POLYDEGREE PROPERTIES OF POLYNOMIAL AUTOMORPHISMS

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ABSTRACT

The group of automorphisms of the affine plane has the structure of an amalgamated free product of the triangular and affine subgroups. This leads us to the polydegree: the unique sequence of degrees of the triangular automorphisms in the amalgamated free product decomposition of the automorphism. This group is also endowed with the structure of an infinite dimensional algebraic variety. The interaction between these two structures is not well understood. We use the Valuation Criterion, due to Furter in [8], to study the interaction between these structures; in particular, it allows us to see if an automorphism $\sigma \in \mathcal{G}$ is also in $\overline{\mathcal{G}_d}$, where d is the polydegree sequence. In this paper, we will discuss a method that gives us new results concerning a class of automorphisms with a polydegree of length one being contained in the closure (in the Zariski topology) of a set of automorphisms with a polydegree of length 2.

DEDICATION

I dedicate this dissertation to my parents and brothers. To Mike and Jenny: Thank you for your unconditional love and support. Without you two, I wouldn't be the person I am today. To Michael, Brian, and Kevin: Thank you for all the laughter, love, and technical support.

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CHAPTER 1

INTRODUCTION

1.1 The Group of Polynomial Automorphisms

Let K be a field. Let $K[X] := K[X_1, ..., X_n]$ denote the polynomial ring over K with n variables. A polynomial map is a map

$$F = (F_1, ..., F_n) : K^n \to K^n$$

of the form

$$(x_1, ..., x_n) \to (F_1(x_1, ..., x_n), ..., F_n(x_1, ..., x_n)),$$

where each $F_i \in K[X]$. We define the degree of F to be: $\deg F = \max\{\deg F_1, ..., \deg F_n\}$.

Example 1.1.1. $F = (X + Y^2, Y)$ is a polynomial map from $\mathbb{C}^2 \to \mathbb{C}^2$.

Such a polynomial map is called invertible if there exists a polynomial map

$$H = (H_1, ..., H_n) : K^n \to K^n$$

such that $x_i = H_i(F_1, ..., F_n)$ for $1 \leq i \leq n$.

Example 1.1.2. Let $H = (X - Y^2, Y)$ and F be defined as in the previous example. Then we have:

$$F \circ H = (X + Y^2, Y) \circ (X - Y^2, Y) = (X, Y).$$

Likewise, $H \circ F = (X, Y)$. Therefore F and H are inverses of each other.

The invertible polynomial maps are in a one-to-one correspondence with K-automorphisms of the polynomial ring $K[X]$, denoted by $\mathrm{Aut_K K[X]}$, via the map $F \to F^*$, where $F^* \in \mathrm{Aut_K K[X]}$ is simply $F^*(g) = g(F)$. We thus identify the group of K-automorphisms of $K[X]$ with the group of all invertible polynomial maps from K^n to K^n. Notice this identification is an anti-isomorphism from the invertible polynomial maps from K^n to K^n to $\mathrm{Aut_K K[X]}$.

We will use $\mathcal{G}(K)$ to denote the group of polynomial automorphisms of the affine plane $\mathbb{A}^2_K = \mathrm{Spec}(K[X,Y])$. An element $\sigma \in \mathcal{G}(K)$ is defined by a pair of polynomials $(f, g) \in K[X,Y]^2$ such that $K[f,g] = K[X,Y]$ and we write $\sigma = (f,g)$. We use $\mathcal{A}(K)$ for the subgroup of affine automorphisms, i.e. automorphisms of degree 1, and $\mathcal{B}(K)$ for the subgroup of triangular automorphisms, i.e. those of the form $(aX + P(Y), bY + c)$ where $a, b \in K^*$, $c \in K$, and $P(Y) \in K[Y]$. Using these subgroups, we can find a structure defined for $\mathcal{G}(K)$ from the following classical theorem:

Theorem 1.1.3 (The Jung-van der Kulk Theorem [14])**.** *$\mathcal{G}(K)$ is the amalgamated free product of $\mathcal{A}(K)$ and $\mathcal{B}(K)$ over their intersection.*

This theorem was first discovered by Jung [11] in 1942 and was later extended to a field of arbitrary characteristic by van der Kulk [15] in 1953. The theorem tells us that if we let $\sigma \in \mathcal{G}(K)$, then there exist $\beta_i \in \mathcal{B}(K)\backslash\mathcal{A}(K)$, $\alpha_0 \in \mathcal{A}(K)$, and $\alpha_i \in \mathcal{A}(K)\backslash\mathcal{B}(K)$ such that

$$\sigma = \alpha_0\beta_1\alpha_1\beta_2\alpha_2 \cdots \beta_n\alpha_n\beta_{n+1}\alpha_{n+1}.$$

This decomposition is not unique, but the degrees of the triangular automorphisms are unique which leads us to the following definition:

Definition 1.1.4 (Polydegree)**.** The polydegree is the unique sequence of the degrees from the triangular automorphisms in the amalgamated free product structure.

In the decomposition above, the polydegree would be: $(\deg \beta_1, \deg \beta_2, ..., \deg \beta_{n+1})$.

Remark 1.1.5. $\deg \beta_i \geq 2$ for all $1 \leq i \leq n+1$ because each β_i is triangular but not affine.

We denote by $\mathcal{G}(K)_d$ the set of all automorphisms of $\mathcal{G}(K)$ whose polydegree is $d = (d_1, d_2, ...d_l)$.

Let us now set $K = \mathbb{C}$. We will denote $\mathcal{G} := \mathcal{G}(\mathbb{C})$, $\mathcal{A} := \mathcal{A}(\mathbb{C})$, $\mathcal{B} := \mathcal{B}(\mathbb{C})$, and $\mathcal{G}_d := \mathcal{G}(\mathbb{C})_d$. The group $\mathcal{G}$ can be endowed with the structure of an infinite-dimensional algebraic variety (cf. [12], [13]). A set H with a fixed sequence of subsets H_n is called an *infinite-dimensional algebraic variety* if the following conditions are satisfied:

1. $H = \bigcup_n H_n$,
2. H_n is a closed algebraic subvariety of H_{n+1}.

Returning to our group $\mathcal{G}$, define $\mathcal{M}$ to be the monoid of K-endomorphisms and set

$$\mathcal{M}_d = \{f \in \mathcal{M}, \deg f \leq d\}.$$

The set $\mathcal{M}_d$ is a K-affine space, that is

$$\mathcal{M}_d \cong \mathbb{A}_K^{2\binom{2+d}{d}},$$

and can be given the structure of a K-algebraic variety. In [1], it is shown that $\mathcal{G}_d$ is a locally closed subvariety of $\mathcal{M}_d$, which gives $\mathcal{G}_d$ the K-algebraic variety structure. Finally, it follows that we will have:

$$\mathcal{M} = \bigcup_{d=0}^{\infty} \mathcal{M}_d$$

which endows $\mathcal{M}$ with the structure of an infinite-dimensional algebraic variety and, in turn, yields $\mathcal{G}$ with the structure of an infinite-dimensional algebraic variety. If $\mathcal{H} \subset \mathcal{G}$, let $\overline{\mathcal{H}}$ denote the closure of $\mathcal{H}$ in $\mathcal{G}$ for the Zariski topology associated with this structure.

We see that this group $\mathcal{G}$ now has two structures: the structure of an infinite-dimensional algebraic variety and the amalgamated free product structure. It is natural to study the interaction between these two structures. Furter, in [8], introduced the Valuation Criterion as a way to study to interaction between these two structures by checking if an automorphism $\sigma \in \mathcal{G}$ is also in $\overline{\mathcal{G}_d}$.

Theorem 1.1.6 (Valuation Criterion). *Let $\sigma \in \mathcal{G}$. Let $\tau = (F_1, F_2) \in \mathcal{G}(\mathbb{C}((Z)))_d$ such that $F_1, F_2 \in \mathbb{C}[Z][X,Y]$ and $\tau \equiv \sigma \mod Z$. Then $\sigma \in \overline{\mathcal{G}(\mathbb{C})_d}$.*

Remark 1.1.7. $\mathbb{C}((Z))$ denotes the field of fractions of $\mathbb{C}[[Z]]$, the ring of formal power series in Z.

Example 1.1.8 (Nagata Map - 1972). Let

$$\tau = (X + \frac{Y^2}{Z}, Y)(X, Y + Z^2X)(X - \frac{Y^2}{Z}, Y) \in \mathcal{G}.$$

Notice that τ is in the amalgamated free product structure form and has the polydegree of $(2,2)$; specifically, $\tau \in \mathcal{G}(\mathbb{C}((Z)))_{(2,2)}$. After composing these polynomial automorphisms on the right, we will simplify and compute modulo Z:

$$\begin{aligned}\tau &= (X - \frac{Y^2}{Z} + \frac{Y^2 + 2YZ(ZX - Y^2) + Z^2(ZX - Y^2)^2}{Z}, Y + Z(ZX - Y^2))\\ &= (X + 2Y(ZX - Y^2) + Z(ZX - Y^2)^2, Y + Z(ZX - Y^2))\\ &\equiv (X - 2Y^3, Y) \mod Z.\end{aligned}$$

Therefore, for $\sigma = (X - 2Y^3, Y) \in \mathcal{G}_3$, we have, by the Valuation Criterion, $\sigma \in \overline{\mathcal{G}_{(2,2)}}$. This can easily be generalized to show $\mathcal{G}_{(3)} \subset \overline{\mathcal{G}_{(2,2)}}$.

1.2 Polydegree Conjecture

The general question to be asked is:

Question 1.2.1. *Let $d = (d_1, ..., d_l)$ and $e = (e_1, ..., e_m)$ be two polydegree sequences. What conditions on d and e guarantee that $\mathcal{G}_d \subset \overline{\mathcal{G}_e}$?*

An obvious necessary condition for the inclusion $\mathcal{G}_d \subset \overline{\mathcal{G}_e}$ is $\mathcal{G}_d \cap \overline{\mathcal{G}_e} \neq \emptyset$. Friedland and Milnor discovered a topological constraint on the dimension which is a necessary condition for the inclusion $\mathcal{G}_d \subset \overline{\mathcal{G}_e}$.

Theorem 1.2.2 (Dimension Constraint [6]). *If $\mathcal{G}_d \subset \overline{\mathcal{G}_e}$ then*

$$d_1 + d_2 + \cdots + d_l \lneq e_1 + e_2 + \cdots + e_m.$$

Another necessary condition due to Furter ([7]) states that the length of an automorphism is lower semi-continuous; this can be reformulated as the following theorem:

Theorem 1.2.3 (Furter [7]). *Let $d = (d_1, ..., d_l)$ and $e = (e_1, ..., e_m)$ be two degree sequences. If $\mathcal{G}_d \cap \overline{\mathcal{G}_e} \neq \emptyset$, then $l \leq m$.*

The case when $l = 1$ and $m = 2$ is of particular interest and leads to the following conjecture:

Conjecture 1.2.4. *Let $d, e \geq 2$ be integers. Then $\mathcal{G}_{(d+e-1)} \subset \overline{\mathcal{G}_{(d,e)}}$.*

The goal is to completely describe the closure of the set of automorphisms of a fixed poly-degree of length two. That is:

Conjecture 1.2.5. *For all integers $d, e \geq 2$, we have:*

$$\overline{\mathcal{G}_{(d,e)}} = \bigcup_{d' \leq d, e' \leq e} \mathcal{G}_{(d',e')} \cup \bigcup_{c \leq d+e-1} \mathcal{G}_{(c)}.$$

Furter's Rigidity Conjecture [9] is an implication of this conjecture. In [5], Edo and van den Essen discovered a link between the Rigidity Conjecture and the Factorial Conjecture, which is related to the famous Jacobian Conjecture. It was stated in [1] that the groups of polynomial automorphisms and their classifications might give a possible approach to proving the Jacobian Conjecture. Furter, Edo, and Lewis have studied this problem and have classified some families of polynomial automorphisms.

As a result of Edo in [2], Conjecture 1.2.4 holds when $d - 1$ divides $e - 1$:

Theorem 1.2.6 (Edo). *Let $d, e \geq 2$ be integers. If $d - 1$ divides $e - 1$, then*

$$\mathcal{G}_{(d+e-1)} \subset \overline{\mathcal{G}_{(d,e)}}.$$

He was also able to extend Theorem 1.2.6 to the following:

Theorem 1.2.7 (Edo)**.** *Let $d, e \geq 2$ be integers, if $e-1$ is even and $d-1 \in \frac{e-1}{2}\mathbb{N}$, then:*

$$\mathcal{G}_{(d+e-1)} \cap \overline{\mathcal{G}_{(d,e)}} \neq \emptyset.$$

To continue with other polydegree properties, we can deduce the following corollary to Theorem 1.2.6:

Corollary 1.2.8. *If we set $d = 2$ in Theorem 1.2.6, then for all integers $e \geq 2$, we will have:*

$$\mathcal{G}_{(e+1)} \subset \overline{\mathcal{G}_{(2,e)}}.$$

We will define the following polydegrees: $d = (d_1, ..., d_l)$, $e = (e_1, ..., e_m)$, and $f = (f_1, ..., f_n)$. Let $d, e := (d_1, ..., d_l, e_1, ..., e_m)$ denote the concatenation of polydegrees.

Lemma 1.2.9. *If $\mathcal{G}_d \subset \overline{\mathcal{G}_f}$, then $\mathcal{G}_{d,e} \subset \overline{\mathcal{G}_{f,e}}$.*

By induction and using the concatenation of polydegrees, we can extend Corollary 1.2.8 to show that:

Corollary 1.2.10. *For all integers $d \geq 2$,*

$$\mathcal{G}_{(d)} \subset \overline{\mathcal{G}_{(2,2,...,2)}} = \overline{\mathcal{G}_{(2)^d}}$$

where $(2)^d$ is the polydegree sequence of 2's of length d.

Proof. Let $d \geq 2$ be an integer. By Corollary 1.2.8, $\mathcal{G}_{(d)} \subset \overline{\mathcal{G}_{(2,d-1)}}$ and $\mathcal{G}_{(d-1)} \subset \overline{\mathcal{G}_{(2,d-2)}}$. By the concatenation of polydegrees, $\mathcal{G}_{(d)} \subset \overline{\mathcal{G}_{(2,2,d-2)}}$. By induction, we will then see that:

$$\mathcal{G}_{(d)} \subset \overline{\mathcal{G}_{(2,2,d-2)}} \subset \overline{\mathcal{G}_{(2,2,2,d-3)}} \subset \cdots \subset \overline{\mathcal{G}_{(2)^d}}.$$

Therefore, $\mathcal{G}_{(d)} \subset \overline{\mathcal{G}_{(2)^d}}$. □

In Chapter 4, we will prove that when we have a polydegree sequence of 2's of length 5 or greater, we will not be able to write it as a union of some $\mathcal{G}_d$ for some polydegree d.

1.3 Main Theorems

Theorems 1.2.6 and 1.2.7 are great improvements on Conjecture 1.2.4; however, it would be ideal to remove the restrictions on d and e in order to improve on this conjecture. In his paper, Edo points out that the next step is to prove that $\mathcal{G}_{(8)} \cap \overline{\mathcal{G}_{(5,4)}} \neq \emptyset$. These theorems and this last challenge gave motivation towards this dissertation.

In Chapter 2, we will be able to improve Conjecture 1.2.4 when $e = 3$:

Theorem 1.3.1 (Main Theorem 1)**.** *For all integers* $d \geq 2$, $\mathcal{G}_{(d+2)} \subset \overline{\mathcal{G}_{(d,3)}}$.

In Chapter 3, using the same method from Chapter 2, we will show the first step in proving the conjecture when $e = 4$ and $2 \leq d \leq 45$.

Theorem 1.3.2 (Main Theorem 2)**.** *For all integers* $2 \leq d \leq 45$, $\mathcal{G}_{(d+3)} \cap \overline{\mathcal{G}_{(d,4)}} \neq \emptyset$.

In Chapter 4, we will discuss more about the union of polynomial automorphisms with a longer polydegree sequence.

Theorem 1.3.3 (Main Theorem 3)**.** *Let* $b \geq 5$ *be an integer. Then* $\overline{\mathcal{G}}_{(2)^b}$ *is not a union of some* $\mathcal{G}_d$ *for some polydegree* d.

CHAPTER 2

PROOF OF MAIN THEOREM 1

In Theorem 1.2.7, if we set $e = 3$, then we know for all integers $d \geq 2$, $\mathcal{G}_{(d+2)} \cap \overline{\mathcal{G}_{(d,3)}} \neq \emptyset$. In this chapter, we will prove that we have inclusion, which will prove that Conjecture 1.2.4 is true when $e = 3$.

Theorem 2.0.1. *For all integers $d \geq 2$, $\mathcal{G}_{(d+2)} \subset \overline{\mathcal{G}_{(d,3)}}$.*

Proof. Let $\sigma \in \mathcal{G}_{(d+2)}$ be a triangular automorphism of degree $d+2$. Then we can write

$$\sigma = (r_1 X + \sum_{k=0}^{d+2} a_k Y^k, r_2 Y + r_3)$$

for some $r_1, r_2, a_{d+2} \in \mathbb{C}^*$ and $r_3, a_i \in \mathbb{C}$ for $0 \leq i \leq d+1$. To prove $\sigma \in \overline{\mathcal{G}_{(d,3)}}$ using the Valuation Criterion (1.1.6), we will explicitly construct an automorphism $\sigma_Z \in \mathcal{G}(\mathbb{C}[Z])$ such that, writing $\sigma_Z = (f_Z, g_Z)$ and working modulo Z, $f_Z \equiv \sigma(X)$ and $g_Z \equiv \sigma(Y)$.

Let $U(Y,Z), V(Y,Z), W(Y)$, and $W_1(Y)$ be arbitrary polynomials for now; we will define them momentarily, but we will assume $\deg_Y U(Y,Z) = 3$, $\deg_Y V(Y,Z) = d$, $\deg W(Y) \leq d$, and $\deg W_1(Y) \leq d$. Let us consider the following polynomial automorphisms:

$$\tau_1 = (r_1 X + \frac{U(Y,Z)}{Z^d}, Y)$$

$$\alpha = (X, Y + Z^{d+1} X)$$

$$\tau_2 = (X - \frac{V(Y,Z)}{Z^d} + \frac{W(Y)}{Z} + W_1(Y), r_2 Y + r_3)$$

Set $\sigma_Z = \tau_2\alpha\tau_1$. Notice that $\sigma_Z \in \mathcal{G}(\mathbb{C}((Z)))_{(d,3)}$. Direct computation shows:

$$f_Z = \sigma_Z(X) = r_1X + \frac{U(Y,Z)}{Z^d} - \frac{V(Y+ZU(Y,Z),Z)}{Z^d} + \frac{W(Y+ZU(Y,Z))}{Z} + W_1(Y)$$
$$g_Z = \sigma_Z(Y) = r_2Y + Z^{d+1}r_1X + ZU(Y,Z) + r_3$$

and we will show $f_Z, g_Z \in \mathbb{C}[Z][X,Y]$, and

$$\sigma_Z(X) \equiv r_1X + \sum_{k=0}^{d+2} a_kY^k \qquad \mod Z$$
$$\sigma_Z(Y) \equiv r_2Y + r_3 \qquad \mod Z.$$

We specify further that $U(Y,Z) = \sum_{i=0}^{2} U_i(Y)Z^i \in \mathbb{C}[Y,Z]$ such that $\deg_{(1,-1)}U(Y,Z) \leq 2$. By the Formal Inverse Function Theorem [14], we define $I(Y,Z) \in \mathbb{C}[[Y,Z]]$ to be the formal inverse of $Y+ZU(Y,Z)$. In other words, $I(Y,Z)$ is the uniquely determined element in $\mathbb{C}[[Y,Z]]$ satisfying $I(Y+ZU(Y,Z),Z) = Y$. We can write

$$U(I(Y,Z),Z) = \sum_{k=0}^{\infty} v_k(Y)Z^k \in \mathbb{C}[[Y,Z]]$$

where, a priori, $v_k(Y) \in \mathbb{C}[[Y]]$, but in fact we will see $v_k(Y) \in \mathbb{C}[Y]$. We now define $V(Y,Z)$ as the truncation of $U(I(Y,Z),Z)$:

$$V(Y,Z) = \sum_{k=0}^{d-2} v_k(Y)Z^k \in \mathbb{C}[Y,Z].$$

Then we will have:

$$U(Y,Z) = U(I(Y+ZU(Y,Z),Z),Z)$$
$$\equiv V(Y+ZU(Y,Z),Z) \qquad \mod Z^{d-1}$$
$$\equiv \sum_{k=0}^{d-2} v_k(Y+ZU(Y,Z))Z^k \qquad \mod Z^{d-1}.$$

Applying Taylor's Formula, we have

$$U(Y,Z)=\sum_{m=0}^{d-2}Z^m\sum_{j=0}^{m}\frac{1}{j!}v_{m-j}^{(j)}(Y)U(Y,Z)^j.$$

Recalling $U(Y,Z)=U_0+ZU_1+Z^2U_2$, we further specify $U_0=c_dY^2$, $U_1=c_{d-1}Y^3$, and $U_2=c_{d-2}Y^3$, where $c_d,c_{d-1},c_{d-2}\in\mathbb{C}$ remain arbitrary for now. Then:

$$U(Y,Z)=\sum_{m=0}^{d-2}Z^m\sum_{j=0}^{m}\sum_{0\le 2a+3b\le j}\frac{1}{a!b!(j-2a-3b)!}v_{m-j}^{(j-a-2b)}U_0^{j-2a-3b}U_1^aU_2^b.$$

Comparing coefficients of Z^k, we obtain the following recursive relation for $v_k(Y)$:

$$\begin{aligned}
v_0&=U_0\\
v_1&=U_1-v_0'U_0\\
v_2&=U_2-[v_1'U_0+\frac{1}{2}v_0''U_0^2+v_0'U_1]\\
v_r&=-\sum_{j=1}^{r}\sum_{0\le 2a+3b\le j}\frac{1}{a!b!(j-2a-3b)!}v_{r-j}^{(j-a-2b)}U_0^{j-2a-3b}U_1^aU_2^b
\end{aligned}\tag{2.0.1}$$

Lemma 2.0.2. $\deg v_r(Y)\le r+2$.

Proof. Proof by induction. Writing $U_0=c_dY^2$, $U_1=c_{d-1}Y^3$, and $U_2=c_{d-2}Y^3$ as above,

$$v_0=U_0=c_dY^2,$$

thus $\deg v_0=2$.

$$v_1=U_1-v_0'U_0=(c_{d-1}-2c_d^2)Y^3,$$

thus $\deg v_1=3$.

$$v_2=U_2-[v_1'U_0+\frac{1}{2}v_0''U_0^2+v_0'U_1]=c_{d-2}Y^3+c_d(-5c_{d-1}+5c_d^2)Y^4,$$

thus $\deg v_2 = 4$.

Now let's suppose $\deg v_r(Y) \leq r+2$. We will show that $\deg v_{r+1}(Y) \leq r+3$. By definition,

$$v_{r+1} = -\sum_{j=1}^{r+1} \sum_{0\leq 2a+3b\leq j} \frac{1}{a!b!(j-2a-3b)!} v_{r+1-j}^{(j-a-2b)} U_0^{j-2a-3b} U_1^a U_2^b$$

Note the following degrees:

$$\deg v_{r+1-j} \leq r+3-j$$

$$\deg v_{r+1-j}^{(j-a-2b)} \leq r+3-2j+a+2b$$

$$\deg U_0^{j-2a-3b} \leq 2j-4a-6b$$

$$\deg U_1^a \leq 3a$$

$$\deg U_2^b \leq 3b$$

When looking at each term in the above summation for v_{r+1}, we will see that

$$\deg(v_{r+1-j}^{(j-a-2b)} U_0^{j-2a-3b} U_1^a U_2^b) \leq r+3-b.$$

When $b = 0$, we will get the maximum degree of $r+3$. □

We can also study the forms of each $v_k(Y)$ through the next lemma.

Lemma 2.0.3. *1. If r is even,* $v_r(Y) = \lambda_r c_d Y^{r+2} + \gamma_r Y^{r+1} + M_r(Y)$, *with* $\lambda_r \in \mathbb{Q}[c_d^2, c_{d-1}]$, $\gamma_r \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$, *and* $M_r(Y) \in \mathbb{C}[Y]$ *such that* $\deg M_r(Y) \leq r$.

2. If r is odd, $v_r(Y) = \lambda_r Y^{r+2} + \gamma_r Y^{r+1} + M_r(Y)$, *with* $\lambda_r \in \mathbb{Q}[c_d^2, c_{d-1}]$, $\gamma_r \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$, *and* $M_r(Y) \in \mathbb{C}[Y]$ *such that* $\deg M_r(Y) \leq r$.

Proof. We proceed by induction from Equation 2.0.1:

$r = 0:\quad v_0(Y) = U_0 = c_d Y^2$. Therefore $\lambda_0 = 1$ and $\gamma_0 = 0$.

$r = 1:\quad v_1(Y) = U_1 - v_0' U_0 = (c_{d-1} - 2c_d^2)Y^3$. Therefore $\lambda_1 = c_{d-1} - 2c_d^2$ and $\gamma_1 = 0$.

$r = 2:\quad v_2(Y) = U_2 - [v_1' U_0 + \frac{1}{2} v_0'' U_0^2 + v_0' U_1] = c_{d-2}Y^3 + c_d(-5c_{d-1} + 5c_d^2)Y^4$.

Therefore $\lambda_2 = -5c_{d-1} + 5c_d^2$ and $\gamma_2 = c_{d-2}$.

Suppose $v_k(Y) = \lambda_k c_d Y^{k+2} + \gamma_k Y^{k+1} + M_k(Y)$ when k is even or $v_k = \lambda_k Y^{k+2} + \gamma_k Y^{k+1} + M_k(Y)$ when k is odd with $\lambda_k \in \mathbb{Q}[c_d^2, c_{d-1}]$, $\gamma_k \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$, and $\deg M_k(Y) \leq k$ for all $0 \leq k \leq r$. We want to show that $v_{r+1}(Y) = \lambda_{r+1} Y^{r+3} + \gamma_{r+1} Y^{r+2} + M_{r+1}(Y)$ when r is even or $v_{r+1}(Y) = \lambda_{r+1} c_d Y^{r+3} + \gamma_{r+1} Y^{r+2} + M_{r+1}(Y)$ when r is odd with $\lambda_{r+1} \in \mathbb{Q}[c_d^2, c_{d-1}]$, $\gamma_{r+1} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$, and $\deg M_{r+1}(Y) \leq r + 1$.

By definition,

$$v_{r+1} = -\sum_{j=1}^{r+1} \sum_{0 \leq 2a+3b \leq j} \frac{1}{a!b!(j-2a-3b)!} v_{r+1-j}^{(j-a-2b)} U_0^{j-2a-3b} U_1^a U_2^b$$

By the induction hypothesis, v_{r+1-j} has the correct form. There are two cases to consider:

1. When $r+1-j$ is even: $v_{r+1-j}(Y) = \lambda_{r+1-j} c_d Y^{r-j+3} + \gamma_{r+1-j} Y^{r-j+2} + M_{r+1-j}(Y)$ where $\lambda_{r+1-j} \in \mathbb{Q}[c_d^2, c_{d-1}]$ and $\gamma_{r+1-j} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$. Each term in the summation for v_{r+1} where $r+1-j$ is even will have the form

$$\beta_1 \lambda_{r+1-j} c_d c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r+3-b} + \beta_2 \gamma_{r+1-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r+2-b} + K(Y)$$

   where $\beta_1, \beta_2 \in \mathbb{C}$ and $\deg K(Y) \leq r + 1 - b$.

   When $b = 0$, we can see that the coefficient of Y^{r+3} will be $\beta_1 \lambda_{r+1-j} c_d c_d^{j-2a} c_{d-1}^a$. When

$b = 1$ in the first term and $b = 0$ in the second, the coefficient of Y^{r+2} will be

$$\beta_1 \lambda_{r+1-j} c_d c_d^{j-2a-3} c_{d-1}^a c_{d-2} + \beta_2 \gamma_{r+1-j} c_d^{j-2a} c_{d-1}^a.$$

2. When $r + 1 - j$ is odd: $v_{r+1-j}(Y) = \lambda_{r+1-j} Y^{r-j+3} + \gamma_{r+1-j} Y^{r-j+2} + M_{r+1-j}(Y)$ where $\lambda_{r+1-j} \in \mathbb{Q}[c_d^2, c_{d-1}]$ and $\gamma_{r+1-j} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$.

   Each term in the summation for v_{r+1} where $r + 1 - j$ is odd will have the form

$$\beta_1 \lambda_{r+1-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r+3-b} + \beta_2 \gamma_{r+1-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r+2-b} + K(Y)$$

   where $\beta_1, \beta_2 \in \mathbb{C}$ and $\deg K(Y) \leq r + 1 - b$.

   When $b = 0$, we can see that the coefficient of Y^{r+3} will be $\beta_1 \lambda_{r+1-j} c_d^{j-2a} c_{d-1}^a$. When $b = 1$ in the first term and $b = 0$ in the second, the coefficient of Y^{r+2} will be

$$\beta_1 \lambda_{r+1-j} c_d^{j-2a-3} c_{d-1}^a c_{d-2} + \beta_2 \gamma_{r+1-j} c_d^{j-2a} c_{d-1}^a.$$

Now we will show:

1. When r is even, $v_{r+1}(Y) = \lambda_{r+1} Y^{r+3} + \gamma_{r+1} Y^{r+2} + M_{r+1}(Y)$.

   If r is even and $r + 1 - j$ is even, then j and $j - 2a$ must be odd. So the coefficient of Y^{r+3}, $\beta_1 \lambda_{r+1-j} c_d^{j-2a+1} c_{d-1}^a \in \mathbb{Q}[c_d^2, c_{d-1}]$. Call this term δ_{r+1-j}.

   If r is even and $r + 1 - j$ is odd, then j and $j - 2a$ must be even. So the coefficient of Y^{r+3}, $\beta_1 \lambda_{r+1-j} c_d^{j-2a} c_{d-1}^a \in \mathbb{Q}[c_d^2, c_{d-1}]$. Call this term δ'_{r+1-j}.

   Therefore, the coefficient of Y^{r+3} in v_{r+1} will be

$$\lambda_{r+1} = \sum_{r+1-j \text{ even}} \delta_{r+1-j} + \sum_{r+1-j \text{ odd}} \delta'_{r+1-j} \in \mathbb{Q}[c_d^2, c_{d-1}].$$

   It is easy to see that the coefficient of Y^{r+2} in v_{r+1} will be in $\mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$.

Therefore, $v_{r+1}(Y) = \lambda_{r+1}Y^{r+3} + \gamma_{r+1}Y^{r+2} + M_{r+1}(Y)$ where $\lambda_{r+1} \in \mathbb{Q}[c_d^2, c_{d-1}]$ and $\gamma_{r+1} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$.

2. When r is odd, $v_{r+1}(Y) = \lambda_{r+1}c_dY^{r+3} + \gamma_{r+1}Y^{r+2} + M_{r+1}(Y)$.

   If r is odd and $r+1-j$ is even, then j and $j-2a$ must be even. So the coefficient of Y^{r+3}, $\beta_1\lambda_{r+1-j}c_dc_d^{j-2a}c_{d-1}^a = c_d(\beta_1\lambda_{r+1-j}c_d^{j-2a}c_{d-1}^a)$. Set $\nu_{r+1-j} = \beta_1\lambda_{r+1-j}c_d^{j-2a}c_{d-1}^a \in \mathbb{Q}[c_d^2, c_{d-1}]$.

   If r is odd and $r+1-j$ is odd, then j and $j-2a$ must be odd. So the coefficient of Y^{r+3}, $\beta_1\lambda_{r+1-j}c_d^{j-2a}c_{d-1}^a = c_d(\beta_1\lambda_{r+1-j}c_d^{j-2a-1}c_{d-1}^a)$. Now, $j-2a-1$ is even. Set $\nu'_{r+1-j} = \beta_1\lambda_{r+1-j}c_d^{j-2a-1}c_{d-1}^a \in \mathbb{Q}[c_d^2, c_{d-1}]$.

   Therefore, the coefficient of Y^{r+3} in v_{r+1} will be $c_d\lambda_{r+1}$ where

   $$\lambda_{r+1} = \sum_{r+1-j \text{ even}} \nu_{r+1-j} + \sum_{r+1-j \text{ odd}} \nu'_{r+1-j} \in \mathbb{Q}[c_d^2, c_{d-1}].$$

   It is easy to see that the coefficient of Y^{r+2} in v_{r+1} will be in $\mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$.

   Therefore, $v_{r+1}(Y) = \lambda_{r+1}c_dY^{r+3} + \gamma_{r+1}Y^{r+2} + M_{r+1}(Y)$ where $\lambda_{r+1} \in \mathbb{Q}[c_d^2, c_{d-1}]$ and $\gamma_{r+1} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$.

□

Using the recursive definition for $v_r(Y)$, Equation 2.0.1, we are able to write a recursive formula for λ_r. Since λ_r is the coefficient of Y^{r+2}, we will take $b = 0$:

$$\begin{aligned} v_r|_{b=0} &= -\sum_{j=1}^{r} \sum_{0 \le 2a \le j} \frac{1}{a!(j-2a)!} v_{r-j}^{(j-a)} U_0^{j-2a} U_1^a \\ &= -\sum_{j=1}^{r} \sum_{0 \le 2a \le j} \frac{1}{a!(j-2a)!} (\lambda_{r-j}Y^{r-j+2})^{(j-a)} c_d^{j-2a} c_{d-1}^a Y^{2j-a} \\ &= -\sum_{j=1}^{r} \sum_{0 \le 2a \le j} \frac{(r-j+2)\cdots(r-2j+a+3)}{a!(j-2a)!} \lambda_{r-j} c_d^{j-2a} c_{d-1}^a Y^{r+2} \end{aligned}$$

Therefore

$$\lambda_r = -\sum_{j=1}^{r} \sum_{0 \le 2a \le j} \frac{(r-j+2)!}{a!(j-2a)!(r-2j+a+2)!} \lambda_{r-j} c_d^{j-2a} c_{d-1}^{a} \tag{2.0.2}$$

Recall that we want to show $\sigma_Z \in \mathbb{C}[Z][X, Y]$ and

$$\sigma_Z \equiv (r_1 X + \sum_{k=0}^{d+2} a_k Y^k, r_2 Y + r_3) \mod Z.$$

For the sake of simplicity, let us write $\sigma_Z(X)$ as

$$\sigma_Z(X) = r_1 X + \sum \mu_{s,t} \frac{Y^s}{Z^t}.$$

Taking a look at the degrees, we obtain the following lemma:

Lemma 2.0.4. *Let $P(Y, Z) \in \mathbb{C}[Y, Z]$ be an arbitrary polynomial. If $\deg_{(1,-1)} P(Y, Z) = r$, then $\deg_{(1,-1)} P(Y + ZU(Y, Z), Z) \le r$.*

Proof. Suppose $\deg_{(1,-1)} P(Y, Z) = r$. Recall that $\deg_{(1,-1)} U(Y, Z) \le 2$. Thus $\deg_{(1,-1)} ZU(Y, Z) \le 1$, which implies that $\deg_{(1,-1)}(Y + ZU(Y, Z)) = 1$. When composing $P(Y, Z)$ with a degree one polynomial the homogeneous degree of $P(Y + ZU(Y, Z), Z)$ will never be increased. Therefore, $\deg_{(1,-1)} P(Y + ZU(Y, Z), Z) \le r$. □

Note a few more homogeneous degrees of some important polynomials for the next step in our computation:

$$\deg_{(1,-1)} V(Y, Z) = 2 \qquad \text{(by Lemma 2.0.2)}$$
$$\deg_{(1,-1)} \frac{V(Y, Z)}{Z^d} = d + 2.$$

Therefore we have

$$\begin{aligned}
&\deg_{(1,-1)}\sigma_Z(X)\\
&\qquad = \deg_{(1,-1)}\left(\frac{U(Y,Z)}{Z^d} - \frac{V(Y+ZU(Y,Z),Z)}{Z^d} + \frac{W(Y+ZU(Y,Z))}{Z} + W_1(Y)\right)\\
&\qquad \leq d+2.
\end{aligned}$$

From this, we can conclude that for $s > d+2$, $\mu_{s,0} = 0$. More generally, for $s+t > d+2$, $\mu_{s,t} = 0$. Since $U(Y,Z) \equiv V(Y+ZU(Y,Z),Z) \mod Z^{d-1}$, $\mu_{s,t} = 0$, for $t > 1$. All that is left to show is what happens when $t = 0$ and $t = 1$.

To make computations easier, we will multiply $\sigma_Z(X)$ by Z^d to get:

$$\begin{aligned}
\sigma_Z(Z^dX) - Z^d r_1 X &= U(Y,Z) - V(Y+ZU,Z) + Z^{d-1}W(Y+ZU) + Z^d W_1(Y+ZU)\\
&\equiv Z^{d-1}v_{d-1}(Y+ZU) + Z^d v_d(Y+ZU) + Z^{d-1}W(Y+ZU)\\
&\qquad + Z^d W_1(Y+ZU) \qquad\qquad \mod Z^{d+1}\\
&\equiv Z^{d-1}[v_{d-1}(Y) + W(Y)]\\
&\qquad + Z^d[U_0 v'_{d-1}(Y) + U_0 W'(Y) + v_d(Y) + W_1(Y)] \qquad \mod Z^{d+1}
\end{aligned}$$

Recall that we have yet to specify $c_d, c_{d-1}, c_{d-2} \in \mathbb{C}$, and $W(Y), W_1(Y) \in \mathbb{C}[Y]$. The next step is to show that we can specify these such that $v_{d-1}(Y) + W(Y) = 0$ and $U_0 v'_{d-1}(Y) + U_0 W'(Y) + v_d(Y) + W_1(Y) = \sum_{k=0}^{d+2} a_k Y^k$.

When d is odd, we have $v_{d-1}(Y) = \lambda_{d-1} c_d Y^{d+1} + \gamma_{d-1} Y^d + M_{d-1}(Y)$, where $\deg M_{d-1}(Y) \leq d-1$. We can set $W(Y) = -\gamma_{d-1}Y^d - M_{d-1}(Y)$ and, therefore, we will need $\lambda_{d-1} c_d Y^{d+1} = 0$. When d is even, we have $v_{d-1}(Y) = \lambda_{d-1} Y^{d+1} + \gamma_{d-1} Y^d + M_{d-1}(Y)$, where $\deg M_{d-1}(Y) \leq d-1$. We can set $W(Y) = -\gamma_{d-1}Y^d - M_{d-1}(Y)$ and again need $\lambda_{d-1} Y^{d+1} = 0$. In either case, we have specified $W(Y)$; suppose we have chosen c_d, c_{d-1} such that $\lambda_{d-1} = 0$. Then

$$U_0[v_{d-1}(Y) + W(Y)]' + v_d(Y) + W_1(Y) = v_d(Y) + W_1(Y).$$

We want to show that we can have

$$v_d(Y) + W_1(Y) = \sum_{k=0}^{d+2} a_k Y^k.$$

That is, when d is odd:

$$\lambda_d Y^{d+2} + \gamma_d Y^{d+1} + M_d(Y) + W_1(Y) = \sum_{k=0}^{d+2} a_k Y^k$$

when d is even:

$$\lambda_d c_d Y^{d+2} + \gamma_d Y^{d+1} + M_d(Y) + W_1(Y) = \sum_{k=0}^{d+2} a_k Y^k.$$

where $\deg M_d(Y) \leq d$. We need $\lambda_d \in \mathbb{C}^*$ and $\gamma_d \in \mathbb{C}[c_{d-2}]\backslash\mathbb{C}$ after choosing c_d, c_{d-1} . In summary, we need to show that we can choose c_d, c_{d-1} such that:

1. $\lambda_{d-1} = 0$
2. λ_d is an arbitrary element of $\mathbb{C}^*$
3. $\gamma_d \in \mathbb{C}[c_{d-2}]\backslash\mathbb{C}$.

Taking a closer look at the λ_i's, we can write

$$\lambda_i = c_{d-1}^{\lfloor \frac{i+1}{2} \rfloor} c_d^{\epsilon_i} Q_i\left(\frac{c_d^2}{c_{d-1}}\right)$$

where $\epsilon_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$

Using the recursive formula for λ_i, Equation 2.0.2, we can derive an explicit formula for Q_r:

Lemma 2.0.5.

$$Q_{n-1}\left(\frac{c_d^2}{c_{d-1}}\right) = \frac{1}{(n+1)!} \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{l+n+1} \frac{(2n-l)!}{l!(n-2l)!} \left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor \frac{n}{2} \rfloor - l}$$

Proof. The proof is by induction on n.

$n = 1$: $Q_0 = \frac{2!}{2!}\left(\frac{c_d^2}{c_{d-1}}\right)^0 = 1$ and by definition of λ_0, $\lambda_0 = c_d$ which would imply that $Q_0 = 1$.

Let's suppose that for all integers $1 \leq n \leq r$,

$$Q_{n-1}\left(\frac{c_d^2}{c_{d-1}}\right) = \frac{1}{(n+1)!}\sum_{l=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^{l+n+1}\frac{(2n-l)!}{l!(n-2l)!}\left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor\frac{n}{2}\rfloor-l}.$$

We will show that

$$Q_r\left(\frac{c_d^2}{c_{d-1}}\right) = \frac{1}{(r+2)!}\sum_{l=0}^{\lfloor\frac{r+1}{2}\rfloor}(-1)^{l+r+2}\frac{(2r+2-l)!}{l!(r+1-2l)!}\left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor\frac{r+1}{2}\rfloor-l}$$

using the recursive formula for λ_r.

$$\lambda_r = -\sum_{j=1}^{r}\sum_{0\le 2a\le j}\frac{(r-j+2)!}{a!(j-2a)!(r-2j+a+2)!}\lambda_{r-j}c_d^{j-2a}c_{d-1}^{a}$$
$$= -\sum_{j=1}^{r}\sum_{0\le 2a\le j}\frac{(r-j+2)!}{a!(j-2a)!(r-2j+a+2)!}(c_{d-1}^{\lfloor\frac{r-j+1}{2}\rfloor}c_d^{\epsilon_{r-j}})Q_{r-j}\left(\frac{c_d^2}{c_{d-1}}\right)c_d^{j-2a}c_{d-1}^{a}$$
$$= -\sum_{j=1}^{r}\sum_{0\le 2a\le j}\sum_{l=0}^{\lfloor\frac{r-j+1}{2}\rfloor}\frac{(-1)^{l+r-j+2}(2r-2j+2-l)!}{a!(j-2a)!(r-2j+a+2)!l!(r-j+1-2l)!}\left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor\frac{r-j+1}{2}\rfloor-l}$$
$$c_{d-1}^{\lfloor\frac{r-j+1}{2}\rfloor+a}c_d^{j-2a+\epsilon_{r-j}}$$
$$= \sum_{j=1}^{r}\sum_{0\le 2a\le j}\sum_{l=0}^{\lfloor\frac{r-j+1}{2}\rfloor}\frac{(-1)^{l+r-j+3}(2r-2j+2-l)!}{a!(j-2a)!(r-2j+a+2)!l!(r-j+1-2l)!}c_d^{r-2l-2a}c_{d-1}^{l+a}$$
$$= -\sum_{l=0}^{\lfloor\frac{r+1}{2}\rfloor}(-1)^{l+r+1}\frac{(2r+2-l)!}{(r+2)!l!(r+1-2l)!}c_d^{2\lfloor\frac{r+1}{2}\rfloor-2l+\epsilon_r}c_{d-1}^{l}$$
$$+\sum_{j=0}^{r}\sum_{0\le 2a\le j}\sum_{l=0}^{\lfloor\frac{r-j+1}{2}\rfloor}\frac{(-1)^{l+r-j+1}(2r-2j+2-l)!}{a!(j-2a)!(r-2j+a+2)!l!(r-j+1-2l)!}c_d^{r-2l-2a}c_{d-1}^{l+a} \qquad \dagger$$
$$= c_d^{\epsilon_r}c_{d-1}^{\lfloor\frac{r+1}{2}\rfloor}\sum_{l=0}^{\lfloor\frac{r+1}{2}\rfloor}(-1)^{l+r+2}\frac{(2r+2-l)!}{(r+2)!l!(r+1-2l)!}c_d^{2\lfloor\frac{r+1}{2}\rfloor-2l}c_{d-1}^{l-\lfloor\frac{r+1}{2}\rfloor}$$
$$= c_d^{\epsilon_r}c_{d-1}^{\lfloor\frac{r+1}{2}\rfloor}\sum_{l=0}^{\lfloor\frac{r+1}{2}\rfloor}(-1)^{l+r+2}\frac{(2r+2-l)!}{(r+2)!l!(r+1-2l)!}\left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor\frac{r+1}{2}\rfloor-l}$$

Since $\lambda_r = c_{d-1}^{\lfloor\frac{r+1}{2}\rfloor}c_d^{\epsilon_r}Q_r\left(\frac{c_d^2}{c_{d-1}}\right)$, we deduce that

$$Q_r\left(\frac{c_d^2}{c_{d-1}}\right) = \sum_{l=0}^{\lfloor\frac{r+1}{2}\rfloor}(-1)^{l+r+2}\frac{(2r+2-l)!}{(r+2)!l!(r+1-2l)!}\left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor\frac{r+1}{2}\rfloor-l},$$

and this completes the induction step and the proof.

†This sum is equal to zero. See Appendix A for proof. □

There is an unexpected link between this problem and the theory of hypergeometric functions. For a general reference on Hypergeometric Functions, see [10].

Lemma 2.0.6 (Gauss hypergeometric polynomials [2])**.** *Let $n \geq 2$ be an integer. Let Q_{n-1} be defined as in Lemma 2.0.5. We have:*

1. $\deg Q_{n-1} = \lfloor \frac{n}{2} \rfloor$ *and* $Q_{n-1}(t) \neq 0$ *for all* $t > 0$*,*

2. Q_{n-1} *has only simple roots.*

3. *There exists a complex number* $u_{n-1} \in \mathbb{C}$ *such that* $Q_n(u_{n-1}) = 0$ *and* $Q_{n+1}(u_{n-1}) \neq 0$*.*

By this lemma, there exists $c_d, c_{d-1} \in \mathbb{C}$ such that $\lambda_{d-1} = 0$ and $\lambda_d = a_{d+2} \neq 0$. In fact, λ_{d-1} and λ_d have no roots in common.

Lemma 2.0.7. λ_{d-1} *and* λ_d *have no roots in common.*

Proof. As we know, we are able to write λ_{d-1} and λ_d in terms of the polynomial Q, which is a hypergeometric polynomial. Let us recall some basic definitions and formulas. Let $a, b, c, z \in \mathbb{C}$, the classical Gauss hypergeometric series is define by:

$$ {}_2F_1(a, b, c|z) = \sum_{k \in \mathbb{N}} \frac{(a)_k (b)_k z^k}{(c)_k k!} $$

where, for a complex number $a \in \mathbb{C}$ and an integer $k \in \mathbb{N}$, the Pochhammer symbol $(a)_k$ is defined by $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$. Using this definition, we can write

$$ Q_{d-1} = (-1)^{d+1} \frac{(2d)!}{d!(d+1)!} {}_2F_1(-\frac{d}{2}, -\frac{d-1}{2}, -2d|4\left(\frac{c_d^2}{c_{d-1}}\right)). $$

Then:

$$ Q_d = (-1)^d \frac{(2d+2)!}{(d+1)!(d+2)!} {}_2F_1(-\frac{d+1}{2}, -\frac{d}{2}, -2d-2|4\left(\frac{c_d^2}{c_{d-1}}\right)) $$

and

$$ Q_{d+1} = (-1)^{d+1} \frac{(2d+4)!}{(d+2)!(d+3)!} {}_2F_1(-\frac{d+2}{2}, -\frac{d+1}{2}, -2d-4|4\left(\frac{c_d^2}{c_{d-1}}\right)). $$

If we let $a = -\frac{d+1}{2}$, $b = -\frac{d}{2}$, $c = -2d-2$, and $z = 4\left(\frac{c_d^2}{c_{d-1}}\right)$, then we will have:

$$\begin{aligned}
Q_{d-1} &= (-1)^{d+1}\frac{(2d)!}{d!(d+1)!}{}_2F_1(b, a+1, c+2|z) \\
&= (-1)^{d+1}\frac{(2d)!}{d!(d+1)!}{}_2F_1(a+1, b, c+2|z) \\
Q_d &= (-1)^{d}\frac{(2d+2)!}{(d+1)!(d+2)!}{}_2F_1(a, b, c|z) \\
Q_{d+1} &= (-1)^{d+1}\frac{(2d+4)!}{(d+2)!(d+3)!}{}_2F_1(b-1, a, c-2|z) \\
&= (-1)^{d+1}\frac{(2d+4)!}{(d+2)!(d+3)!}{}_2F_1(a, b-1, c-2|z)
\end{aligned}$$

In fact,

$$\begin{aligned}
{}_2F_1(a+1, b, c+2|z) &= \frac{d!(d+1)!}{(-1)^{d+1}(2d)!}Q_{d-1} \\
{}_2F_1(a, b, c|z) &= \frac{(d+1)!(d+2)!}{(-1)^{d}(2d+2)!}Q_{d} \\
{}_2F_1(a, b-1, c-2|z) &= \frac{(d+2)!(d+3)!}{(-1)^{d+1}(2d+4)!}Q_{d+1}
\end{aligned}$$

There is a linear relation between three hypergeometric polynomials. That is: there exists functions f,g,h such that

$$f(z){}_2F_1(a, b-1, c-2|z) + g(z){}_2F_1(a, b, c|z) + h(z){}_2F_1(a+1, b, c+2|z) = 0.$$

When looking at Gauss Contiguous Relations, we can figure out what f, g, h have to be. We will use the following relations:

1. $c{}_2F_1(a, b, c|z) - a{}_2F_1(a+1, b, c+1|z) + (a-c){}_2F_1(a, b, c+1|z) = 0$

2. $(c+1){}_2F_1(a, b, c+1|z) + (b-c-a){}_2F_1(a+1, b, c+2|z) + (z-1)(c-1){}_2F_1(a+1, b, c+1|z) = 0$

3. $(c-1){}_2F_1(a, b-1, c-1|z) + (a-c+1)z{}_2F_1(a, b, c|z) + (z-1)(c-1){}_2F_1(a, b, c-1|z) = 0$

4. $(c-2){}_2F_1(a, b-1, c-2|z) - (b-1){}_2F_1(a, b, c-1|z) + (b-c+1){}_2F_1(a, b-1, c-1|z) = 0$

5. $(c-1)(z-1)c_2F_1(a,b,c-1|z)+(a-c)(b-c)_2F_1(a,b,c+1|z)+c(c+(a+b-2c+1)z-1)_2F_1(a,b,c|z)=0$

We will solve for ${}_2F_1(a+1,b,c+1|z)$ in relation 1 and substitute it into relation 2 to get:

$$
\begin{aligned}
{}_2F_1(a,b,c+1|z) = &-\frac{a(b-c-1)}{a(c+1)+(z-1)(c-1)(a-c)}{}_2F_1(a+1,b,c+2|z) \\
&-\frac{c(c-1)(z-1)}{a(c+1)+(z-1)(c-1)(a-c)}{}_2F_1(a,b,c|z)
\end{aligned}
$$

Then, we will solve for ${}_2F_1(a,b-1,c-1|z)$ in relation 3 and substitute it into relation 4 to get:

$$
\begin{aligned}
{}_2F_1(a,b,c-1|z) = &\frac{(a-c+1)(b-c+1)z}{(c-1)((1-b)-(b-c-1)(z-1))}{}_2F_1(a,b,c|z) \\
&-\frac{c-2}{(c-1)((1-b)-(b-c-1)(z-1))}{}_2F_1(a,b-1,c-2|z)
\end{aligned}
$$

Now, we will substitute these two new relations into relation 5 to see that:

$$
f(z)_2F_1(a,b-1,c-2|z)+g(z)_2F_1(a,b,c|z)+h(z)_2F_1(a+1,b,c+2|z)=0
$$

or

$$
f(z)\frac{(d+2)!(d+3)!}{(-1)^{d+1}(2d+4)!}Q_{d+1}+g(z)\frac{(d+1)!(d+2)!}{(-1)^{d}(2d+2)!}Q_d+h(z)\frac{d!(d+1)!}{(-1)^{d+1}(2d)!}Q_{d-1}=0
$$

where:

$$
\begin{aligned}
&f(z)=(2-c)(-1+c)c(a(1+c)+(a-c)(-1+c)(-1+z))(-1+z), \\
&g(z)=(-1+c)(-1+z)(2+c(-1+z)-(1+b)z)+(1+a-c)(1+b-c)c(-1+z)z \\
&\qquad (a(2+(-1+c)z)+c(-1+c+z-cz))+c(2+c(-1+z)-(1+b)z) \\
&\qquad (-1+c+(1+a+b)z-2cz)(a(2+(-1+c)z)+c(-1+c+z-cz))
\end{aligned}
$$

and

$$h(z) = -a(a-c)(-b+c)(1-b+c)z(2+c(-1+z)-(1+b)z)$$

Let us suppose that Q_d and Q_{d+1} have a common zero. Then either Q_{d-1} or $h(z)$ would also have this zero.

Claim: $h(z)$ does not have a common zero. The roots of $h(z)$ are $z = 0$ and

$$z = \frac{-2+c}{-1-b+c} = \frac{-2(d+2)}{\frac{-3}{2}(d+2)} = \frac{4}{3}.$$

We will show that these roots are not roots of Q_d. For any hypergeometric function,

$$_2F_1(a, b, c|0) = 1.$$

Thus, when $z = 0$, we have

$$Q_d(0) = (-1)^d \frac{(2d+2)!}{(d+1)!(d+2)!} \neq 0.$$

When $z = \frac{4}{3}$, we have $4\left(\frac{c_d^2}{c_{d-1}}\right) = \frac{4}{3}$; that is $\frac{c_d^2}{c_{d-1}} = \frac{1}{3}$. When evaluating, we will deduce that

$$\begin{aligned} Q_d\left(\frac{1}{3}\right) &= \frac{1}{(d+2)!} \sum_{l=0}^{\lfloor \frac{d+1}{2} \rfloor} (-1)^{l+d} \frac{(2d+2-l)!}{l!(d+1-2l)!} \left(\frac{1}{3}\right)^{\lfloor \frac{d+1}{2} \rfloor - l} \\ &= \frac{(-1)^d}{(d+2)!} \left(\frac{1}{3}\right)^{\lfloor \frac{d+1}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{d+1}{2} \rfloor} (-3)^l \frac{(2d+2-l)!}{l!(d+1-2l)!}. \end{aligned}$$

It is easy to verify that

$$\sum_{l=0}^{\lfloor \frac{d+1}{2} \rfloor} (-3)^l \frac{(2d+2-l)!}{l!(d+1-2l)!} \neq 0.$$

Thus $Q_d(\frac{1}{3}) \neq 0$ and we can conclude that Q_d and $h(z)$ do not have a common zero. Therefore, Q_{d-1} must have a common zero with Q_d and Q_{d+1}. By induction, Q_0, Q_1, and Q_2 would have common zeros. By definition, $Q_0 = 1$, $Q_1 = 1 - 2\frac{c_d^2}{c_{d-1}}$, $Q_2 = -5 + 5\frac{c_d^2}{c_{d-1}}$. Clearly, these do not have the same zeros, so we have reached a contradiction. Therefore, λ_{d-1} and λ_d have no roots in common. $\square$

Since λ_{d-1} and λ_d have no roots in common, it suffices to show that there is a root of λ_{d-1} that is not a root of γ_d. That is, we need to check that with our choice of c_d and c_{d-1}, $\gamma_d \in \mathbb{C}[c_{d-2}]\backslash\mathbb{C}$.

From Equation 2.0.1, we can derive γ_r:

$$\begin{aligned}
v_r &= -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{1}{a!b!(j-2a-3b)!} v_{r-j}^{(j-a-2b)} U_0^{j-2a-3b} U_1^a U_2^b \\
&= -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(\lambda_{r-j}Y^{r+2-j} + \gamma_{r-j}Y^{r+1-j} + K_{r-j}(Y))^{(j-a-2b)}}{a!b!(j-2a-3b)!} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{2j-a-3b} \\
&= -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(r+2-j)\cdots(r-2j+a+2b+3)}{a!b!(j-2a-3b)!} \lambda_{r-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r+2-b} \\
&\quad -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(r+1-j)\cdots(r-2j+a+2b+2)}{a!b!(j-2a-3b)!} \gamma_{r-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r+1-b} \\
&\quad -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(r-j)\cdots(r-2j+a+2b+1)}{a!b!(j-2a-3b)!} \delta_{r-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r-b}
\end{aligned}$$

where $\delta_{r-j} \in \mathbb{C}$. To calculate γ_r, we will use $b = 1$ in the first summation and $b = 0$ in the second because γ_r is the coefficient of Y^{r+1}.

$$\begin{aligned}
\gamma_r &= -\sum_{j=3}^{r} \sum_{0\le 2a+3\le j} \frac{(r+2-j)\cdots(r-2j+a+5)}{a!(j-2a-3)!} \lambda_{r-j} c_d^{j-2a-3} c_{d-1}^a c_{d-2} \\
&\quad -\sum_{j=1}^{r} \sum_{0\le 2a\le j} \frac{(r+1-j)\cdots(r-2j+a+2)}{a!(j-2a)!} \gamma_{r-j} c_d^{j-2a} c_{d-1}^a \\
&= -\sum_{j=3}^{r} \sum_{0\le 2a+3\le j} \frac{(r+2-j)!}{a!(j-2a-3)!(r-2j+a+4)!} \lambda_{r-j} c_d^{j-2a-3} c_{d-1}^a c_{d-2} \\
&\quad -\sum_{j=1}^{r} \sum_{0\le 2a\le j} \frac{(r+1-j)!}{a!(j-2a)!(r-2j+a+1)!} \gamma_{r-j} c_d^{j-2a} c_{d-1}^a
\end{aligned} \tag{2.0.3}$$

Claim: $\gamma_d \in \mathbb{C}[c_{d-2}]\backslash\mathbb{C}$. To see this, it suffices to show that

$$T_d = -\sum_{j=3}^{d} \sum_{0\leq 2a+3\leq j} \frac{(d+2-j)!}{a!(j-2a-3)!(d-2j+a+4)!} \lambda_{d-j} c_d^{j-2a-3} c_{d-1}^{a} \notin \mathbb{C}.$$

In fact, by Appendix B, we have:

$$\begin{aligned} T_d &= -\sum_{j=3}^{d} \sum_{0\leq 2a+3\leq j} \frac{(d+2-j)!}{a!(j-2a-3)!(d-2j+a+4)!} \lambda_{d-j} c_d^{j-2a-3} c_{d-1}^{a} \\ &= c_{d-1}^{\lfloor \frac{d-2}{2} \rfloor} \sum_{\alpha=0}^{\lfloor \frac{d-2}{2} \rfloor} (-1)^{\alpha+d-1}(-3)^{\alpha}(-2)^{d-2\alpha-2} \left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor \frac{d-2}{2} \rfloor - \alpha}. \end{aligned}$$

Now, we will show that there exists a root of Q_{d-1} that is not a root of T_d. Note $\deg Q_{d-1} = \lfloor \frac{d}{2} \rfloor$ and $\deg T_d = \lfloor \frac{d-2}{2} \rfloor$. Let's assume all the roots of Q_{d-1} are also roots of T_d. We will prove by contradiction that they do not have the same roots. Since $\deg Q_{d-1} = \deg T_d + 1$, there exist complex numbers $s, t \in \mathbb{C}$ such that $Q_{d-1}\left(\frac{c_d^2}{c_{d-1}}\right) = \left(s\left(\frac{c_d^2}{c_{d-1}}\right) + t\right) T_d\left(\frac{c_d^2}{c_{d-1}}\right)$. We will solve for s and t by comparing coefficients and then we will show a contradiction.

Comparing coefficients of $\left(\frac{c_d^2}{c_{d-2}}\right)^{\lfloor \frac{d}{2} \rfloor}$, we can explicitly compute s:

$$s = \frac{(2d)!}{(-2)^{d-2}(d+1)!d!}$$

Comparing coefficients of $\left(\frac{c_d^2}{c_{d-2}}\right)^{\lfloor \frac{d-2}{2} \rfloor}$, we can explicitly compute t:

$$t = -\frac{(2d-1)!}{(-2)^{d-2}(d+1)!d-2!}$$

Comparing coefficients of $\left(\frac{c_d^2}{c_{d-2}}\right)^{\lfloor \frac{d-2}{2} \rfloor - 1}$, we obtain:

$$\begin{aligned} \frac{(2d-2)!}{2(d+1)!(d-4)!} &= \frac{(2d)!(-3)^2(-2)^{d-6}}{(d+1)!d!(-2)^{d-2}} + \frac{(2d-1)!(-3)(-2)^{d-4}}{(d+1)!(d-2)!(-2)^{d-2}} \\ &= \frac{144(2d)! - 6(2d)!(d-1)}{(d+1)!d!} \end{aligned}$$

which is impossible. Therefore, with our choice of c_d and c_{d-1}, we will see that $\gamma_d \in \mathbb{C}[c_{d-2}]\backslash\mathbb{C}$. In summary, we have shown that we can choose c_d and c_{d-1} so that:

1. $\lambda_{d-1} = 0$,

2. λ_d is an arbitrary element of $\mathbb{C}^*$, and

3. $\gamma_d \in \mathbb{C}[c_{d-2}]\backslash\mathbb{C}$.

We can choose c_{d-2} so that $\gamma_d = a_{d+1} \in \mathbb{C}^*$.

Therefore, we have:

$$\sigma_Z(Z^d X) - Z^d r_1 X \equiv Z^d[v_d(Y) + W_1(Y)] \mod Z^{d+1}$$

and dividing by Z^d:

$$\begin{aligned}\sigma_Z(X) &\equiv r_1 X + v_d(Y) + W_1(Y) \mod Z \\ &\equiv r_1 X + \lambda_d Y^{d+2} + \gamma_d Y^{d+1} + M_d(Y) + W_1(Y) \mod Z\end{aligned}$$

From our choice of $c_d, c_{d-1}, c_{d-2} \in \mathbb{C}$, $\lambda_d = a_{d+2}$ and $\gamma_d = a_{d+1}$. We can finally specify $M_d(Y) + W_1(Y) = \sum_{k=0}^{d} a_k Y^k$.

When looking at $\sigma_Z(Y)$, after computing, we will have:

$$\begin{aligned}\sigma_Z(Y) &= r_2 Y + Z^{d+1} r_1 X + ZU(Y,Z) + r_3 \\ &\equiv r_2 Y + r_3 \mod Z\end{aligned}$$

Therefore,

$$\sigma \equiv (r_1 X + \sum_{k=0}^{d+2} a_k Y^k, r_2 Y + r_3) \mod Z$$

and we can conclude that $\mathcal{G}_{(d+2)} \subset \overline{\mathcal{G}_{(d,3)}}$. $\square$

CHAPTER 3

PROOF OF MAIN THEOREM 2

It is natural to examine what will happen when we increase the value for e. In this chapter, we will use the same method as chapter 3 in order to prove the following theorem:

Theorem 3.0.1. *For all integers* $2 \leq d \leq 45$*,* $\mathcal{G}_{(d+3)} \cap \overline{\mathcal{G}_{(d,4)}} \neq \emptyset$.

Proof. Let $\sigma \in \mathcal{G}_{(d+3)}$ be a triangular automorphism of degree $d+3$. Then we can write:

$$\sigma = (X + \sum_{k=0}^{d+3} a_k Y^k, Y)$$

for some $a_{d+3} \in \mathbb{C}^*$ and $a_i \in \mathbb{C}$ for $0 \leq i \leq d+2$. To prove $\sigma \in \overline{\mathcal{G}_{(d,4)}}$ using the Valuation Criterion (1.1.6), we will explicitly construct an automorphism $\sigma_Z \in \mathcal{G}(\mathbb{C}[Z])$ such that, writing $\sigma_Z = (f_Z, g_Z)$ and working modulo Z, $f_Z \equiv \sigma(X)$ and $g_Z \equiv \sigma(Y)$.

Let $U(Y,Z), V(Y,Z), W_1(Y)$, and $W_2(Y)$ be arbitrary polynomials for now; we will define them momentarily, but we will assume $\deg_Y U(Y,Z) = 4$, $\deg_Y V(Y,Z) = d$, and $\deg W_1(Y)$, $W_2(Y) \leq d$. The degree of $U(Y,Z)$ here is the first change from the previous case. Consider the following polynomial automorphisms:

$$\tau_1 = (X + \frac{U(Y,Z)}{Z^{d+1}}, Y)$$

$$\alpha = (X, Y + Z^{d+2}X)$$

$$\tau_2 = (X - \frac{V(Y,Z)}{Z^{d+1}} + \frac{W_1(Y)}{Z^2} + \frac{W_2(Y)}{Z}, Y)$$

Set $\sigma_Z = \tau_2\alpha\tau_1$. Notice that $\sigma_Z \in \mathcal{G}(\mathbb{C}((Z)))_{(d,4)}$. Direct computation shows:

$$\begin{aligned} f_Z &= \sigma_Z(X) \\ &= X + \frac{U(Y,Z)}{Z^{d+1}} - \frac{V(Y+ZU(Y,Z),Z)}{Z^{d+1}} + \frac{W_1(Y+ZU(Y,Z))}{Z^2} + \frac{W_2(Y+ZU(Y,Z))}{Z} \\ g_Z &= \sigma_Z(Y) = Y + Z^{d+2}X + ZU(Y,Z) \end{aligned}$$

and we will show that $f_Z, g_Z \in \mathbb{C}[Z][X,Y]$, and

$$\sigma_Z(X) \equiv X + \sum_{k=0}^{d+3} a_k Y^k \qquad \mod Z$$

$$\sigma_Z(Y) \equiv Y \qquad \mod Z.$$

We specify further that $U(Y,Z) = \sum_{i=0}^{3} U_i(Y)Z^i \in \mathbb{C}[Y,Z]$ such that $\deg_{(1,-1)} U(Y,Z) \leq 2$. By the Formal Inverse Function Theorem [14], we define $I(Y,Z) \in \mathbb{C}[[Y,Z]]$ to be the formal inverse of $Y + ZU(Y,Z)$. In other words, $I(Y,Z)$ is the uniquely determined element in $\mathbb{C}[[Y,Z]]$ satisfying $I(Y+ZU(Y,Z),Z) = Y$. We can write

$$U(I(Y,Z),Z) = \sum_{k=0}^{\infty} v_k(Y)Z^k \in \mathbb{C}[[Y,Z]]$$

where, a priori, $v_k(Y) \in \mathbb{C}[[Y]]$, but in fact we will see $v_k(Y) \in \mathbb{C}[Y]$. We now define $V(Y,Z)$ as the truncation of $U(I(Y,Z),Z)$:

$$V(Y,Z) = \sum_{k=0}^{d-2} v_k(Y)Z^k \in \mathbb{C}[Y,Z].$$

Then we will have:

$$\begin{aligned} U(Y,Z) &= U(I(Y+ZU(Y,Z),Z),Z) \\ &\equiv V(Y+ZU(Y,Z),Z) && \mod Z^{d-1} \\ &\equiv \sum_{k=0}^{d-2} v_k(Y+ZU(Y,Z))Z^k && \mod Z^{d-1} \end{aligned}$$

Applying Taylor's Formula, we have:

$$U(Y,Z) = \sum_{m=0}^{d-2} Z^m \sum_{j=0}^{m} \frac{1}{j!} v_{m-j}^{(j)}(Y) U(Y,Z)^j$$

and substituting in $U(Y,Z) = U_0 + ZU_1 + Z^2U_2 + Z^3U_3$, where $U_0 = c_dY^2$, $U_1 = c_{d-1}Y^3$, $U_2 = c_{d-2}Y^4$, and $U_3 = c_{d-3}Y^4$, then:

$$U(Y,Z) = \sum_{m=0}^{d-2} Z^m \sum_{j=0}^{m} \sum_{0\le 2a+3b+4c\le j} \frac{1}{a!b!c!(j-2a-3b-4c)!} v_{m-j}^{(j-a-2b-3c)} U_0^{j-2a-3b-4c} U_1^a U_2^b U_3^c.$$

Comparing coefficients of Z^k, we obtain the following recursive relation for $v_k(Y)$:

$$\begin{aligned} v_0 &= U_0 \\ v_1 &= U_1 - v_0'U_0 \\ v_2 &= U_2 - [v_1'U_0 + \frac{1}{2}v_0''U_0^2 + v_0'U_1] \\ v_3 &= U_3 - [v_2'U_0 + \frac{1}{2}v_1''U_0^2 + v_1'U_1 + v_0''U_0U_1 + v_0'U_2] \\ v_r &= -\sum_{j=1}^{r} \sum_{0\le 2a+3b+4c\le j} \frac{1}{a!b!c!(j-2a-3b-4c)!} v_{r-j}^{(j-a-2b-3c)} U_0^{j-2a-3b-4c} U_1^a U_2^b U_3^c \end{aligned} \tag{3.0.1}$$

Lemma 3.0.2. $\deg v_r(Y) \le r+2$.

Proof. Proof by induction. Writing $U_0 = c_dY^2$, $U_1 = c_{d-1}Y^3$, $U_2 = c_{d-2}Y^4$, and $U_3 = c_{d-3}Y^4$ as above,

$$v_0(Y) = U_0 = c_d Y^2$$

thus $\deg v_0 = 2$.

$$v_1(Y) = U_1 - v_0' U_0 = (c_{d-1} - 2c_d^2)Y^3$$

thus $\deg v_1 = 3$.

$$v_2(Y) = U_2 - [v_1' U_0 + \frac{1}{2} v_0'' U_0^2 + v_0' U_1] = (c_{d-2} - 5c_{d-1}c_d + 5c_d^3)Y^4$$

thus $\deg v_2 = 4$.

$$\begin{aligned} v_3(Y) &= U_3 - [v_2' U_0 + \frac{1}{2} v_1'' U_0^2 + v_1' U_1 + v_0'' U_0 U_1 + v_0' U_2] \\ &= (-3c_{d-1}^2 - 6c_{d-2}c_d + 21c_{d-1}c_d^2 - 14c_d^4)Y^5 + c_{d-3}Y^4 \end{aligned}$$

thus $\deg v_3 = 5$.

Now let's suppose $\deg v_r(Y) \leq r + 2$. We will show that $\deg v_{r+1}(Y) \leq r + 3$. By definition,

$$v_{r+1} = -\sum_{j=1}^{r+1} \sum_{0 \leq 2a+3b+4c \leq j} \frac{1}{a!b!c!(j-2a-3b-4c)!} v_{r+1-j}^{(j-a-2b-3c)} U_0^{j-2a-3b-4c} U_1^a U_2^b U_3^c.$$

Note the following degrees:

$$\begin{aligned}
\deg v_{r+1-j} &\leq r+3-j \\
\deg v_{r+1-j}^{(j-a-2b-3c)} &\leq r+3-2j+a+2b+3c \\
\deg U_0^{j-2a-3b-4c} &\leq 2j-4a-6b-8c \\
\deg U_1^a &\leq 3a \\
\deg U_2^b &\leq 4b \\
\deg U_3^c &\leq 4c
\end{aligned}$$

When looking at each term in the above summation for v_{r+1}, we will see that

$$\deg\left(v_{r-j}^{(j-a-2b-3c)} U_0^{j-2a-3b-4c} U_1^a U_2^b U_3^c\right) \leq r+3-c$$

and when $c=0$, we will get the maximum degree of $r+3$. □

We can also study the forms of each $v_k(Y)$ through the next lemma.

Lemma 3.0.3. *$v_r(Y) = \lambda_r Y^{r+2} + \gamma_r Y^{r+1} + \delta_r Y^r + M_r(Y)$, with $\lambda_r \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$, $\gamma_r, \delta_r \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}, c_{d-3}]$, and $M_r(Y) \in \mathbb{C}[Y]$ such that $\deg M_r(Y) \leq r-1$.*

Proof. We proceed by induction from Equation 3.0.1:

$$\begin{aligned}
r=0: \quad & v_0(Y) = U_0 = c_d Y^2 \\
r=1: \quad & v_1(Y) = U_1 - v_0' U_0 = (c_{d-1} - 2c_d^2) Y^3 \\
r=2: \quad & v_2(Y) = U_2 - [v_1' U_0 + \frac{1}{2} v_0'' U_0^2 + v_0' U_1] = (c_{d-2} - 5c_{d-1}c_d + 5c_d^3) Y^4 \\
r=3: \quad & v_3(Y) = U_3 - [v_2' U_0 + \frac{1}{2} v_1'' U_0^2 + v_1' U_1 + \frac{1}{2} v_0'' U_0 U_1 + v_0' U_2] \\
& \qquad = (-3c_{d-1}^2 - 6c_{d-2}c_d + 21c_{d-1}c_d^2 - 14c_d^4) Y^5 + c_{d-3} Y^4
\end{aligned}$$

Suppose $v_k(Y) = \lambda_k Y^{k+2} + \gamma_k Y^{k+1} + \delta_k Y^k + M_k(Y)$ with $\lambda_k \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$,

$\gamma_k, \delta_k \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}, c_{d-3}]$, and $M_k(Y) \in \mathbb{C}[Y]$ such that $\deg M_k(Y) \leq k-1$ for all $0 \leq k \leq r$. We want to show that $v_{r+1}(Y) = \lambda_{r+1}Y^{r+3} + \gamma_{r+1}Y^{r+2} + \delta_{r+1}Y^{r+1} + M_{r+1}(Y)$ with $\lambda_{r+1} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$, $\gamma_{r+1}, \delta_{r+1} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}, c_{d-3}]$, and $\deg M_{r+1}(Y) \leq r$. By definition,

$$v_{r+1}(Y) = -\sum_{j=1}^{r+1} \sum_{0 \leq 2a+3b+4c \leq j} \frac{1}{a!b!c!(j-2a-3b-4c)!} v_{r+1-j}^{(j-a-2b-3c)} U_0^{j-2a-3b-4c} U_1^a U_2^b U_3^c$$

By our induction hypothesis, we know that each $v_{r+1-j}(Y)$ have the correct form with $\lambda_{r+1-j} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$. We can deduce that each term in the summation will look like:

$$\theta_{j,a,b,c}\lambda_{r+1-j}c_d^{j-2a-3b-4c}c_{d-1}^a c_{d-2}^b c_{d-3}^c Y^{r+3-c}$$

where $\theta_{j,a,b,c} \in \mathbb{Q}$. For the coefficient of Y^{d+3} term, set $c = 0$ to see that each term will have the form:

$$\theta_{j,a,b,0}\lambda_{r+1-j}c_d^{j-2a-3b}c_{d-1}^a c_{d-2}^b.$$

$$\lambda_{r+1} = \sum_{j=1}^{r+1} \sum_{0 \leq 2a+3b \leq j} \theta_{j,a,b,0}\lambda_{r+1-j}c_d^{j-2a-3b}c_{d-1}^a c_{d-2}^b \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}].$$

since each $\lambda_{r+1-j} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}]$. It is easy to see that

$$\gamma_{r+1}, \delta_{r+1} \in \mathbb{Q}[c_d, c_{d-1}, c_{d-2}, c_{d-3}]$$

and when $c \geq 3$, we will have a polynomial $M_{r+1}(Y)$ such that $\deg M_{r+1} \leq r$. □

Using the recursive definition for $v_r(Y)$, Equation 3.0.1, we are able to write a recursive formula for λ_r. Since λ_r is the coefficient of Y^{r+3}, we will take $c = 0$:

$$
\begin{aligned}
v_r|_{c=0} &= -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{1}{a!b!(j-2a-3b)!} v_{r-j}^{(j-a-2b)} U_0^{j-2a-3b} U_1^a U_2^b \\
&= -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{1}{a!b!(j-2a-3b)!} (\lambda_{r-j} Y^{r-j+2})^{(j-a-2b)} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{2j-a-2b} \\
&= -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(r-j+2)\cdots(r-2j+a+2b+3)}{a!b!(j-2a-3b)!} \lambda_{r-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b Y^{r+2}
\end{aligned}
$$

Therefore

$$
\lambda_r = -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(r-j+2)!}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!} \lambda_{r-j} c_d^{j-2a-3b} c_{d-1}^a c_{d-2}^b \qquad (3.0.2)
$$

Recall that we want to show $\sigma_Z \in \mathbb{C}[Z][X, Y]$ and

$$
\sigma_Z \equiv (X + \sum_{k=0}^{d+3} a_k Y^k, Y) \mod Z
$$

or, for the sake of simplicity, let us write $\sigma_Z(X)$ as

$$
\sigma_Z(X) = X + \sum \mu_{s,t} \frac{Y^s}{Z^t}.
$$

Using Lemma 2.0.4, we can take a look at the degrees of the following polynomials:

$$
\begin{aligned}
&\deg_{(1,-1)} V(Y, Z) = 2 \qquad \text{(by Lemma 3.0.3)} \\
&\deg_{(1,-1)} \frac{V(Y, Z)}{Z^d} = d + 2.
\end{aligned}
$$

Therefore we have:

$$\begin{aligned}&\deg_{(1,-1)} \sigma_Z(X) = \\ &\deg_{(1,-1)} \left(\frac{U(Y,Z)}{Z^{d+1}} - \frac{V(Y+ZU(Y,Z),Z)}{Z^{d+1}} + \frac{W_1(Y+ZU(Y,Z))}{Z^2} + \frac{W_2(Y+ZU(Y,Z))}{Z}\right) \\ &\qquad \le d+3\end{aligned}$$

From this we can conclude that for $s > d+3$, $\mu_{s,0} = 0$. More generally, for $s+t > d+3$, $\mu_{s,t} = 0$. Since $U(Y,Z) \equiv V(Y+ZU(Y,Z),Z) \mod Z^{d-1}$, $\mu_{s,t} = 0$ for $t > 2$. All that is left to show is what happens when $t = 0,1,2$.

To make computations easier, we will multiply $\sigma_Z(X)$ by Z^{d+1} to get:

$$\begin{aligned}\sigma_Z(Z^{d+1}X) - Z^{d+1}X &= U(Y,Z) - V(Y+ZU,Z) + Z^{d-1}W_1(Y+ZU) + Z^dW_2(Y+ZU) \\ &\equiv Z^{d-1}v_{d-1}(Y+ZU) + Z^dv_d(Y+ZU) + Z^{d+1}v_{d+1}(Y+ZU) \\ &\quad + Z^{d-1}W_1(Y+ZU) + Z^dW_2(Y+ZU) \qquad \mod Z^{d+2} \\ &\equiv Z^{d-1}[v_{d-1}(Y) + W_1(Y)] + Z^d[U(Y,Z)v'_{d-1}(Y) + v_d(Y) + W_2(Y) \\ &\quad + W'_1(Y)U(Y,Z)] + Z^{d+1}[\frac{1}{2}v''_{d-1}(Y)U^2(Y,Z) + v'_d(Y)U(Y,Z) \\ &\quad + v_{d+1}(Y) + W'_2(Y)U(Y,Z) + \frac{1}{2}W''_1(Y)U^2(Y,Z)] \qquad \mod Z^{d+2} \\ &\equiv Z^{d-1}[v_{d-1}(Y) + W_1(Y)] + Z^d[U_0v'_{d-1}(Y) + v_d(Y) + W_2(Y) + W'_1(Y)U_0] \\ &\quad + Z^{d+1}[\frac{1}{2}v''_{d-1}(Y)U_0^2 + v'_d(Y)U_0 + v_{d+1}(Y) + W'_2(Y)U_0 + \frac{1}{2}W''_1(Y)U_0^2 \\ &\quad + U_1v'_{d-1}(Y) + U_1W'_1(Y)] \qquad \mod Z^{d+2}\end{aligned}$$

Recall that we have yet to specify $c_d, c_{d-1}, c_{d-2}, c_{d-3} \in \mathbb{C}$, and $W_1(Y), W_2(Y) \in \mathbb{C}[Y]$. The next step is to show that we can specify these such that $v_{d-1}(Y) + W_1(Y) = 0$, $U_0v'_{d-1}(Y) + v_d(Y) + W_2(Y) + W'_1(Y)U_0 = 0$, and

$$\frac{1}{2}v''_{d-1}(Y)U_0^2 + v'_d(Y)U_0 + v_{d+1}(Y) + W'_2(Y)U_0 + \frac{1}{2}W''_1(Y)U_0^2 + U_1v'_{d-1}(Y) + U_1W'_1(Y) \neq 0.$$

Specifically, we want it to be $\sum_{k=0}^{d+3} a_k Y^k$.

By definition, $v_{d-1} = \lambda_{d-1}Y^{d+1} + \gamma_{d-1}Y^d + \delta_{d-1}Y^{d-1} + M_{d-1}(Y)$, where $\deg M_{d-1} \le d-2$. We can set $W_1(Y) = -\gamma_{d-1}Y^d - \delta_{d-1}Y^{d-1} - M_{d-1}(Y)$ and we will therefore need $\lambda_{d-1}Y^{d+1} = 0$. Then:

$$\begin{aligned} U_0 v'_{d-1}(Y) + v_d(Y) + W_2(Y) + W'_1(Y)U_0 &= U_0[v'_{d-1}(Y) + W'_1(Y)] + v_d(Y) + W_2(Y) \\ &= v_d(Y) + W_2(Y) \\ &= \lambda_d Y^{d+2} + \gamma_d Y^{d+1} + \delta_d Y^d + M_d(Y) + W_2(Y). \end{aligned}$$

We can set $W_2(Y) = -\delta_d Y^d - M_d(Y)$ and we will need $\lambda_d = 0$ and $\gamma_d = 0$. In summary, we have specified $W_1(Y)$ and $W_2(Y)$. Suppose we have chosen c_d, c_{d-1}, c_{d-2} such that $\lambda_{d-1} = 0$, $\lambda_d = 0$ and $\gamma_d = 0$. Then:

$$\frac{1}{2}U_0^2[v''_{d-1}(Y) + W''_1(Y)] + U_0[v'_d(Y) + W'_2(Y)] + U_1[v'_{d-1}(Y) + W'_1(Y)] + v_{d+1}(Y) = v_{d+1}(Y).$$

We want to show that we can have

$$v_{d+1}(Y) = \lambda_{d+1}Y^{d+3} + \gamma_{d+1}Y^{d+2} + \delta_{d+1}Y^{d+1} + M_{d+1}(Y) = \sum_{k=0}^{d+3} a_k Y^k,$$

with $a_{d+3} \neq 0$. To show $\mathcal{G}_{(d+3)} \cap \overline{\mathcal{G}_{(d,4)}} \neq \emptyset$, we only need to show that after we have chosen c_d, c_{d-1}, and c_{d-2}, $\lambda_{d+1} \neq 0$.

Taking a closer look at the λ_i's, we can write:

$$\lambda_i = c_{d-1}^{\lfloor \frac{i+1}{2} \rfloor} c_d^{\epsilon_i} K_i\left(\frac{c_d^2}{c_{d-1}}, c_{d-2}\right).$$

where $\epsilon_i = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$

Using the recursive formula for λ_r, Equation 3.0.3, we can derive an explicit formula for K_i:

Lemma 3.0.4.

$$K_{n-1} = \frac{1}{(n+1)!} \sum_{0 \leq 2l+3m \leq n} (-1)^{n+l+2m+1} \frac{(2n-l-2m)!}{l!m!(n-2l-3m)!} \left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor \frac{n-3m}{2} \rfloor - l} c_{d-2}^m$$

Proof. Proof by induction.

$n = 1$: $K_0 = \frac{2!}{2!} = 1$ and by definition, $\lambda_0 = c_d$ which would imply that $K_0 = 1$.

Let's suppose that for all integers $1 \leq n \leq r$,

$$K_{n-1} = \frac{1}{(n+1)!} \sum_{0 \leq 2l+3m \leq n} (-1)^{n+l+2m+1} \frac{(2n-l-2m)!}{l!m!(n-2l-3m)!} \left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor \frac{n-3m}{2} \rfloor - l} c_{d-2}^m.$$

We will show that

$$K_r = \frac{1}{(r+2)!} \sum_{0 \leq 2l+3m \leq r+1} (-1)^{r+l+2m+2} \frac{(2r-l-2m+2)!}{l!m!(r-2l-3m+1)!} \left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor \frac{r-3m+1}{2} \rfloor - l} c_{d-2}^m$$

by the recursive formula for λ_r.

$$\lambda_r = -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(r-j+2)!}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!} \lambda_{r-j} c_d^{j-2a-3b} c_{d-1}^{a} c_{d-2}^{b}$$

$$= -\sum_{j=1}^{r} \sum_{0\le 2a+3b\le j} \frac{(r-j+2)!}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!} c_{d-1}^{\lfloor \frac{r-j+1}{2} \rfloor} c_d^{\epsilon_{r-j}} K_{r-j} c_d^{j-2a-3b} c_{d-1}^{a} c_{d-2}^{b}$$

$$= -\sum_{j=1}^{r} \sum_{2a+3b\le j} \sum_{2l+3m\le r-j+1} \frac{(-1)^{r-j+l+2m+2}(2r-2j-l-2m+2)!}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!l!m!(r-j-2l-3m+1)!}$$
$$c_d^{r-2a-2l-3b-3m} c_{d-1}^{a+l} c_{d-2}^{b+m}$$

$$= -\sum_{2l+3m\le r+1} \frac{(-1)^{r+l+2m+2}(2r-2j-l-2m+2)!}{(r+2)!l!m!(r-2l-3m+1)!} c_d^{r-2l-3m} c_{d-1}^{l} c_{d-2}^{m}$$

$$-\sum_{j=0}^{r} \sum_{2a+3b\le j} \sum_{2l+3m\le r-j+1} \frac{(-1)^{r-j+l+2m+2}(2r-2j-l-2m+2)!}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!l!m!(r-j-2l-3m+1)!}$$
$$c_d^{r-2a-2l-3b-3m} c_{d-1}^{a+l} c_{d-2}^{b+m} \qquad \dagger$$

$$= -\sum_{2l+3m\le r+1} \frac{(-1)^{r+l+2m+2}(2r-2j-l-2m+2)!}{(r+2)!l!m!(r-2l-3m+1)!} c_d^{r-2l-3m} c_{d-1}^{l} c_{d-2}^{m}$$

$$= c_{d-1}^{\lfloor \frac{r+1}{2} \rfloor} c_d^{\epsilon_r} \sum_{2l+3m\le r+1} \frac{(-1)^{r+l+2m+2}(2r-2j-l-2m+2)!}{(r+2)!l!m!(r-2l-3m+1)!} c_d^{2\lfloor \frac{r-3m+1}{2} \rfloor -2l} c_{d-1}^{l-\lfloor \frac{r-3m+1}{2} \rfloor} c_{d-2}^{m}$$

$$= c_{d-1}^{\lfloor \frac{r+1}{2} \rfloor} c_d^{\epsilon_r} \sum_{2l+3m\le r+1} \frac{(-1)^{r+l+2m+2}(2r-2j-l-2m+2)!}{(r+2)!l!m!(r-2l-3m+1)!} \left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor \frac{r-3m+1}{2} \rfloor -l} c_{d-2}^{m}$$

Since $\lambda_r = c_{d-1}^{\lfloor \frac{r+1}{2} \rfloor} c_d^{\epsilon_r} K_r \left(\frac{c_d^2}{c_{d-1}}, c_{d-2}\right)$, we deduce that

$$K_r = \sum_{2l+3m\le r+1} \frac{(-1)^{r+l+2m+2}(2r-2j-l-2m+2)!}{(r+2)!l!m!(r-2l-3m+1)!} \left(\frac{c_d^2}{c_{d-1}}\right)^{\lfloor \frac{r-3m+1}{2} \rfloor -l} c_{d-2}^{m}$$

and this completes the induction step and the proof.

†This term is equal to zero. See Appendix C for proof.

□

Notice when $m = 0$, K_{n-1} is equivalent to Q_{n-1} from Lemma 2.0.5.

We need to show that there exists $c_d, c_{d-1}, c_{d-2} \in \mathbb{C}$ such that $\lambda_{d-1} = 0$, $\lambda_d = 0$ and $\lambda_{d+1} \neq 0$.

In the previous chapter we used a nice property of hypergeometric polynomials in Lemma 2.0.6 to show that there exists complex numbers such that $\lambda_d = 0$ and $\lambda_{d+1} \neq 0$; however, this lemma states that if there exists a complex number such that $\lambda_{d-1} = 0$, then $\lambda_d \neq 0$. So we are unable to use this method here. Using Gröbner basis in Mathematica, we are able to check that for $2 \leq d \leq 45$, we are able to have $\lambda_d = 0, \lambda_{d-1} = 0$ and $\lambda_{d+1} = a_{d+3}$.

Therefore, we have:

$$\sigma_Z(Z^d X) - Z^d X \equiv Z^{d+1}[v_{d+1}(Y)] \mod Z^{d+1}$$

and dividing by Z^{d+1}:

$$\begin{aligned} \sigma_Z(X) &\equiv X + v_{d+1}(Y) && \mod Z \\ &\equiv X + \lambda_{d+1} Y^{d+3} + \gamma_{d+1} Y^{d+2} + \delta_{d+1} Y^{d+1} + M_{d+1}(Y) && \mod Z \end{aligned}$$

From our choice of $c_d, c_{d-1}, c_{d-2} \in \mathbb{C}$, $\lambda_{d+1} = a_{d+3} \neq 0$.

When looking at $\sigma_Z(Y)$, after the composition we will have:

$$\begin{aligned} \sigma_Z(Y) &= Y + Z^{d+2} X + ZU(Y, Z) \\ &\equiv Y \mod Z \end{aligned}$$

Therefore,

$$\sigma_Z \equiv (X + \sum_{k=0}^{d+3} a_k Y^k, Y) \mod Z$$

where $a_{d+3} \in \mathbb{C}^*$. Hence, $\sigma \in \mathcal{G}_{(d+3)} \cap \overline{\mathcal{G}_{(d,4)}}$ for $2 \leq d \leq 45$.

□

Example 3.0.5. Notice that when $d = 5$, we have shown $\mathcal{G}_{(8)} \cap \overline{\mathcal{G}_{(5,4)}} \neq \emptyset$.

To show this is true, we need to show that we can choose $c_d, c_{d-1}, c_{d-2} \in \mathbb{C}$ so that $\lambda_4 = 0, \lambda_5 = 0$, and $\lambda_6 \neq 0$. To simplify computations, define $x = \frac{c_d^2}{c_{d-1}}$ and $y = c_{d-2}$. By our recursive formula for λ_r in Equation 3.0.2,

$$\lambda_4 = 28 - 84x + 42x^2 - 7y + 28xy$$

$$\lambda_5 = -12 + 180x - 330x^2 + 132x^3 - 72y + 120xy + 4y^2$$

$$\lambda_6 = -165 + 990x - 1287x^2 + 429x^3 + 45y - 495xy + 495x^2y + 45y^2$$

When we set $\lambda_4 = 0$ and $\lambda_5 = 0$, we will have a system of equations in two variables:

$$28 - 84x + 42x^2 - 7y + 28xy = 0$$

$$-12 + 180x - 330x^2 + 132x^3 - 72y + 120xy + 4y^2 = 0$$

Solving the first equation yields:

$$y = \frac{6x^2 - 12x + 4}{1 - 4x}.$$

We can substitute this into the second equation. With the help of Mathematica, we will get a few approximate solutions. For instance, $x \approx 0.21203733715929782$ and $y \approx 11.361893628390607$. The last step to show is that these values for x and y will lead to a non-zero value for λ_6, the coefficient of Y^8.

$$\lambda_6 \approx 5371.93$$

Here, we see that if we choose $\frac{c_d^2}{c_{d-1}} \approx 0.21203733715929782$ and $c_{d-2} \approx 11.361893628390607$, then all our conditions will be satisfied.

CHAPTER 4

UNION OF POLYNOMIAL AUTOMORPHISMS

The Rigidity Conjecture is related to the union of groups of polynomial automorphisms. In [9], Furter states the following theorem:

Theorem 4.0.1 (Furter). *If $d = (d_1, d_2)$ is a polydegree with d_1 or $d_2 \leq 3$, then*

$$\overline{\mathcal{G}_d} = \bigcup_{e \preceq d} \mathcal{G}_e,$$

where $\preceq$ is defined in the following way:

Definition 4.0.2. The partial order $\preceq$ is induced by the following relations:

1. $\emptyset \preceq d$ (for any polydegree d),

2. $(d_1, ..., d_k) \preceq (e_1, ..., e_k)$ when $d_j \leq e_j$ for any j,

3. $(d_1, ..., d_{j-1}, d_j + d_{j+1} - 1, d_{j+2}, ..., d_k) \preceq (e_1, ..., e_k)$ when $1 \leq j \leq k - 1$.

Example 4.0.3. $\overline{\mathcal{G}_{(2,2)}} = \mathcal{G}_{(2,2)} \cup \mathcal{G}_{(3)} \cup \mathcal{G}_{(2)} \cup \mathcal{G}_{(1)}$.

This example allows us to write $\overline{\mathcal{G}_{(2,2)}}$ as a union of certain $\mathcal{G}_d$'s. Looking at this example, we can gather some information such as:

1. $\mathcal{G}_{(3)} \cap \overline{\mathcal{G}_{(2,2)}} \neq \emptyset$ (Corollary 1.2.8 when $e = 2$),

2. $\mathcal{G}_{(4)} \cap \overline{\mathcal{G}_{(2,2)}} = \emptyset$.

Being able to write the closure of a set of automorphisms as a union of groups of polynomial automorphisms makes it easier to see the subgroups. It would be ideal to show that Theorem 4.0.1 can be extended to larger values of d_1 or d_2 or even longer polydegree sequences.

Recall from Corollary 1.2.8, we have $\mathcal{G}_{(e+1)} \subset \overline{\mathcal{G}_{(2,e)}}$ which can be extended to the following property:

$$\mathcal{G}_{(d)} \subset \overline{\mathcal{G}_{(2,2,\ldots,2)}} = \overline{\mathcal{G}_{(2)^d}}$$

where $(2)^d$ is the polydegree sequence of 2's of length d.

If we have a polydegree of length $b \geq 5$, then the following theorem holds true:

Theorem 4.0.4. *Let $b \geq 5$ be an integer. Then $\overline{\mathcal{G}}_{(2)^b}$ is not a union of some $\mathcal{G}_d$ for some polydegree d.*

The following theorem by Edo and Furter is essential to the proof of Theorem 4.0.4.

Theorem 4.0.5 (Edo, Furter [3])**.** *Let $a, b \geq 2$, $c \geq 1$ be integers. Set $d = ab - 1$. Then*

$$\mathcal{G}_{(cd+a)} \cap \overline{\mathcal{G}_{(a+(c-1)d,b,a)}} \neq \emptyset$$

Proof of Theorem 4.0.4. Let $a = 2, b \geq 5$, and $c = 1$. Then using Theorem 4.0.5, we have

$$\mathcal{G}_{(2b+1)} \cap \overline{\mathcal{G}_{(2,b,2)}} \neq \emptyset$$

Let's consider $\mathcal{G}_{(b)}$. Note that $\mathcal{G}_{(b)} \subset \overline{\mathcal{G}_{(2,b-1)}}$. Therefore

$$\mathcal{G}_{(b)} \subset \overline{\mathcal{G}_{(2,b-1)}} \subset \overline{\mathcal{G}_{(2,2,b-2)}} \subset \cdots \subset \overline{\mathcal{G}_{(2,\ldots,2,5)}}$$

with $(b-5)$ 2's. Since $\mathcal{G}_{(5)} \subset \overline{\mathcal{G}_{(2)^3}}$ ([4]), we have

$$\mathcal{G}_{(b)} \subset \overline{\mathcal{G}_{(2)^{b-2}}}$$

Hence, $\mathcal{G}_{(2b+1)} \cap \overline{\mathcal{G}_{(2)^b}} \neq \emptyset$. However, $\mathcal{G}_{(2b+1)} \not\subset \overline{\mathcal{G}_{(2)^b}}$ because of the dimension constraint: $2b+1 \nleq 2b$. Thus for $b \geq 5$, $\overline{\mathcal{G}_{(2)^b}}$ is not a union of some $\mathcal{G}_d$ for some polydegree d. □

Remark 4.0.6. This is still open for $b = 3, 4$.

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APPENDIX A

SUMMATION IN PROOF OF LEMMA 2.0.5

Claim: Let

$$S = \sum_{j=0}^{r} \sum_{0 \le 2a \le j} \sum_{l=0}^{\lfloor \frac{r-j+1}{2} \rfloor} (-1)^{l+r-j} \frac{(2r-2j+2-l)!}{a!(j-2a)!(r-2j+a+2)!l!(r-j+1-2l)!} x^{r-2l-2a} y^{l+a}.$$

Then $S = 0$.

Proof. Let $\alpha = l + a$.

$$\begin{aligned} S &= \sum_{j=0}^{r} \sum_{a=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{\alpha=a}^{a+\lfloor \frac{r-j+1}{2} \rfloor} \frac{(-1)^{\alpha+r-a-j}(2r-2j+2-\alpha+a)!}{a!(j-2a)!(r-2j+a+2)!(\alpha-a)!(r-j+1-2\alpha+2a)!} x^{r-2\alpha} y^{\alpha} \\ &= \sum_{\alpha=0}^{\lfloor \frac{r+1}{2} \rfloor} \sum_{a=0}^{\alpha} \sum_{j=2a}^{2a+r-2\alpha+1} \frac{(-1)^{\alpha+r-a-j}(2r-2j+2-\alpha+a)!}{a!(j-2a)!(r-2j+a+2)!(\alpha-a)!(r-j+1-2\alpha+2a)!} x^{r-2\alpha} y^{\alpha} \end{aligned}$$

Let $b = j - 2a$ and $\beta = r - 2\alpha + 1$.

$$\begin{aligned} S &= \sum_{\alpha=0}^{\lfloor \frac{r+1}{2} \rfloor} (-1)^{\alpha+r} x^{r-2\alpha} y^{\alpha} \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \frac{(-1)^{a+b}(2(\beta-b)+3(\alpha-a))!}{a!(\alpha-a)!b!(\beta-b)!(\beta-2b+2\alpha-3a+1)!} \\ &= \sum_{\alpha=0}^{\lfloor \frac{r+1}{2} \rfloor} \frac{(-1)^{\alpha+r}}{\alpha+\beta} x^{r-2\alpha} y^{\alpha} \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{2(\beta-b)+3(\alpha-a)}{\alpha+\beta-1} \end{aligned}$$

It suffices to show that

$$\sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{2(\beta-b)+3(\alpha-a)}{\alpha+\beta-1} = 0$$

and if we replace $\alpha - a$ with a and $\beta - b$ with b, it suffices to show that

$$S' = \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{2b+3a}{\alpha+\beta-1} = 0.$$

Applying the Egorychev Method:

$$\begin{aligned}
S' &= \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \oint \frac{1}{2\pi i} \frac{(1+z)^{3a+2b}}{z^{\alpha+\beta}}\, dz \\
&= \frac{1}{2\pi i} \oint \frac{1}{z^{\alpha+\beta}} \left(\sum_{a=0}^{\alpha} (-1)^a \binom{\alpha}{a} [(1+z)^3]^a \right) \left(\sum_{b=0}^{\beta} (-1)^b \binom{\beta}{b} [(1+z)^2]^b \right) dz \\
&= \frac{1}{2\pi i} \oint \frac{1}{z^{\alpha+\beta}} \left(1-(1+z)^3\right)^{\alpha} \left(1-(1+z)^2\right)^{\beta} dz \\
&= \frac{1}{2\pi i} \oint \frac{(-3z-3z^2-z^3)^{\alpha}(-2z-z^2)^{\beta}}{z^{\alpha+\beta}}\, dz \\
&= 0
\end{aligned}$$

since

$$\frac{(-3z-3z^2-z^3)^{\alpha}(-2z-z^2)^{\beta}}{z^{\alpha+\beta}}$$

is an analytic function. □

APPENDIX B

GAMMA IN EQUATION 2.0.3

Claim: Let

$$T_d = -\sum_{j=3}^{d} \sum_{0 \le 2a+3 \le j} \frac{(d+2-j)!}{a!(j-2a-3)!(d-2j+a+4)!} \lambda_{d-j} c_d^{j-2a-3} c_{d-1}^{a}.$$

Then $T_d \neq 0$.

Proof. Let $k = j - 3$ and $r = d - 3$ so that

$$T = -\sum_{k=0}^{r} \sum_{0 \le 2a \le k} \frac{(r-k+2)!}{a!(k-2a)!(r-2k+a+1)!} \lambda_{r-k} c_d^{k-2a} c_{d-1}^{a}.$$

Recall that we can write

$$\lambda_{r-k} = c_{d-1}^{\lfloor \frac{r-k+1}{2} \rfloor} c_d^{\epsilon_{r-k}} Q_{r-k}\left(\frac{c_d^2}{c_{d-1}}\right)$$

where Q_{r-k} is defined in Lemma 2.0.5. After substitution and simplifying, we will see that:

$$T = \sum_{k=0}^{r} \sum_{0 \le 2a \le k} \sum_{l=0}^{\lfloor \frac{r-k+1}{2} \rfloor} \frac{(-1)^{l+r-k+1}(2r-2k+2-l)!}{a!(k-2a)!(r-2k+a+1)!l!(r-k-2l+1)!} c_d^{r-2l-2a} c_{d-1}^{l+a}.$$

From the previous appendix, we proved that:

$$S = \sum_{j=0}^{r} \sum_{0 \le 2a \le j} \sum_{l=0}^{\lfloor \frac{r-j+1}{2} \rfloor} \frac{(-1)^{l+r-j}(2r-2j+2-l)!}{a!(j-2a)!(r-2j+a+2)!l!(r-j+1-2l)!} x^{r-2l-2a} y^{l+a} = 0$$

Notice the only difference between T and S is that S has $(r-2j+a+2)$ in the denominator. We will use the same outline and substitutions as in the proof in Appendix A.

Let $\alpha = l + a$. Thus: $T =$

$$\sum_{\alpha=0}^{\lfloor \frac{r+1}{2} \rfloor} \sum_{a=0}^{\alpha} \sum_{k=2a}^{2a+r-2\alpha+1} \frac{(-1)^{\alpha+r-a-k}(2r-2k+2-\alpha+a)!}{a!(k-2a)!(r-2k+a+1)!(\alpha-a)!(r-k+1-2\alpha+2a)!} c_d^{r-2\alpha} c_{d-1}^{\alpha}$$

Let $b = k - 2a$ and $\beta = r - 2\alpha + 1$.

$$\begin{aligned} T &= \sum_{\alpha=0}^{\lfloor \frac{r+1}{2} \rfloor} (-1)^{\alpha+r} c_d^{r-2\alpha} c_{d-1}^{\alpha} \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \frac{(-1)^{a+b}(2(\beta-b)+3(\alpha-a))!}{a!(\alpha-a)!b!(\beta-b)!(\beta-2b+2\alpha-3a)!} \\ &= \sum_{\alpha=0}^{\lfloor \frac{r+1}{2} \rfloor} (-1)^{\alpha+r} c_d^{r-2\alpha} c_{d-1}^{\alpha} \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{2(\beta-b)+3(\alpha-a)}{\alpha+\beta} \end{aligned}$$

It suffices to show that

$$\sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{2(\beta-b)+3(\alpha-a)}{\alpha+\beta} \neq 0$$

and if we replace $\alpha - a$ with a and $\beta - b$ with b, it suffices to show that

$$T' = \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{2b+3a}{\alpha+\beta} \neq 0.$$

Applying the Egorychev Method:

$$\begin{aligned} T' &= \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} (-1)^{a+b} \binom{\alpha}{a} \binom{\beta}{b} \oint \frac{1}{2\pi i} \frac{(1+z)^{3a+2b}}{z^{\alpha+\beta+1}} \, dz \\ &= \frac{1}{2\pi i} \oint \frac{1}{z^{\alpha+\beta+1}} \left(\sum_{a=0}^{\alpha} (-1)^a \binom{\alpha}{a} [(1+z)^3]^a \right) \left(\sum_{b=0}^{\beta} (-1)^b \binom{\beta}{b} [(1+z)^2]^b \right) dz \\ &= \frac{1}{2\pi i} \oint \frac{1}{z^{\alpha+\beta+1}} \left(1-(1+z)^3\right)^{\alpha} \left(1-(1+z)^2\right)^{\beta} dz \\ &= \frac{1}{2\pi i} \oint \frac{(-3z-3z^2-z^3)^{\alpha}(-2z-z^2)^{\beta}}{z^{\alpha+\beta+1}} \, dz \\ &= (-3)^{\alpha}(-2)^{\beta}. \end{aligned}$$

□

APPENDIX C

SUMMATION IN PROOF OF LEMMA 3.0.4

Claim: Let $R =$

$$\sum_{j=0}^{r} \sum_{0 \leq 2a+3b \leq j} \sum_{2l+3m \leq r-j+1} \frac{(-1)^{r-j+l+2m+2}(2r-2j+2-l-2m)!x^{r-2l-2a-3b-3m}y^{l+a}v^{m+b}}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!l!m!(r-j+1-2l-3m)!}.$$

Then $R = 0$.

Proof. Let $\alpha = l + a$ and $\beta = b + m$. $q = a + \lfloor \frac{r-j+1-3m}{2} \rfloor$, $p = b + \lfloor \frac{r-j+1}{3} \rfloor$

$$\sum_{j=0}^{r} \sum_{a=0}^{\lfloor \frac{j-3b}{2} \rfloor} \sum_{b=0}^{\lfloor \frac{j}{3} \rfloor} \sum_{\alpha=a}^{q} \sum_{\beta=b}^{p}$$
$$\frac{(-1)^{r-j+\alpha-a+2\beta-2b}(2r-2j+2-\alpha+a-2\beta+2b)!x^{r-2\alpha-3\beta}y^{\alpha}v^{\beta}}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!(\alpha-a)!(\beta-b)!(r-j+1-2\alpha+2a-3\beta+3b)!}$$
$$= \sum_{\alpha=0}^{\lfloor \frac{r+1-3m}{2} \rfloor} \sum_{\beta=0}^{\lfloor \frac{r+1}{3} \rfloor} \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \sum_{j=2a+3b}^{2a+3b+r-2\alpha-3\beta+1}$$
$$\frac{(-1)^{r-j+\alpha-a+2\beta-2b}(2r-2j+2-\alpha+a-2\beta+2b)!x^{r-2\alpha-3\beta}y^{\alpha}v^{\beta}}{a!b!(j-2a-3b)!(r-2j+a+2b+2)!(\alpha-a)!(\beta-b)!(r-j+1-2\alpha+2a-3\beta+3b)!}$$

Let $u = j - 2a - 3b$ and $w = r - 2\alpha - 3\beta + 1$.

$$R = \sum_{\alpha=0}^{\lfloor \frac{r+1-3m}{2} \rfloor} \sum_{\beta=0}^{\lfloor \frac{r+1}{3} \rfloor} (-1)^{u+a+b} x^{r-2\alpha-3\beta} y^{\alpha} v^{\beta}$$

$$\sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \sum_{u=0}^{w} \frac{(-1)^{a+b+u}(2(w-u)+3(\alpha-a)+4(\beta-b))!}{a!b!u!(\alpha-a)!(\beta-b)!(w-u)!(w-2u+2\alpha-3a+3\beta-4b+1)!}$$

$$= \sum_{\alpha=0}^{\lfloor \frac{r+1-3m}{2} \rfloor} \sum_{\beta=0}^{\lfloor \frac{r+1}{3} \rfloor} (-1)^{u+a+b} x^{r-2\alpha-3\beta} y^{\alpha} v^{\beta}$$

$$\sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \sum_{u=0}^{w} (-1)^{u+a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{w}{u} \binom{2(w-u)+3(\alpha-a)+4(\beta-b)}{w+\alpha+\beta-1}$$

It suffices to show that

$$\sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \sum_{u=0}^{w} (-1)^{u+a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{w}{u} \binom{2(w-u)+3(\alpha-a)+4(\beta-b)}{w+\alpha+\beta-1} = 0$$

and if we replace $\alpha - a$ with a, $\beta - b$ with b, and $w - u$ with u, it suffices to show that

$$R' = \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \sum_{u=0}^{w} (-1)^{u+a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{w}{u} \binom{2u+3a+4b}{w+\alpha+\beta-1} = 0.$$

Applying the Egorychev Method:

$$R' = \sum_{a=0}^{\alpha} \sum_{b=0}^{\beta} \sum_{u=0}^{w} (-1)^{u+a+b} \binom{\alpha}{a} \binom{\beta}{b} \binom{w}{u} \oint \frac{1}{2\pi i} \frac{(1+z)^{2u+3a+4b}}{z^{\alpha+\beta+w}} \, dz$$

$$= \frac{1}{2\pi i} \oint \frac{1}{z^{\alpha+\beta+w}} \left(\sum_{a=0}^{\alpha} (-1)^a \binom{\alpha}{a} [(1+z)^3]^a \right) \left(\sum_{b=0}^{\beta} (-1)^b \binom{\beta}{b} [(1+z)^4]^b \right)$$

$$\left(\sum_{u=0}^{w} (-1)^u \binom{w}{u} [(1+z)^2]^u \right) dz$$

$$= \frac{1}{2\pi i} \oint \frac{1}{z^{\alpha+\beta+w}} \left(1-(1+z)^3\right)^{\alpha} \left(1-(1+z)^4\right)^{\beta} \left(1-(1+z)^2\right)^{w} \, dz$$

$$= \frac{1}{2\pi i} \oint \frac{(-3z-3z^2-z^3)^{\alpha}(-4z-6z^2-4z^3-z^4)^{\beta}(-2z-z^2)^{w}}{z^{\alpha+\beta+w}} \, dz$$

$$= 0$$

since

$$\frac{(-3z-3z^2-z^3)^\alpha(-4z-6z^2-4z^3-z^4)^\beta(-2z-z^2)^w}{z^{\alpha+\beta+w}}$$

is an analytic function. □