It is known that the spin can be included with the addition of an $SL(2, C)$ invariant Wess-Zumino term [19]. It is first order in time derivatives of an $SL(2, C)$ -valued matrix. The term can easily be generalized to a $GL(2, C)$ invariant \mathcal{L}_{WZ} , which is first order in time derivatives of the $GL(2, C)$ -valued matrices N and N^\dagger . It is

$$\mathcal{L}_{WZ} = -\frac{1}{4}\text{tr}(WN^{-1}D_\tau N + W^\dagger(D_\tau N)^\dagger N^{\dagger-1}), \quad (6.1)$$

²(5.30) is still valid when \mathcal{V} is singular and (5.31) is valid when $\bar{\mathcal{V}}$ is singular.

where W is a constant 2×2 complex matrix. The covariant derivative in (6.1) is given by

$$D_\tau N = \frac{dN}{d\tau} + i\mathcal{A}_\mu(z)N\dot{z}^\mu, \quad (6.2)$$

where \mathcal{A}_μ is again the $GL(2, C)$ connection (5.16). W contains the six spin degrees of freedom of the particle in a fixed frame, along with two $U(1)$ charges. One can apply a similarity transformation on W to go to an arbitrary reference frame:

$$\Sigma = NWN^{-1} = (-i\epsilon_{ijk}s^{ij} + 2s^{0k})\sigma_k + 2(\tilde{q} + iq)\mathbb{1}, \quad (6.3)$$

where $s^{ab} = -s^{ba}$ are the spin variables and q and \tilde{q} are the two $U(1)$ charges. The former are dynamical quantities dependent on the traceless parts of N , while q and \tilde{q} are constants, since $\text{Tr } \Sigma = \text{Tr } W = 4(\tilde{q} + iq)$. The spin variables are unaffected by the action of the $U(1)$ subgroups of $GL(2, C)$. So a general $GL(2, C)$ variation of s^{ab} is just a $SL(2, C)$ variation

$$\delta_\Lambda s^{ab} = s^{ac}\lambda_c^b - s^{bc}\lambda_c^a. \quad (6.4)$$

Two Pauli-Lubanski-type vectors can be constructed for this theory

$$w_a = \frac{1}{2}\epsilon_{abcd}p^b s^{cd}, \quad \tilde{w}_a = \frac{1}{2}\epsilon_{abcd}\tilde{p}^b s^{cd}, \quad (6.5)$$

which transform under $GL(2, C)$ transformations as p^a and \tilde{p}^a , respectively, in (4.4). It follows that two additional invariants can then be constructed from w_a and \tilde{w}_a which are analogous to (4.7) and (4.8):

$$\begin{aligned} \mathcal{C}_w^{(2)} &= w_a w^a - \tilde{w}_a \tilde{w}^a, \\ \mathcal{C}_w^{(4)} &= (w_a w^a + \tilde{w}_a \tilde{w}^a)^2 - 4(w_a \tilde{w}^a)^2. \end{aligned} \quad (6.6)$$

The former generalizes the usual invariant for a relativistic spinning particle, while the second one is new. An additional invariant can be constructed from w_a , \tilde{w}_a , p^a , and \tilde{p}^a ,

$$\mathcal{C}_{p,w}^{(2)} = \tilde{p}_a w^a - p_a \tilde{w}^a = \epsilon_{abcd}\tilde{p}^a p^b s^{cd}. \quad (6.7)$$

Notice that the invariant $p_a w^a - \tilde{p}_a \tilde{w}^a$ is identically zero. (6.6) and (6.7), along with (4.7) and (4.8), can be used to classify spinning particles in this theory. More nontrivial invariants using other combinations of w_a , \tilde{w}_a , p^a , and \tilde{p}^a are may also be possible.

The full Lagrangian for spinning particles is obtained by adding (6.1) to the Lagrangian \mathcal{L}_K . The corresponding action

$$\mathcal{S} = \int d\tau \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_K + \mathcal{L}_{WZ} \quad (6.8)$$

is gauge invariant and reparametrization invariant. The gauge symmetries include transformations by the little group (4.19), where now elements $\{n\}$ of the little group have to satisfy

$$nWn^{-1} = W, \quad (6.9)$$

which leaves (6.3) invariant, in addition to the conditions (4.20). If we treat the gauge fields dynamically, then the action (6.8) is invariant under $GL(2, C)$ gauge transformations (5.9) and (5.15). In addition to the $GL(2, C)$ gauge symmetries, the total action is invariant under independent $U(1) \times U(1)$ transformations, where the connections \mathcal{A}_μ transform as

$$\mathcal{A}_\mu \rightarrow \mathcal{A}'_\mu = \mathcal{A}_\mu + \partial_\mu \chi, \quad (6.10)$$

for complex function χ , while N , \mathcal{E}_μ , and $\tilde{\mathcal{E}}_\mu$ are unchanged.³ The Wess-Zumino Lagrangian picks up a τ derivative under such transformations.

The Euler-Lagrange equations for the particle are again obtained from variations of N , N^\dagger , and z^μ . The equations of motion which follow from variations of N in the total action (6.8) are

$$\mathcal{P} \bar{\mathcal{V}} - \mathcal{V} \bar{\mathcal{P}} - D_\tau \Sigma = 0, \quad (6.11)$$

generalizing (5.10), while variations of N^\dagger give its Hermitian conjugate. Here

$$D_\tau \Sigma = \frac{d\Sigma}{d\tau} + i[\mathcal{A}_\mu(z), \Sigma]\dot{z}^\mu. \quad (6.12)$$

Upon expanding in terms of components, (6.11) gives the particle's spin precession

$$v^{[a} p^{b]} - \tilde{v}^{[a} \tilde{p}^{b]} + \dot{s}^{ab} + \omega^a_c s^{cb} - s^a_c \omega^{cb} = 0, \quad (6.13)$$

while (5.12) is unchanged. The Euler-Lagrange equations following from variations of z^μ in the total action (6.8) state that

$$\begin{aligned} &\text{tr} \left(\frac{d\mathcal{P}}{d\tau} \tilde{\mathcal{E}}_\mu + \frac{d\bar{\mathcal{P}}}{d\tau} \mathcal{E}_\mu + i \frac{d\Sigma}{d\tau} \mathcal{A}_\mu - i \frac{d\Sigma^\dagger}{d\tau} \mathcal{A}_\mu^\dagger \right) \\ &= \text{tr} (\bar{\mathcal{P}} \partial_{[\mu} \mathcal{E}_{\nu]} + \mathcal{P} \partial_{[\mu} \tilde{\mathcal{E}}_{\nu]} + i \Sigma \partial_{[\mu} \mathcal{A}_{\nu]} \\ &\quad - i \Sigma^\dagger \partial_{[\mu} \mathcal{A}_{\nu]}^\dagger) \dot{z}^\nu, \end{aligned} \quad (6.14)$$

or equivalently, in the explicitly gauge invariant form

$$\begin{aligned} \text{tr} (D_\tau \mathcal{P} \tilde{\mathcal{E}}_\mu + D_\tau \bar{\mathcal{P}} \mathcal{E}_\mu) &= \text{tr} (\bar{\mathcal{P}} \mathcal{T}_{\mu\nu} + \mathcal{P} \tilde{\mathcal{T}}_{\mu\nu} + i \Sigma \mathcal{F}_{\mu\nu} \\ &\quad - i \Sigma^\dagger \mathcal{F}_{\mu\nu}^\dagger) \dot{z}^\nu, \end{aligned} \quad (6.15)$$

generalizing (5.17). We have used the equations of motion (6.11) in deriving (6.15). $\mathcal{T}_{\mu\nu}$ and $\tilde{\mathcal{T}}_{\mu\nu}$ again correspond to the $GL(2, C)$ torsion tensors (5.18), while $\mathcal{F}_{\mu\nu}$ is the $GL(2, C)$ curvature, here expressed in the defining representation, i.e.,

³Alternatively, we can keep the connections \mathcal{A}_μ fixed, while N , \mathcal{E}_μ , and $\tilde{\mathcal{E}}_\mu$ undergo the transformations

$$N \rightarrow e^\chi N, \quad \mathcal{E}_\mu \rightarrow e^{\chi+\chi^*} \mathcal{E}_\mu, \quad \tilde{\mathcal{E}}_\mu \rightarrow e^{-\chi-\chi^*} \tilde{\mathcal{E}}_\mu.$$

$$\mathcal{F}_{\mu\nu} = \frac{1}{4}(\epsilon_{ijk}R_{\mu\nu}^{ij} + 2iR_{\mu\nu}^{0k})\sigma_k + \frac{1}{2}(\partial_{[\mu}a_{\nu]} + i\partial_{[\mu}b_{\nu]})\mathbb{1}, \quad (6.16)$$

where $R_{\mu\nu}^{ab}$ is the standard Lorentz curvature. (6.15) generalizes the Mathisson-Papapetrou equations [21,22], by including interactions with the two $U(1)$ gauge fields along with spin curvature and torsion.

VII. FLAT SPACE-TIME

Flat space-time corresponds to the choice (3.16) for the vierbein matrix L_μ , or equivalently

$$e_\mu^a = \cosh\chi\delta_\mu^a, \quad f_\mu^a = \sinh\chi\delta_\mu^a, \quad (7.1)$$

for some real constant χ . So in this case \mathcal{V} and $\tilde{\mathcal{V}}$ are given by

$$\begin{aligned} \mathcal{V} &= (\cosh\chi + \sinh\chi)(z^0\mathbb{1} + z^i\sigma_i), \\ \tilde{\mathcal{V}} &= (\cosh\chi - \sinh\chi)(z^0\mathbb{1} - z^i\sigma_i), \end{aligned} \quad (7.2)$$

and then $\mathcal{V}\tilde{\mathcal{V}} = -z^a z_a \mathbb{1}$. It follows that $\text{tr}(\mathcal{V}\tilde{\mathcal{V}}) = -2z^a z_a$, $\det(\mathcal{V}\tilde{\mathcal{V}}) = (z^a z_a)^2$ and furthermore that the condition (5.32) is satisfied.

By going to flat space-time we are breaking the $GL(2, C)$ gauge invariance of the Lagrangian. Although this gauge invariance is broken, a number of symmetries survive. They correspond to global Poincaré transformations, reparametrizations, and the local transformations (4.19). Of course, the discrete symmetries, parity, and time reversal are present as well. If we treat the two $U(1)$ potentials a_μ and b_μ dynamically, then the additional $U(1) \times U(1)$ gauge symmetry (6.10) can also be included.

Below we first consider the case of spinless and chargeless particles and then remark on the inclusion of the Wess-Zumino term.

A. Spinless and chargeless particles

In flat space-time the Lagrangian (5.6) reduces to

$$\mathcal{L}_K = \pi_a \dot{z}^a, \quad \pi_a = \cosh\chi p_a - \sinh\chi \tilde{p}_a. \quad (7.3)$$

π_a are the canonical momenta (4.13), and from the equations of motion (5.13), π_a also serves as the conserved energy-momentum vector. Here the equations of motion (5.11) imply that $z^{[a}\pi^{b]}$ are the conserved angular momenta, and by Noether's theorem these two conservation laws are associated with the Poincaré symmetry. We thus recover the standard dynamics for a free spinless particle. On the other hand, this system, at first glance, contains additional degrees of freedom, as there are two momentum variables p^a and \tilde{p}^a , or equivalently π^a and

$$\tilde{\pi}_a = \cosh\chi \tilde{p}_a - \sinh\chi p_a. \quad (7.4)$$

The dynamics of the latter is constrained by the additional equation of motion

$$\tilde{\pi}_a \dot{z}^a = 0, \quad (7.5)$$

which follows from (5.12). π^a and $\tilde{\pi}^a$ transform under $GL(2, C)$ transformations as p^a and \tilde{p}^a , respectively, in (4.4). The quadratic and quartic invariants (4.7) and (4.8) may be expressed directly in terms of π_a and $\tilde{\pi}_a$,

$$\mathcal{C}^{(2)} = \pi_a \pi^a - \tilde{\pi}_a \tilde{\pi}^a, \quad (7.6)$$

$$\mathcal{C}^{(4)} = (\pi_a \pi^a + \tilde{\pi}_a \tilde{\pi}^a)^2 - 4(\pi_a \tilde{\pi}^a)^2.$$

From the conservation of angular momentum, it follows that $\pi_a \propto \dot{z}_a$, and then from (7.5) that $\mathcal{C}^{(4)} \geq 0$ [assuming $\dot{z} \neq 0$]. Furthermore,

$$\begin{aligned} \pi_a \pi^a &= \frac{1}{2}(\mathcal{C}^{(2)} \pm \sqrt{\mathcal{C}^{(4)}}), \\ \tilde{\pi}_a \tilde{\pi}^a &= \frac{1}{2}(-\mathcal{C}^{(2)} \pm \sqrt{\mathcal{C}^{(4)}}), \\ \pi_a \tilde{\pi}^a &= 0. \end{aligned} \quad (7.7)$$

We next examine these constraints for the various orbits discussed in Sec. IV.

For orbits i(b), ii(b), iii, and iv, we have $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = 0$ and so all scalar products in (7.7) vanish. It follows that $\tilde{\pi}^a = \lambda \pi^a$, and hence that $\tilde{\pi}^a$ are not independent variables. The independent degrees of freedom are those of a massless particle, and this is valid for all four types of orbits. The different orbits are distinguished by their values for λ . These values can be determined from the expressions for p^a and \tilde{p}^a ,

$$\begin{aligned} p_a &= (\cosh\chi + \lambda \sinh\chi)\pi_a, \\ \tilde{p}_a &= (\sinh\chi + \lambda \cosh\chi)\pi_a. \end{aligned} \quad (7.8)$$

The orbit i(b) is recovered for $\lambda = -\tanh\chi$ and ii(b) is recovered for $\lambda = -1/\tanh\chi$. $\lambda = 1$ for iii and $\lambda = -1$ for iv. All of them correspond to the solutions (5.35) with $\mathcal{V}\tilde{\mathcal{V}} = 0$ with different values for ς and $\bar{\varsigma}$ in the four cases.

For orbits ia, ic, iia, and iic where $\mathcal{C}^{(2)}$ or $\mathcal{C}^{(4)}$ differ from zero, the choice of the sign in front of the $\sqrt{\mathcal{C}^{(4)}}$ terms in (7.7) may be determined from the signs of the determinants of \mathcal{P} and $\tilde{\mathcal{P}}$. From (7.7) and (7.8) one gets

$$\begin{aligned} \det\mathcal{P} &= \mp(\cosh\chi + \sinh\chi)^2 \sqrt{\mathcal{C}^{(4)}}, \\ \det\tilde{\mathcal{P}} &= \mp(\cosh\chi - \sinh\chi)^2 \sqrt{\mathcal{C}^{(4)}}. \end{aligned} \quad (7.9)$$

The signs of $\det\mathcal{P}$ and $\det\tilde{\mathcal{P}}$ agree as is the case with all four classes of orbits. For orbits i(a) and ii(a) we must choose the lower sign and for orbits i(c) and ii(c) we must choose the upper sign. We now examine the flat space-time dynamics for the four different orbits.

For the case i(a) of a massive particle where $\mathcal{C}^{(2)} = -m^2$ and $\mathcal{C}^{(4)} = m^4$, the choice of the lower sign in (7.7) leads to the physically reasonable results, i.e., $\pi_a \pi^a = -m^2$, $\pi_a \tilde{\pi}^a = 0$, and $\tilde{\pi}_a \tilde{\pi}^a = 0$. The latter two equations

mean that $\tilde{\pi}$ must vanish. [This is easily seen by going to the particle rest frame $\pi = (m, 0, 0, 0)$.] Hence

$$\pi^a = \pm \frac{m\dot{z}^a}{\sqrt{-\dot{z}^b \dot{z}_b}}, \quad \tilde{\pi}^a = 0, \quad (7.10)$$

and so the independent degrees of freedom are those of a massive particle. The result also follows from the solution (5.20). Upon substituting (7.2) we get

$$\mathcal{P} = \pm(\cosh\chi + \sinh\chi) \frac{m\dot{z}^a}{\sqrt{-\dot{z}^b \dot{z}_b}} (\dot{z}^0 \mathbb{1} + \dot{z}^i \sigma_i), \quad (7.11)$$

$$\bar{\mathcal{P}} = \pm(\cosh\chi - \sinh\chi) \frac{m\dot{z}^a}{\sqrt{-\dot{z}^b \dot{z}_b}} (\dot{z}^0 \mathbb{1} - \dot{z}^i \sigma_i),$$

from which follows

$$p^a = \pm \cosh\chi \frac{m\dot{z}^a}{\sqrt{-\dot{z}^b \dot{z}_b}}, \quad \tilde{p}^a = \pm \sinh\chi \frac{m\dot{z}^a}{\sqrt{-\dot{z}^b \dot{z}_b}}, \quad (7.12)$$

and hence (7.10).

For case ii(a) the invariants are $\mathcal{C}^{(2)} = \tilde{m}^2$ and $\mathcal{C}^{(4)} = \tilde{m}^4$. So now $\pi_a \pi^a = 0$ and $\tilde{\pi}_a \tilde{\pi}^a = -\tilde{m}^2$. The former implies that \dot{z}^a is lightlike or zero, while the latter means that $\tilde{\pi}^a$ is timelike. But along with $\tilde{\pi}_a \pi^a = 0$, this implies that $\pi^a = 0$ and $\dot{z}^a = 0$. [For this one can transform to the frame where $\tilde{\pi} = (\tilde{m}, 0, 0, 0)$.] This therefore appears to be a pathological case.

For case i(c) one has $\mathcal{C}^{(2)} = \kappa^2$ and $\mathcal{C}^{(4)} = \kappa^4$. Upon choosing the upper sign in (7.15), $\pi_a \pi^a = \kappa^2$ and $\tilde{\pi}_a \tilde{\pi}^a = 0$. The result is a tachyon, but here $\tilde{\pi}^a$ need not vanish.

For case ii(c), $\mathcal{C}^{(2)} = -\tilde{\kappa}^2$ and $\mathcal{C}^{(4)} = \tilde{\kappa}^4$. Again choosing the upper sign in (7.7), one gets $\pi_a \pi^a = 0$ and $\tilde{\pi}_a \tilde{\pi}^a = \tilde{\kappa}^2$. The former implies that the particle velocity vector is lightlike or zero, while the latter means that $\tilde{\pi}^a$ is spacelike. Here $\tilde{\pi}^a$ does not vanish, and also π^a need not vanish. The system therefore describes a massless particle. This result is unexpected since the orbits here have the same values for the invariants $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(4)}$ as with the case of the massive particle i(a). Unlike the massless particle orbits i(b), ii(b), iii, and iv, extra degrees of freedom are present for case iv(a), which are associated with the orthogonal spacelike vector $\tilde{\pi}^a$.

B. Inclusion of the Wess-Zumino term

The addition of the Wess-Zumino term (6.1), with $\mathcal{A}_\mu = 0$, to the total Lagrangian does not affect the equations of motion $\dot{\pi}^a = 0$ or (7.5). On the other hand, the addition of the Wess-Zumino term does lead to the inclusion of spin in the conserved angular momentum

$$j^{ab} = z^{[a} \pi^{b]} + s^{ab}. \quad (7.13)$$

Infinitesimal Lorentz variations of j^{ab} are as usual,

$$\delta_\Lambda j^{ab} = j^{ac} \lambda_c^b - j^{bc} \lambda_c^a. \quad (7.14)$$

Thus when $\mathcal{A}_\mu = 0$, the $GL(2, C)$ Wess-Zumino term (6.1) is equivalent to the $SL(2, C)$ Wess-Zumino term, and it only gives dynamics to the spin variables [19]. This is evident because the Wess-Zumino term does not depend on the determinant of the $GL(2, C)$ matrix N . N can be decomposed according to $N = \zeta \hat{N}$, with $\hat{N} \in SL(2, C)$ and the terms in (6.1) (with $\mathcal{A}_\mu = 0$) involving ζ are τ derivatives. If one assumes that the spin and orbital angular momentum are separately conserved then the analysis of the motion for orbits i–iv is identical to what was found in Sec. VII A.

Lastly, if we again consider flat space-time but now drop the restriction that $\mathcal{A}_\mu = 0$, the particle can feel the presence of Lorentz forces. Upon allowing for the two $U(1)$ potentials; i.e., $\mathcal{A}_\mu = \frac{1}{2}(a_\mu + ib_\mu)$, then the Wess-Zumino action will contain the minimal coupling terms (3.26). Although (7.5) still holds, the momenta π_a and angular momenta j^{ab} are not in general conserved. Rather, a Lorentz force equation results from the two gauge fields

$$\dot{\pi}_\mu = (q\partial_{[\mu} a_{\nu]} + \tilde{q}\partial_{[\mu} b_{\nu]})\dot{z}^\nu. \quad (7.15)$$

VIII. QUANTUM THEORY

Standard constraint Hamiltonian formalism can be applied for the Lagrangians of Secs. V and VI in order to obtain the quantum theory. The analysis proceeds in a similar fashion to that carried out in [19] (second reference). As a multitude of constraints on the phase space result in this case and the orbits have to be studied separately, the procedure is quite lengthy. Here we instead write down the quantum algebra which should result for all orbits and sketch their representations on momentum eigenstates. The algebra is a 16-dimensional generalization of the Poincaré algebra, spanned by the two sets of momenta and the $GL(2, C)$ generators. Unitary representations of the algebra can be constructed along the lines of induced representations.

For the quantum theory we replace the two momentum vectors p^a and \tilde{p}^a , respectively, with the Hermitian operators \mathbf{p}^a and $\tilde{\mathbf{p}}^a$, acting on a Hilbert space \mathcal{H} . Additional observables are the $GL(2, C)$ generators $\mathbf{j}^{ab} = -\mathbf{j}^{ba}$, \mathbf{Y} , and \mathbf{Z} , where \mathbf{j}^{ab} are the Lorentz generators and \mathbf{Y} and \mathbf{Z} are the $U(1)$ generators. Since these generators are Hermitian, we can construct the unitary operators

$$\mathbf{U}(\Lambda) = \exp\left\{\frac{i}{2} \lambda_{ab} \mathbf{j}^{ab} + i\alpha \mathbf{Y} + \beta \mathbf{Z}\right\}, \quad (8.1)$$

for real parameters $\Lambda = (\lambda_{ab}, \alpha, \beta)$. The adjoint action can then be utilized to induce $GL(2, C)$ transformations on the space of observables $\{\mathbf{A}\}$,

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{U}(\Lambda)^\dagger \mathbf{A} \mathbf{U}(\Lambda). \quad (8.2)$$

For infinitesimal Λ , the transformations on \mathbf{p}^a , $\tilde{\mathbf{p}}^a$, and \mathbf{j}^{ab}

are given in (4.4) and (7.14). Thus

$$\begin{aligned} \mathbf{U}(\Lambda)^\dagger \mathbf{j}^{ab} \mathbf{U}(\Lambda) &= \mathbf{j}^{ab} + \mathbf{j}^{ac} \lambda_c^b - \mathbf{j}^{bc} \lambda_c^a, \\ \mathbf{U}(\Lambda)^\dagger \mathbf{p}^a \mathbf{U}(\Lambda) &= \mathbf{p}^a + \mathbf{p}^b \lambda_b^a + 2\tilde{\mathbf{p}}^a \beta, \\ \mathbf{U}(\Lambda)^\dagger \tilde{\mathbf{p}}^a \mathbf{U}(\Lambda) &= \tilde{\mathbf{p}}^a + \tilde{\mathbf{p}}^b \lambda_b^a + 2\mathbf{p}^a \beta, \end{aligned} \quad (8.3)$$

while \mathbf{Y} and \mathbf{Z} are invariant under $GL(2, C)$ transformations. From this we then get the quantum algebra for the observables \mathbf{p}^a , $\tilde{\mathbf{p}}^a$, \mathbf{j}^{ab} , \mathbf{Y} , and \mathbf{Z} . The nonvanishing commutators are

$$\begin{aligned} i[\mathbf{j}^{ab}, \mathbf{j}^{cd}] &= \mathbf{j}^{ac} \eta^{bd} - \mathbf{j}^{bc} \eta^{ad} - \mathbf{j}^{ad} \eta^{bc} + \mathbf{j}^{bd} \eta^{ac}, \\ i[\mathbf{p}^a, \mathbf{j}^{bc}] &= \mathbf{p}^b \eta^{ac} - \mathbf{p}^c \eta^{ab}, \\ i[\tilde{\mathbf{p}}^a, \mathbf{j}^{bc}] &= \tilde{\mathbf{p}}^b \eta^{ac} - \tilde{\mathbf{p}}^c \eta^{ab}, \\ i[\mathbf{p}^a, \mathbf{Z}] &= 2\tilde{\mathbf{p}}^a, \\ i[\tilde{\mathbf{p}}^a, \mathbf{Z}] &= 2\mathbf{p}^a. \end{aligned} \quad (8.4)$$

The subgroup generated by \mathbf{j}^{ab} and any linear combination of \mathbf{p}^a and $\tilde{\mathbf{p}}^a$ is the Poincaré group. [From (7.3), the generator of translations in flat space-time is the linear combination $\cosh \chi \mathbf{p}_a - \sinh \chi \tilde{\mathbf{p}}_a$.] \mathbf{Y} is a central element, while \mathbf{Z} , \mathbf{p}^a , and $\tilde{\mathbf{p}}^a$ form a Euclidean algebra for fixed a . The operator analogues of $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(4)}$ defined in (4.7) and (4.8), along with $\mathcal{C}_w^{(2)}$, $\mathcal{C}_w^{(4)}$, and $\mathcal{C}_{p,w}^{(2)}$ in (6.6) and (6.7), are Casimir operators whose values are fixed in any unitary irreducible representation.

Quantum states can be expressed in terms of eigenvectors $\Psi_{(p,\tilde{p}),\sigma,q}$ of \mathbf{p}^a , $\tilde{\mathbf{p}}^a$, and \mathbf{Y} . (We cannot include \mathbf{Z} in the set of commuting operators, and so the momentum eigenstates are labeled by a single charge.)

$$\begin{aligned} \mathbf{p}^a \Psi_{(p,\tilde{p}),\sigma,q} &= p^a \Psi_{(p,\tilde{p}),\sigma,q}, \\ \tilde{\mathbf{p}}^a \Psi_{(p,\tilde{p}),\sigma,q} &= \tilde{p}^a \Psi_{(p,\tilde{p}),\sigma,q}, \\ \mathbf{Y} \Psi_{(p,\tilde{p}),\sigma,q} &= q \Psi_{(p,\tilde{p}),\sigma,q}. \end{aligned} \quad (8.5)$$

σ denote degeneracy indices. Following the usual procedure for induced representations (see, for example, [24,25]), one can determine the values of σ by going to a fiducial point $(p, \tilde{p}) = (k, \tilde{k})$ on an orbit and then act on the state $\Psi_{(k,\tilde{k}),\sigma,q}$ with elements of the little group $G_{k,\tilde{k}} = \{n\}$ defined in (4.19). Let $\{D_{\sigma'\sigma}^q\}$ be a unitary irreducible representation of $G_{k,\tilde{k}}$, which we define using

$$\mathbf{U}(n) \Psi_{(k,\tilde{k}),\sigma,q} = D_{\sigma'\sigma}^q(n) \Psi_{(k,\tilde{k}),\sigma',q}. \quad (8.6)$$

One can then transform from the states at the fiducial point (k, \tilde{k}) to states at an arbitrary point (p, \tilde{p}) on the orbit using the analogue of a Wigner boost $L_{(p,\tilde{p})} \in GL(2, C)$, where $(p, \tilde{p}) = L_{(p,\tilde{p})} \circ (k, \tilde{k})$ and $L_{(k,\tilde{k})}$ is the identity map. Here \circ denotes the action of $GL(2, C)$ on (p, \tilde{p}) given by (4.6). The states can be defined to transform as

$$\Psi_{(p,\tilde{p}),\sigma,q} = \mathcal{N}_{(p,\tilde{p})} \mathbf{U}(L_{(p,\tilde{p})}) \Psi_{(k,\tilde{k}),\sigma,q}, \quad (8.7)$$

where $\mathcal{N}_{(p,\tilde{p})}$ is a normalization factor. An arbitrary transformation by $\Lambda \in GL(2, C)$ on this state is given by

$$\mathbf{U}(\Lambda) \Psi_{(p,\tilde{p}),\sigma,q} = \frac{\mathcal{N}_{(p,\tilde{p})}}{\mathcal{N}_{\Lambda \circ (p,\tilde{p})}} D_{\sigma'\sigma}^q(n_{\Lambda,(p,\tilde{p})}) \Psi_{\Lambda \circ (p,\tilde{p}),\sigma',q}, \quad (8.8)$$

where $n_{\Lambda,(p,\tilde{p})} \in G_{k,\tilde{k}}$ is the analogue of a Wigner rotation,

$$n_{\Lambda,(p,\tilde{p})} = L_{\Lambda \circ (p,\tilde{p})}^{-1} \circ \Lambda \circ L_{(p,\tilde{p})}. \quad (8.9)$$

IX. DISCUSSION

Using the quadratic invariants for $GL(2, C)$ we wrote down a general coupling of particles to an extended theory of gravity based on $GL(2, C)$ gauge theory [5]. Two momentum variables p^a and \tilde{p}^a were needed to couple to the two sets of vierbein fields e_μ^a and f_μ^a . We classified the orbits of these momenta using the values of the three invariants $\mathcal{C}^{(2)}$, $\mathcal{C}^{(4)}$, and $\text{sign det } \mathcal{P}$, in addition to their little groups $G_{k,\tilde{k}}$. Various orbits were examined, which are summarized in the table at the end of Sec. IV. Further divisions of orbits can also be made based on the sign of the energy. The orbits i(a) were identified with massive particles, while i(c) represented tachyons. A degeneracy in the classification was found when $\mathcal{C}^{(2)} = \mathcal{C}^{(4)} = \text{det } \mathcal{P} = 0$ and $G_{k,\tilde{k}} = U(1) \otimes E(2)$, as i(b), ii(b), iii(b), and iv(b) represent disconnected regions in the space of orbits. These four classes of orbits also could not be distinguished at the level of dynamics in flat space-time, as all of them describe massless particles (with only one independent momentum vector). Surprisingly, the orbits ii(c) also describe massless particles, despite their invariants $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(4)}$ taking the same values as those of massive particles i(a). Unlike with i(b), ii(b), iii, and iv, the massless particles ii(c) possess extra momentum degrees of freedom, as \tilde{p}^a is not fully determined from p^a . The physical meaning of these extra degrees of freedom is not clear and worth further investigation. Moreover, the list of orbits given in the table at the end of Sec. IV is by no means complete. A more complete classification could be of interest—in particular—with regards to the pursuit of dark matter candidates.

In Sec. VC we obtained the general solutions to the equations of motion (for particles with no spin or charge) in an arbitrary background, which is characterized by e_μ^a and f_μ^a . For orbits i(a) and ii(c) we obtained an effective Lagrangian (5.27) which contained corrections to the naive Lagrangian (3.24). The corrections vanished in the limit where the vierbein fields f_μ^a vanish, and so one recovers geodesic motion in this limit. On the other hand, corrections to geodesic motion do occur in the more general setting. Therefore deviations from geodesic motion, such as the reported Pioneer anomaly [26], could signal the

presence of additional fields in gravity like the vierbeins f_{μ}^a .

The particle spin, along with two $U(1)$ charges were taken into account in Sec. VI. There we found three more independent $GL(2, C)$ invariants $C_w^{(2)}$, $C_w^{(4)}$, and $C_{p,w}^{(2)}$ in (6.6) and (6.7), which were constructed using two Pauli-Lubanski vectors (6.5). It then follows that at least six invariants (not including the two charges) are needed to classify spinning particles in this theory—as opposed to the usual two, i.e., mass and the square of the Pauli-Lubanski vector. The dynamical equations for this system, including the interactions with the two $U(1)$ fields and two sets of torsion tensors, were obtained using the Wess-Zumino term for $GL(2, C)$. Their solutions should lead to further deviations from geodesic motion.

We wrote down the algebra of quantum mechanical observables in Sec. VIII and showed that the usual method of induced representations can be applied to construct the Hilbert space. It remains to develop the n -particle interacting system and also the field theory associated with the various particle representations. (The coupling to fermions and generalization to supergravity was recently studied in [18].)

Since the $GL(2, C)$ gauge theory has the advantage of being amenable to a noncommutative generalization, it is then natural to also promote the particle dynamics to the noncommutative setting. One approach would be to take the infinite field limit of one of the $U(1)$ gauge fields, as

nontrivial space-time commutation relations result from canonical quantization.⁴ Alternatively, or in addition, one can search for solutions to the noncommutative field equations, and then apply a Seiberg-Witten map back to the commutative theory. One can thereby obtain corrections in the metric tensor (3.22), as well as other $GL(2, C)$ invariant quantities, for known gravity solutions, such as black hole and cosmological solutions.⁵ In this case, the fields degrees of freedom a_{μ} , b_{μ} , and f_{μ}^a associated with the $GL(2, C)$ central extension, which are zero for the familiar gravity solutions, will in general pick up nonvanishing contributions after applying the Seiberg-Witten map from the noncommutative solutions. Moreover, it is expected that these contributions are first order in the noncommutativity parameter. First order corrections to geodesic motion may then result and are computable using the results found here.

⁴I thank A. Pinzul for this remark.

⁵This differs from previous approaches to finding noncommutative corrections to the solutions of general relativity [7–16]. For example, in some of those works a noncommutative analogue of the metric tensor had to be defined in order to make a physical interpretation. On the other hand, the approach here utilizes the commutative metric tensor (3.13). What we are proposing is similar in spirit to [27], where, in the context of Maxwell theory, noncommutative corrections were found to the Coulomb solution. It is also similar to [28], where first order noncommutative corrections were found for pp -wave solutions to the coupled Einstein-Maxwell equations.

-
- [1] R. Utiyama, Phys. Rev. **101**, 1597 (1956).
 - [2] T. W. B. Kibble, J. Math. Phys. (N.Y.) **2**, 212 (1961).
 - [3] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).
 - [4] For a recent review see, A. H. Chamseddine, Int. J. Geom. Methods Mod. Phys. **3**, 149 (2006).
 - [5] A. H. Chamseddine, Phys. Rev. D **69**, 024015 (2004).
 - [6] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp, and J. Wess, Classical Quantum Gravity **22**, 3511 (2005).
 - [7] A. Pinzul and A. Stern, Classical Quantum Gravity **23**, 1009 (2006).
 - [8] A. P. Balachandran, T. R. Govindarajan, K. S. Gupta, and S. Kurkcuoglu, Classical Quantum Gravity **23**, 5799 (2006).
 - [9] B. P. Dolan, K. S. Gupta, and A. Stern, Classical Quantum Gravity **24**, 1647 (2007).
 - [10] M. Chaichian, A. Tureanu, and G. Zet, Phys. Lett. B **660**, 573 (2008); M. Chaichian, M. R. Setare, A. Tureanu, and G. Zet, J. High Energy Phys. 04 (2008) 064.
 - [11] P. Mukherjee and A. Saha, Phys. Rev. D **77**, 064014 (2008).
 - [12] A. Kobakhidze, arXiv:0712.0642.
 - [13] R. Banerjee, B. R. Majhi, and S. Samanta, Phys. Rev. D **77**, 124035 (2008); R. Banerjee, B. R. Majhi, and S. K. Modak, Classical Quantum Gravity **26**, 085010 (2009).
 - [14] M. Buric and J. Madore, Eur. Phys. J. C **58**, 347 (2008).
 - [15] P. Nicolini, Int. J. Mod. Phys. A **24**, 1229 (2009).
 - [16] S. Fabi, B. Harms, and A. Stern, Phys. Rev. D **78**, 065037 (2008).
 - [17] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032.
 - [18] P. Aschieri and L. Castellani, arXiv:0902.3817; arXiv:0902.3823.
 - [19] A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, Phys. Lett. B **89**, 199 (1980); Lect. Notes Phys. **188**, 1 (1983).
 - [20] P. Horvathy, J. Math. Phys. (N.Y.) **20**, 49 (1979).
 - [21] M. Mathisson, Acta Phys. Pol. **6**, 163 (1937).
 - [22] A. Papapetrou, Proc. R. Soc. A **209**, 248 (1951).
 - [23] C. J. Isham, A. Salam, and J. A. Strathdee, Lett. Nuovo Cimento Soc. Ital. Fis. **5**, 969 (1972).
 - [24] A. P. Balachandran and C. G. Trahern, Lectures on Group Theory for Physicists, Monographs and Textbooks In Physical Science Vol. 3 (Bibliopolis, Naples, Italy, 1984), p. 176.

- [25] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1995), Vol. 1, p. 609.
- [26] J.D. Anderson, P.A. Laing, E.L. Lau, A.S. Liu, M.M. Nieto, and S.G. Turyshev, Phys. Rev. Lett. **81**, 2858 (1998); Phys. Rev. D **65**, 082004 (2002); S.G. Turyshev, V.T. Toth, L.R. Kellogg, E.L. Lau, and K.J. Lee, Int. J. Mod. Phys. D **15**, 1 (2006).
- [27] A. Stern, Phys. Rev. D **78**, 065006 (2008); Phys. Rev. Lett. **100**, 061601 (2008).
- [28] S. Marculescu and F.R. Ruiz, Phys. Rev. D **74**, 105004 (2006).