

A NON-GEOMETRIC BROWNIAN MOTION MODEL  
ESTIMATED BY MARKOV CHAIN APPROXIMATION

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## ABSTRACT

The pricing of most contingent claims is continuously monitored the movement of the underlying assets that follow geometric Brownian motion. However, for exotic options, the pricing of the underlying assets is difficult to be obtained analytically. In reality, numerical methods are employed to monitor discretized path-dependent options since complexity of exotic options increases the difficulty of obtaining the closed-form solutions.

In this dissertation, we propose a Markov chain method to discretely monitor the underlying asset pricing of a European knock-out call option with time-varying barriers. Markov chain method provides some advantages in computation since the discretized time step can be partitioned to match with the number of the underlying non-dividend paying asset prices. Compared to Monte Carlo simulation, Markov chain method can not only efficiently handle the case where the initial asset price is close to a barrier level but also effectively improve the accuracy of obtaining the price of a barrier option.

We study a European knock-out call option with either constant or time-varying barriers. Under risk-neutral measure, the movement of the underlying stock price is said to follow a non-geometric Brownian motion. Furthermore, we are interested to estimate the parameter  $p$  value that generates optimal payoff of a knock-out option with time-varying barriers. However, implied volatility is an essential factor that affects the movement of the underlying asset price and determines whether the barrier option is knocked out or not during the lifetime of the option.

## DEDICATION

This dissertation is dedicated to everyone who provided assistance and contribution in creating this manuscript. Particularly, my committee members devoted their effort and time to guide me throughout this work.

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## CHAPTER 1

# INTRODUCTION

In the 1970s, the most important development in the field of financial mathematics, was the Black-Scholes model for option pricing, a theoretical framework that was widely adopted by practitioners. Bachelier (1870-1946) expanded this time-continuous model to develop some modern economic theories: Brownian motion and martingales on efficient markets. Option pricing theory became one of the pillars in developing complex financial derivatives such as exotic options where payoffs depend on the paths of their underlying assets.

For most path-dependent options, such as Asian options, barrier options and look-back options, it is difficult to obtain analytical expressions since the pricing of the underlying assets is constructed by complex partial differential equations. Therefore, practical applications through the design of efficient numerical procedures becomes essentially important to discretely monitor the pricing of the underlying assets for exotic options. For instance, lattice (binomial or trinomial tree), finite difference, and Monte Carlo simulation are the most frequently used numerical methods studied for pricing barrier options.

For option pricing, each numerical method has its strengths and weaknesses. The lattice method lacks accuracy due to the difficulty of matching the assumed barrier (in the nodes) with the true barrier. However, the rate of convergence can be improved by changing the tree structure. On the contrary, the Crank-Nicolson finite difference method is stable and convergent when a set of difference equations lying on a grid are solved iteratively by averaging the implicit and explicit finite difference methods. Lastly, the flexibility of Monte Carlo simulation gives it the advantage of being easily modified to accommodate complicated processes governing the underlying asset pricing and its standard error of estimate can be

reduced by an increase in the number of realizations. However, it is very time-consuming with respect to computation and cannot handle early exercise American options.

In this dissertation, we propose Markov chain approximation to estimate a European knock-out call option with time-varying barriers with the pricing of its discretely monitored underlying stock following a so-called non-geometric Brownian motion  $dS_t = S_t^p(rdt + \sigma dW_t)$  in a risk-neutral measure, where  $p \in \mathbf{R}^+ \setminus \{0, 1\}$  is a parameter. Since the time step of a time-homogeneous Markov chain can be matched with the discretely monitoring frequency of the barriers as well as its underlying stock prices, it performs accurately in the constant-volatility (Black-Scholes) and time-varying volatility (GARCH) pricing models. An analysis of pricing accuracy and computing time also suggests that Markov chain approximation is superior to the Boyle and Tian [4] finite difference method. In addition to computational advance, the Markov chain method can efficiently handle the case where the initial asset price is close to a barrier level (Duan, Dudley, Gauthier, and Simonato [10]).

The aim of our work is to study the relationship between parameter  $p$  and the underlying stock price governed by non-geometric Brownian motion. We are interested in finding the estimate  $p$  value in the case that generates the optimal price of the barrier option and maximizes the instantaneous rate of change in option price with respect to the  $p$  value; for instance,  $\frac{\partial C_{barrier}}{\partial p} > 0$ . Furthermore, we observe the effect imposed by implied volatility  $\sigma$  and other parameters on option valuation.

The dissertation is organized as follows: Chapter 2 highlights the important theories governing continuous-time models, Chapter 3 introduces discrete-time Markov chain approximation and its applications, while Chapter 4 discusses the empirical results of discretely monitored barrier options by the Markov chain method and its superior computational advantages to Monte Carlo simulation.

## CHAPTER 2

# STOCHASTIC CALCULUS

### 2.1. Continuous-time process

The general methodology to price financial derivatives, such as options on a stock that pays continuous dividends, path-dependent options, options under stochastic volatility, and barrier options, is to monitor the continuous-time movement of the underlying assets. The dynamic of the Black-Scholes option pricing model is widely applied in the finance industry; this model is based on the development of a continuous-time process that follows the fundamentals of the Girsanov Theorem and martingale representation, which are the core developments of derivative pricing and will be explained in the following sections.

**DEFINITION 2.1.1.** *Suppose  $\Omega$  is a nonempty set and  $t \in [0, T]$  is a parameter, where  $T$  is a fixed positive number. Then there exists a  $\sigma$ -algebra  $\mathcal{F}_t$  such that it models the acquisition of information until  $t$ . The collection of  $\sigma$ -algebras  $\mathcal{F}_t$  is called a filtration.*

**DEFINITION 2.1.2.** *The  $\sigma$ -algebra  $\mathcal{F}$  on a nonempty set  $\Omega$  is a family of subsets of  $\Omega$  satisfying the following properties:*

- (1)  $\phi \in \mathcal{F}$
- (2)  $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$ , where  $F^c = \Omega \setminus F$  is the complement of  $F$  in  $\Omega$
- (3)  $N_1, N_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} N_i \in \mathcal{F}$ .

We now assume a filtration is *complete* (i.e.  $\phi \in \mathcal{F}$ ) and *right continuous* (i.e.  $\mathcal{F} = \bigcap_{s>t} \mathcal{F}_s$  for  $t \in [0, T]$ ). The right continuity assumption shows that a filtration is an increasing family of sub- $\sigma$ -fields of  $\mathcal{F}$  (i.e. for  $t \leq s$ ,  $\mathcal{F}_t \subseteq \mathcal{F}_s$ ).

**DEFINITION 2.1.3.** *A stopping time is a random variable  $\tau: (\Omega, \mathcal{F}) \rightarrow \overline{\mathcal{R}} = \{0, 1, \dots, \infty\}$  such that for all  $t \in \mathcal{R}$ ,  $\{\tau \leq t\} \in \mathcal{F}_t$ .*

This means that the event  $\{\tau \leq t\}$  depends only on the history up to time  $t$ . The fundamental of stopping time is applied to some financial derivatives. For example, a barrier option is expired if the price of the underlying asset reaches a certain price level— a barrier. In this case, the stopping time is also called *the first passage time*.

**THEOREM 1.** *Suppose  $U, V$  are stopping times.*

- (1)  $U \leq V$ , then  $\mathcal{F}_U \subseteq \mathcal{F}_V$ .
- (2)  $G \in \mathcal{F}_U$ , then  $G \cap \{U \leq V\} \in \mathcal{F}_V$ .

**PROOF.** (1) Suppose that  $H \in \mathcal{F}_U$ . Then, for  $t \in [0, T]$ ,

$$H \cap \{V \leq t\} = H \cap \{U \leq t\} \cap \{V \leq t\} \in \mathcal{F}_t.$$

(2) Suppose that  $G \in \mathcal{F}_U$ . For  $t \in [0, T]$ , we have

$$G \cap \{U \leq V\} \cap \{V \leq t\} = (G \cap \{U \leq t\}) \cap \{V \leq t\} \cap \{U \wedge t \leq V \wedge t\} .$$

On the right-hand side of above equation, the first set  $(G \cap \{U \leq t\})$  is in  $\mathcal{F}_t$  since  $G \in \mathcal{F}_U$ ; the second set is in  $\mathcal{F}_t$  since  $T$  is a stopping time; the last set is also in  $\mathcal{F}_t$  since  $U \wedge t$  and  $V \wedge t$  are  $\mathcal{F}_t$ -measurable random variables.

□

## 2.2. Brownian motion

Brownian motion, named for Robert Brown, a Scottish botanist in the early nineteenth century, originally defined the erratic motion of a particle on the surface of a fluid, caused by tiny impulses of molecules. Later Bachelier applied Brownian motion to model the motion of stock prices, which instantly respond to the numerous incoming information similar to a particle reacting to the impacts of molecules. Nowadays many financial models are developed with the fundamental concepts of Brownian motion.

**DEFINITION 2.2.1.** *A standard Brownian motion is a real-valued, continuous stochastic process  $(W_t)_{t \geq 0}$ , with independent and stationary increments. In other words,  $(W_t)_{t \geq 0}$*

is a collection of random variables defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , satisfying the following conditions:

- (1)  $W_0 = 0$ ;
- (2)  $t \mapsto W_t(\omega)$  is almost surely continuous;
- (3) independent increments: if  $0 \leq t_1 < t_2 \leq t_3 < t_4$ ,  
 $W_{t_2} - W_{t_1}$  is independent of  $W_{t_4} - W_{t_3}$ ;
- (4) stationary increments: if  $0 \leq t_1 < t_2$ ,  
 $W_{t_2} - W_{t_1}$  and  $W_{t_2-t_1} - W_0$  have the same probability law.

For  $t > 0$ , a Brownian motion  $W_t$  is normally distributed with mean 0 and variance  $\sigma^2 t$  (i.e.  $W_t \sim \mathcal{N}(0, \sigma^2 t)$ ), where  $\mathcal{N}(\mu, \sigma^2)$  denotes the normal distribution with mean  $\mu$  and variance  $\sigma^2$ .

PROPOSITION 2.2.1. Suppose  $(W_t)_{t \geq 0}$  is a standard Brownian motion. We then have  
(1)  $\text{Var}(W_t - W_s) = t - s$ ; (2)  $\text{Cov}[W_s, W_t] = \min(s, t) = s$  for  $0 \leq s < t$ .

PROOF. (1) Since  $\mathbb{E}(W_t) = \mathbb{E}(W_t - W_0) = 0$ , the variance of  $W_t$  is

$$\text{Var}(W_t) = \text{Var}(W_t - W_0) = \mathbb{E}[(W_t - W_0)^2] = t. \quad (2.2.1)$$

For  $0 \leq s < t$ , the covariance of  $W_s$  and  $W_t$  is

$$\begin{aligned} \mathbb{E}(W_s W_t) &= \mathbb{E}[W_s(W_t - W_s) + (W_s)^2] \\ &= \mathbb{E}(W_s)\mathbb{E}(W_t - W_s) + \mathbb{E}[(W_s)^2] \\ &= 0 + \text{Var}(W_s) = s. \end{aligned}$$

Given that  $W_0 = 0$ , the expectation of  $W_s W_t$  can be rewritten as

$$\mathbb{E}(W_s W_t) = \mathbb{E}[(W_s - W_0)(W_t - W_0)].$$

Since  $E[W_t] = 0$  and  $\text{Var}[W_t] = t$ , we then have  $E[(W_t - W_0)^2] = E[W_t^2] = \text{Var}[W_t] - [E(W_t)]^2 = t$ . By the same token,  $E[(W_s - W_0)^2] = E[W_s^2] = s$  for  $s \geq 0$ .

By observation, we have the expectation of  $(W_t - W_s)^2$  in the form of

$$\begin{aligned} E[(W_t - W_s)^2] &= E[(W_t - W_0)^2] + E[(W_s - W_0)^2] - 2E[(W_s - W_0)(W_t - W_0)] \\ &= t + s - 2s = t - s. \end{aligned}$$

Therefore, we have the variance of  $W_t - W_s$  in the form of

$$\begin{aligned} \text{Var}(W_t - W_s) &= E[(W_t - W_s)^2] - [E(W_t - W_s)]^2 \\ &= E[(W_t - W_s)^2] = t - s. \end{aligned}$$

(2) Using the property  $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$  where  $X, Y$  are any arbitrary random variables, then

$$\begin{aligned} \text{Cov}[W_s, W_t] &= \text{Cov}[W_s, W_s + (W_t - W_s)] \\ &= E[W_s\{W_s + (W_t - W_s)\}] - E(W_s)E[W_s + (W_t - W_s)] \\ &= E(W_s^2) + E[W_s(W_t - W_s)] - E(W_s)E(W_s) - E(W_s)E(W_t - W_s) \\ &= [E(W_s^2) - E(W_s)E(W_s)] + [E[W_s(W_t - W_s)] - E(W_s)E(W_t - W_s)] \\ &= \text{Cov}[W_s, W_s] + \text{Cov}[W_s, W_t - W_s] = \text{Var}[W_s] + \text{Cov}[W_s, W_t - W_s] \\ &= s + \text{Cov}[W_s, W_t - W_s] \end{aligned}$$

Since  $W_s - W_0$  and  $W_t - W_s$  are independent random variables,  $\text{Cov}[W_s, W_t - W_s] = \text{Cov}[W_s - W_0, W_t - W_s] = 0$ . Therefore, the covariance of  $W_s$  and  $W_t$  is

$$\text{Cov}[W_s, W_t] = \min(s, t) = s. \quad (2.2.2)$$

□

For  $t_{j-1} \leq t < t_j$  and  $0 = t_0 < t_1 < \dots < t_N = T$ , the *first variation* of  $W_t$ , which is the limit of

$$\sum_{j=1}^N |W_{t_j} - W_{t_{j-1}}| \quad (2.2.3)$$

is not bounded even in case  $t_j - t_{j-1} \rightarrow 0$  for  $N \rightarrow \infty$ . Then

$$\sum_{j=1}^N |W_{t_j} - W_{t_{j-1}}|^2 \leq \max_j (|W_{t_j} - W_{t_{j-1}}|) \sum_{j=1}^N |W_{t_j} - W_{t_{j-1}}| \quad (2.2.4)$$

holds. The left-hand sum in Eq. (2.2.4) is the *quadratic variation* of  $W_t$  with the expectation

$$\begin{aligned} \mathbf{E} \sum_{j=1}^N (W_{t_j} - W_{t_{j-1}})^2 &= \sum_{j=1}^N \mathbf{E}[(W_{t_j} - W_{t_{j-1}})^2] \\ &= \sum_{j=1}^N (t_j - t_{j-1}) = t_N - t_0 = T. \end{aligned} \quad (2.2.5)$$

and the variance

$$\begin{aligned} \text{Var} \sum_{j=1}^N (W_{t_j} - W_{t_{j-1}})^2 &= \sum_{j=1}^N \text{Var}(W_{t_j} - W_{t_{j-1}})^2 \\ &= \sum_{j=1}^N \mathbf{E}[(W_{t_j} - W_{t_{j-1}})^2 - \mathbf{E}(W_{t_j} - W_{t_{j-1}})^2]^2 = \sum_{j=1}^N \mathbf{E}[(W_{t_j} - W_{t_{j-1}})^2 - (t_{j+1} - t_j)]^2 \\ &= \sum_{j=1}^N \{\mathbf{E}[(W_{t_j} - W_{t_{j-1}})^4] - 2\mathbf{E}[(W_{t_j} - W_{t_{j-1}})^2](t_{j+1} - t_j) + (t_{j+1} - t_j)^2\} \\ &= \sum_{j=1}^N [3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2] = 2 \sum_{j=1}^N (t_{j+1} - t_j)^2 \quad (2.2.6) \\ &\leq 2 \sum_{j=1}^N M(t_{j+1} - t_j) = 2MT, \quad \text{where } M = \|\max_j (t_{j+1} - t_j)\|. \end{aligned}$$

As  $M \rightarrow 0$ ,  $\text{Var} \sum_{j=1}^N (W_{t_j} - W_{t_{j-1}})^2 = 0$ . Hence the quadratic variation of Brownian motion  $W_t$  is

$$\lim_{M \rightarrow 0} \sum_{j=1}^N (W_{t_j} - W_{t_{j-1}})^2 = T. \quad (2.2.7)$$



The part of the derivation in (2.2.5) and (2.2.6) can be expressed as  $\mathbb{E}((\Delta W_t)^2 - \Delta t) = 0$  and  $\text{Var}(\Delta W_t)^2 = 2(\Delta t)^2$ . Consequently,  $(\Delta W_t)^2 \approx \Delta t$  which implies

$$(dW_t)^2 = dt. \quad (2.2.8)$$

By the same token, the variation of  $W_t$  with  $t$  and the variation of  $t$  with itself are

$$\lim_{M \rightarrow 0} \sum_{j=1}^N (W_{t_j} - W_{t_{j-1}})(t_j - t_{j-1}) = 0, \quad (2.2.9)$$

$$\lim_{M \rightarrow 0} \sum_{j=1}^N (t_j - t_{j-1})^2 = 0. \quad (2.2.10)$$

Respectively, Eq.(2.2.9) and Eq.(2.2.10) imply

$$dW_t dt = 0, \quad dt dt = 0. \quad (2.2.11)$$

Now we consider the random process  $dX_t$  with the influence of volatility  $\sigma$  such that  $dX_t = \sigma dW_t$ , where  $dW_t \sim \mathcal{N}(0, dt)$  and  $dX_t \sim \mathcal{N}(0, \sigma^2 dt)$ . Equivalently,  $dX_t = \sigma \sqrt{dt} dZ$ , where  $dZ$  is a random variable normally distributed with mean 0 and variance 1 (i.e.  $dZ \sim \mathcal{N}(0, 1)$ ). The time interval  $dt = T/n$  is defined for  $0 = t_0 < t_1 < \dots < t_n = T$ . Suppose  $X_{t_0} = X_0 = 0$  and that the Brownian increment is changed by amount  $dX_i = \sigma \sqrt{dt} dZ_i$  over the  $i$ th time interval  $dt$ . Hence  $X_T = \sum_{i=1}^n (\sigma \sqrt{dt} \Delta Z_i) = \sigma \sqrt{dt} \sum_{i=1}^n \Delta Z_i$  where  $\Delta Z_i \approx dZ_i$ . Since each  $dZ_i$  is an *i.i.d.* random variable with  $dZ_i \sim \mathcal{N}(0, 1)$ , by the Law of Large Numbers, the expectation of  $X_T$  is

$$\mathbb{E}[X_T] = \sigma \sqrt{dt} \mathbb{E} \left[ \sum_{i=1}^n \Delta Z_i \right] = 0.$$

Since  $\text{Var}[X_T] = 1$  and  $\text{Var}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \text{Var}[X_i]$ , the variance of  $X_T$  is

$$\begin{aligned}\text{Var}[X_T] &= \text{Var}\left[\sigma\sqrt{dt}\sum_{i=1}^n \Delta Z_i\right] = \sigma^2 dt \text{Var}\left[\sum_{i=1}^n \Delta Z_i\right] \\ &= \sigma^2 dt \sum_{i=1}^n \text{Var}[\Delta Z_i] = \sigma^2 dt \sum_{i=1}^n 1 \\ &= \sigma^2 ndt = \sigma^2 T.\end{aligned}$$

Hence,  $X_T \sim \mathcal{N}(0, \sigma^2 T)$  for  $dX_t = \sigma dW_t$ , where  $dW_t \sim \mathcal{N}(0, dt)$ .

If a constant drift  $\mu$  is considered to alter the direction of the movement in the process  $dX_t$ , then we have a different structure of the process  $dX_t = \mu dt + \sigma\sqrt{dt}dZ_i$ , where  $dZ_i \sim \mathcal{N}(0, 1)$ .

Hence,

$$\begin{aligned}X_T &= \sum_{i=1}^n (\mu\Delta t + \sigma\sqrt{dt}\Delta Z_i) \\ &= \mu \sum_{i=1}^n \Delta t + \sigma\sqrt{dt} \sum_{i=1}^n \Delta Z_i \\ &= \mu T + \sigma\sqrt{dt} \sum_{i=1}^n \Delta Z_i,\end{aligned}$$

where  $\Delta Z_i \approx dZ_i$ .

The expected value of  $X_T$  is computed as

$$\begin{aligned}\mathbb{E}[X_T] &= \mathbb{E}\left[\mu T + \sigma\sqrt{dt} \sum_{i=1}^n \Delta Z_i\right] \\ &= \mu T + \sigma\sqrt{dt} \mathbb{E}\left[\sum_{i=1}^n \Delta Z_i\right] \\ &= \mu T.\end{aligned}$$

Since  $\text{Var}[X_T] = 1$  and  $\text{Var}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \text{Var}[X_i]$ , the variance of  $X_T$  is

$$\begin{aligned}\text{Var}[X_T] &= \text{Var}\left[\mu T + \sigma\sqrt{dt}\sum_{i=1}^n \Delta Z_i\right] \\ &= \text{Var}\left[\sigma\sqrt{dt}\sum_{i=1}^n \Delta Z_i\right] = \sigma^2 dt \sum_{i=1}^n \text{Var}[\Delta Z_i] \\ &= \sigma^2 dt \sum_{i=1}^n 1 = \sigma^2 ndt = \sigma^2 T.\end{aligned}$$

Hence,  $X_T \sim \mathcal{N}(\mu T, \sigma^2 T)$  for  $dX_t = \mu dt + \sigma dW_t$ , where  $dW_t \sim \mathcal{N}(0, dt)$ . Therefore, addition of a drift in Brownian motion does not change its volatility as in the case of zero drift Brownian motion. This important property is extensively used in the theory of derivative pricing.

### 2.3. Martingale Properties

DEFINITION 2.3.1. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\mathcal{F}_t$  be a filtration for  $0 \leq t \leq T$ . A stochastic process  $\{X_t : 0 \leq t \leq T\}$  is a martingale if

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s \tag{2.3.1}$$

for all  $0 \leq s \leq t \leq T$ .

PROPOSITION 2.3.1. Using the martingale property (2.3.1) in Definition 2.3.1, we can show that (1)  $W_t$ ; (2)  $W_t^2 - t$ ; (3)  $\exp(\sigma W_t - \frac{1}{2}\sigma^2 t)$  follow a martingale process.

PROOF. (1) Let  $W_t$  be a Brownian motion for  $t \geq 0$  and  $\mathcal{F}_t$  be a filtration that contains the information learned by observing the process up to time  $t$ .

Then for  $0 \leq s \leq t \leq T$ ,

$$\begin{aligned}\mathbb{E}[W_t | \mathcal{F}_s] &= \mathbb{E}[(W_t - W_s) + W_s | \mathcal{F}_s] \\ &= \mathbb{E}[W_t - W_s | \mathcal{F}_s] + W_s = W_s.\end{aligned}$$

Since  $\mathbb{E}[W_t|\mathcal{F}_s] = W_s$ , the Brownian motion  $W_t$  is a martingale.

(2) For  $(W_t^2 - t)$ , we have

$$\begin{aligned}\mathbb{E}[W_t^2 - W_s^2|\mathcal{F}_s] &= \mathbb{E}[(W_t - W_s)^2 + 2W_t(W_t - W_s)|\mathcal{F}_s] \\ &= \mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + 2W_s\mathbb{E}[(W_t - W_s)|\mathcal{F}_s].\end{aligned}$$

Since  $\mathbb{E}[(W_t - W_s)|\mathcal{F}_s] = 0$ , independence implies that

$$\mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] = \mathbb{E}[(W_t - W_s)^2] = t - s.$$

Therefore,  $\mathbb{E}[W_t^2 - t|\mathcal{F}_s] = W_s^2 - s$  is a martingale.

(3) Let  $\sigma$  be a constant. For  $0 \leq s \leq t$ , we have

$$\begin{aligned}\mathbb{E}[\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t\}|\mathcal{F}_s] &= \mathbb{E}[\exp\{\sigma(W_t - W_s)\}\exp\{\sigma W_s - \frac{1}{2}\sigma^2 t\}|\mathcal{F}_s] \\ &= \exp\{\sigma W_s - \frac{1}{2}\sigma^2 t\}\mathbb{E}[\exp\{\sigma(W_t - W_s)\}|\mathcal{F}_s] \\ &= \exp\{\sigma W_s - \frac{1}{2}\sigma^2 t\}\mathbb{E}[\exp\{\sigma(W_t - W_s)\}].\end{aligned}$$

Since  $\sigma(W_t - W_s) \sim \mathcal{N}(0, \sigma^2(t - s))$ , then by the Central Limit Theorem (CLT) we obtain  $\mathbb{E}[\exp\{\sigma(W_t - W_s)\}] = \exp\{\frac{1}{2}\sigma^2(t - s)\}$ . Therefore,

$$\begin{aligned}\mathbb{E}[\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t\}|\mathcal{F}_s] &= \exp\{\sigma W_s - \frac{1}{2}\sigma^2 t\}\exp\{\frac{1}{2}\sigma^2(t - s)\} \\ &= \exp\{\sigma W_s - \frac{1}{2}\sigma^2 s\}.\end{aligned}\tag{2.3.2}$$

From the above result, we can say  $\mathbb{E}[\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t\}]$  is a martingale.  $\square$

The notion of martingale means that given all the available information  $\mathcal{F}_s$ , the best approximation of  $X_t$  is  $X_s$ . In financial markets, the best way to predict a future price is to use the current price. The third property of the previous proposition is useful for studying the dynamics of financial asset prices; for example, the expectation of the underlying asset price for a barrier option is often written as  $\mathbb{E}[S_t] = \mathbb{E}[\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t\}]$ .

## 2.4. Geometric Brownian motion

Geometric Brownian motion (GBM) is preferred for modeling the pricing process of the underlying assets of financial derivatives, whereas a random walk in Brownian motion (BM) sometimes goes negative. The exponential distribution of GBM is a nonnegative variation of BM, so that its applications have been widely employed in modeling financial derivatives. For pricing a financial asset, geometric Brownian motion is defined by

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad t \in [0, \infty) \quad (2.4.1)$$

where  $S_t$  is the asset price at time  $t$ ,  $\mu$  and  $\sigma$  are the mean (*drift*) and the standard deviation (*volatility*), respectively, of the asset price  $S_t$ , and  $W_t$  is a Brownian motion.

For  $W_0 = 0$ , the solution to the stochastic differential equation (2.4.1) for the asset price  $S_t$  is

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \quad (2.4.2)$$

So the expected value of  $S_t$  is

$$\begin{aligned} \mathbb{E}[S_t] &= \mathbb{E} \left\{ S_0 \exp \left[ \left( \mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right] \right\} \\ &= S_0 \exp \left[ \left( \mu - \frac{1}{2}\sigma^2 \right) t \right] \mathbb{E} \{ \exp [\sigma W_t] \} \\ &= S_0 \exp \left[ \left( \mu - \frac{1}{2}\sigma^2 \right) t \right] \exp \left[ \frac{1}{2}\sigma^2 t \right] \\ &= S_0 e^{\mu t} \end{aligned} \quad (2.4.3)$$

and the variance of  $S_t$  can be obtained from

$$\begin{aligned}
\text{Var} [S_t] &= \mathbf{E} [S_t^2] - [\mathbf{E}S_t]^2 \\
&= \mathbf{E} \left[ S_0^2 \exp \left\{ 2\left(\mu - \frac{1}{2}\sigma^2\right)t + 2\sigma W_t \right\} \right] - [S_0 \exp \{\mu t\}]^2 \\
&= S_0^2 \exp \left\{ 2\left(\mu - \frac{1}{2}\sigma^2\right)t + 2\sigma^2 t \right\} - S_0^2 \exp \{2\mu t\} \\
&= S_0^2 e^{\sigma^2 t}
\end{aligned} \tag{2.4.4}$$

The mean and volatility of a derivative with a single asset, based on geometric Brownian motion, are  $S_0 e^{\mu t}$  and  $S_0 e^{\sigma\sqrt{t}}$ , respectively. In addition, GBM has a Markov property, which means the future given the present state is independent of the past. In accordance to no-arbitrage opportunities, no one should be able to make a profit with certainty by just observing the past values of the asset. In other words, the future asset price  $S_{t+\Delta t}$  is independent of the past values  $\{S_i : 0 \leq i < t\}$  before time  $t$ , given the present value  $S_t$  at time  $t$ . Suppose the future asset price is defined as

$$\begin{aligned}
S_{t+\Delta t} &= S_0 e^{W_{t+\Delta t}} = S_0 e^{W_t + (W_{t+\Delta t} - W_t)} \\
&= [S_0 e^{W_t}] e^{W_{t+\Delta t} - W_t} = S_t e^{W_{t+\Delta t} - W_t}.
\end{aligned}$$

Hence, the future  $S_{t+\Delta t}$ , given  $S_t$ , only depends on the future increment of the BM  $\{W_{t+\Delta t} - W_t : \Delta t \geq 0\}$ . Since BM has independent increments, this future asset value is independent of the past asset values. Moreover, since  $W_{t+\Delta t} - W_t$  has the same drift  $\mu$  and standard deviation  $\sigma$ , the future process  $\{S_t e^{W_{t+\Delta t} - W_t} : \Delta t \geq 0\}$  has the same GBM distribution.

## 2.5. Itô Formula

Suppose that a Brownian motion  $W_t$  is a representation of the movement of an asset and let  $s(t)$  be the shares of the asset held in a portfolio at time  $t$ . We assume that trading is only allowed at discrete time instances  $t_j$  for a partition of  $0 = t_0 < t_1 < \dots < t_n = T$ . Then trading strategy  $s_t = s_{t_{j-1}}$  for  $t_{j-1} \leq t < t_j$  is a random variable. The **Itô integral**

over time  $t \geq 0$  is

$$I_t = \int_{t_0}^t s_u dW_u := \sum_{j=1}^n s_{t_{j-1}} (W_{t_j} - W_{t_{j-1}}). \quad (2.5.1)$$

The differential form of Eq.(2.5.1) is

$$dI_t = s_t dW_t, \quad (2.5.2)$$

and its square is

$$dI_t dI_t = s_t^2 dW_t dW_t = s_t^2 dt. \quad (2.5.3)$$

**PROPOSITION 2.5.1.** *If  $I_t$  is an Itô integral over time  $t \geq 0$ , then  $I_t$  has the following properties*

- (1)  $I_t$  is a martingale.
- (2)  $[I, I](t) = \int_0^t s_u^2 du$ .
- (3)  $\mathbb{E}[I_t^2] = \mathbb{E} \int_0^t s_u^2 du < \infty$ .
- (4) If  $I_0 = 0$ , then  $\mathbb{E}(I_t) = 0$  and  $\text{Var}(I_t) < \infty$ .

In addition to *Stratonovich integral*, we can use Itô integral to compute

$$\int_0^t W_u dW_u = \frac{1}{2} W_t^2 - \frac{1}{2} t, \quad \text{for } t \geq 0. \quad (2.5.4)$$

Now we want to use Itô formula to verify Eq.(2.5.4). Let  $f(W_t)$  be a differentiable function, where  $W_t$  is a Brownian motion with  $W_0 = 0$ . The **Itô formula** in *integral* form is defined by

$$f(W_t) - f(W_0) = \int_0^t f'(W_u) dW_u + \frac{1}{2} \int_0^t f''(W_u) du. \quad (2.5.5)$$

Let  $f(x) = \frac{1}{2}x^2$ , where  $x = W_t$ . Inserting  $f'(x) = x$  and  $f''(x) = 1$  into Eq.(2.5.5), we can obtain

$$\begin{aligned} \frac{1}{2} W_t^2 - 0 &= \int_0^t W_u dW_u + \frac{1}{2} \int_0^t du \\ \Rightarrow \int_0^t W_u dW_u &= \frac{1}{2} W_t^2 - \frac{1}{2} t. \end{aligned}$$

Different from the ordinary integrand, the extra term  $-\frac{1}{2}t$  in Eq.(2.5.4) is generated from the nonzero quadratic variation of Brownian motion. By property (1) in Definition 2.5.1, the Itô integral (2.5.4) is a martingale with constant expectation.

### 2.5.1. Itô Lemma.

DEFINITION 2.5.1. (**Itô Process**) An Itô stochastic differential equation is

$$dX_t = \alpha(X_t, t)dt + \beta(X_t, t)dW_t, \quad (2.5.6)$$

which is a form of the integral equation

$$X_t = X_0 + \int_0^t \alpha(X_u, u)du + \int_0^t \beta(X_u, u)dW_u. \quad (2.5.7)$$

where  $X_0$  is a constant;  $\alpha(X_t, t)$  and  $\beta(X_t, t)$  are adapted stochastic processes; and  $W_t$  is a Brownian motion for  $t \geq 0$ .

Let  $f(X_t, t)$  be a differentiable function and  $X_t$  follow an Itô process as defined in Definition 2.5.1. By the Taylor series expansion of  $f(X_t, t)$ , the **Itô formula** in differential form for Eq.(2.5.6) is

$$df(X_t, t) = f_t(X_t, t)dt + f_x(X_t, t)dX_t + \frac{1}{2}f_{xx}(X_t, t)dX_t dX_t. \quad (2.5.8)$$

Since  $dW_t dt = 0$ ,  $dt dt = 0$ , and  $dW_t dW_t = dt$ , then

$$dX_t dX_t = \beta^2(X_t, t)dt. \quad (2.5.9)$$

Substituting Eq. (2.5.6) and Eq.(2.5.9) into Eq. (2.5.8), we obtain the **Itô Lemma**

$$\begin{aligned} df(X_t, t) &= f_t(X_t, t)dt + f_x(X_t, t)\beta(X_t, t)dW_t \\ &\quad + f_x(X_t, t)\alpha(X_t, t)dt + \frac{1}{2}f_{xx}(X_t, t)\beta^2(X_t, t)dt. \end{aligned} \quad (2.5.10)$$



Eq.(2.5.10) can be rewritten as

$$df = \left[ \left( \frac{\partial f}{\partial x} \right) \alpha + \left( \frac{\partial f}{\partial t} \right) + \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} \right) \beta^2 \right] dt + \left( \frac{\partial f}{\partial x} \right) \beta dW. \quad (2.5.11)$$

The discretized version of (2.5.11) is

$$\Delta f = \left[ \left( \frac{\partial f}{\partial x} \right) \alpha + \left( \frac{\partial f}{\partial t} \right) + \frac{1}{2} \left( \frac{\partial^2 f}{\partial x^2} \right) \beta^2 \right] \Delta t + \left( \frac{\partial f}{\partial x} \right) \beta \Delta W, \quad (2.5.12)$$

where the derivatives of  $f$  as well as the coefficient functions  $\alpha$  and  $\beta$  in general depend on the arguments  $(X_t, t)$ .

EXAMPLE 2.1. (1) Use the Itô lemma for a differentiable function  $f(S_t, t)$  when the dynamics of stock prices are described by the stochastic differential equation (2.4.1):  $dS_t = S_t \mu dt + S_t \sigma dW_t$ , where  $\mu$  and  $\sigma$  are constant; (2) Use the result in part (1) to compute  $d(S_t^p)$ , where  $p$  is a positive constant. (Note:  $d(S_t^p)$  denotes the differential of  $S_t$  raised to the power  $p$ .)

(1) Let  $f = \ln S$ . Then  $\frac{\partial f}{\partial t} = 0$ ,  $\frac{\partial f}{\partial S} = \frac{1}{S}$ , and  $\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$ . Inserting  $\frac{\partial f}{\partial t}$ ,  $\frac{\partial f}{\partial S}$ ,  $\frac{\partial^2 f}{\partial S^2}$  and  $(dS)^2 = \sigma^2 S^2 dt$  into the Itô lemma (2.5.11), we can obtain

$$df = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW. \quad (2.5.13)$$

Integrating Eq.(2.5.13) from 0 to  $t$ , we then have the solution

$$\begin{aligned} \int_0^t df &= \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \\ \Rightarrow \ln S_t &= \ln S_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \\ \Rightarrow S_t &= S_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}. \end{aligned} \quad (2.5.14)$$

(2) For a shortcut, we can just replace  $S_t$  with  $S_t^p$  in Eq. (2.5.14) for the solution of  $d(S_t^p)$ . Therefore, the solution for the stochastic differential equation  $d(S_t^p)$  is  $S_t^p = S_0^p \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}$ .

Alternatively, again we can apply the Ito lemma for  $d(S_t^p)$ . Let  $G := G(S_t, t) = \ln(S_t^p)$ , then  $G_t = 0$ ,  $G_{S_t} = \frac{p}{S_t}$  and  $G_{S_t S_t} = -\frac{p}{S_t^2}$ . Using Eq. (2.5.11) we obtain

$$\ln(S_t^p) - \ln(S_0^p) = \left[ \frac{p}{S_t} S_t \mu - \frac{1}{2} \frac{p}{S_t^2} S_t^2 \sigma^2 \right] dt + \frac{p}{S_t} S_t \sigma dW_t = p \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right].$$

Consequently, the solution for  $d(S_t^p)$  is  $S_t^p = S_0^p \exp \left\{ p \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}$ .

**2.5.2. Itô Product and Quotient Rules.** Now we derive expressions for the product and quotient of two Itô processes (i.e.  $f \rightarrow f(X_1, X_2)$ ):

$$dX_1 = \alpha_1 dt + \beta_1 dW_1 \quad \text{and} \quad dX_2 = \alpha_2 dt + \beta_2 dW_2$$

where  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are adapted stochastic processes; and  $W_1$  and  $W_2$  are two independent Brownian motions. By the Taylor series expansion of  $f(X_1, X_2)$ , we can extend Itô lemma 2.5.11 to a two-dimensional version of the Itô lemma:

$$df = \frac{\partial f}{\partial X_1} dX_1 + \frac{\partial f}{\partial X_2} dX_2 + \frac{1}{2} \mathbb{E} \left[ \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial^2 f}{\partial X_i \partial X_j} dX_i dX_j \right] \quad (2.5.15)$$

where  $\frac{\partial f}{\partial t} = 0$  is neglected.

### Itô Product Rule

Let  $f = f(X_1, X_2)$ . Then the partial derivatives of  $f$  are:

$$\frac{\partial f}{\partial X_1} = X_2, \quad \frac{\partial f}{\partial X_2} = X_1, \quad \frac{\partial^2 f}{\partial X_1^2} = \frac{\partial^2 f}{\partial X_2^2} = 0, \quad \frac{\partial^2 f}{\partial X_1 \partial X_2} = \frac{\partial^2 f}{\partial X_2 \partial X_1} = 1.$$

By Eq.(2.5.15), we have

$$\begin{aligned} df &= X_2 dX_1 + X_1 dX_2 + \frac{2\mathbb{E}[dX_1 dX_2]}{2} \\ &= X_2 dX_1 + X_1 dX_2 + \mathbb{E}[dX_1 dX_2]. \end{aligned} \quad (2.5.16)$$

Since  $\mathbb{E}[dX_1 dX_2] = \beta_1 \beta_2 \mathbb{E}[dW_1 dW_2] = \beta_1 \beta_2 \rho_{12} dt$ , then we obtain

$$df = X_2 dX_1 + X_1 dX_2 + \beta_1 \beta_2 \rho_{12} dt \quad (2.5.17)$$

where  $\rho_{12}$  is the correlation coefficient between the 1<sup>st</sup> and the 2<sup>nd</sup> assets.

### Itô Quotient Rule

Let  $f = f\left(\frac{X_1}{X_2}\right)$ . Then the partial derivatives of  $f$  are:

$$\frac{\partial f}{\partial X_1} = \frac{1}{X_2}, \quad \frac{\partial f}{\partial X_2} = -\frac{X_1}{X_2^2}, \quad \frac{\partial^2 f}{\partial X_1^2} = 0, \quad \frac{\partial^2 f}{\partial X_2^2} = \frac{2X_1}{X_2^3}, \quad \frac{\partial^2 f}{\partial X_1 \partial X_2} = \frac{\partial^2 f}{\partial X_2 \partial X_1} = -\frac{1}{X_2^2}.$$

By Eq.(2.5.15), we have a stochastic differential equation:

$$\begin{aligned} d\left(\frac{X_1}{X_2}\right) &= \frac{dX_1}{X_2} - \frac{X_1}{X_2^2}dX_2 + \frac{1}{2}\mathbb{E}\left[\left\{\frac{2X_1}{X_2^3}dX_2^2\right\} - \left\{\frac{2}{X_2^2}dX_1dX_2\right\}\right] \\ &= \left(\frac{X_1}{X_2}\right)\left\{\left(\frac{dX_1}{X_1}\right) - \left(\frac{dX_2}{X_2}\right) + \mathbb{E}\left[\left(\frac{dX_2}{X_2}\right)\left(\frac{X_1}{X_2}\right)\right] - \mathbb{E}\left[\left(\frac{dX_1}{X_2}\right)\left(\frac{X_1}{X_2}\right)\right]\right\} \end{aligned} \quad (2.5.18)$$

PROPOSITION 2.5.2. *Suppose there exist two Itô processes:*

$$dX_1 = X_1\mu_1dt + X_1\sigma_1dW_1 \quad \text{and} \quad dX_2 = X_2\mu_2dt + X_2\sigma_2dW_2$$

or equivalently,

$$\frac{dX_1}{X_1} = \mu_1dt + \sigma_1dW_1 \quad \text{and} \quad \frac{dX_2}{X_2} = \mu_2dt + \sigma_2dW_2$$

Then

$$\begin{aligned} \mathbb{E}\left[\left(\frac{dX_2}{X_2}\right)\left(\frac{dX_2}{X_2}\right)\right] &= \mathbb{E}[(\mu_2dt + \sigma_2dW_2)(\mu_2dt + \sigma_2dW_2)] \\ &= \mathbb{E}[\mu_2^2dt^2] + \mathbb{E}[\sigma_2^2(dW_2)^2] + 2\mathbb{E}[\sigma_2dtdW_2] \\ &= \mu_2^2dt^2 + \sigma_2^2dt + \sigma_2dt\mathbb{E}[dW_2]. \end{aligned} \quad (2.5.19)$$

Since  $dt \rightarrow 0$ , we neglect all terms in  $dt$  with order higher than 1 in Eq.(2.5.19). Use the fact that  $\mathbb{E}[dW_2] = 0$ , so Eq.(2.5.19) becomes

$$\mathbb{E}\left[\left(\frac{dX_2}{X_2}\right)\left(\frac{dX_2}{X_2}\right)\right] = \sigma_2^2dt. \quad (2.5.20)$$

By the same token, we have

$$\begin{aligned}
\mathbb{E} \left[ \left( \frac{dX_1}{X_1} \right) \left( \frac{dX_2}{X_2} \right) \right] &= \mathbb{E} [(\mu_1 dt + \sigma_1 dW_1) (\mu_2 dt + \sigma_2 dW_2)] \\
&= \mathbb{E} [\mu_1^2 \mu_2^2 dt^2] + \mathbb{E} [\mu_2 dt \sigma_1 dW_1] + \mathbb{E} [\mu_1 dt \sigma_2 dW_2] + \mathbb{E} [\sigma_1 \sigma_2 dW_1 dW_2] \\
&= \mu_1 \mu_2 dt^2 + \sigma_1 \mu_2 dt \mathbb{E} [dW_1] + \sigma_2 \mu_1 dt \mathbb{E} [dW_2] + \sigma_1 \sigma_2 \mathbb{E} [dW_1 dW_2] \\
&= \sigma_1 \sigma_2 \rho_{12} dt. \tag{2.5.21}
\end{aligned}$$

Substituting Eq.(2.5.20) and Eq.(2.5.21) into Eq.(2.5.18), we obtain a stochastic differential equation:

$$\begin{aligned}
d \left( \frac{X_1}{X_2} \right) &= \left( \frac{X_1}{X_2} \right) \left\{ \frac{dX_1}{X_1} - \frac{dX_2}{X_2} + \sigma_2^2 dt - \sigma_1 \sigma_2 \rho_{12} dt \right\} \\
&= \left( \frac{X_1}{X_2} \right) \left\{ \mu_1 dt + \sigma_1 dW_1 - \mu_2 dt - \sigma_2 dW_2 + \sigma_2^2 dt - \sigma_1 \sigma_2 \rho_{12} dt \right\} \\
&= \left( \frac{X_1}{X_2} \right) \left\{ \mu_1 - \mu_2 + \sigma_2^2 - \sigma_1 \sigma_2 \rho_{12} \right\} dt + \left( \frac{X_1}{X_2} \right) \left\{ \sigma_1 dW_1 - \sigma_2 dW_2 \right\}.
\end{aligned}$$

## 2.6. Risk-neutral measure

Since the drift  $\mu$  in a partial differential equation makes itself difficult to be solved under the original measure  $\mathbb{P}$ , a new measure  $\mathbb{Q}$  is mandatory to replace the drift term so that the differential equation becomes a martingale. The risk-neutral measure  $\mathbb{Q}$  is powerful for computing prices of derivative securities, such as Black-Scholes-Merton-based models. For Eq. (2.4.1), risk neutrality implies that the expected total return on the asset  $\frac{dS_t}{S_t}$  is the product of risk-free rate  $r$  and time interval  $dt$ :  $\mathbb{E}^* \left[ \frac{dS_t}{S_t} \right] = r dt$ .

**THEOREM 2. Girsanov Theorem** For any stochastic process  $g_t$ ,  $0 \leq t \leq T$ , such that  $\int_0^t g_s^2 ds < \infty$ , the Radon-Nikodým derivative  $\frac{d\mathbb{Q}}{d\mathbb{P}} = \rho_t$  is

$$\rho_t = \exp \left\{ \int_0^t g_s dW_s^{\mathbb{P}} - \frac{1}{2} g_s^2 ds \right\} \tag{2.6.1}$$

where  $dW_s^{\mathbb{P}}$  is Brownian motion under probability measure  $\mathbb{P}$ .

Under probability measure  $\mathbb{Q}$  we have Brownian motion  $W_t^{\mathbb{Q}}$  defined as

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \int_0^t g_s ds. \quad (2.6.2)$$

The purpose of applying the Girsanov theorem in the risk-neutral measure is to provide a mechanism for changing the drift of a Brownian motion.

Now we use GBM to model a stock price process  $S_t$  for  $0 \leq t \leq T$ , which has the differential equation (2.4.1) with solution (2.5.14). For the case that  $\mu_t$  and  $\sigma_t$  are time-dependent adapted processes, we define the discount process  $D_t$  by adapted interest rate process  $R_t$  as the following:

$$D_t = \exp \left\{ - \int_0^t R_u du \right\} \quad \text{or} \quad dD_t = -R_t D_t dt. \quad (2.6.3)$$

Then the discounted stock price is

$$D_t S_t = S_0 \exp \left[ \int_0^t \left( \mu_u - \frac{1}{2} \sigma_u^2 - R_u \right) du + \int_0^t \sigma_u dW_u^{\mathbb{P}} \right]. \quad (2.6.4)$$

Differentiating (2.6.4) by the Itô product rule, we have

$$d(D_t S_t) = (\mu_t - R_t) D_t S_t dt + \sigma_t D_t S_t dW_t^{\mathbb{P}}. \quad (2.6.5)$$

The removal of  $\mu_t$  can be achieved by substituting the Sharpe ratio

$$\theta_t = \frac{\mu_t - R_t}{\sigma_t} \quad (2.6.6)$$

into Eq.(2.6.5) such that

$$d(D_t S_t) = \sigma_t D_t S_t [\theta_t dt + dW_t^{\mathbb{P}}]. \quad (2.6.7)$$

In order to make  $D_t S_t$  as a martingale, we use Girsanov Theorem 2 to replace the original measure  $\mathbb{P}$  by the risk-neutral measure  $\mathbb{Q}$ . Hence after applying the Girsanov Theorem, the

differential form of the discounted stock price process is

$$d(D_t S_t) = \sigma_t D_t S_t dW_t^{\mathbb{Q}} \quad (2.6.8)$$

with integral

$$D_t S_t = S_0 + \int_0^t \sigma_t D_t S_t dW_t^{\mathbb{Q}}, \quad (2.6.9)$$

which is a martingale since  $\int_0^t \sigma_t D_t S_t dW_t^{\mathbb{Q}}$  is an Itô integral. Since  $\mu_t$ , under measure  $\mathbb{P}$ , is equivalent to  $\theta_t$  under risk-neutral measure  $\mathbb{Q}$ , we can just replace  $W_t^{\mathbb{P}}$  and  $\mu_t$ , respectively, by  $W_t^{\mathbb{Q}}$  and  $R_t$  in Eq. (2.5.14) to obtain the solution

$$S_t = S_0 \exp \left\{ \left( R_t - \frac{1}{2} \sigma_t^2 \right) t + \sigma_t W_t^{\mathbb{Q}} \right\} \quad (2.6.10)$$

and by Eq. (2.5.14), its differential is computed as

$$dS_t = R_t S_t dt + \sigma_t S_t dW_t^{\mathbb{Q}} \quad (2.6.11)$$

where  $R_t$  is risk-free interest rate under risk-neutral measure  $\mathbb{Q}$ .

Since both  $D_t$  and  $S_t$  are Itô processes with zero quadratic variation,  $D_t S_t$  is a martingale (by Definition 2.5.1). Under risk-neutral  $\mathbb{Q}$ , the discounted stock price is

$$D_t S_t = \mathbb{E}^* [D_T S_T | \mathcal{F}_t] \quad (2.6.12)$$

where  $S_T$ , the payoff of a stock price at time  $T$ , is an  $\mathcal{F}_T$ -measurable and path-dependent random variable for  $0 \leq t \leq T$ . Consequently, the price of the stock  $S_t$  at time  $t$  can be computed by

$$S_t = \mathbb{E}^* \left[ \frac{D_T}{D_t} S_T | \mathcal{F}_t \right] \quad (2.6.13)$$

Then using the discount factor  $D_t$  defined in Eq. (2.6.3), for  $0 \leq t \leq T$  we obtain the risk-neutral formula for pricing a stock as the following:

$$S_t = \mathbb{E}^* \left[ \exp \left\{ - \int_t^T R_u du \right\} S_T | \mathcal{F}_t \right] \quad (2.6.14)$$

In a similar manner, for  $0 \leq t \leq T$  a risk-neutral formula for pricing a derivative security with a payoff  $Z_T$  can be derived by

$$Z_t = \mathbb{E}^* [ \exp \{ -r(T-t) \} Z_T | \mathcal{F}_t ] \quad (2.6.15)$$

In Black-Scholes framework, the European call option with its payoff function  $C_{s,T} = (S_T - K)^+$  is priced at time  $t$  by

$$C_{s,t} = \mathbb{E}^* [ \exp \{ -r(T-t) \} (S_T - K)^+ | \mathcal{F}_t ] \quad (2.6.16)$$

where  $S_t$  is the price of the underlying asset for  $0 \leq t \leq T$  and  $K$  is the strike price.

However, the valuation of European options with a large number of the underlying assets is more complex but can conveniently be achieved by using Monte Carlo simulation or Markov Chain approximation to compute the multi-dimensional definite integrands.

## 2.7. Non-Geometric Brownian Motion

In sections 2.2 and 2.4, we discussed the properties of Brownian motion  $dS_t = \mu_t dt + \sigma_t dW_t$  and the properties of geometric Brownian motion  $dS_t = S_t(\mu_t dt + \sigma_t dW_t)$ . Now we consider a non-geometric Brownian motion defined by the partial differential equation  $dS_t = S_t^p(\mu_t dt + \sigma_t dW_t)$ , where  $S_t^p$  is a time-dependent random variable raised to a constant power  $p \in \mathbf{R}^+ \setminus \{0, 1\}$ ; the drift  $\mu_t$  and the volatility  $\sigma_t$  are adapted processes;  $W_t$  is a standard Brownian motion.

To solve the partial differential equation of the non-geometric Brownian motion

$$dS_t = S_t^p(\mu dt + \sigma dW_t), \quad (2.7.1)$$

we apply Itô lemma 2.5.11 for  $0 \leq t \leq T$  to find any possible analytical expression of Eq. (2.7.1). In our case, we assume  $\mu$  and  $\sigma$  are constants. Let  $f(S_t, t) = \frac{1}{1-p} S_t^{1-p}$ , then we have  $f_t = 0$ ,  $f_{S_t} = \frac{1}{S_t^p}$ ,  $f_{S_t S_t} = -p S_t^{-(1+p)}$ . By Itô lemma we can obtain

$$\begin{aligned}
df &= f_t dt + f_{S_t} dS_t + \frac{1}{2} f_{S_t S_t} (dS_t)^2 \\
&= 0 + \frac{1}{S_t^p} S_t^p (\mu dt + \sigma dW_t) - \frac{1}{2} p S_t^{-(1+p)} S_t^{2p} \sigma^2 dt \\
&= \left[ \frac{1}{S_t^p} S_t^p \mu - \frac{1}{2} p S_t^{-(1+p)} S_t^{2p} \sigma^2 \right] dt + \frac{1}{S_t^p} S_t^p \sigma dW_t \\
&= \left[ \mu - \frac{1}{2} p \sigma^2 S_t^{p-1} \right] dt + \sigma dW_t
\end{aligned} \tag{2.7.2}$$

Integrating Eq. (2.7.2) from 0 to  $T$ , we obtain

$$\begin{aligned}
\int_0^T df &= \int_0^T \left( \mu - \frac{1}{2} p \sigma^2 S_t^{p-1} \right) dt + \int_0^T \sigma dW_t \\
\Rightarrow \frac{1}{1-p} [S_T^{1-p} - S_0^{1-p}] &= \mu T + \sigma W_T - \frac{1}{2} p \sigma^2 \int_0^T S_t^{p-1} dt \\
\Rightarrow S_T^{1-p} &= S_0^{1-p} + (1-p) \left[ \mu T + \sigma W_T - \frac{1}{2} p \sigma^2 \int_0^T S_t^{p-1} dt \right]
\end{aligned} \tag{2.7.3}$$

$$\Rightarrow S_T = \left\{ S_0^{1-p} + (1-p) \left[ \mu T + \sigma W_T - \frac{1}{2} p \sigma^2 \int_0^T S_t^{p-1} dt \right] \right\}^{\frac{1}{1-p}}. \tag{2.7.4}$$

On the right-hand side of Eq. (2.7.2), the power term  $\frac{1}{1-p}$  causes difficulty in computing the expectation of  $S_T$ . Instead, we take the expectation of Eq. (2.7.3)

$$\mathbb{E} [S_T^{1-p}] = \mathbb{E} \left\{ S_0^{1-p} + (1-p) \left[ \mu T + \sigma W_T - \frac{1}{2} p \sigma^2 \int_0^T S_t^{p-1} dt \right] \right\} \tag{2.7.5}$$

Since  $p$ ,  $T$ ,  $\mu$ , and  $\sigma$  are deterministic values and  $\mathbb{E} [W_T] = 0$ , we then have

$$\mathbb{E} [S_T^{1-p}] = S_0^{1-p} + (1-p) \mu T - \frac{1}{2} p (1-p) \sigma^2 \int_0^T \mathbb{E} [S_t^{p-1}] dt. \tag{2.7.6}$$



Let  $h(t, p)$  be the function of  $t$  and  $p$  such that  $\mathbb{E}[S_t^p] = h(t, p)$ ,  $\mathbb{E}[S_t^{p-1}] = h(t, p-1)$ , and  $\mathbb{E}[S_t^{1-p}] = h(t, 1-p)$ . Thus the expectation of  $S_T^{1-p}$  can be expressed as

$$h(T, 1-p) = S_0^{1-p} + (1-p)\mu T - \frac{1}{2}p(1-p)\sigma^2 \int_0^T h(t, p-1)dt. \quad (2.7.7)$$

For discretization, Eq. (2.7.7) can be written as

$$h(T, 1-p) = S_0^{1-p} + (1-p)\mu T - \frac{1}{2}p(1-p)\sigma^2 \sum_{j=1}^n h(t_j, p-1)dt \quad (2.7.8)$$

where  $t_j = \left\{ \frac{t_{j-1} + t_{j+1}}{2} : j \geq 1 \right\}$  for time interval  $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$ .

In order to solve the implicit integrand at the end of Eq. (2.7.7), we use substitution to form nonhomogeneous second-order differential equations. Given initial and boundary conditions, we can solve the differential equations to obtain the corresponding solutions.

Let  $t^* = T$  and  $q = 1-p$  such that  $h(T, 1-p)$  and  $h(T, p-1)$  can be defined as

$$x(t^*) \triangleq h(t^*, q) = S_0^q + q\mu t^* - \frac{1}{2}q(1-q)\sigma^2 \int_0^{t^*} \mathbb{E}[S_t^{-q}] dt, \quad (2.7.9)$$

$$y(t^*) \triangleq h(t^*, -q) = S_0^{-q} - q\mu t^* + \frac{1}{2}q(1+q)\sigma^2 \int_0^{t^*} \mathbb{E}[S_t^q] dt, \quad (2.7.10)$$

with initial conditions  $x(0) = S_0^q$  and  $y(0) = S_0^{-q}$ .

We take first derivative of Eq. (2.7.9) and Eq. (2.7.10) with respect to time  $t^*$

$$x'(t^*) = q\mu - \frac{1}{2}q(1-q)\sigma^2 y(t^*), \quad (2.7.11)$$

$$y'(t^*) = -q\mu + \frac{1}{2}q(1+q)\sigma^2 x(t^*), \quad (2.7.12)$$

with  $x'(0) = q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}$  and  $y'(0) = -q\mu - \frac{1}{2}q(1+q)\sigma^2 S_0^q$ .

Then we take second derivative of Eq. (2.7.9) and Eq. (2.7.10) with respect to time  $t^*$

$$x''(t^*) + \frac{1}{4}q^2(1 - q^2)\sigma^4 x(t^*) = \frac{1}{2}q^2(1 - q)\mu\sigma^2 \quad (2.7.13)$$

$$y''(t^*) + \frac{1}{4}q^2(1 - q^2)\sigma^4 y(t^*) = \frac{1}{2}q^2(1 + q)\mu\sigma^2. \quad (2.7.14)$$

Now Eq. (2.7.13) is an nonhomogeneous second-order differential equation with deterministic values  $q$ ,  $\mu$  and  $\sigma$ . Since  $\frac{1}{2}q^2(1 - q)\mu\sigma^2$  is a constant, we let  $x_p(t^*)=A$  (where  $A$  is a constant) be a particular solution of Eq. (2.7.13). Then  $x_p''(t^*) = x_p'(t^*) = 0$ . Using Eq. (2.7.13) we obtain  $\frac{1}{4}q^2(1 - q^2)\sigma^4 x_p(t^*) = \frac{1}{2}q^2(1 - q)\mu\sigma^2$ . Hence  $x_p(t^*) = \frac{2\mu}{(1+q)\sigma^2}$ .

**Case 1:**  $q = -1$ . Inserting  $q = -1$  into the differential equation (2.7.13), we have  $x''(t^*) = \mu\sigma^2$ . Now we integrate  $x''(t^*) = \mu\sigma^2$  with respect to  $t^*$  to obtain the following equations

$$x'(t) = \mu\sigma^2 t^* + a_1$$

$$x(t) = \frac{1}{2}\mu\sigma^2 t^{*2} + a_1 t^* + a_2$$

where  $a_1$  and  $a_2$  are corresponding coefficients.

Given the conditions  $x'(0) = -\mu + \sigma^2 S_0$  and  $x(0) = S_0^{-1}$ , we can obtain  $a_1 = S_0^{-1}$  and  $a_2 = -\mu + \sigma^2 S_0$ . The solution to the differential equation (2.7.13) is

$$x(t) = \frac{1}{2}\mu\sigma^2 t^{*2} + S_0^{-1} t^* + \sigma^2 S_0 - \mu. \quad (2.7.15)$$

**Case 2:**  $q = 0$  or  $q = 1$ . There exist double roots  $R = 0$  for the characteristic equation of  $x''(t^*) = 0$ . Hence the nonhomogeneous equation has the solution in the form of

$$x(t^*) = b_1 + b_2 t^* + \frac{2\mu}{(1 + q)\sigma^2}. \quad (2.7.16)$$

Given that  $x(0) = S_0^q$  and  $x'(0) = q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}$ , we can solve for the corresponding coefficients  $b_1$  and  $b_2$  in the following equations

$$\begin{aligned} x(0) &= b_1 + \frac{2\mu}{(1+q)\sigma^2} = S_0^q \\ x'(0) &= b_2 = q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}. \end{aligned}$$

So  $b_1 = S_0^q - \frac{2\mu}{(1+q)\sigma^2}$  and the solution to the nonhomogeneous equation is

$$x(t^*) = S_0^q + \left[ q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q} \right] t^*. \quad (2.7.17)$$

For  $q = 0$ , the corresponding solution is  $x(t^*) = 1$ .

For  $q = 1$ , the corresponding solution is  $x(t^*) = S_0 + \mu t^*$ .

**Case 3:**  $|q| > 1$ . The characteristic equation of the associated homogeneous equation in Eq. (2.7.13) has two distinct real roots  $r_1 = -\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2$  and  $r_2 = \frac{1}{2}|q|\sqrt{q^2-1}\sigma^2$ . The solution to the nonhomogeneous equation can be represented in the form of

$$x(t^*) = c_1 e^{-\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2 t^*} + c_2 e^{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2 t^*} + \frac{2\mu}{(1+q)\sigma^2}. \quad (2.7.18)$$

Given the conditions that  $x(0) = S_0^q$  and  $x'(0) = q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}$ , we can obtain the corresponding coefficients  $c_1$  and  $c_2$

$$\begin{aligned} c_1 &= \frac{1}{2} \left( S_0^q - \frac{2\mu}{(1+q)\sigma^2} - \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2} \right) \\ c_2 &= \frac{1}{2} \left( S_0^q - \frac{2\mu}{(1+q)\sigma^2} + \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2} \right). \end{aligned}$$

Hence the solution to Eq. (2.7.13) is

$$\begin{aligned}
x(t^*) &= \frac{1}{2} \left( S_0^q - \frac{2\mu}{(1+q)\sigma^2} - \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2} \right) e^{-\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2 t^*} \\
&+ \frac{1}{2} \left( S_0^q - \frac{2\mu}{(1+q)\sigma^2} + \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2} \right) e^{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2 t^*} \\
&+ \frac{2\mu}{(1+q)\sigma^2}.
\end{aligned} \tag{2.7.19}$$

**Case 4:**  $|q| < 1$ . Since the characteristic equation has two distinct complex roots  $I_{1,2} = \pm \frac{i}{2}|q|\sqrt{1-q^2}\sigma^2$  for  $|q| < 1$ , the solution to the nonhomogeneous differential equation (2.7.13) can be written in the form of

$$x(t^*) = d_1 \cos \left( \frac{1}{2}|q|\sqrt{1-q^2}\sigma^2 t^* \right) + d_2 \sin \left( \frac{1}{2}|q|\sqrt{1-q^2}\sigma^2 t^* \right) + \frac{2\mu}{(1+q)\sigma^2} \tag{2.7.20}$$

where  $d_1 = S_0^q - \frac{2\mu}{(1+q)\sigma^2}$  and  $d_2 = \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}}{\frac{1}{2}|q|\sqrt{1-q^2}\sigma^2}$  are the corresponding coefficients.

As  $|q| < 1$ , the solution to the nonhomogeneous differential equation (2.7.13) is

$$\begin{aligned}
x(t^*) &= \left[ S_0^q - \frac{2\mu}{(1+q)\sigma^2} \right] \cos \left( \frac{1}{2}|q|\sqrt{1-q^2}\sigma^2 t^* \right) \\
&+ \left[ \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}}{\frac{1}{2}|q|\sqrt{1-q^2}\sigma^2} \right] \sin \left( \frac{1}{2}|q|\sqrt{1-q^2}\sigma^2 t^* \right) \\
&+ \frac{2\mu}{(1+q)\sigma^2}.
\end{aligned} \tag{2.7.21}$$

## CHAPTER 3

### DISCRETE-TIME MARKOV CHAIN

#### 3.1. Definitions and Properties

Let  $I$  be a countable set such that each  $i \in I$ , where  $i$  is called a *state* and  $I$  is called the *state space*. In addition,  $\lambda = (\lambda_i : i \in I)$  is a *measure* on  $I$  if  $0 \leq \lambda_i < \infty$  for all  $i \in I$  and  $\lambda$  is a distribution if *total mass*  $\sum_{i \in I} \lambda_i$  equals 1. For a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , consider a random variable  $X$  with values in  $I$  such that  $X : \Omega \rightarrow I$ .

Suppose we set

$$\lambda_i = \mathbb{P}(X = i) = \mathbb{P}(\omega : X(\omega) = i).$$

Then  $\lambda$  defines the distribution of  $X$  where a random state  $X$  takes the value  $i$  with probability  $\lambda_i$ . In addition, an  $N \times N$  *transition* probability matrix  $\mathbf{P} = (p_{ij} : i, j \in I)$  is *stochastic* if  $\mathbf{P}_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$  for all  $n = 0, 1, 2, \dots, N - 1$  has the following properties: (i)  $0 \leq \mathbf{P}_{i,j} \leq 1$  for  $1 \leq i, j \leq N$ , (ii)  $\sum_{j=1}^N \mathbf{P}_{i,j} = 1$  for  $1 \leq i \leq N$ .

By a definition in terms of the corresponding matrix, we say that  $(X_n)_{n \geq 0}$  is a **Markov chain** with *initial distribution*  $\lambda$  and *transition* probability matrix  $\mathbf{P}$  for  $n \geq 0$  and  $i_0, i_1, \dots, i_{n+1} \in I$ , if

- $\mathbb{P}(X_0 = i_0) = \lambda_{i_0}$ ;
- $\mathbb{P}(X_{n+1} = i_{n+1} | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = p_{i_n i_{n+1}}$ .

In other words, if  $(X_n)_{0 \leq n \leq N}$  is a finite sequence of random variables satisfying the above conditions for  $n = 0, 1, \dots, N - 1$ , then we say  $(X_n)_{0 \leq n \leq N}$  is *Markov*  $(\lambda, \mathbf{P})$ .

**THEOREM 3.** *A discrete-time random process  $(X_n)_{0 \leq n \leq N}$  is Markov  $(\lambda, \mathbf{P})$  if and only if for all  $i_0, i_1, \dots, i_N \in I$*

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_N = i_N) = \lambda_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{N-1} i_N} \quad (3.1.1)$$

**PROOF.** Suppose  $(X_n)_{0 \leq n \leq N}$  is Markov  $(\lambda, \mathbf{P})$ , then

$$\begin{aligned} & \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_N = i_N) \\ = & \mathbb{P}(X_0 = i_0) \mathbb{P}(X_1 = i_1 | X_0 = i_0) \cdots \mathbb{P}(X_N = i_N | X_0 = i_0, X_1 = i_1, \dots, X_{N-1} = i_{N-1}) \\ = & \lambda_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{N-1} i_N}. \end{aligned}$$

On the other hand, if Eq. (3.1.1) holds for  $N$ , then using  $\sum_{j \in I} p_{ij} = 1$  over  $i_N \in I$ , Eq. (3.1.1) holds for  $N-1$  and, by induction

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = \lambda_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

for all  $n = 0, 1, \dots, N$ . In particular,  $\mathbb{P}(X_0 = i_0) = \lambda_{i_0}$  and, for  $n = 0, 1, \dots, N-1$ ,

$$\begin{aligned} & \mathbb{P}(X_{n+1} = i_{n+1} | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) \\ = & \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n, X_{n+1} = i_{n+1}) / \mathbb{P}(X_0 = i_0, \dots, X_n = i_n) \\ = & p_{i_n i_{n+1}} \end{aligned}$$

So  $(X_n)_{0 \leq n \leq N}$  is Markov  $(\lambda, \mathbf{P})$ . □

We consider distributions and measures  $\lambda$  as row vectors whose components are indexed by  $I$ , whereas  $\mathbf{P}$  is a matrix whose entries are indexed by  $I \times I$ . In general, for  $I$  is finite, we define states  $n = 1, 2, \dots, N$ ; then  $\lambda$  will be an  $N$ -vector and  $\mathbf{P}$  is an  $N \times N$ -matrix. Furthermore, matrix multiplications  $\lambda \mathbf{P}$  and a new matrix  $\mathbf{P}^2$  are defined by

$$(\lambda \mathbf{P})_j = \sum_{i \in I} \lambda_i p_{ij}, \quad \mathbf{P}_{ik}^2 = \sum_{j \in I} p_{ij} p_{jk}.$$

Similarly, for any  $n$  we define  $\mathbf{P}^n$  and  $\mathbf{P}^0$  is the identity matrix  $I$ , where  $(I)_{ij} = \delta_{ij}$  ( $\delta_{ij} = 1$  for  $i = j$ ; otherwise,  $\delta_{ij} = 0$ ). So we can write  $p_{ij}^{(n)} = (\mathbf{P}^n)_{ij}$  for the  $(i, j)$  entry in  $\mathbf{P}^n$  which represents  $n$ -step transition probability from  $i$  to  $j$ .

**DEFINITION 3.1.1.** *A Markov chain  $\{X_n : n \in \mathbb{N}\}$  is homogeneous if*

$$\mathbb{P}[X_{n+1} = j | X_n = i] = \mathbb{P}[X_1 = j | X_0 = i]$$

for all states  $i, j \in I$  and for all time  $n \in \mathbb{N}$ .

**THEOREM 4.** *Let  $(X_n)_{n \geq 0}$  be Markov  $(\lambda, \mathbf{P})$ . Then, for all  $n, m \geq 0$ ,*

- (1)  $\mathbb{P}(X_n = j) = (\lambda \mathbf{P}^n)_j$ ;
- (2)  $\mathbb{P}_i(X_n = j) = \mathbb{P}(X_{n+m} = j | X_m = i) = p_{ij}^{(n)}$ .

**PROOF.** (1) By Theorem (3), we can obtain that

$$\begin{aligned} \mathbb{P}(X_n = j) &= \sum_{i_0 \in I} \cdots \sum_{i_{n-1} \in I} \mathbb{P}(X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = j) \\ &= \sum_{i_0 \in I} \cdots \sum_{i_{n-1} \in I} \lambda_{i_0} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} j} \\ &= (\lambda \mathbf{P}^n)_j. \end{aligned}$$

- (2) Condition on  $X_m = i$  with Markov property,  $(X_{m+n})_{n \geq 0}$  is Markov  $(\delta_i, \mathbf{P})$ ; in other words, we can just substitute  $\delta_i$  for  $\lambda$  in (1).

□

The following example uses linear difference equations to calculate  $p_{ij}^{(n)}$ .

**EXAMPLE 3.1.** *Given a general two-state chain has transition matrix of the form*

$$\mathbf{P} = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}$$

where  $0 \leq \alpha, \beta \leq 1$ .

The relation  $\mathbf{P}_{n+1} = \mathbf{P}^n \mathbf{P}$  is used to form

$$p_{11}^{(n+1)} = p_{12}^{(n)} \beta + p_{11}^{(n)} (1 - \alpha). \quad (3.1.2)$$

By the Markov property, we know that

$$\mathbb{P}_1(X_n = 1 \text{ or } 2) = p_{11}^{(n)} + p_{12}^{(n)} = 1. \quad (3.1.3)$$

So using Eq. (3.1.2) and Eq. (3.1.3) to eliminate  $p_{12}^{(n)}$ , we have a recurrence relation for  $p_{11}^{(n)}$ :

$$P_{11}^{(n+1)} = (1 - \alpha - \beta)p_{11}^{(n)} + \beta, \quad p_{11}^{(0)} = 1.$$

After algebraic calculations by solving linear difference equations, we obtain a unique solution:

$$p_{11}^{(n)} = \begin{cases} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1 - \alpha - \beta)^n & \text{for } \alpha + \beta > 0 \\ 1 & \text{for } \alpha + \beta = 0. \end{cases}$$

### 3.2. Invariant Distributions

The long-time property of a Markov chain is connected with the invariant measure of its transition probability matrix  $\mathbf{P}^n$  for large  $n$ . For instance, the transition probability matrix forms a limited matrix, defined by

$$\mathbb{P} = \lim_{n \rightarrow \infty} \mathbf{P}^n,$$

and it has identical rows  $\bar{\pi}$ . If  $\bar{r}$  is any probability vector with non-negative entries, then

$$\lim_{n \rightarrow \infty} \bar{r} \mathbf{P}^n = \bar{\pi}$$

where  $\bar{\pi}$  is one of the rows of  $\mathbb{P}$ .

We say  $\bar{\pi}$  is an *invariant* probability distribution for  $\mathbf{P}$  if

$$\bar{\pi} = \bar{\pi} \mathbf{P}$$

The terms *equilibrium* and *stationary* have the same meaning.

**THEOREM 5.** *Suppose for some  $i \in I$ , where  $I$  is finite, that*

$$p_{ij}^{(n)} \rightarrow \pi_j \quad \text{as } n \rightarrow \infty \text{ for all } j \in I$$

*Then  $\pi = (\pi : j \in I)$  is an invariant distribution.*

**PROOF.** Observe that

$$\sum_{j \in I} \pi_j = \sum_{j \in I} \lim_{n \rightarrow \infty} p_{ij}^{(n)} = \lim_{n \rightarrow \infty} \sum_{j \in I} p_{ij}^{(n)} = 1$$

so

$$\pi_j = \lim_{n \rightarrow \infty} p_{ij}^{(n)} = \lim_{n \rightarrow \infty} \sum_{k \in I} p_{ik}^{(n)} p_{kj} = \sum_{k \in I} \lim_{n \rightarrow \infty} p_{ik}^{(n)} p_{kj} = \sum_{k \in I} \pi_k p_{kj}$$

Hence  $\pi$  is an invariant distribution. □



### 3.3. Hitting time and absorption probabilities

For a Markov chain  $(X_n)_{n \geq 0}$  with transition probability matrix  $\mathbf{P}$ , the *hitting time* of a subset  $A$  of  $I$  is the random variable  $H^A : \Omega \rightarrow \{0, 1, 2, \dots\} \cup \{\infty\}$  given by

$$H^A(\omega) = \inf\{n \geq 0 : X_n(\omega) \in A\}$$

where  $\inf\{\emptyset\} = \infty$ . Also, the probability starting from  $i$  that  $(X_n)_{n \geq 0}$  hits  $A$  is

$$h_i^A = \mathbb{P}_i(H^A < \infty). \quad (3.3.1)$$

For a closed class  $A$ ,  $h_i^A$  is called the *absorption probability*. The expected time for  $(X_n)_{n \geq 0}$  to hit  $A$  is given by

$$m_i^A = \mathbb{E}_i(H^A) = \sum_{n < \infty} n \mathbb{P}(H^A = n) + \infty \mathbb{P}(H^A = \infty). \quad (3.3.2)$$

For abbreviated expressions, Eq. (3.3.1) and Eq. (3.3.2) can be rewritten as

$$h_i^A = \mathbb{P}_i(\text{hit } A), \quad m_i^A = \mathbb{E}_i(\text{time to hit } A)$$

The hitting time and absorption probability can be computed by linear equations associated with the transition probability matrix  $\mathbf{P}$ .

**EXAMPLE 3.2.** *The transition probability matrix  $\mathbf{P}$  is given as*

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

*Starting from 2, what is the probability of absorption in 4? How long does it take until the chain is absorbed in 1 or 4?*

*First, we need to find the probability hitting 4, then the mean time for hitting 4*

$$h_i = \mathbb{P}_i(\text{hit } 4), \quad m_i = \mathbb{E}_i(\text{time to hit } \{1, 4\}).$$

Obviously,  $h_1 = 0$ ,  $h_4 = 1$ , and  $m_1 = 0 = m_4$ . Now we consider we start from 2 and then make one step. The Markov chain could reach 1 with probability  $\frac{1}{2}$  or reach 3 with probability  $\frac{1}{2}$ . Therefore,

$$h_2 = \frac{1}{2}h_1 + \frac{1}{2}h_3, \quad m_2 = 1 + \frac{1}{2}m_1 + \frac{1}{2}m_3.$$

The 1 in the second formula is the time counted for the first step. By the same token,

$$h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4, \quad m_3 = 1 + \frac{1}{2}m_2 + \frac{1}{2}m_4.$$

After some substitutions, we have

$$\begin{aligned} h_2 &= \frac{1}{2}h_3 = \frac{1}{2}\left(\frac{1}{2}h_2 + \frac{1}{2}\right), \\ m_2 &= 1 + \frac{1}{2}m_3 = 1 + \frac{1}{2}\left(1 + \frac{1}{2}m_2\right). \end{aligned}$$

Hence, we can obtain  $h_2 = \frac{1}{3}$  and  $m_2 = 2$ . In other words, starting from 2, the probability of hitting 4 is  $\frac{1}{3}$  and the mean time to absorption is 2.

**THEOREM 6.** *The vector of hitting probabilities  $h^A = (h_A : i \in I)$  is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} h_i^A = 1 & \text{for } i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{for } i \notin A \end{cases}$$

**PROOF.** First, we need to show that  $h_i^A$  satisfies the above conditions.

If  $X_0 = i \in A$ , then  $H^A = 0$  and  $h_i^A = 1$ . If  $X_0 = i \notin A$ , then  $H^A \geq 1$ , so by the Markov property

$$\mathbb{P}_i(H^A < \infty | X_1 = j) = \mathbb{P}_j(H^A < \infty) = h_j^A$$

and

$$\begin{aligned} h_i^A &= \mathbb{P}_i(H^A < \infty) = \sum_{j \in I} \mathbb{P}_i(H^A < \infty, X_1 = j) \\ &= \sum_{j \in I} \mathbb{P}_i(H^A < \infty | X_1 = j) = \sum_{j \in I} p_{ij} h_j^A. \end{aligned}$$

Now suppose that  $x = (x_i : i \in I)$  is any solution to the above conditions. Then, we have  $h_i^A = x_i = 1$  for  $i \in I$ . Suppose  $i \notin A$ , then

$$x_i = \sum_{j \in I} p_{ij} x_j = \sum_{j \in A} p_{ij} + \sum_{i \notin A} p_{ij} x_j.$$

Substitute for  $x_j$ , we obtain

$$\begin{aligned} x_i &= \sum_{j \in A} p_{ij} + \sum_{j \notin A} p_{ij} \left( \sum_{k \in A} p_{jk} + \sum_{k \notin A} p_{jk} x_k \right) \\ &= \mathbb{P}_i(X_1 \in A) + \mathbb{P}_i(X_1 \notin A, X_2 \in A) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} x_k. \end{aligned}$$

Continue the substitution for  $x$  in the final term. After  $n$  steps, we have

$$\begin{aligned} x_i &= \mathbb{P}_i(X_1 \in A) + \dots + \mathbb{P}_i(X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A) \\ &\quad + \sum_{j_1 \notin A} \dots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \dots p_{j_{n-1} j_n} x_{j_n}. \end{aligned} \tag{3.3.3}$$

Since

$$\mathbb{P}_i(H^A \leq n) = \mathbb{P}_i(X_1 \in A) + \dots + \mathbb{P}_i(X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A),$$

Eq. (3.3.3) can be rewritten as

$$x_i = \mathbb{P}_i(H^A \leq n) + \sum_{j_1 \notin A} \dots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \dots p_{j_{n-1} j_n} x_{j_n}. \tag{3.3.4}$$

If  $x$  is non-negative, so is  $\sum_{j_1 \notin A} \dots \sum_{j_n \notin A} p_{ij_1} p_{j_1 j_2} \dots p_{j_{n-1} j_n} x_{j_n}$ .

By Eq. (3.3.4), we obtain  $x_i \geq \mathbb{P}_i(H^A \leq n)$  for all  $x$ .

Hence

$$x_i \geq \lim_{n \rightarrow \infty} \mathbb{P}_i(H^A \leq n) = P_i(H^A < \infty) = h_i$$

□

### 3.4. Recurrence and Transience

Let  $(X_n)_{n \geq 0}$  be a homogeneous Markov chain with a discrete infinite state  $I = \{0, 1, \dots\}$  with a transition matrix  $\mathbf{P} = (p_{ij} : i, j \in I)$ .

We say  $i$  leads  $j$ , written as  $i \rightarrow j$ , if

$$\mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0.$$

**THEOREM 7.** *For distinct states  $i$  and  $j$  the following are equivalent:*

- (1)  $i \rightarrow j$ ;
- (2)  $p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} > 0$  for some states  $i_0, i_1, i_2, \dots, i_n$  with  $i_0 = i$  and  $i_n = j$ ;
- (3)  $p_{ij}^{(n)} > 0$  for some  $n \geq 0$ .

**PROOF.** Obviously,

$$p_{ij}^{(n)} \leq \mathbb{P}_i(X_n = j \text{ for some } n \geq 0) \leq \sum_{n=0}^{\infty} p_{ij}^{(n)}$$

proves the equivalence of (1) and (3). In addition,

$$p_{ij}^{(n)} = \sum_{i_1, \dots, i_{n-1}} p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} j}$$

shows the equivalence of (2) and (3). □

When states  $i$  and  $j$  are accessible to each other, we say  $i$  *communicates* with  $j$ , and we write  $i \leftrightarrow j$ . On the other hand, a class  $C$  is *closed* if

$$i \in C, i \rightarrow j \quad \text{imply} \quad j \in C.$$

A state  $i$  is *absorbing* if  $\{i\}$  is a closed class. The Markov chain where  $I$  is a single class is called *irreducible*. In general, state  $i$  in a single class is *recurrent* since  $i$  will come back with probability

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1.$$

Otherwise, state  $i$  is *transient* since  $i$  will eventually leave with probability

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 0.$$

Whether state  $i$  is transient or recurrent, we can compute the time for a Markov chain to reach state  $i$ . The hitting time to state  $i$  is the random variable  $T_i$  defined by

$$T_i(\omega) = \inf \{n \geq 1 : X_n(\omega) = i\}$$

where  $\inf \{\emptyset\} = \infty$ . If state  $i$  is transient, inductively we have

$$T_i^{(0)}(\omega) = 0, T_i^{(1)}(\omega) = T_i(\omega), \dots, T_i^{(r)}(\omega) = T_i^{(1)}(\omega) + T_i^{(2)}(\omega) + \dots + T_i^{(r-1)}(\omega).$$

For the  $(r+1)$ th hitting time  $T_i^{(r+1)}$ , we define

$$T_i^{(r+1)} = \inf \{n \geq T_i^{(r)}(\omega) + 1 : X_n(\omega) = i\}.$$

and the length of the  $r$ th excursion to  $i$  is

$$L_i^{(r)} = \begin{cases} T_i^{(r)} - T_i^{(r-1)} & \text{if } T_i^{(r-1)} < \infty \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we can calculate the number of visits  $V_i$  to  $i$  in terms of indication functions as

$V_i = \sum_{n=0}^{\infty} 1_{\{X_n=i\}}$ . The *expected* number of visits to  $i$  would be

$$\begin{aligned} \mathbb{E}_i(V_i) &= \mathbb{E}_i \sum_{n=0}^{\infty} 1_{\{X_n=i\}} = \sum_{n=0}^{\infty} \mathbb{E}_i(1_{\{X_n=i\}}) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \sum_{n=0}^{\infty} p_{ii}^{(n)}. \end{aligned}$$

With *return* probability defined as  $R_i = \mathbb{P}_i(T_i < \infty)$ , we can compute the distribution of  $V_i$  under  $\mathbb{P}_i$ .

**THEOREM 8.** *For  $r = 0, 1, 2, \dots$ , we have  $\mathbb{P}_i(V_i > r) = R_i^r$ .*

PROOF. If  $X_0 = i$ , then  $\{V_i > r\} = \{T_i^{(r)} < \infty\}$ . It is true for  $r = 0$ . Inductively, suppose that it is true for  $r$ , then

$$\begin{aligned}
\mathbb{P}_i(V_i > r + 1) &= \mathbb{P}_i(T_i^{(r+1)} < \infty) \\
&= \mathbb{P}_i(T_i^{(r)} < \infty \text{ and } S_i^{(r+1)} < \infty) \\
&= \mathbb{P}_i(S_i^{(r+1)} < \infty | T_i^{(r)} < \infty) \mathbb{P}_i(T_i^{(r)} < \infty) \\
&= R_i R_i^{(r)} = R_i^{(r+1)}
\end{aligned}$$

Hence, by induction it is true for all  $r$ . □

**THEOREM 9.** *Every state is either recurrent or transient. Equivalently,*

- (1) *if  $\mathbb{P}_i(T_i < \infty) = 1$ , then  $i$  is recurrent and  $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$  ;*
- (2) *if  $\mathbb{P}_i(T_i < \infty) < 1$ , then  $i$  is transient and  $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$ .*

PROOF. If  $\mathbb{P}_i(T_i < \infty) = 1$ , by Theorem 8, we have

$$\mathbb{P}_i(V_i = \infty) = \lim_{r \rightarrow \infty} \mathbb{P}_i(V_i > r) = 1$$

Then  $i$  is recurrent and

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i(V_i) = \infty.$$

By the same token, if  $R_i = \mathbb{P}_i(T_i < \infty) < 1$ , by Theorem 8, we have

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i(V_i) = \sum_{r=0}^{\infty} \mathbb{P}_i(V_i > r) = \sum_{r=0}^{\infty} R_i^r = \frac{1}{1-R_i} < \infty.$$

Then  $i$  is transient and  $\mathbb{P}_i(V_i = \infty) = 0$ . □

To say that states  $i$  and  $j$  are *communicative* (written as  $i \leftrightarrow j$ ) means that each state has a positive probability of eventually being reached by a chain starting in the other state. This communicative relation has the following properties: (1) *reflexive*:  $i \leftrightarrow i$ ; (2) *symmetric*:  $i \rightarrow j$  implies that  $j \rightarrow i$ ; (3) *transitive*:  $i \rightarrow k$  and  $k \rightarrow j$  implies that  $i \rightarrow j$ . These communicative properties partition the state space into disjoint sets— *communication classes*. To exploit this advantage, a transition matrix can be reduced into transient communication classes and

recurrent communication classes. *Reducibility* in state space is a successful way to increase efficiency in computation by solving the complexity of matrix multiplications  $\mathbf{P}^n$  for large  $n$ ,

Now consider that  $\mathbf{P}$  is the matrix for a reducible Markov chain consisting of recurrent communication classes  $R_1, R_2, \dots, R_k$  and transient communication classes  $T_1, T_2, \dots, T_l$ . So a series of submatrices of  $\mathbf{P}^n$  is associated with each recurrent class  $R_k^n$  in the transition matrix  $\mathbf{P}^n$  and constructed in the following form:

$$\mathbf{P}^n = \left[ \begin{array}{ccc|c} \mathbf{P}_1^n & & & \\ & \mathbf{P}_2^n & & 0 \\ & & \mathbf{P}_3^n & \\ & 0 & & \ddots \\ & & & & \mathbf{P}_k^n \\ \hline & & & & \mathbf{S}^n & \mathbf{Q}^n \end{array} \right]$$

for some matrix  $\mathbf{S}^n$  and transient matrix  $\mathbf{Q}^n$ , which only consists of transient classes  $T_1, T_2, \dots, T_l$ .

Now suppose for large  $n$ , a matrix  $\mathbf{P}^n$  consists of some **absorbing** boundaries so that each submatrix  $\mathbf{P}_j^n$  for  $0 \leq j \leq k$  of the matrix  $\mathbf{P}^n$  has one absorbing state  $i$  being the first site. Then new matrix  $\bar{\mathbf{P}}^n$  can be re-arranged in a canonical form:

$$\bar{\mathbf{P}}^n = \left[ \begin{array}{c|c} p_n(i, i) & \mathbf{0} \\ \hline \mathbf{S}^n & \mathbf{Q}^n \end{array} \right] = \left[ \begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{S}^n & \mathbf{Q}^n \end{array} \right]$$

where  $\mathbf{I}$  is an identity matrix and matrix  $\mathbf{Q}^n$  consists of only rows and columns of transient states. Hence some transient state  $l$  in submatrix  $\mathbf{Q}^n$  will eventually end in some absorbing state  $p_n(i, i)$ . In other words, a Markov Chain  $(X_n)_{n \geq 0}$  with *absorbing* boundaries is eventually terminated for  $n < \infty$ .

## CHAPTER 4

# FINANCIAL DERIVATIVES

### 4.1. Introduction

Financial derivatives are instruments whose values and characteristic features depend on the more basic underlying assets. The main types of derivatives are *forwards*, *futures*, *swaps*, and *options*. Such an instrument is so attractive to investors because the future value of the underlying asset is usually unknown. At maturity, the agreed price may differ drastically from the spot price associated with influenced factors in the market. In general, an instrument with a larger volatile (or riskier) underlying asset has the potential to generate larger fluctuations in its payoff. Hence *hedging* is an important strategy to avoid or limit risks in order to achieve optimal investment returns.

*Forwards* and *Futures* are agreements between two parties to buy or sell an asset at maturity in the future for a designated delivery price. There is a binding commitment between two parties with nothing to choose at a later time. Prior to maturity, no premiums are required and no money changes hands. A basic difference between futures and forwards is that futures contracts are traded on exchanges and are more formalized, whereas forwards are traded in the over-the-counter market (OTC).

*Swaps* are contracts regulating an exchange of cash flows in the future. A common type of swap is the *interest rate swap*, in which an exchange of a fixed interest rate and a floating rate on the same principal (itself not exchanged) is executed between two parties. In addition, variance swaps became popular in recent years as they provide a better way to observe the volatility of an underlying stock or index than regular options since options have the disadvantage of having Vega and Gamma exposures weighted around the strike price of the option.



*Options* are rights with no obligations to buy or sell underlying assets for an *exercise* (or *strike*) price  $K$ , which is fixed through the terms of the option contract. The standard options are *European* and *American* calls and puts, whose payoffs depend only on the final value of the underlying asset; on the other hand, *exotic* options, such as *Asian*, *barrier*, and *lookback* options, are *path-dependent* since their payoffs depend on the path of the underlying asset.

Options are worthless or expired after *maturity* date  $T$ . The call (or *put*) option gives the holder the right to buy (or *sell*) the underlying asset for an agreed price  $K$  by the maturity date  $T$ . The holder of the option paid the writer a premium which compensates for the writer's potential future liabilities. In contrast, the writer (or issuer) of the option has the obligation to deliver or buy the underlying asset for the price  $K$ , in case the holder chooses to exercise it.

**4.1.1. Path-independent options.** The standard options are European and American calls and puts, whose payoffs depend on the final values of the underlying assets. The major difference between European options and American options is their exercise flexibility. For European options exercise is only permitted at expiry date  $T$ . American options can be exercised at any time  $t \leq T$ . The value  $V$  of the option depends on the price  $S_t$  per share of the underlying asset with time  $t$  and the remaining time to expiry  $T - t$ . Hence the dependence of  $V$  on  $S_t$  and  $t$  can be expressed as  $V(S_t, t)$ .

The payoff of a European call option at  $t = T$  is the difference of the underlying asset price  $S_T$  and the strike price  $K$ . In other words, the holder will exercise the call at maturity  $T$  for  $S_T > K$  so that the value of a European call option is  $V = S_T - K$ . On the other hand, when  $S_T < K$ , the value of a European call option is worthless since the asset can be purchased on the market for the cheaper price  $S_T$  and the holder will not exercise. Hence the payoff of a European option at maturity  $T$  can be expressed as

$$V(S_T, T) = (S_T - K)^+. \quad (4.1.1)$$

For a European put, the holder can make a gain only when  $S_T < K$ . Hence at expiry date  $T$  the payoff  $V(S_T, T)$  of a European put is

$$V(S_T, T) = (K - S_T)^+. \quad (4.1.2)$$

The speculation of the options is an indication of future movement of the underlying asset. When an unusual amount of call options are overbought, investors consider that the underlying asset is likely to rally when the options approach the maturity  $T$ . In general, this implication is bullish on the current price of the underlying asset despite other macroeconomic factors. On the other hand, the value of  $V(S, t)$  also depends on the strike price  $K$ , the maturity  $T$ , and other market parameters such as the risk-free interest rate  $r$ , the volatility  $\sigma$  of the underlying asset price  $S_t$  and dividends, in the case of a dividend-paying asset. The volatility  $\sigma$ , defined as the standard deviation of the fluctuations in  $S_t$ , measures the uncertainty in the asset.

In the complex reality of the financial world, some mathematical models can be served for pricing financial derivatives by discrete approximations or stochastic integrations. The most recognized and accepted model for pricing financial options was originally developed by Black, Merton and Scholes in 1973. The Black-Scholes option pricing model became one of the fundamental pricing theories in modern finance, especially for the hedging of contingent claims. For the Black-Scholes-Merton (BSM) model, there is a series of assumptions to characterize the functions  $V(S, t)$  as solutions of certain partial differential equations:

- (1) No arbitrage opportunities.
- (2) The dividend-exclusive option is European with constant  $r$  and  $\sigma$   
for  $0 \leq t \leq T$ .
- (3) The asset price follows a geometric Brownian motion.
- (4) No transaction costs (fees or taxes). Equal interest rate for borrowing and lending.  
No restrictions in accessibility of entering the market. All securities and credits are

available at any time and any size. The market is a perfect one without friction binding.

Based on the above assumptions, the *Black-Scholes equation* derived from the partial differential equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + rS_t \frac{\partial V}{\partial S_t} = rV \quad (4.1.3)$$

for  $t \in [0, T)$ ,  $S_t > 0$ , and that satisfies the terminal condition  $V(S_T, T) = (S_T - K)^+$ . By Eq. (4.1.3), the partial differential equation for a European call  $c(t, S_t)$  option is

$$c_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{S_t S_t}(t, S_t) + rS_t c_{S_t}(t, S_t) = rc(t, S_t) \quad (4.1.4)$$

where  $c_t(t, S_t) = \frac{\partial c}{\partial t}$ ,  $c_{S_t}(t, S_t) = \frac{\partial c}{\partial S_t}$ ,  $c_{S_t S_t}(t, S_t) = \frac{\partial^2 c}{\partial S_t^2}$ , and that satisfies Eq. (4.1.1).

The boundary condition at  $S_t = 0$  for the partial differential equation (4.1.4) is

$$c(t, 0) = (0 - K)^+ = 0 \quad (4.1.5)$$

for all  $t \in [0, T]$ . Hence the solution to the partial differential equation (4.1.4) of a European call without dividend payments is

$$c(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2), \quad (4.1.6)$$

where  $N(\cdot)$  is the standard normal distribution, and  $d_1, d_2$  are defined as

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad (4.1.7)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \quad (4.1.8)$$

for constant  $\mu$  and  $\sigma$ ,  $S_t > 0$ ,  $0 \leq t < T$ .

On the other hand, the terminal condition for a European put option is  $p(t, S_T) = (K - S_T)^+$  and the boundary condition at  $S_t = 0$  is  $p(t, 0) = (K - 0)^+ = K$  for all  $t \in [0, T]$ . By solving Eq. (4.1.3) with the boundary and terminal conditions, the value of a European

put option is evaluated at

$$p(t, S_t) = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1). \quad (4.1.9)$$

Different from European options, American options can be exercised any time before or at expiration date  $T$ . Although this flexibility advantage provides room for optimal payoffs, the value of American put options are priced lower than European put options in a fair market. Similarly, the payoffs of American options are generated by solving Eq. (4.1.3) with their corresponding boundary and terminal conditions.

**4.1.2. Path-dependent options.** *Exotic* options are options that cannot be replicated by some standard options due to the complexity of their composite structures. Consequently, the difficulty of pricing exotic options can be troublesome since the variation of the underlying assets is altered in the complex structures of exotic options at each unit time. Most of the time exotic options can only be priced by numerical methods, such as Monte Carlo simulation or Markov Chain approximation because it is difficult to solve their analytical integrations. Besides American options, the payoffs of exotic options depend on all values of the underlying assets across time up to maturity. The following is a list of important types of exotic options which are traded outside the exchanges (OTC):

- (1) *Binary Options*: The payoff of a binary option is the derivative of an option with its strike price. For example, the value of a vanilla binary call is the negative derivate of a vanilla European call:

$$V_b = \begin{cases} -\frac{\partial C}{\partial K} & \text{for } S_T > K \\ 0 & \text{for } S_T \leq K \end{cases}$$

- (2) *Chooser Options*: After a certain period of time the buyer of the option has the right to decide whether the option is call or put. A *simple* chooser option is the option with the same strike prices for call and put. The value of a chooser option at exercise is  $\max\{V_C, V_P\}$ , where  $V_C$  and  $V_P$  are the values of call and put options, respectively.

- (3) *Compound Options*: Compound options are options on options and depend on whether the options are put or calls. There are four types of compound options; for example, the option may be a call on a put or a put on a call.
- (4) *Digital Options*: A digital option gives a fixed payoff when the price of the underlying asset goes above or below a certain point; otherwise, the payoff is zero in other cases. Since digital options cannot be replicated by a set of standard options, Monte Carlo simulation is applied to price digital options.

For some exotic options, their payoffs depend on the path of  $S_t$  up to the exercise or at maturity. The important types of *path-dependent* options are:

- (1) *Asian Options*: The payoff of an Asian option depends on the average price of the underlying asset over the period of time between the initiation or later and expiration. The mechanism of an Asian option is difficult to manipulate by altering the underlying asset price.

If the prices  $S_t$  of the underlying asset are from *continuous sampling*, the average in the time period  $0 \leq t \leq T$  is

$$A =: \frac{1}{T} \int_0^T S_t dt. \quad (4.1.10)$$

The price of an Asian option can be derived through partial differential equations with corresponding boundary and terminal conditions.

If the price  $S_t$  is *discretely sampled* in equidistant time instances  $t_i$ , the arithmetic mean for a time series  $S_{t_1}, S_{t_2}, \dots, S_{t_n}$  is

$$A =: \frac{1}{n} \sum_{i=1}^n S_{t_i} \quad (4.1.11)$$

for  $0 < t_1 < t_2 < \dots < t_n = T$ . For discretely sampled asset prices  $S_t$ , Monte Carlo simulation is applied to price an Asian option.

(2) *Lookback Options*: The buyer of a lookback option receives the optimal payoff at the maximum level of the underlying asset price over some period of time prior to maturity. Typically, there are four different types of lookback options:

- (a) *Fixed strike call* lookback option: The option pays the difference between the highest value of the underlying price and the strike price over some time before expiration. [i.e.  $C_{fixed} = (\max_{0 < t < T} (S_t) - K)^+$ ]
- (b) *Fixed strike put* lookback option: The option pays the difference between the strike price and the lowest value of the underlying price over some time before expiration. [i.e.  $P_{fixed} = (K - \min_{0 < t < T} (S_t))^+$ ]
- (c) *Floating strike call* lookback option: The option pays the difference between the value of the underlying price at maturity and the lowest value of the underlying price before expiration. [i.e.  $C_{float} = (S_T - \min_{0 < t < T} (S_t))^+$ ]
- (d) *Floating strike put* lookback option: The option pays the difference between the highest value of the underlying price before expiration and the value of the underlying price at maturity. [i.e.  $P_{float} = (\max_{0 < t < T} (S_t) - S_T)^+$ ]

Either fixed strike lookback options or floating strike lookback options can be priced in a Monte Carlo process. The difference is that the floating strike lookback options cannot be replicated by a set of barrier (knock-out) options since the strike of a floating strike lookback call option resets if the underlying asset price goes down. Instead, the replication of a floating strike lookback call option can be done by a set of forward starting options.

(3) *Barrier Options*: For a barrier option the payoff depends on whether the price of the underlying  $S_t$  reaches a specified level  $B$ , called barrier. Classification of barrier options depends on whether  $S_t$  reaches  $B$  from above (*down*) or from below (*up*). A barrier option is considered being *knocked out* when the price of the underlying

$S_t$  reaches  $B$ . On the other hand, *knock-in* options come into existence when  $B$  is reached. The existence of a barrier option gives an initial condition, that is, for a down option,  $S_0 > B$  and for an up option,  $S_0 < B$ . A number of different types of barrier options are developed, based on whether they are calls or puts. For an European call, there are four types of barrier options:

- (a) *down-to-out*: The value  $c_{do}$  of a knock-out vanilla option ceases to exist when the price of the underlying asset goes below the barrier level. [i.e.  $c_{do} = (S_T - K)^+$  for  $S_t > B$ ; otherwise, 0 for  $S_t \leq B$ . ]
- (b) *up-to-out*: The value  $c_{uo}$  of a knock-out vanilla option ceases to exist when the price of the underlying asset goes above the barrier level. [i.e.  $c_{uo} = (S_T - K)^+$  for  $S_t < B$ ; otherwise, 0 for  $S_t \geq B$ . ]
- (c) *down-to-in*: The value  $c_{di}$  of a knock-in vanilla option comes into existence when the price of the underlying asset goes below the barrier level. [i.e.  $c_{di} = (S_T - K)^+$  for  $S_t \leq B$ ; otherwise, 0 for  $S_t > B$ . ]
- (d) *up-to-in*: The value  $c_{ui}$  of a knock-in vanilla option comes into existence when the price of the underlying asset goes above the barrier level. [i.e.  $c_{ui} = (S_T - K)^+$  for  $S_t \geq B$ ; otherwise, 0 for  $S_t < B$ . ]

The *in-out* parity of an European call option can be expressed as

$$c = c_{ui} + c_{uo} = c_{di} + c_{do}. \quad (4.1.12)$$

Because of this parity between knock-out and knock-in options, it is generally sufficient to study one of the two types. In our case, we consider single and double knock-out call options without rebate payment. In other words, those calls pay the option holders if the underlying asset price does not cross a barrier level.

Based on the assumption of continuous monitoring in the underlying asset price, the first analytical closed-form formula for pricing a down and out call was derived by Merton [1] and subsequently, the formulas for all types of barrier options with assumption

that the underlying asset price follows a lognormal process were developed by Reiner & Rubinstein [13]. Also, the extended work of pricing continuously monitored barrier options were contributed by Kunitomo and Ikeda [12], and Rich [14].

On the other hand, some real contracts with barrier features are discretely monitored at fixed times, for instance, on daily basis. Since analytical expressions of exotic options are difficult to be obtained in most cases, lattice techniques, finite-difference methods, Monte Carlo simulation, or Markov Chain approximation are frequently used numerical methods in the valuation of discretely monitored barrier options.

For efficiency of handling the case where the initial asset price is close to a barrier level, numerical evidence indicates that Markov Chain approximation is accurate enough to price discretely monitored barrier options. Monte Carlo simulation and other numerical methods are considered less accurate and more time-consuming, compared to Markov Chain approximation. Furthermore, we will discuss different types of barrier options, such as double barrier options and power assets options with time-varying barriers by Markov chain approximation.

## 4.2. Double barrier options

The analysis throughout this section is based on Black-Scholes assumptions. Given an initial asset price  $S_0$ , the asset price  $\{S_t, t \geq 0\}$  follows the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad (4.2.1)$$

where  $\mu$  and  $\sigma > 0$  are constants and  $W_t$  is a standard Brownian motion process.

Under the risk-neutral measure  $\mathbb{Q}$ , the price of a claim contingent on  $S_t$  is the expected present value of its cash flows under the equivalent martingale measure  $\mathbb{P}$

$$dS_t = S_t(r dt + \sigma dW_t^{\mathbb{Q}}) \quad (4.2.2)$$

where  $r$  is a constant risk-free interest rate.



Now we consider a European knock-out call with strike price  $K$ , up-and-out barrier  $B_t$  and down-and-out barrier  $b_t$ . The underlying asset price of this option, which expires at time  $T$ , follows a geometric Brownian motion (4.2.2) with constant volatility  $\sigma$ . The solution to the stochastic differential equation (4.2.2) is

$$S_t = S_0 e^{\sigma W_t^{\mathbb{Q}} + (r - \frac{1}{2}\sigma^2)t}. \quad (4.2.3)$$

We apply Girsanov theorem 2 in Eq. (4.2.2) under probability measure  $\mathbb{Q}^*$ . Let  $\beta = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$  such that  $W_t^{\mathbb{Q}^*} = \beta t + W_t^{\mathbb{Q}}$ . The solution to the stochastic differential equation (4.2.2) can be re-written as  $S_t = S_0 e^{\sigma W_t^{\mathbb{Q}^*}}$ .

In order to have in-the-money payoff for the option, we assume  $K < B_t$  and  $K > b_t$ . Now let  $M_t = \max W_t^{\mathbb{Q}^*}$  and  $m_t = \min W_t^{\mathbb{Q}^*}$  for  $0 \leq t \leq T$  so that  $\max S_t = S_0 e^{M_t}$  and  $\min S_t = S_0 e^{m_t}$ .

The option pays off  $(S_T - K)^+ = (S_0 e^{\sigma W_T^{\mathbb{Q}^*}} - K)^+$  if and only if  $S_0 e^{M_t} < B_t$  and  $S_0 e^{m_t} > b_t$  for  $0 \leq t \leq T$ . Under all conditions, the payoff of the double barrier option can be computed by

$$\begin{aligned} Y(T, S_T) &= (S_0 e^{W_T^{\mathbb{Q}^*}} - K)^+ \mathbb{I}_{\{S_0 e^{M_t} < B_t, S_0 e^{m_t} > b_t\}} \\ &= (S_0 e^{W_T^{\mathbb{Q}^*}} - K) \mathbb{I}_{\{W_T^{\mathbb{Q}^*} \geq J, M_t < G, m_t > g\}} \end{aligned} \quad (4.2.4)$$

where  $J = \frac{1}{\sigma} \log \frac{K}{S_0}$ ,  $G = \log \frac{B_t}{S_0}$ , and  $g = \log \frac{b_t}{S_0}$ .

In order to be in-the-money, the payoff function has to satisfy the boundary conditions

$$\begin{aligned} Y(t, b_t) &= 0, \quad 0 \leq t \leq T \\ Y(t, B_t) &= 0, \quad 0 \leq t \leq T \\ Y(T, S_T) &= (S_T - K)^+, \quad b_T < S_T < B_T. \end{aligned} \quad (4.2.5)$$

Given initial asset price  $S_0$ , the price of the option at time  $t \in [0, T]$  is given by

$$Y(t) = \mathbb{E}[e^{-r(T-t)} Y(T) | \mathcal{F}(t)]. \quad (4.2.6)$$

### 4.3. Power asset options

In the Black-Scholes framework, Ronald C. Heynen and Harry M. Katt [15] provided the closed-form formula for so-called power options, composite of  $n$  calls which underlying assets follow the stochastic differential equation

$$d \left[ \ln \left( \frac{S_t}{S_0} \right) \right] = \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t$$

and the options pays off at the expiration  $T$

$$C_T = \left( \sum_{j=1}^n w_j S_T^j - w_j K^j, 0 \right)^+, \quad w_j \geq 0$$

where the risk-neutral interest rate  $r$  and the volatility  $\sigma$  are known constants;  $W_t$  follows a standard Brownian motion;  $S_t$  is an underlying asset price;  $w_i$  is a non-negative integer and  $K$  is the corresponding strike price.

The initial price of a power option at time  $t = 0$  can be evaluated by the expected value of the payoff

$$C_0 = e^{-rT} \mathbb{E} \left[ \left( \sum_{j=1}^n w_j S_T^j - w_j K^j, 0 \right)^+ \right]. \quad (4.3.1)$$

After some algebra calculations, a closed-form formula of pricing a power option in the money can be expressed in the form of

$$C_0 = e^{-rT} \left[ \sum_{j=1}^n w_j S_T^j e^{(jrT + \frac{1}{2}(j^2-j)\sigma^2 T)} N[d_j] - \sum_{j=1}^n w_j K^j N[d_0] \right] \quad (4.3.2)$$

where  $N[\cdot]$  is the cumulative standard normal distribution function and

$$d_j = \frac{\ln(S_T/K) - (r - \frac{1}{2}\sigma^2)T + j\sigma^2 T}{\sigma\sqrt{T}}, \quad j = 0, 1, 2, \dots$$

Now we define our so-called *power asset option* where the underlying asset price, with a power  $p$ , is a non-geometric Brownian motion in the stochastic differential equation

$$dS_t = S_t^p (\mu dt + \sigma dW_t), \quad (4.3.3)$$

where  $p \in \mathbf{R}^+ \setminus \{0, 1\}$ , the mean  $\mu$  and the volatility  $\sigma$  are deterministic constants, and  $W_t$  is a standard Brownian motion. (Note: for  $p = 0$ ,  $dS_t = \mu dt + \sigma dW_t$  follows a Brownian motion and for  $p = 1$ ,  $dS_t = S_t(\mu dt + \sigma dW_t)$  is a geometric Brownian motion.)

The initial price of a power asset option at time  $t = 0$  is defined as the expected value of the payoff at expiration  $T$

$$\begin{aligned} C_{p,0} &= e^{-rT} \mathbb{E} [(S_T^{1-p} - K)^+] \\ &= e^{-rT} \mathbb{E} [S_T^{1-p} - K] \mathbb{I}_{\{S_T^{1-p} \geq K\}} \\ &= e^{-rT} \mathbb{E} [S_T^{1-p} - K] \mathbb{I}_{\{p \leq 1 - \frac{\log K}{\log S_T}\}}. \end{aligned} \quad (4.3.4)$$

Let  $q = 1 - p$  so the condition  $p \leq 1 - \frac{\log K}{\log S_T}$  is equivalent to  $q \geq \frac{\log K}{\log S_T}$ . Hence, Eq. (4.3.4) can be rewritten in the form of

$$C_{q,0} = e^{-rT} \mathbb{E} [S_T^q - K] \mathbb{I}_{\{q \geq \frac{\log K}{\log S_T}\}}. \quad (4.3.5)$$

For the non-geometric Brownian motion model, we consider  $q \in \mathbf{R}^+ \setminus \{0, 1\}$ . In order to have in-the-money payoff for the power asset option,  $q \neq 1$  gives the assumption that  $S_T \neq K$ .

Since strike price  $K$  is a pre-determined constant, the initial price of the power asset option at time  $t = 0$  is

$$C_{q,0} = \begin{cases} e^{-rT} \mathbb{E} [S_T^q] - K e^{-rT} & , \quad q \geq \frac{\log K}{\log S_T} \\ 0 & , \quad \textit{otherwise}. \end{cases}$$

Inserting the closed-form formulae of  $\mathbb{E} [S_T^q]$ <sup>1</sup> derived in section 2.7 into Eq. (4.3.5), the initial price of the power asset option at time  $t = 0$  can be calculated by different  $q$  values.

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<sup>1</sup>MATLAB codes [Appendix 0.1]

For  $q = -1$ ,  $\mathbb{E}[S_T^{-1}] = \frac{1}{2}\mu\sigma^2t^2 + S_0^{-1}t + \sigma^2S_0 - \mu$ . The initial price of the power asset option at time  $t = 0$  is

$$\begin{aligned} C_{-1,0} &= e^{-rT}\mathbb{E}[S_T^{-1}] - Ke^{-rT} \\ &= e^{-rT}\left(\frac{1}{2}\mu\sigma^2t^2 + S_0^{-1}t + \sigma^2S_0 - \mu\right) - Ke^{-rT}. \end{aligned}$$

For the non-geometric Brownian motion model,  $q = 0$  and  $q = 1$  cases are neglected.

Now for  $|q| > 1$ , the expected value of  $S_T^q$  is

$$\begin{aligned} \mathbb{E}[S_T^q] &= \frac{1}{2}\left(S_0^q - \frac{2\mu}{(1+q)\sigma^2} - \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2}\right)e^{-\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2T} \\ &\quad + \frac{1}{2}\left(S_0^q - \frac{2\mu}{(1+q)\sigma^2} + \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2}\right)e^{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2T} \\ &\quad + \frac{2\mu}{(1+q)\sigma^2}. \end{aligned} \tag{4.3.6}$$

The initial price of the power asset option at time  $t = 0$  is

$$\begin{aligned} C_{|q|>1,0} &= e^{-rT}\mathbb{E}[S_T^q] - Ke^{-rT} \\ &= \frac{1}{2}\left(S_0^q - \frac{2\mu}{(1+q)\sigma^2} - \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2}\right)e^{-(r+\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2)T} \\ &\quad + \frac{1}{2}\left(S_0^q - \frac{2\mu}{(1+q)\sigma^2} + \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2S_0^{-q}}{\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2}\right)e^{(\frac{1}{2}|q|\sqrt{q^2-1}\sigma^2-r)T} \\ &\quad + \left[\frac{2\mu}{(1+q)\sigma^2} - K\right]e^{-rT}. \end{aligned}$$

For  $|q| < 1$  (specifically,  $q \in (-1, 1) \setminus \{0\}$ ), we have the expected value of  $S_T^q$

$$\begin{aligned} \mathbb{E}[S_T^q] &= \left[S_0^q - \frac{2\mu}{(1+q)\sigma^2}\right]\cos\left(\frac{1}{2}|q|\sqrt{1-q^2}\sigma^2T\right) \\ &\quad + \left[\frac{q\mu - \frac{1}{2}q(1-q)\sigma^2S_0^{-q}}{\frac{1}{2}|q|\sqrt{1-q^2}\sigma^2}\right]\sin\left(\frac{1}{2}|q|\sqrt{1-q^2}\sigma^2T\right) \\ &\quad + \frac{2\mu}{(1+q)\sigma^2}. \end{aligned}$$

With  $|q| < 1$ , the initial price of the power asset option at time  $t = 0$  is

$$\begin{aligned}
C_{|q|<1,0} &= e^{-rT} \left[ S_0^q - \frac{2\mu}{(1+q)\sigma^2} \right] \cos \left( \frac{1}{2}|q|\sqrt{1-q^2}\sigma^2 T \right) \\
&+ e^{-rT} \left[ \frac{q\mu - \frac{1}{2}q(1-q)\sigma^2 S_0^{-q}}{\frac{1}{2}|q|\sqrt{1-q^2}\sigma^2} \right] \sin \left( \frac{1}{2}|q|\sqrt{1-q^2}\sigma^2 T \right) \\
&+ e^{-rT} \left[ \frac{2\mu}{(1+q)\sigma^2} \right].
\end{aligned}$$

#### 4.4. Monte Carlo Simulation approach

In daily practice, the pricing of a financial derivative may be made so that, as the prices of the underlying asset do not move continuously, discrete-time models are more appropriate. Boyle [2] initially applied the Monte Carlo simulation method to price a range of one-factor and multi-factor options in Quantitative Finance. When the time evolution of a stochastic process is not easy to describe analytically, Monte Carlo simulation may well be robust, in comparison with other numerical methods, such as lattice and finite difference methods.

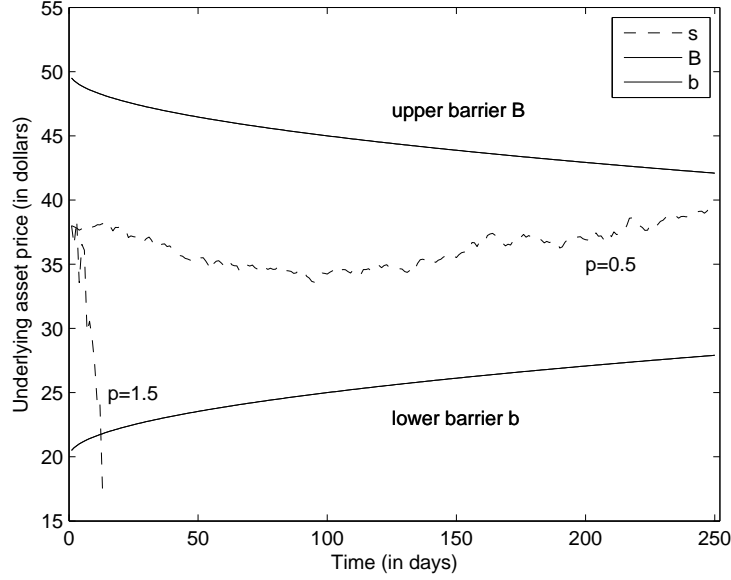
For the non-geometric Brownian motion case, we model the price of the non-dividend-paying stock of a European knock-out call option with two time-varying barriers and the discretized stock price follows stochastic process  $dS_t = S_t^p(\mu dt + \sigma z\sqrt{t})$ ,  $z \sim N(0, 1)$ . The initial stock price  $S_0$  is fixed but the succeeding stock prices  $S_1, S_2, S_3, \dots, S_T$ , at times  $t_1, t_2, t_3, \dots, T$ , can be generated by

$$S_{t_{i+1}} = S_{t_i} + S_{t_i}^p(r dt + \sigma z_i \sqrt{dt}), \quad (4.4.1)$$

where  $r$  is the risk-neutral interest rate,  $dt$  is the time interval,  $\sigma$  is a constant volatility, and  $z_i$  is a standard normal random variable [i.e.  $z_i \sim N(0, 1)$ ].

Set the maturity  $T$  of this barrier option to be a year of 250 trading days for  $0 < t_1 < t_2 < \dots < t_i < \dots < t_{250} = 1$ , where  $1 \leq i \leq 250$  with time interval  $dt = \frac{1}{250}$ . Then there are  $\frac{T}{dt} = 250$  iterations for each realization.

FIGURE 4.1. Underlying Asset Price with Parameter  $p$

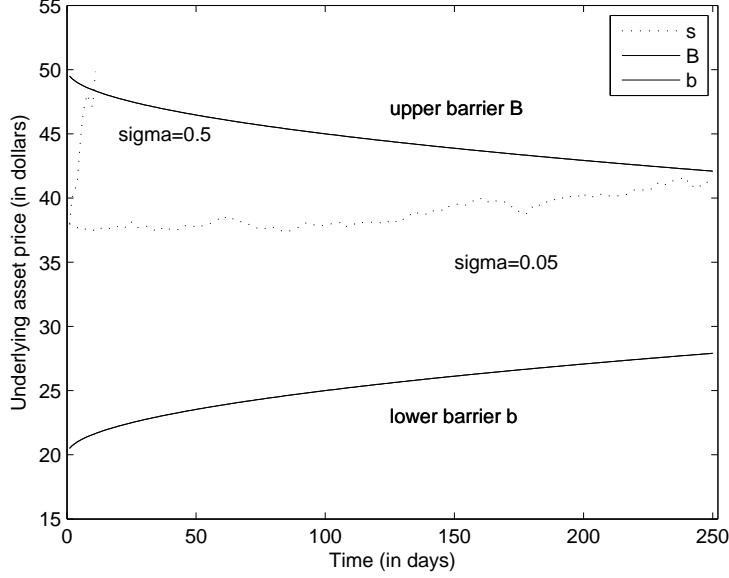


The barrier option is assumed to have two time-varying barriers: upper barrier  $B_{t_i}$  and lower barrier  $b_{t_i}$ . If two barriers are far from one another, the more precision would be gained in Monte Carlo simulation. Therefore, we set upper barrier  $B_{t_i} = 50 - \frac{1}{2}\sqrt{t_i}$  and lower barrier  $b_{t_i} = 20 + \frac{1}{2}\sqrt{t_i}$  for estimating the underlying stock price in Eq. (4.4.1).

In order to be in the money, the stock price cannot cross either barrier level for all  $t_i \in [0, T]$  and the strike price  $K$  should be pre-determined at a lower level than upper barrier  $B_{t_i}$ . In other words,  $K < B_{t_i}$  for all  $t_i \in [0, T]$ .

Now we observe the influence of parameter value  $p$  on stock price  $S_{t_{i+1}}$  in Eq. (4.4.1) with  $S_0 = 38$ ,  $r = 0.05$ ,  $K = 35$ ,  $T = 1$ ,  $dt = \frac{1}{250}$ ,  $B_{t_i} = 50 - \frac{1}{2}\sqrt{t_i}$ ,  $b_{t_i} = 20 + \frac{1}{2}\sqrt{t_i}$ : (1) For  $p \leq 0$ , we have  $S_{t_i} \gg S_{t_i}^p$ ; consequently,  $S_{t_i}$  acts as a dominant term in approximating stock price  $S_{t_{i+1}}$ . As a result, a change in parameter value  $p$  only causes a relatively small impact on the simulated stock prices  $S_{t_{i+1}}$  for  $p \leq 0$ . The graph of the simulated stock prices  $S_{t_{i+1}}$  for  $0 < t_1 < t_2 < \dots < t_{250} = 1$  is considered, relatively, a horizontal line from the initial stock price  $S_0$ .

FIGURE 4.2. Underlying Asset Price with Volatility  $\sigma$

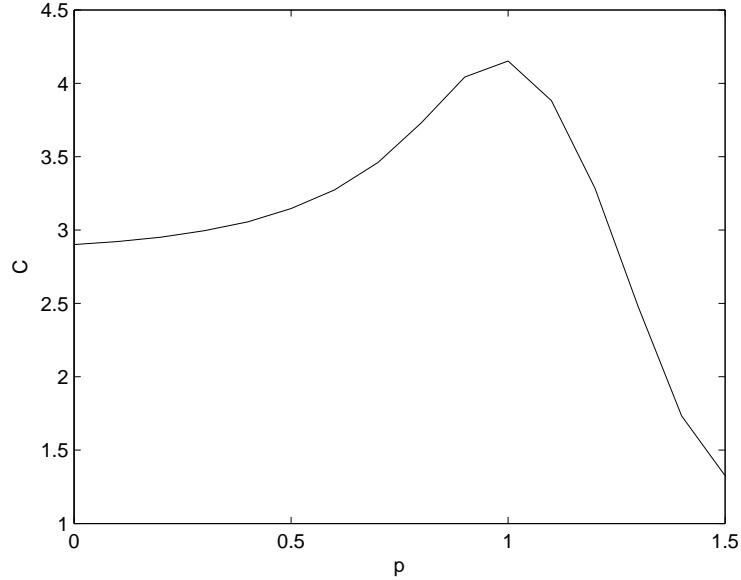


(2) Let  $p = 1$  [geometric Brownian motion case:  $dS_t = S_t(rdt + \sigma z\sqrt{t})$ ]. In Eq. (4.4.1)  $\sigma z\sqrt{dt}$  is the dominant term that affects the stock price  $S_{t_{i+1}}$  since  $\sqrt{dt}(=0.063)$  is larger than  $dt(=0.04)$ . Consequently, a small change in the volatility  $\sigma$  can make a huge difference in the stock price  $S_{t_{i+1}}$  since the risk-neutral interest rate  $r$  is relatively small.

In Figure 4.2, stock price  $S_{11} = 49.84$  with volatility  $\sigma = 0.5$  crosses the upper barrier  $B_{11} = 48.34$  on the 11<sup>th</sup> trading day. In comparison, stock price  $S_{250} = 41.12$  with volatility  $\sigma = 0.05$  remains within the bounds until maturity. As a result, we can conclude that larger volatility  $\sigma$  may cause greater probability that the stock price crosses a barrier for  $p = 1$ .

(3) Now let  $p \in \mathbf{R}^+ \setminus \{0, 1\}$  [non-geometric Brownian motion case:  $dS_t = S_t^p(rdt + \sigma z\sqrt{dt})$ ]. For  $p \rightarrow \infty$ ,  $S_t^p(rdt + \sigma z\sqrt{t})$  is considered more weighted than  $S_{t_i}$  in the process of simulating stock price  $S_{t_{i+1}}$ . In Figure 4.1, for  $p = 1.5$ , stock price  $S_{13} = 17.30$  crosses the lower barrier  $b_{13} = 21.8$  on the 13<sup>th</sup> trading day. Therefore, we can conclude that stock price with higher  $p$  value has greater probability of being 'knocked-out' than the one with lower  $p$  value.

FIGURE 4.3. Knock-out Option with Parameter  $p$



If the option does not cross any barrier levels until maturity  $T(= t_{250})$ , the pricing of each barrier option is evaluated by

$$C_T^{(j)} = e^{-rT}[\max(S_T - K, 0)], \quad 1 \leq j \leq n$$

where  $n$  is the number of realizations. As  $n \rightarrow \infty$ , Monte Carlo simulation becomes very slow but robust. Therefore, we set  $n = 10^5$  for all estimations.

Consequently, we can estimate the price of a barrier option, composed of a simple average across realizations of each option, of which stock prices do not cross over barriers during the lifetime of the options:

$$C_{barrier} = \frac{1}{n} \sum_{j=1}^n C_T^{(j)}. \quad (4.4.2)$$

Monte Carlo simulation can be used to value virtually any European-style derivative security [11]; however, a standard Monte Carlo simulation is inappropriate for American options because of early exercise: it is difficult to determine the early-exercise point based on one single path. Although later Tilley [16] showed that modified Monte Carlo simulation can be used to price American options, the estimate has some biased numerical results.



TABLE 1. Parameter  $p$  with the Price  $C$  of a Knock-out Option

$p$	$C$	$n$
1.5	1.3249	88181
1.4	1.7335	79129
1.3	2.4765	69596
1.2	3.2856	56524
1.1	3.8815	37525
1.0	4.1521	16486
0.9	4.0428	2791
0.8	3.7298	57
0.7	3.4602	0
0.6	3.2734	0
0.5	3.1466	0
0.4	3.0556	0
0.3	2.9950	0
0.2	2.9513	0
0.1	2.9217	0
0.0	2.9008	0

Therefore, for our case, the pricing of the European call option with time-varying barriers has been discretely monitored with respect to the parameter  $p$  in Eq. (4.4.1). [Figure 4.3.]

Monte Carlo simulation is used  $10^5$  realizations to estimate the price of the barrier option, which is based on the movement of the asset price  $S_t$  in the stochastic process  $dS_t = S_t^p(rdt + \sigma z\sqrt{t})$ . Parameter values for pricing the knock-out option with time-dependent barriers are  $S_0 = 38$ ,  $r = 0.05$ ,  $\sigma = 0.05$ ,  $K = 35$ ,  $T = 1$ ,  $dt = \frac{1}{250}$ ,  $B_t = 50 - 0.5\sqrt{t}$ ,  $b_t = 20 + 0.5\sqrt{t}$ . The number of realizations  $n$  that the stock prices cross over barriers is proportionally increased with the parameter  $p$  from 0 to 1.5; however, there are no barrier options knocked out ( $n = 0$ ) for  $p \in [0, 0.7]$ . [Table 1] In other words, a barrier option has greater probability of being knocked out as  $p : 1^+ \rightarrow \infty$ .

Figure 4.3 shows that there exists an optimal price ( $C_{barrier} \approx 4.15$ ) of the barrier option as  $p \rightarrow 1$  and the instantaneous rate of change in  $C_{barrier}$  with respect to  $p$  is positive as  $p \rightarrow 1^-$  (i.e.  $\frac{\partial C_{barrier}}{\partial p} > 0$ ). On the contrary,  $\frac{\partial C_{barrier}}{\partial p} < 0$  for  $p : 1^+ \rightarrow 1.5$ . As a result, the pricing of the barrier option is lognormally distributed with respect to the parameter  $p$ .

Accordingly, the pricing of the barrier option converges as  $p : 1^- \rightarrow -\infty$  since  $S_{t_{i+1}} \cong S_{t_i}$  for all  $t_i \in [0, T]$  in Eq. (4.4.1). (As  $p : 1^- \rightarrow -\infty$ ,  $S_{t_i}^p(rdt + \sigma z_i \sqrt{dt}) \rightarrow 0$ .) On the other hand, as  $p : 1^+ \rightarrow \infty$  the pricing of the barrier option is decreased by an increasing number of realizations of which stock prices cross over barriers. (Referring to Eq. (4.4.2), the summation of *total* option prices is decreased. However, the *total* number of realizations remains constant.) Therefore,  $C_{barrier} \rightarrow 0$  as  $p : 1^+ \rightarrow \infty$ .

Monte Carlo simulation is simple and flexible to obtain numerical solutions in pricing exotic options where analytical expressions are difficult to be found. For instance, the stochastic process  $dS_t = S_t^p(\mu dt + \sigma z \sqrt{dt})$  determining a set of terminal stock prices  $S_T$  for  $n$  realizations can compute the price of a power asset option by taking the average of overall values generated by  $n$  realizations. The optimal price of the option is estimated when the parameter  $p \approx 1$  and the instantaneous rate of change in  $C_{barrier}$  with respect to  $p$  is positive:  $\frac{\partial C_{barrier}}{\partial p} > 0$  for  $p \rightarrow 1^-$  (non-geometric Brownian motion). Furthermore, implied volatility  $\sigma$  is more effective than the risk-neutral interest rate  $r$  in the stock price process  $S_{t+1} = S_t + S_t^p(rdt + \sigma z \sqrt{dt})$  since the time interval is much less than its square (i.e.  $dt \ll \sqrt{dt}$  for  $dt = \frac{1}{250}$ ).

## 4.5. Markov Chain Approximation approach

In this section, we propose a time-homogeneous Markov chain method to approximate the price of the barrier option. Since it is difficult to obtain the analytical expression of the partial differential equation  $dS_t = S_t^p(rdt + \sigma dW_t)$  governing the underlying stock process, the Markov chain method can be modified to adapt the stock price process and efficiently handle the case where the initial stock price is close to a barrier level. In addition to the computational advance, the time-varying barriers in a Markov chain can be discretely monitored at each time step that is partitioned for pricing the underlying stock.

**4.5.1. Plain Vanilla Options:  $p=1$ .** We consider a European knock-out call option without paying dividends as the benchmark for option valuation. Let the underlying stock

price  $S_t$  of this non-dividend option follow a geometric Brownian motion with constant risk-free interest rate  $r$  and volatility  $\sigma$  in a Black-Scholes framework. Given initial stock price  $S_0$ , we have the following partial differential equation governing the pricing of the stock process under the risk-neutral measure  $\mathbb{Q}^*$

$$dS_t = S_t(rdt + \sigma dW_t) \quad (4.5.1)$$

where  $r$  is a risk-neutral interest rate and  $W_t$  is a standard Brownian motion.

For the valuation of a discretely monitored knock-out option, a Markov chain  $(X_t)_{t \geq 0}$  with state space  $I = \{1, 2, \dots, m\}$  can bridge the number of stock prices with the number of time steps in the approximation. For sufficiently large  $m$ , a time-homogeneous Markov chain can be constructed for the convergence to the stochastic process, and option prices computed by this Markov chain are convergent to the theoretical values of the option (Duan and Simonato [7]).

Over the time index set  $\{t = 0, 1, \dots, T\}$ , we construct a time-homogeneous Markov chain with a set of discrete prices  $\{s_1, s_2, \dots, s_m\}$  which can be represented in a vector form  $\vec{s} = [s_1, s_2, \dots, s_m]$ . The associated one-step  $m \times m$  transition probability matrix  $\mathbf{P}$  of this Markov chain is stochastically constructed and each entry  $p_{ij}$  is a representation of the conditional probability  $\mathbb{P}\{X_{t+1} = j | X_t = i\} = \mathbb{P}\{X_1 = j | X_0 = i\}$  for  $i, j \in I$  (*homogeneity* by Definition 3.1.1).

All entries  $p(i, j)$  of the transition probability matrix  $\mathbf{P}$  are analytically computed through the standard lognormal distribution of the underlying stock price process [Eq.4.5.1]. Let  $\Delta t$  be the length of time steps  $n$  and  $T$  be the maturity of the knock-out option. The overall interval  $[0, T]$  can be partitioned into equal-sized  $m$  cells such that the state values  $s_1, s_2, \dots, s_m$  are the middle points of these cells. The  $m$  cells are constructed as  $U_1 = (u_1, u_2)$  and  $U_i = [u_i, u_{k+1})$  for  $k \in \{2, \dots, m\}$ , where  $u_1 = -\infty$ ,  $u_i = \frac{s_k + s_{k-1}}{2}$  for  $k = \{2, 3, \dots, m\}$

and  $u_{m+1} = \infty$ . Using the cells and state values, we can obtain the transition probabilities

$$\begin{aligned} p(i, j) &= \mathbb{P}\{s(t + \Delta t) \in [u_j, u_{j+1}) | s(t) = s_i\} \\ &= \Phi\left[\frac{u_{j+1} - s_i - s_i r \Delta t}{s_i \sigma \sqrt{\Delta t}}\right] - \Phi\left[\frac{u_j - s_i - s_i r \Delta t}{s_i \sigma \sqrt{\Delta t}}\right], \end{aligned} \quad (4.5.2)$$

where  $\Phi(\cdot)$  denotes the cumulative standard normal distribution function.

Consequently, the transition probability matrix  $\mathbf{P}$  can be constructed as

$$\mathbf{P} = \begin{bmatrix} p(1, 1) & \dots & p(1, m) \\ \vdots & \ddots & \vdots \\ p(m, 1) & \dots & p(m, m) \end{bmatrix}.$$

In order for a knock-out option with constant barriers to be in-the-money, at time  $t = \{0, 1, \dots, T\}$  the underlying stock price cannot cross either upper barrier  $B$  or lower barrier  $b$ . On the other hand, when the Markov chain reaches an absorbing state, the barrier option is considered being knocked-out. For time-independent barriers  $B$  and  $b$ , we set the boundary conditions of the transition probability matrix  $\tilde{\mathbf{P}}$  as

$$\begin{aligned} \tilde{p}(1, 1) &= 1, \quad \tilde{p}(1, k) = 0, \quad k = \{2, 3, \dots, m\} \\ \tilde{p}(m, k) &= 0, \quad \tilde{p}(m, m) = 1, \quad k = \{1, 2, \dots, m-1\} \end{aligned}$$

where  $\tilde{\mathbf{P}}$  can be constructed as

$$\tilde{\mathbf{P}} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ \tilde{p}(2, 1) & \dots & \dots & \dots & \tilde{p}(2, m) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \tilde{p}(m-1, 1) & \dots & \dots & \dots & \tilde{p}(m-1, m) \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

and each  $\tilde{p}(i, j)$  for  $2 \leq i \leq m-1$  and  $1 \leq j \leq m$  is computed by Eq. (4.5.2). When the Markov chain  $(X_t)_{t \geq 0}$  reaches an absorbing state [either state 1 for lower barrier  $b$  or state  $m$  for upper barrier  $B$ ] at time  $t < T$ , the barrier option is knocked out and its value becomes worthless.

Now we define the initial probability distribution  $\vec{\lambda}^{(0)}$  in a row vector form

$$\vec{\lambda}^{(0)} = [\lambda^{(0)}(1), \lambda^{(0)}(2), \dots, \lambda^{(0)}(m)] \quad (4.5.3)$$

and  $\vec{\lambda}^{(0)}\vec{\mathbf{1}} = 1$ . Each  $\lambda^{(0)}(k)$  for  $k = \{1, 2, \dots, m\}$  is defined by a delta function

$$\delta(X_0) = \begin{cases} 1 & \text{if } X_0 = s(0) \\ 0 & \text{if } X_0 \neq s(0) \end{cases}$$

which is determined by the probabilities  $\mathbb{P}\{X_0 = s(0)\} = 1$  and  $\mathbb{P}\{X_0 \neq s(0)\} = 0$ .

For instance, if  $\lambda^{(0)}(2)$  is the index location of the initial stock price  $s(0)$  among the set of discrete prices  $s_1, s_2, \dots, s_m$ , then the initial probability distribution  $\vec{\lambda}^{(0)}$  can be represented as

$$\vec{\lambda}^{(0)} = [0 \quad 1 \quad 0 \dots 0].$$

Accordingly, the  $n$ -step transition probability  $\tilde{p}^{(n)}(i, j)$  is in fact the  $(i, j)$  entry in the transition probability matrix  $\tilde{\mathbf{P}}^n$  defined by

$$\tilde{p}^{(n)}(i, j) = \tilde{\mathbb{P}}\{X_n = j | X_0 = i\}. \quad (4.5.4)$$

where time step  $n$  is computed by  $\frac{T}{\Delta t}$ . If the initial probability distribution  $\vec{\lambda}^{(0)}$  is given [Eq. (4.5.3)], the probability distribution  $\lambda^{(n)}$  is defined by

$$\lambda^{(n)} = \vec{\lambda}^{(0)}\tilde{\mathbf{P}}^n = \sum_{i \in I} \lambda^{(0)}(i)\tilde{\mathbb{P}}\{X_n = j | X_0 = i\} \quad (4.5.5)$$

where  $\lambda^{(n)}(j) = \mathbb{P}\{X_n = j\}$ .

For time-independent barriers  $B$  and  $b$ , the payoff of the knock-out option with discretely monitored stock prices  $b < s_k < B$  for  $k = \{1, 2, \dots, m\}$  can be obtained by

$$C_b = \sum_{k=1}^m e^{-rT} (s_k - K)^+ \lambda^{(n)}(s_k) \quad (4.5.6)$$

where  $\lambda^{(n)}(s_k) = \mathbb{P}\{X_n = s_k\}$  is determined by Eq. (4.5.5),  $r$  is the annual risk-free interest rate,  $T$  is the maturity of the option, and  $K$  is the pre-determined strike price.

In comparison with time-independent barrier options, pricing a knock-out option with time-varying barriers  $B(t)$  and  $b(t)$  is more complicated since the transition probability matrix  $\mathbf{P}$  is replicated by

$$\bar{\mathbf{P}} = \left[ \begin{array}{c|c|c} \mathbf{lb} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{Qb} & \mathbf{Q} & \mathbf{QB} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{UB} \end{array} \right] = \left[ \begin{array}{c|c|c} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{Qb} & \mathbf{Q} & \mathbf{QB} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I} \end{array} \right],$$

where  $[\mathbf{UB}]$  and  $[\mathbf{lb}]$  are identity submatrices containing absorbing (or recurrent) states,  $[\mathbf{Q}]$ ,  $[\mathbf{QB}]$  and  $[\mathbf{Qb}]$  are submatrices with transient states, and  $[\mathbf{0}]$  are submatrices with 0 entries. The composite matrix  $[\bar{\mathbf{Q}}]$  of the submatrices  $[\mathbf{Q}]$ ,  $[\mathbf{QB}]$  and  $[\mathbf{Qb}]$  is a stochastic matrix with entries whose row sums are equal to 1.

Suppose  $[\mathbf{lb}]$  is an  $l \times l$  probability matrix generated by time-dependent lower barrier  $b(t)$  and  $[\mathbf{UB}]$  is a  $(m - v + 1) \times (m - v + 1)$  probability matrix generated by time-dependent upper barrier  $B(t)$ , where  $1 \leq l < v \leq m$ . Therefore, the boundary conditions of the transition probability matrix  $\bar{\mathbf{P}}$  are

$$\begin{aligned} \bar{p}(1 : l, :) &= [\mathbf{I}_{l \times l}, \bar{p}(1 : l, l + 1 : m) = 0] \quad \text{and} \\ \bar{p}(v : m, :) &= [\bar{p}(v : m, 1 : v - 1) = 0, \mathbf{I}_{(m-v+1) \times (m-v+1)}], \end{aligned}$$

where  $\mathbf{I}_{l \times l}$  and  $\mathbf{I}_{(m-v+1) \times (m-v+1)}$  are identity matrices in which each 1 in diagonal entries is a representation of absorption probability  $[\bar{p}(k, k) = 1 \text{ for } k \in \{1, 2, \dots, l\} \cup \{v, v + 1, \dots, m\}]$ . When the Markov chain  $(X_t)_{t \geq 0}$  reaches an absorbing state at time  $t$ , the knock-out option with time-varying barriers is considered being executed and its value is set to zero.

However, the submatrix  $[\bar{\mathbf{Q}}]_{(v-l-1) \times m}$  of  $\bar{\mathbf{P}}$  contains only the rows and columns for the transient states which are determined the movement of the stock prices. Due to time-dependent barriers, the number of absorbing states in  $\bar{\mathbf{P}}$  are varied at different time  $t$ .

Consequently, the dimensions of  $[\overline{\mathbf{Q}}]$  are modified by the changing dimensions of identity submatrices  $[\mathbf{Ib}]$  and  $[\mathbf{UB}]$  at different time  $t$ .

The initial probability distribution  $\vec{\phi}^{(0)}$  is a row vector defined as

$$\vec{\phi}^{(0)} = [\phi_b^{(0)}, \phi_Q^{(0)}, \phi_B^{(0)}] \quad (4.5.7)$$

and  $\vec{\phi}^{(0)}\vec{\mathbf{1}} = 1$ . The probability distribution at  $t = 1$  is

$$\phi^{(1)} = \vec{\phi}^{(0)}\overline{\mathbf{P}}^{(1)} = [\phi_b^{(1)}, \phi_Q^{(1)}, \phi_B^{(1)}],$$

where  $\phi_b^{(1)} = \phi_b^{(0)}[\mathbf{Ib}]$ ,  $\phi_Q^{(1)} = \phi_Q^{(0)}[\overline{\mathbf{Q}}]$ ,  $\phi_B^{(1)} = \phi_B^{(0)}[\mathbf{UB}]$ , and  $\overline{\mathbf{P}}^{(1)}$  is conditioned by time-dependent barriers  $B_1$  and  $b_1$ .

Iteratively, the probability distribution at  $t = n$  can be obtained by

$$\phi^{(n)} = \vec{\phi}^{(0)}\overline{\mathbf{P}}^{(n)} = [\phi_b^{(n)}, \phi_Q^{(n)}, \phi_B^{(n)}], \quad (4.5.8)$$

where  $\overline{\mathbf{P}}^{(n)}$  is the transition probability matrix modified by barriers  $B_n$  and  $b_n$  ( $\overline{\mathbf{P}}^{(n)} \neq \overline{\mathbf{P}}^n$ ),  $\phi_Q^{(n)}$  is the transient probability determining the movement of the stock price  $s_k$ , and  $\phi_b^{(n)}$  and  $\phi_B^{(n)}$  are the absorption probabilities of the stock price  $s_n$  hitting lower barrier  $b_n$  and upper barrier  $B_n$ , respectively.

Accordingly, at  $t = n$  the probability of the stock price  $s_k$  not crossing over any barrier is

$$\mathbf{P}_Q^{(n)}(s_k) = \mathbb{P}\{b_t < s_k < B_t, t = n\} = \phi_Q^{(n)}\vec{\mathbf{1}}_Q, \quad (4.5.9)$$

and the probability of the stock price  $s_k$  being knocked out is

$$\mathbf{P}_A^{(n)}(s_k) = \mathbb{P}\{b_t \geq s_k \quad \text{or} \quad B_t \leq s_k, t = n\} = \phi_b^{(n)}\vec{\mathbf{1}}_b + \phi_B^{(n)}\vec{\mathbf{1}}_B. \quad (4.5.10)$$

(Note:  $\phi_b^{(n)}$  and  $\phi_B^{(n)}$  may be different dimensional row vectors. So are the column vectors  $\vec{\mathbf{1}}_b$  and  $\vec{\mathbf{1}}_B$ .)

Furthermore, we can compute the transient probability of the stock prices  $s_k$  for  $l - 1 \leq k \leq v - 1$  by

$$\mathbf{P}_{\bar{Q}}^{(t)}(s_k) = \mathbb{P}\{b_t < s_k < B_t, 0 \leq t \leq n\} = \sum_{t=0}^n \sum_{k=l+1}^{v-1} \phi_{\bar{Q}}^{(t)} \bar{\mathbf{1}}_{\bar{Q}} \quad (4.5.11)$$

and the absorption probability of the stock price  $s_k$  for  $1 \leq k \leq l$  or  $v \leq k \leq m$  can be generated by

$$\mathbf{P}_A^{(t)}(s_k) = \mathbb{P}\{b_t \geq s_k \quad \text{or} \quad B_t \leq s_k, 0 \leq t \leq n\} = \sum_{t=0}^n \sum_{k=1}^l \phi_b^{(t)} \bar{\mathbf{1}}_b + \sum_{t=0}^n \sum_{k=v}^m \phi_B^{(t)} \bar{\mathbf{1}}_B. \quad (4.5.12)$$

If the discretely monitored stock prices  $s_k$  do not cross over either upper barrier  $B_t$  or lower barrier  $b_t$  until maturity  $T(=n)$ , the valuation of the knock-out option at  $t = n$  can be obtained by

$$C_{bt} = \sum_{k=l+1}^{v-1} e^{-rT} (s_k - K)^+ \phi_{\bar{Q}}^{(n)}(s_k) \quad (4.5.13)$$

where  $\phi_{\bar{Q}}^{(n)}(s_k)$  is determined by Eq. (4.5.8),  $r$  is the annual risk-free interest rate,  $T$  is the maturity of the option, and  $K$  is the pre-determined strike price.

**4.5.2. Power Asset Options:  $\mathbf{p} \in \mathbf{R}^+ \setminus \{0, 1\}$ .** Suppose a power asset option without dividend payout consists of a European call option embedded by barrier boundaries. Under the risk-neutral measure  $\mathbb{Q}^*$ , its underlying stock price  $S_t$  follows the non-geometric Brownian motion

$$dS_t = S_t^p (r dt + \sigma dW_t) \quad (4.5.14)$$

with constant risk-free interest rate  $r$  and volatility  $\sigma$ . The parameter  $p$ , defined as  $p \in \mathbf{R}^+ \setminus \{0, 1\}$ , is the power of the stock price and its existence in the partial differential equation (4.5.14) has influenced the movement of the underlying stock which determines the payoff of the power asset option.

In this section, we use a Markov chain  $(X_t)_{t \geq 0}$  with state space  $s_1, s_2, \dots, s_m$  to approximate discretely monitored stock prices over time index set  $t = \{0, 1, \dots, T\}$ , where  $T$  is the maturity of the power asset option. We partition the period  $[0, T]$  into equal-sized  $m$



cells such that the state values  $s_1, s_2, \dots, s_m$  are the middle points of these cells. The  $m$  cells are constructed as  $U_1 = (u_1, u_2)$  and  $U_i = [u_i, u_{k+1})$  for  $k \in \{2, \dots, m\}$ , where  $u_1 = -\infty$ ,  $u_i = \frac{s_k + s_{k-1}}{2}$  for  $k = \{2, 3, \dots, m\}$  and  $u_{m+1} = \infty$ .

Each  $s_k$  for  $k = \{1, 2, \dots, m\}$  is an independent and identically distributed (i.i.d.) random variable so that it follows the same distribution with mean  $s_k^p r \Delta t$  and variance  $s_k^p \sigma^2 \Delta t$ , where  $\Delta t$  is the length of time steps  $n$ . Moreover, each entry  $p(i, j)$  in the transition probability matrix  $\mathbf{P}$  can be computed by

$$\begin{aligned} p(i, j) &= \mathbb{P}\{s(t + \Delta t) \in [u_j, u_{j+1}) | s(t) = s_i\} \\ &= \Phi\left[\frac{u_{j+1} - s_i - s_i^p r \Delta t}{s_i^p \sigma \sqrt{\Delta t}}\right] - \Phi\left[\frac{u_j - s_i - s_i^p r \Delta t}{s_i^p \sigma \sqrt{\Delta t}}\right], \end{aligned} \quad (4.5.15)$$

where  $\Phi(\cdot)$  denotes the cumulative standard normal distribution function.

Since  $p$  is a critical parameter that has influential effect in computing the transition probabilities  $p(i, j)$ , we are interested in the relationship between the value of  $p$  and the price of the power asset option. In other words, what is the  $p$  value that generates the maximal value of the power asset option?

In Eq. (4.5.15),  $s_i^p$  is the dominant term in the denominator  $s_i^p \sigma \sqrt{\Delta t}$  since  $\sigma$  and  $\sqrt{\Delta t}$  are small values (for instance,  $\sigma, \sqrt{\Delta t} \ll 1$ ). Relatively,  $s_i$  in general is a value greater than one because most initial stock price  $s(0)$  is priced above \$4 on the first trading day. Moreover, the numerator can be affected by the term  $s_i^p r \Delta t$  in the computation of  $\Phi(\cdot)$ . In conclusion, it is difficult to select the critical value of  $p$  in order to maximize the price of the power asset option.

Now we analyze the computation of  $\Phi[z_j]$  by decomposition, where  $z_j = \frac{u_j - s_i - s_i^p r \Delta t}{s_i^p \sigma \sqrt{\Delta t}}$ . Let  $z_j = w_j + x_j$ , where  $w_j = \frac{u_j - s_i}{s_i^p \sigma \sqrt{\Delta t}}$  and  $x_j = \frac{-r \sqrt{\Delta t}}{\sigma}$ . Since  $r, \sqrt{\Delta t}$ , and  $\sigma$  are small positive values,  $x_j$  is also a small negative value. On the other hand,  $s_i^p$  is the dominant term for computing  $w_j$  since  $\frac{u_j - s_i}{\sigma \sqrt{\Delta t}}$  is a relatively small value in comparison to  $s_i^p$  for  $s_i \gg 1$ . If  $p : 1^+ \rightarrow \infty$ , the value of  $w_j$  would be decreased proportionally with  $s_i^p$ . However, for  $p : 0^- \rightarrow -\infty$ ,  $\lim_{z_j \rightarrow \infty} \Phi(z_j) \rightarrow 1$  since  $w_j \rightarrow \infty$ . Consequently, transition probabilities  $p(i, j)$

approach zero as  $p : 0^- \rightarrow -\infty$ . Therefore, as  $p \in \mathbf{R}^+ \setminus \{0, 1\}$  the transition probabilities  $p(i, j)$  could possibly generate the maximal value of the power asset option. In addition to the above discussion, there will be some results that provide the numerical analysis of  $p$  effect on price of the power asset option in the next section.

The settings of boundary conditions and probability distributions for a power asset options are the same as the ones for plain vanilla options. Overall, we can use the utility functions Eq. (4.5.6) or Eq. (4.5.13) to approximate the price of power asset option embedded with constant barriers  $B$  and  $b$ , or time-dependent barriers  $B(t)$  and  $b(t)$ , respectively.

**4.5.3. Optimal stopping time.** For a knock-out option, its price depends on whether the underlying asset price hits either upper or lower barrier up to maturity  $T$ . The corresponding barriers are set as the boundaries for the movement of the underlying asset price. These boundaries can be interpreted as absorbing states in a Markov chain  $(X_t)_{t \geq 0}$ . Furthermore, absorption probabilities of the transition probability matrix  $\mathbf{P}$  are the probabilities of some state  $X_t$  hitting an absorbing state where the chain executes at time  $t = y$ . The so-called *optimal stopping time*,  $y$ , is defined as

$$T_y = \min\{y > 0 : X_y \in I_a\}, \quad (4.5.16)$$

where  $I_a$  is the set of absorbing states.

In section 4.5.1, absorbing states are defined in the boundary conditions for the transition probability matrix  $\mathbf{P}_{m \times m}$ . For time-independent barriers, the transition probabilities for absorbing boundaries are  $\tilde{p}(1, 1) = 1$  and  $\tilde{p}(m, m) = 1$  where we set state 1 and  $m$  as absorbing states. If  $X_t$  reaches either state 1 or  $m$ , the Markov chain will be executed at  $t = k$ . Therefore, we can compute the stopping time for the chain reaching either state 1 or  $m$  by

$$T_c = \min\{k > 0 : X_k \in \{1, m\}\}. \quad (4.5.17)$$

Furthermore, the corresponding probability of  $X_t$  reaching either absorbing state 1 or  $m$  is defined by

$$\tilde{\mathbf{P}}_A = \mathbb{P}\{s_k \leq b \text{ or } s_k \geq B\} = \mathbb{P}\{T_c = k\}. \quad (4.5.18)$$

On the other hand, the probability of  $X_t$  continuing on transient states is computed by

$$\tilde{\mathbf{P}}_T = \mathbb{P}\{b < s_k < B, 1 \leq k \leq m\} = 1 - \mathbb{P}\{T_c = k\}. \quad (4.5.19)$$

However, for time-dependent barriers, absorbing states are varied due to time-varying boundary conditions, which are defined in the submatrices  $[\mathbf{lb}]_{l \times l}$  and  $[\mathbf{UB}]_{(m-v+1) \times (m-v+1)}$  of the transition probability matrix  $\mathbf{P}$ . The  $[\mathbf{lb}]$  and  $[\mathbf{UB}]$  are identity matrices and each diagonal entry  $p(i, i)$  for  $i \in \{1, 2, \dots, l\} \cup \{v, v+1, \dots, m\}$  is the probability of the chain reaching an absorbing state  $i$ .

Since the dimensions of  $[\mathbf{lb}]$  and  $[\mathbf{UB}]$  are varied by time-dependent barriers at different  $t$ , the probability of the underlying asset price being knocked out in the period up to maturity  $T(=n)$  is the probability of the chain reaching an absorbing state at time  $t=n$ . If the stopping time of the chain hitting an absorbing state at time  $t=n$  is

$$T_v = \min\{n > 0 : X_n \in \{1, 2, \dots, l\} \cup \{v, v+1, \dots, m\}\}, \quad (4.5.20)$$

its corresponding probability of the underlying asset price  $s_k$  crossing over a barrier level is computed by

$$\begin{aligned} \mathbf{P}_A^{(t)}(s_k) &= \mathbb{P}\{T_v = n\} = \mathbb{P}\{b_t \geq s_k \text{ or } B_t \leq s_k, 0 \leq t \leq n\} \\ &= \sum_{t=0}^n \sum_{k=1}^l \phi_b^{(t)} \vec{\mathbf{1}}_b + \sum_{t=0}^n \sum_{k=v}^m \phi_B^{(t)} \vec{\mathbf{1}}_B, \end{aligned} \quad (4.5.21)$$

where  $\phi_b^{(n)}$  and  $\phi_B^{(n)}$  are the absorption probabilities of the asset price  $s_k$  hitting lower barrier  $b_n$  and upper barrier  $B_n$ , respectively.

On the contrary, we can compute the probability of  $X_t$  reaching only transient states in the period up to maturity  $T$  by

$$\begin{aligned} \mathbf{P}_{\bar{Q}}^{(t)}(s_k) &= \mathbb{P}\{b(t) < s_k < B(t), 0 \leq t \leq n\} \\ &= \sum_{t=0}^n \sum_{k=l+1}^{v-1} \phi_{\bar{Q}}^{(t)} \vec{\mathbf{1}}_{\bar{Q}} = 1 - \mathbb{P}\{T_v = n\} \end{aligned} \quad (4.5.22)$$

where  $\phi_{\bar{Q}}^{(t)}$  is the transient probability determining the movement of asset price  $s_k$ .

In addition to time-dependent barriers, there are other factors, such as initial asset price  $S(0)$ , the parameter  $p$ , volatility  $\sigma$ , and the length  $\Delta t$  of time steps, having effect on the optimal stopping time of the Markov chain which determines the lifetime of a power asset option. In the next section 4.6, we will discuss the effect of these factors on the price of a knock-out option by numerical results.

#### 4.6. Comparison of numerical methods

The closed-form expressions of most exotic options are difficult to be derived analytically due to complex SDEs. Some numerical methods, such as lattice (trinomial tree), finite-difference, and Monte Carlo simulation can solve the utility functions of exotic options. Each numerical method has its own advantages and downsides in the process of pricing an exotic option.

The adaptive mesh model in restricted trinomial tree provides the flexibility of placing the initial asset price but the convergence of this method is too slow, especially for an initial asset price close to a barrier level; moreover, recursion of this tree results a lack of accuracy. Given the length  $\Delta t$  of time step, the space step is of order  $\sqrt{\Delta t}$  which tends to be the error's order as well.

The implicit finite difference is more stable than the explicit finite difference when it comes to convergence; however, the accuracy of finite difference is considered relatively lower than other numerical methods. Compared to lattice and finite difference methods, Monte Carlo simulation provides flexibility of computing multiple assets, time-varying volatility or

TABLE 2. Plain Vanilla Option Valuation

Initial asset price $S_0$	Analytic value $C^A$	Monte Carlo $C^M$	Markov chain $C^V$
33	4.4639	4.4672	4.4630
32	3.4688	3.4697	3.4686
31	2.4934	2.4915	2.4933
30	1.5850	1.5811	1.5850
29	0.8348	0.8296	0.8349
28	0.3356	0.3375	0.3356
27	0.0939	0.0933	0.0939
26	0.0166	0.0166	0.0166

Parameters:  $\Delta s = 0.01$ ,  $K = 30$ ,  $r = 0.05$ ,  $\sigma = 0.05$ ,  $dt = \frac{1}{250}$ ,  $T = 1$

interest rate on the options. Nevertheless, the simulation cannot handle the cases for early exercise of American options and when an initial asset price is close to a barrier level.

The downsides of these numerical methods are costly and time-consuming for pricing exotic options; however, Markov chain approximation is an efficient and accurate numerical tool to estimate the price of the underlying asset due to sparsity of the transition probability matrix  $\mathbf{P}$ . In addition to computational advance, Markov chain method can handle technical difficulties— for example, early exercise for American options and initial asset price which is very close to a barrier of a knock-out option. Furthermore, we will discuss how Markov chain method approximates a financial derivative, of which underlying asset price is based on a non-geometric Brownian motion.

The analytical formula for the price of an European knock-out call option with time-independent barriers is derived by Elliot and Kopp [8]; however, it raises the level of difficulty to obtain the payoff function when we deal with time-dependent barriers. In this section, Monte Carlo simulation and Markov chain approximation are preferred numerical methods to price a knock-out option with time-dependent barriers and later, we will apply Markov chain approximation to discretely monitor the non-geometric Brownian motion model – ”power asset option.”

Now let us compare the numerical results generated by Monte Carlo simulation and Markov chain approximation with the analytical value of an European call option in Black-Scholes framework. Compared to Monte Carlo simulation, the convergence of the Markov

chain approximation appears to be fairly fast and it only requires a far smaller number of discrete asset prices for Markov chain method to achieve the same level of accuracy (Duan and Simonato [7]). In Table 2 the numerical results, generated by MATLAB-code implementation, show that both methods achieve penny accuracy to the analytical value of the option. Even Markov chain approximation has better results than Monte Carlo simulation since some of realizations in Monte Carlo simulation are neglected when computing the average of the values generated by overall realizations. Therefore, we employ Markov chain approximation to compute the values of knock-out options and power asset options with time-varying barriers.

The analytical formulae for barrier options were derived by Reiner and Rubinstein [13]; however, pricing knock-out options with time-varying barriers so far could only be implemented by numerical methods. We use a Markov chain to approximate the underlying asset price of an European knock-out call option embedded with time-dependent barriers<sup>2</sup>. In addition to the supplementary risk proposed by time-dependent barriers, the transient probabilities  $\mathbf{P}_Q^{(t)}(s_k)$  [Eq. (4.5.11)] of the transition probability matrix can be affected by the length  $\Delta t$  of time steps  $n$ , the interval  $\Delta s$  of discrete asset prices, annual risk-free interest rate  $r$ , volatility  $\sigma$ .

Recall that transition probabilities  $p(i, j)$  are generated by

$$p(i, j) = \Phi\left[\frac{u_{j+1} - s_i - s_i r \Delta t}{s_i \sigma \sqrt{\Delta t}}\right] - \Phi\left[\frac{u_j - s_i - s_i r \Delta t}{s_i \sigma \sqrt{\Delta t}}\right], \quad (4.6.1)$$

where  $\Phi(\cdot)$  denotes the cumulative standard normal distribution function.

(*Note:* The values of  $u_j$  and  $s_i$  are defined in Section 4.5.1.)

Without loss of generality, we discuss the effect of each factor:  $\Delta s$ ,  $\Delta t$ ,  $r$ ,  $\sigma$  on the transition probabilities  $p(i, j)$ . We first analyze the computation of  $\Phi[a_j]$  by decomposition, where  $a_j = \frac{u_j - s_i - s_i r \Delta t}{s_i \sigma \sqrt{\Delta t}}$ . Let  $a_j = q_j + k_j$ , where  $q_j = \frac{u_j - s_i}{s_i \sigma \sqrt{\Delta t}}$  and  $k_j = \frac{-r \sqrt{\Delta t}}{\sigma}$ . As  $\Delta t \uparrow$ ,  $a_j \downarrow$  since  $q_j \downarrow$  and  $k_j \downarrow$ ; consequently,  $\Phi[a_j] \downarrow$ . Despite the absorption probabilities, the transient

<sup>2</sup> $B_t = 50 - 0.5\sqrt{t}$ ,  $b_t = 20 + 0.5\sqrt{t}$  [Appendix 4.3]

TABLE 3. The Prices of Knock-out Options Determined by Different  $\Delta t$  and  $\Delta s$

Call prices	$\Delta t = \frac{1}{2}$	$\Delta t = \frac{1}{4}$	$\Delta t = \frac{1}{12}$	$\Delta t = \frac{1}{50}$	$\Delta t = \frac{1}{250}$
$\Delta s=1$	9.2184	8.7921	8.4900	7.4102	6.6591
$\Delta s=0.5$	9.2185	8.7921	8.5077	8.3758	7.1474
$\Delta s=0.1$	9.2185	8.7921	8.5078	8.3996	8.2879
$\Delta s=0.05$	9.2185	8.7921	8.5078	8.3997	8.3057
$\Delta s=0.01$	9.2185	8.7921	8.5078	8.3997	8.3132

Parameters:  $S_0 = 35$ ,  $K = 28$ ,  $r = 0.05$ ,  $\sigma = 0.05$ ,  $T = 1$ ,  $B_t = 50 - 0.5\sqrt{t}$ ,  $b_t = 20 + 0.5\sqrt{t}$

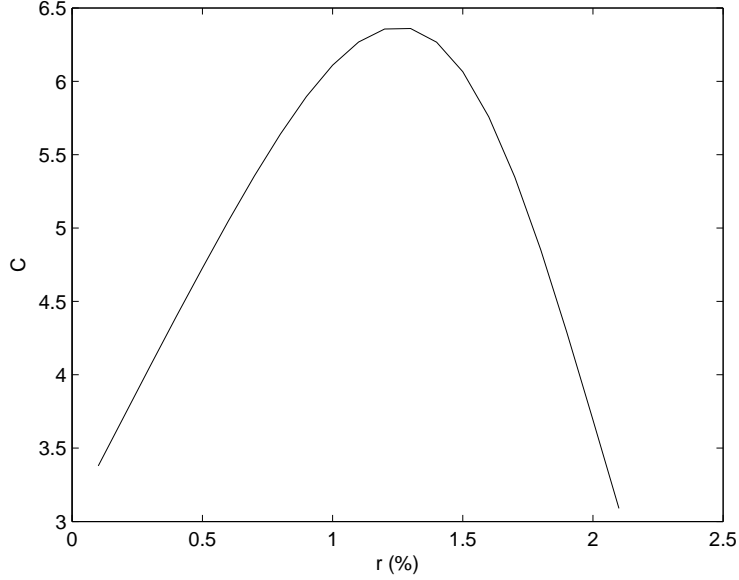
probabilities  $\phi_{\overline{Q}}(s_k)$  are increased when the length  $\Delta t$  of time steps increases. Consequently, using Eq. 4.5.13 this leads an increase in the value of the knock-out option. In Table 3, the values of the knock-out option in each row are increased as  $\Delta t \uparrow$  from the right to the left.

On the other hand, as  $\Delta s \uparrow$ ,  $a_j \uparrow$  since  $q_j \uparrow$ . Consequently,  $\Phi[a_j] \uparrow$  leads a decrease in the transient probabilities  $\phi_{\overline{Q}}(s_k)$ . The value of the knock-out option is decreased when  $\Delta s$  is increased from the bottom to the up in column values of Table 3. As  $\Delta t \downarrow$ , the number of time steps  $n$  is increased since  $n = \frac{T}{\Delta t}$ , where maturity  $T$  is pre-determined value at the initial time. In addition, as  $\Delta s \downarrow$  the dimension of the transition probability matrix  $\mathbf{P}$  expands. Therefore, in order to achieve better accuracy, we choose  $\Delta t$  and  $\Delta S$  values as small as possible for Markov chain approximation. (Note:  $\Delta t = \frac{1}{250}$  means that the pricing process is monitored daily since we assume a year has 250 trading days. Therefore,  $\Delta t = \frac{1}{2}, \frac{1}{4}, \frac{1}{12}$ , and  $\frac{1}{50}$  mean that the knock-out options are monitored semi-annually, quarterly, monthly, and weekly, respectively.)

The annual risk-free interest rate  $r$  would also affect the transient probabilities. The  $a_j$  value decreases as  $k_j \downarrow$  since  $r \uparrow$ . Consequently, the transient probabilities are increased because of a decrease in  $\Phi[a_j]$ . If  $r$  is large enough to make  $a_j$  a negative value, we would have an increase in  $\Phi[a_j]$ , which causes a decrease in the transient probabilities. Using the payoff function [Eq. 4.5.13] we can obtain the price of the knock-out option with respect to the annual risk-free interest rate. In Figure 4.4, the price of the knock-out option<sup>3</sup> is increased as  $r \rightarrow 1.3^-$ , then after  $r = 1.3$  the option price begins to drop.

<sup>3</sup>parameters:  $S_0 = 38$ ,  $\Delta s = 0.1$ ,  $\sigma = 0.05$ ,  $K = 35$ ,  $T = 1$ ,  $dt = \frac{1}{50}$ ,  $B_t = 50 - 0.5\sqrt{t}$ ,  $b_t = 20 + 0.5\sqrt{t}$

FIGURE 4.4. Knock-out Option with Risk-free Interest Rate  $r$

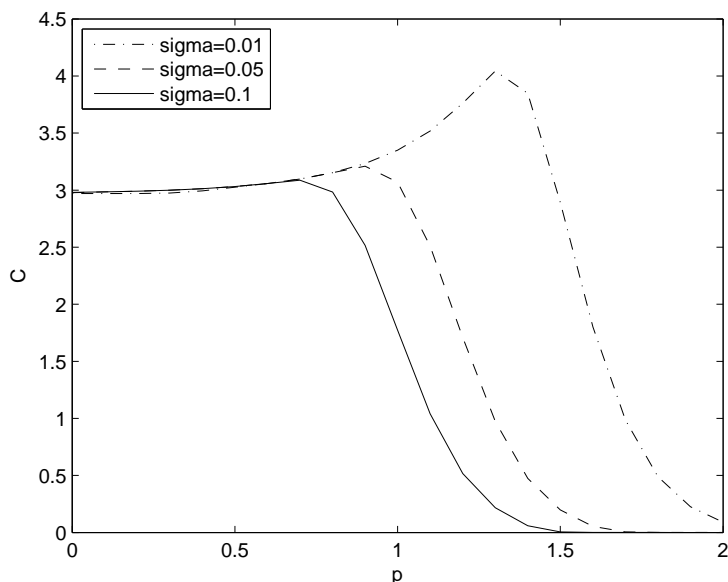


Measuring volatility is an important hedging for pricing financial derivatives since implied volatility is the key factor making the movement of the underlying asset. In addition to the risk imposed by the time-dependent barriers, implied volatility can essentially impact the transient probabilities by order of  $\mathcal{O}(\frac{1}{s_i \sigma \sqrt{\Delta t}})$ . When the volatility  $\sigma$  increases, the transient probabilities are decreased. In other words, as  $\sigma \rightarrow \infty$  transient states have probability  $\mathbb{P}(\lim_{\sigma \rightarrow \infty} |\Phi[a_{j+1}] - \Phi[a_j]| \geq \epsilon) = 0$ . Consequently, the value of the knock-out option with time-varying barriers<sup>4</sup> becomes worthless when the volatility  $\sigma$  decreases. In Figure 4.5, the prices of the knock-out option at  $p = 1$  with  $\sigma = 0.1$ ,  $\sigma = 0.05$ , and  $\sigma = 0.1$  are 1.7724, 3.0693, and 3.3498, respectively.

Moreover, the time-dependent barriers provide the boundary condition that is changed at different time  $t$  during the lifetime of the option. This restricted feature can influence the movement of the asset price since at each time  $t$  the dimensions of absorbing submatrices **[lb]** and **[UB]** of the transition probability matrix **P** are modified by the boundary condition. Therefore, the selection of time-varying barriers can be one of the factors that influence the value of the knock-out option. In our case, we let  $B_t = 50 - \frac{1}{2}\sqrt{t}$  and  $\overline{B}_t = 50 - \frac{1}{2}\sqrt{t}, b_t = 20 + \frac{1}{2}\sqrt{t}$



FIGURE 4.5. Power Asset Knock-out Option with Volatility  $\sigma$

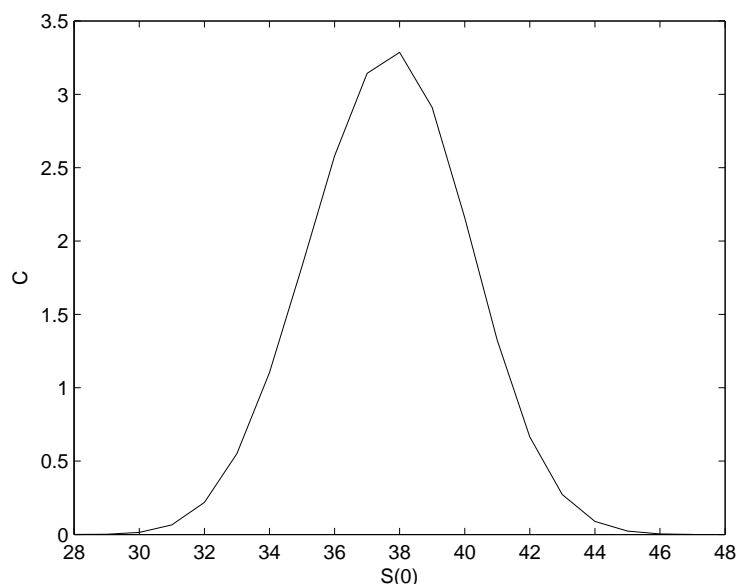


$b_t = 20 + \frac{1}{2}\sqrt{t}$  be the upper and lower barrier functions [Figure4.2]. The gap between two barriers is shrinking when time continues running. In addition to the effect of  $e^{-rT}$  in the payoff function [Eq. (4.5.13)], the longer the maturity of the knock-out option lasts, the less value the knock-out option becomes. For maturity  $T=0.5, 1.0, 1.5$  and  $2.0$ , the prices of the knock-out option<sup>5</sup> are 3.8708, 3.4298, 1.1236 and 0.2141, respectively.

Apparently, the knock-out option with time-varying barriers at longer maturity has less value than the one at shorter maturity. Using Eq. (4.5.21) and Eq. (4.5.22) we can determine the probabilities of the underlying asset price crossing a barrier level and of the one remaining valuable until maturity. For the knock-out option with time-varying barriers, the probabilities of the asset prices being knocked out are  $1.3973 \times 10^{-4}$  (for  $T = \frac{1}{2}$ ), 0.1694 (for  $T = 1$ ), 0.6445 (for  $T = \frac{3}{2}$ ) and 0.8930 (for  $T = 2$ ). The corresponding probabilities of the asset prices not hitting a barrier level are 0.9999, 0.8306, 0.3555 and 0.1070, respectively. That means at longer maturity the price of its underlying asset has greater probability of hitting a barrier level (in other words, shorter stopping time). In conclusion, the price of

<sup>5</sup>parameters:  $S_0 = 38$ ,  $\Delta s = 0.01$ ,  $\sigma = 0.05$ ,  $r = 0.05$ ,  $K = 35$ ,  $dt = \frac{1}{250}$

FIGURE 4.6. Knock-out Option with Initial Asset Price  $S_0$



the knock-out option with time-dependent barriers does depend on its maturity and barrier levels.

From investors' perspective, how do we know whether the way of setting up the time-varying barriers for an European knock-out call option is fair for investors, according to the initial price of the underlying asset? It depends on all the factors mentioned above. In Figure 4.6, the knock-out option has the maximal value  $C \approx 3.2859$  when the initial price  $S_0$  of the underlying asset is \$38, located at almost the middle of the gap between upper barrier  $B_0 = 50$  and lower barrier  $b_0 = 20$ . Conversely, when the initial prices of the underlying asset are at  $S_0 = 28$  and  $S_0 = 48$ , the knock-out option have values of  $C \approx 1.9104 \times 10^{-4}$  and  $C \approx 3.4256 \times 10^{-5}$ , respectively.

Similarly to the numerical method proposed by Cheuk, T. and T. Vorst [6], Markov chain approximation can handle the case where the initial price  $S_0 = 48$  of the underlying asset is close to a barrier level  $B_0 = 50$ . This technical difficulty of computation cannot be achieved by Monte Carlo simulation since the prices of underlying asset in most realizations are crossing over a barrier level (Boyle, Broadie and Glasserman [3]).

Compared to time-varying barriers, constant barriers provide the knock-out option with more flexibility for the movement of the underlying asset price. On the other hand, the shrinking gap between two time-varying barriers causes an expansion in the dimensions of the submatrices  $[\mathbf{lb}]$  and  $[\mathbf{UB}]$  by the shrinking gap between two time-varying barriers. Therefore, the price ( $C \approx 1.8491$ ) of an knock-out option with time-independent barriers is more expensive than the one ( $C \approx 1.8251$ ) with time-varying barriers (parameters:  $S_0 = 35$ ,  $\Delta s = 0.01$ ,  $K = 35$ ,  $dt = \frac{1}{250}$ ,  $\sigma = 0.05$ ,  $r = 0.05$ ,  $T = 1$ ; time-varying barriers:  $B_t = 50 - \frac{1}{2}\sqrt{t}$ ,  $b_t = 20 + \frac{1}{2}\sqrt{t}$ ; time-independent barriers:  $B = 50$  and  $b = 20$ ).

According to discretely monitoring feature, Markov chain approximation can price not only the knock-out options but also the knock-in options. Even for the early exercise of American options, the discretely monitored asset prices are checked at each time  $t$  for whether the asset price is executed or not. Overall, Markov chain approximation is a fast convergent numerical method to price exotic options of which analytical expressions are difficult to be found and to tackle computational difficulties when the initial price of the underlying asset is close to a barrier level and American options exercise prior to maturity.

#### 4.7. The Parameter- $p$

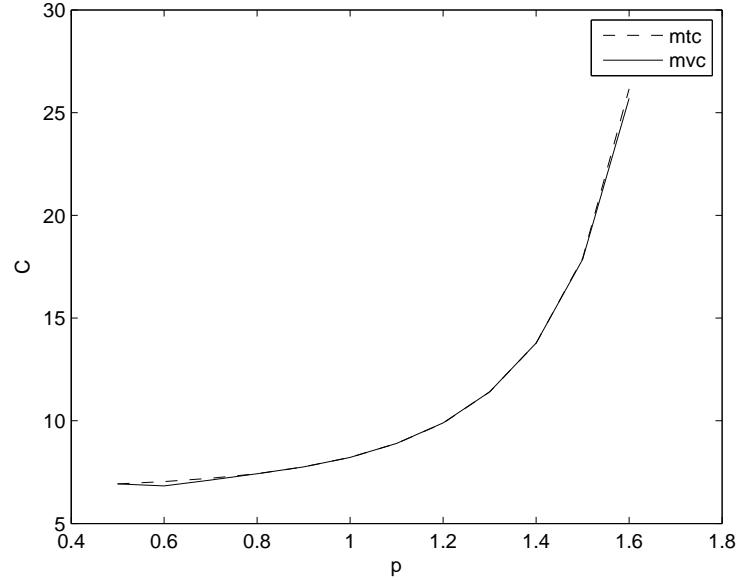
A power asset option assumes that asset prices are governed by the following process:

$$dS_t = S_t^p (rdt + \sigma dW_t), \quad (4.7.1)$$

where the parameter  $p$  is one of the key parameters driving the movement of the underlying asset price. The process above is a generalization of geometric Brownian motion when the parameter  $p = 1$ ; however, it is the representation of a Brownian motion when  $p = 0$ .

In this session, we are interested in observing the effect of the parameter  $p$  when  $p \in \mathbf{R}^+ \setminus \{0, 1\}$ . This opposes the geometric Brownian motion model which would assume the volatility remains constant, no matter what external events may be happening in the market. In other words, we say the pricing process [Eq. (4.7.1)] of the underlying asset follows the non-geometric Brownian motion. We assume that the economy experiences a

FIGURE 4.7. Power Asset Option with Parameter  $p$



blooming stage which causes a rapid change in the underlying asset price  $dS_t$ . Since the flow of money in the market is frequently exchanged in hands, the prices of stocks are spontaneously changed in proportional to  $S_t^p$  due to sufficient demands from investors. The volatility parameter  $S_t^p \sigma \sqrt{\Delta t}$  may be more realistic for generating the stock returns since investors are interested in highly potential returns of the underlying stocks.

Exotic options in general exclude penny stocks (which values less than one dollar) for the underlying assets. The price of the underlying asset for a power asset option, therefore, is assumed to be greater or equal to one dollar. According to the exponential growth  $S_t^p$ , the value of a call option, of which asset price follows the non-geometric Brownian motion defined as Eq. (4.7.1), is determined by the value of the asset price  $S_t$  and the parameter  $p$ . In Figure 4.7, using Markov chain approximation the option price<sup>6</sup> increases exponentially as  $p \rightarrow 1.6$  but converges to  $C \approx 6.93$  as  $p \rightarrow 0.05$ . Because of this phenomenon, we can only focus on  $p = 1 + \delta$  for  $0 \leq |\delta| \leq 1$  and its effect on the price of an power asset option.

<sup>6</sup>The solid 'mvc' line represents the values of the power asset option evaluated by Markov chain approximation at different parameter  $p$  values. On the other hand, the 'mtc' values on the dash line are the prices of the power asset option estimated by Monte Carlo simulation with  $10^5$  realizations at different parameter  $p$  values. Parameters:  $S_0 = 32$ ,  $\Delta s = 0.1$ ,  $r = 0.05$ ,  $\sigma = 0.05$ ,  $K = 25$ ,  $T = 1$ ,  $dt = \frac{1}{250}$ .

How does  $p$  affect the price of an power asset option in Markov chain approximation? First, let us look at the effect of  $p$  on the transition probabilities  $p(i, j)$ , which are defined as

$$p(i, j) = \Phi\left[\frac{u_{j+1} - s_i - s_i^p r \Delta t}{s_i^p \sigma \sqrt{\Delta t}}\right] - \Phi\left[\frac{u_j - s_i - s_i^p r \Delta t}{s_i^p \sigma \sqrt{\Delta t}}\right], \quad (4.7.2)$$

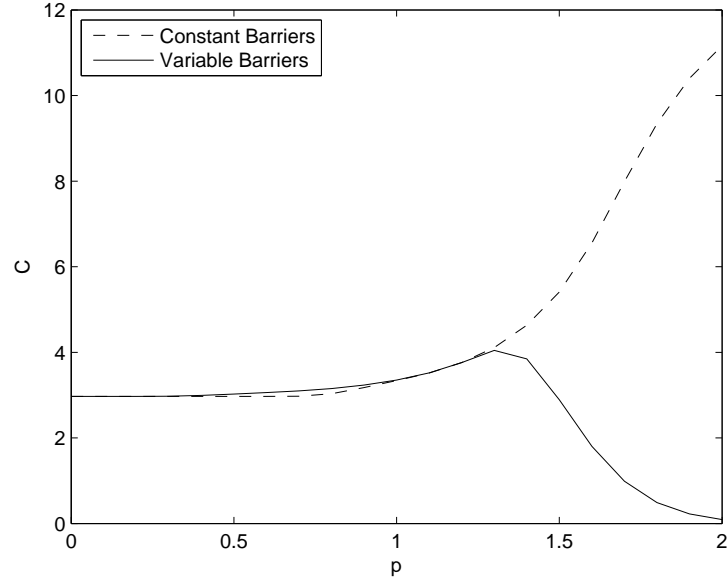
where  $\Phi(\cdot)$  denotes the cumulative standard normal distribution function.

By decomposition, we let  $z_j = \frac{u_j - s_i - s_i^p r \Delta t}{s_i^p \sigma \sqrt{\Delta t}} = w_j + x_j$ , where  $w_j = \frac{u_j - s_i}{s_i^p \sigma \sqrt{\Delta t}}$  and  $x_j = \frac{-r \sqrt{\Delta t}}{\sigma}$  so that the transition probabilities are computed by  $p(i, j) = \Phi[z_{j+1}] - \Phi[z_j]$ . Mentioned in the previous session,  $p(i, j)$  of a knock-out option (parameter  $p = 1$ ) decreases when  $\sigma$  increases. For a power asset option with time-varying barriers, the values of  $z_{j+1}$  and  $z_j$  are decreased by order of  $\mathcal{O}\left(\frac{1}{s_i^p \sigma \sqrt{\Delta t}}\right)$ , which is more rapidly than the ones of a knock-out option (order of  $\mathcal{O}\left(\frac{1}{s_i \sigma \sqrt{\Delta t}}\right)$ ). Figure 4.5 shows that the price of the power asset option drops after reaching the maximal value as  $p \rightarrow 2$ . As the parameter  $p$  increases, for the power asset option with smaller  $\sigma$  the transition probability  $p(i, j)$  increases until the price of the underlying asset is very close to a barrier level. The price of the power asset option then reverses its direction after reaching the maximum because of supplementary risk imposed by the barrier level. In Figure 4.5, the power asset options with  $\sigma = 0.01$ ,  $\sigma = 0.05$  and  $\sigma = 0.1$  have maximal values  $C \approx 4.0436$  (at  $p = 1.3$ ),  $C \approx 3.2093$  (at  $p = 0.9$ ), and  $C \approx 3.0868$  (at  $p = 0.7$ ), respectively. As a result, the greater volatility  $\sigma$  a power asset option has, the larger the optimal price it will obtain as the parameter  $p$  increases to a certain point.

Nevertheless, the prices of power asset knock-out options<sup>7</sup> depend on the barrier levels which determine the dimensions of the submatrices  $[\mathbf{lb}]$  and  $[\mathbf{UB}]$ . For time-varying barriers  $B_t = 50 - \frac{1}{2}\sqrt{t}$  and  $b_t = 20 + \frac{1}{2}\sqrt{t}$ , the gap between the two barriers is shrinking when time continues running. This results an expansion in the dimensions of  $[\mathbf{lb}]$  and  $[\mathbf{UB}]$  since more absorbing states are created. The probability of the asset price hitting a barrier level increases since the number of transient states is decreased. Furthermore, in Figure 4.8 as

<sup>7</sup>Parameters:  $S_0 = 38$ ,  $\Delta s = 0.01$ ,  $K = 35$ ,  $r = 0.01$ ,  $\sigma = 0.01$ ,  $dt = \frac{1}{250}$ ,  $T = 1$   
Time-varying barriers:  $B_t = 50 - 0.5\sqrt{t}$ ,  $b_t = 20 + 0.5\sqrt{t}$   
Constant barriers:  $B = 50$ ,  $b = 20$

FIGURE 4.8. Power Asset Knock-out Option with Constant and Time-varying Barriers



$p : 1.3 \rightarrow 2$  the transition probabilities  $p(i, j)$  decrease rapidly for the power asset knock-out option with time-varying barriers due to an increase in the number of absorbing states. On the other hand, the boundary condition for time-independent barriers remains the same over time period. For small volatility ( $\sigma = 0.01$ ), the movement of the underlying asset is less fluctuated in amplitude. As  $p : 1 \rightarrow 2$  the prices of the power asset options embedded with constant barriers increase steadily since the asset prices are considered in 'relatively' safe distance from the barrier levels because of its small volatility ( $\sigma = 0.01$ ).

We now discuss the computational advance of Markov chain approximation since it can handle the case where the initial price of the underlying asset is close to a barrier level. We assume that power asset knock-out options have a time-varying barrier  $B_t = 10 - (0.5t)^{-0.05}$ . Numerical evidence in Table 4 shows that Markov chain method can take small values of the initial asset prices, even for  $S_0 = 2$  close to zero (boundary condition) to price power asset knock-out options with the time-varying barrier  $B_t$ . The implicit boundary condition [Eq. (4.1.5)] is set by the payoff function of the options  $C_{uo} = (S_T - K)^+$ . For the price of the underlying asset following a Brownian motion  $dS_t = rdt + \sigma dW_t$  as  $p = 0$ , the

TABLE 4. Power Asset *Up-to-out* Call Options

Call prices (C)	$p = 0$	$p = 0.5$	$p = 1.0$	$p = 1.5$
$S_0 = 2.0, K = 2.0$	$5.5179 \times 10^{-15}$	$4.0493 \times 10^{-8}$	0.0001	0.0054
$S_0 = 3.0, K = 3.0$	$5.5179 \times 10^{-15}$	$8.3955 \times 10^{-6}$	0.0076	0.0485
$S_0 = 4.0, K = 4.0$	$5.5179 \times 10^{-15}$	$1.2385 \times 10^{-4}$	0.0246	0.0896
$S_0 = 5.0, K = 5.0$	$5.5179 \times 10^{-15}$	$6.1694 \times 10^{-4}$	0.0441	0.1246

Parameters:  $\Delta s = 0.01, r = 0.01, \sigma = 0.01, dt = \frac{1}{250}, T = 1, B_t = 10 - (\frac{1}{2}t)^{-\frac{1}{20}}$

values of the power asset knock-out option remain the same ( $C \approx 5.5179 \times 10^{-15}$ ) regardless of the initial prices  $S_0$  of the underlying asset. On the contrary, the price of the underlying asset following the stochastic differential equation  $dS_t = S_t^p(rdt + \sigma dW_t)$  and the change in the value of the power asset option without barriers are increased by the order of  $\mathcal{O}(S_t^p)$  as  $p : 0.5 \rightarrow 1.5$  [Figure 4.7]. Because of the supplementary risk imposed by the barriers, the values of the power asset options with time-dependent barriers in Table 4 are cheaper than the ones without barriers attached.

Overall, the parameter  $p$  value has some influence on the volatility, which is the key to drive the movement of the underlying asset price. For time-dependent barriers, the volatility parameter  $S_t^p \sigma \sqrt{\Delta t}$  in power asset options become more influential than the ordinary implied volatility because  $p$  can cause the price of the power asset option drop dramatically due to the effect of the parameter  $p$  and supplementary risk by the barriers. Consequently, the prices of the power asset knock-out options are cheaper than the ones of standard knock-out options.

## CHAPTER 5

# SUMMARY, CONCLUSION AND FUTURE RESEARCH

## SUMMARY

Most of the time the prices of exotic options are difficult to derive analytically due to their complex structure. In option pricing theory, fast and accurate numerical methods are recommended to tackle the technical difficulties in computation. We proposed a Markov chain method to price discretely monitored knock-out options with either constant or time-varying barriers. Furthermore, the time-homogeneous Markov chain was used to approximate the pricing process of the underlying asset, which follows a non-geometric Brownian motion  $dS_t = S_t^p(\mu dt + \sigma dW_t)$ , in so-called power asset options. We study the effects of the parameters in pricing knock-out options and the parameter  $p$  in power asset options with either time-dependent or moving barriers.

## CONCLUSION

Compared with Monte Carlo simulation, Markov chain approximation is a better numerical method to price discretely monitored knock-out options because of its fast convergence and accurate estimation. In addition to computational advances, the Markov chain method can handle the case when the initial price of the underlying asset is close to a barrier level. Numerical evidence shows that the parameter  $p$  has essential impact in options pricing: when  $p = 1 + \delta$  for  $0 \leq |\delta| \leq 1$ , there exists a maximal price of the power asset knock-out option. Thereafter, the power asset option would dramatically decrease when  $p$  continues to increase due to supplementary risk imposed by time-varying barriers.



## FUTURE RESEARCH

The research in this dissertation has covered the pricing process of one-dimensional underlying assets of knock-out options. However, in reality investors may hold a portfolio of exotic options containing multi-dimensional underlying assets. In order to select the most potential assets that are able to generate possible maximal payoffs, investors would like to find correlations among these assets. For our defined power asset options with time-varying barriers, analytic expression of the covariance becomes more complicated to be derived. In addition to risk management, portfolio managers are interested in variance swaps for a basket of potential assets. How does the parameter  $p$  affect the payoff of the portfolio, especially for those exotic options embedded with irregularly time-varying barriers? Future investigation is recommended to analyze the payoff function of a financial portfolio consisting of multi-dimensional assets that follow the non-geometric Brownian motions.

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# APPENDIX

## Matlab codes

### 0.1. Expectation of $[S(t)]^p$

---

This MATLAB expect.m file generates expectation of  $S(t)^q$ .

Inputs:

s0     -represents the initial asset price (e.g.  $s(0)=10$ )  
mu     -represents the mean (e.g.  $\mu=0.001$ )  
sigma -represents the volatility (e.g.  $\sigma=0.001$ )  
t     -represents the time in the period up to maturity T  
      (e.g.  $t=1/12:1/12:1$ )  
q     -represents the parameter in the power of  $S(t)$   
      (e.g.  $q=1$ )

Outputs:

E     -represents the expectation of  $S(t)^q$   
      (e.g.  $E[S(t)^q]$ )

---

```
function E=expect(s0,mu,sigma,t,q)

switch q
case -1
    disp('q=-1')
    E=0.5*mu*sigma^2*(t.^2)+s0^(-1)*t+s0*sigma^2-mu;
    plot(t,E,'+',t,E,':');
    title('E[S_t^q] v.s. t');
    xlabel('t');
    ylabel('E[S_t^q]');
case 1
    %repeated roots for x''(t)=0
```

```

disp('q=1')
E=s0 + mu*t;
plot(t,E,'+',t,E,':');
title('E[S_t^q] v.s. t');
xlabel('t');
ylabel('E[S_t^q]');
case 0      %repeated roots for x''(t)=0
disp('q=0')
E=1;
plot(t,E,'+',t,E,':');
title('E[S_t^q] v.s. t');
xlabel('t');
ylabel('E[S_t^p]');
otherwise
if abs(q)<1      %for |q|<1, we have distinct imaginary roots
disp('|q|<1')
E=(s0^(q)-2*mu/((1+p)*sigma^2))*cos(0.5*abs(q)*sqrt(1-q^2)*sigma^2*t)
+(q*mu-0.5*q*(1-q)*sigma^2*s0^(-q))/(0.5*abs(q)*sqrt(1-q^2)*sigma^2)
*sin(0.5*abs(q)*sqrt(1-q^2)*sigma^2*t)+2*mu/((1+q)*sigma^2);
plot(t,E,'+',t,E,':');
title('E[S_t^q] v.s. t');
xlabel('t');
ylabel('E[S_t^q]');
else      %for |q|>1, we have distinct real roots
disp('|q|>1')
E=0.5*(s0^(p)-2*mu/((1+q)*sigma^2)-
(q*mu-0.5*p*(1-q)*sigma^2*s0^(-q))/
(0.5*abs(p)*sqrt(p^2-1)*sigma^2))*
exp(-0.5*abs(q)*sqrt(q^2-1)*sigma^2*t)
+0.5*(s0^(q)-2*mu/((1+q)*sigma^2)
+(q*mu-0.5*q*(1-q)*sigma^2*s0^(-p))/
(0.5*abs(q)*sqrt(q^2-1)*sigma^2))

```

```
    *exp(0.5*abs(q)*sqrt(q^2-1)*sigma^2*t)
    +2*mu/((1+q)*sigma^2);
plot(t,E,'+',t,E,':');
title('E[S_t^q] v.s. t');
xlabel('t');
ylabel('E[S_t^q]');
end
end
```

## 0.2. Pricing a knock-out Option with *Time-dependent* Barriers by Monte Carlo Simulation

---

This MATLAB `barriers.m` file generates time-dependent barrier  $B(t)$  and  $b(t)$ .

Inputs:

`t` -represents the time in the period up to maturity  $T$

Outputs:

`B` -represents the time-dependent upper barrier

`b` -represents the time-dependent lower barrier

---

```
function [B,b]=barriers(t)
```

```
B=50-0.5*t.^(1/2);           %time-dependent upper barrier
```

```
b=20+0.5*t.^(1/2);          %time-dependent lower barrier
```

-----  
The MATLAB carlo.m file is used Monte Carlo simulation to price an knock-out option with time-dependent upper barrier  $B(t)$  and lower barrier  $b(t)$ .

Inputs:

T -represents the maturity (i.e.  $T=1$  for one year)  
e.g.  $T=1$

dt -represents the length of time steps  
e.g.  $dt=1/250$ , a day in a year of 250 trading days

sigma -represents the standard deviation  
e.g.  $sigma=0.5$

r -represents the annual risk-free interest rate  
e.g.  $r=0.05$

K -represents strike price  
e.g.  $K=30$

B -represents the time-dependent upper barrier  
e.g.  $B=50-0.5*t.^{(1/2)}$

b -represents the time-dependent lower barrier  
e.g.  $b=20+0.5*t.^{(1/2)}$

Output:

c -represents the price of the knock-out option

-----

```
function carlo(T,dt,sigma,r,K,B,b)
```

```
s(1)=30;          %initial asset price
```

```
for n=1:100000
```

```
    for t=1:250
```

```
        z=randn();
```

```
        if (s(t)>B(t) || K>B(t) || s(t)<b(t))
```

```
            disp('choose correct s,B,k,b to satisfy conditions  $B \geq s \geq b, K < B$ ');
```

```
            return;
```

```
else
    s(t);
end
s(t+1)=s(t)+(s(t)^p)*(r*dt+sigma*z*sqrt(dt));
end
c(n)=exp(-r*T)*max(s(251)-K,0);
end
c=mean(c);
```



### 0.3. Pricing a knock-out Option with *Time-independent* Barriers by Markov Chain Approximation

---

This MATLAB ms.m file generates the mean (m) and the standard deviation (std) of discretely monitored stock prices.

Inputs:

r        -represents the annual risk-free interest rate  
sigma -represents the standard deviation  
s        -represents the sequence of discrete asset prices  
dt       -represents the length of time steps  
          e.g. dt=1/250, a day in a year of 250 trading days

Outputs:

m        -represents the mean of the normal distribution  
std      -represents the standard deviation of the normal  
          distribution

---

```
function [m,std]=ms(r,sigma,s,dt,p)
m=[ ];
std=[ ];
n=length(s);

for i=1:n
    if s(i)<0
        disp('choose positive value for asset price')
        return;
    else
        m(i)=s(i)^p*r*dt;
        std(i)=s(i)^p*sigma*sqrt(dt);
    end
end
end
```

-----  
This MATLAB generatemat.m file is used Markov process property  
to generate a transition probability matrix P.

Inputs:

s     -represents the sequence of discrete asset prices  
m     -represents the mean of the normal distribution  
std   -represents the standard deviation for the normal  
      distributiion

Output:

P     -represents the transition matrix  $P(s(t+1)=j|s(t)=i)$   
      Given condition  $s(t)=i$ , finding the probability of  $s(t+1)=j$

-----  
function P=generatemat(s,m,std)

P=[ ];

if std<=0

    disp('error in sigma')

    return;

end

n=length(s);

for i=1:n-1

    u(i)=(s(i)+s(i+1))/2;

end

for i=1:n

    cdf=[0,normcdf(u-s(i),m(i),std(i)),1];

    P(i,:)=diff(cdf);

end

-----  
This MATLAB index.m generates the index location of the initial asset price among discretely monitored asset prices.

Input:

s        -represents the sequence of discrete asset prices

Output:

s0       -represents the index location of the initial asset price  
-----

```
function k=index(s,s0)
```

```
n=length(s);
```

```
for i=1:n
```

```
    if (s(i)==s0||(s(i)<s0 && s(i+1)>s0))
```

```
        k=i;
```

```
    elseif (s0<s(1)||s0>s(n)||s0<0)
```

```
        disp('choose positive s0 within the bound')
```

```
    return;
```

```
end
```

```
end
```

---

This MATLAB knockconst.m file is used Markov chain approximation to price an European call (knock-out) option with time-independent upper barrier B and lower barrier b.

Inputs:

s -represents the sequence of asset prices (e.g. s=20:0.05:50)  
r -represents the annual risk-free interest rate (e.g. r=0.05)  
sigma -represents the standard deviation (e.g. sigma=0.5)  
s0 -represents the initial stock price (e.g. s0=30)  
K -represents strike price (e.g. K=30)  
T -represents the maturity (i.e. T=1 for one year)  
dt -represents the length of time step  
(e.g. dt=1/250, a day in a year of 250 trading days)  
p -represents the parameter in the power of asset prices

Output:

c -represents the price of the knock-out option

---

```
function c=knockconst(s,r,sigma,T,dt,s0,K,p)
disp('pricing knock-out option with CONSTANT barriers B and b');

[m,std]=ms(r,sigma,s,dt,p);
P=generatemat(s,m,std);
k=index(s,s0);
phi=zeros(size(s));
phi(k)=1;
nsteps=T/dt;

P(1,1)=1;           %lower barrier boundary
P(1,2:end)=0;
P(end,end)=1;      %upper barrier boundary
```

```
P(end,1:(end-1))=0;
ST=phi*P^nsteps;

c=0;
for i=1:length(s)
    c=c+max(s(i)-K,0)*ST(i)*exp(-r*T);    %call price
end
```

#### 0.4. Pricing a knock-out Option with *Time-dependent* Barriers by Markov Chain Approximation

---

This MATLAB barriers.m file generates time-dependent barrier  $B(t)$  and  $b(t)$ .

Inputs:

t -represents the time in the period up to maturity T

Outputs:

B -represents the time-dependent upper barrier

b -represents the time-dependent lower barrier

---

```
function [B,b]=barriers(t)
```

```
B=50-0.5*t.^(1/2);           %time-dependent upper barrier
```

```
b=20+0.5*t.^(1/2);          %time-dependent lower barrier
```

---

This MATLAB findupperindex.m file finds the index location of time-dependent upper barrier  $B(t)$  in the transition probability matrix  $P$

Inputs:

$s$  -represents the sequence of discrete asset prices  
 $B$  -represents the time-dependent upper barrier  $B(t)$

Output:

$iB$  -represents the index location of the upper barrier  $B(t)$

---

```
function iB=findupperindex(s,B)
```

```
n=length(B);
```

```
for k=1:n
```

```
    iB(k)=index(s,B(k));
```

```
end
```

---

This MATLAB findlowerindex.m file finds the index location of time-dependent upper barrier  $b(t)$  in the transition probability matrix  $P$

Inputs:

$s$  -represents the sequence of discrete asset prices  
 $b$  -represents the time-dependent lower barrier  $b(t)$

Output:

$ib$  -represents the index location of the lower barrier  $b(t)$

---

```
function ib=findlowerindex(s,b)
```

```
n=length(b);
```

```
for k=1:n
```

```
    ib(k)=index(s,b(k));
```

```
end
```



-----  
This MATLAB ms.m file generates the mean (m) and the standard deviation (std)  
of discretely monitored asset prices.

Inputs:

r        -represents the annual risk-free interest rate  
sigma -represents the standard deviation  
s        -represents the sequence of discrete asset prices  
dt       -represents the length of time steps  
          e.g. dt=1/250, a day in a year of 250 trading days

Outputs:

m        -represents the mean of the normal distribution  
std      -represents the standard deviation of the normal  
          distribution

-----  
function [m,std]=ms(r,sigma,s,dt,p)

m=[ ];

std=[ ];

n=length(s);

for i=1:n

  if s(i)<0

    disp('choose positive value for stock price')

    return;

  else

    m(i)=s(i)^p\*r\*dt;

    std(i)=s(i)^p\*sigma\*sqrt(dt);

  end

end

-----  
This MATLAB generatemat.m file is used Markov process property  
to generate a transition probability matrix P.

Inputs:

s     -represents the sequence of discrete asset prices  
m     -represents the annual risk-free interest rate  
std   -represents the standard deviation for the normal  
       distributiion

Output:

P     -represents the transition matrix  $P(s(t+1)=j|s(t)=i)$   
       Given condition  $s(t)=i$ , finding the probability of  $s(t+1)=j$

-----  
function P=generatemat(s,m,std)

P=[ ];

if std<=0

    disp('error in sigma')

    return;

end

n=length(s);

for i=1:n-1

    u(i)=(s(i)+s(i+1))/2;

end

for i=1:n

    cdf=[0,normcdf(u-s(i),m(i),std(i)),1];

    P(i,:)=diff(cdf);

end

-----  
This MATLAB index.m generates the index location of the initial stock price among discretely monitored asset prices.

Input:

s        -represents the sequence of discrete asset prices

Output:

s0       -represents the index location of the initial asset price  
-----

```
function k=index(s,s0)

n=length(s);

for i=1:n
    if (s(i)==s0||(s(i)<s0 && s(i+1)>s0))
        k=i;
    elseif (s0<s(1)||s0>s(n)||s0<0)
        disp('choose positive s0 within the bound')
    end
end
return;
end
end
```

---

This MATLAB onestep.m file generates new transition probability matrix after being replicated by upper-left submatrix  $i_b$  for lower barrier  $b(t)$  and lower-right submatrix  $i_B$  for upper barrier  $B(t)$ . Both submatrices are identity matrices and their diagonal entries are the probabilities of the Markov chain reaching absorbing states.

Inputs:

P -represents the transition probability matrix generated by  
P=generatemat(s,m,std)  
 $i_b$  -represents the upper-left submatrix for lower barrier  
 $i_B$  -represents the lower-right submatrix for upper barrier

Output:

Pt -represents the transition probability matrix after being  
replicated by submatrices  $i_b$  and  $i_B$

---

```
function Pt=onestep(P,ib,iB)
Pt=P;
[n,n]=size(P);
Pt(1:ib,:)=eye(ib),zeros(ib,n-ib)];
Pt(iB:end,:)=zeros(n-iB+1,iB-1),eye(n-iB+1)];
```

---

This MATLAB knockvar.m file is used Markov chain approximation to price an European call (knock-out) option with time-dependent upper barrier  $B(t)$  and lower barrier  $b(t)$ .

Inputs:

s -represents the sequence of asset prices (e.g.  $s=20:0.05:50$ )  
ds -represents the interval of asset prices (e.g.  $ds=0.5$ )  
r -represents the annual risk-free interest rate (e.g.  $r=0.05$ )  
sigma -represents the standard deviation (e.g.  $\sigma=0.5$ )  
s0 -represents the initial asset price (e.g.  $s_0=30$ )  
K -represents strike price (e.g.  $K=30$ )  
T -represents the maturity (i.e.  $T=1$  for one year)  
dt -represents the length of time step  
(e.g.  $dt=1/250$ , a day in a year of 250 trading days)  
p -represents the parameter in the power of discrete prices

Output:

c -represents the price of the knock-out option

---

```
function c=knockvar(dt,T,s,ds,sigma,s0,K,r,p)
disp('pricing knock-out option with VARIABLE barriers B(t) & b(t)');

t=0:T/dt;
[B,b]=barriers(t);           %time-dependent barriers B(t) and b(t)
s=min(b):ds:max(B);
ib=findlowerindex(s,b);
iB=findupperindex(s,B);

[m,std]=ms(r,sigma,s,dt,p);
P=generatemat(s,m,std);     %transition probability matrix
```

```

num=index(s,s0);
phi=zeros(size(s));

phiB=phi;
phi(num)=1;
nt=length(t);           %time steps
phiBt=zeros(1,nt);

for i=1:nt
pt=onestep(P,ib(i),iB(i));
pt=phi*pt;
block=0;
for j=1:length(s)
    if ib(i)<j&&j<iB(i)
        phi(j)=+pt(j);           %probabilities for transient states
        phiB(j)=0;
    else
        phiB(j)=pt(j);           %probabilities for absorbtion states
        phi(j)=0;
        block=block+phiB(j);
    end
end
phiBt(i)=block;           %absorbption probability at time i
end

c=0;
for g=1:length(s)
    c=c+max(s(g)-K,0)*phi(g)*exp(-r*T);   %option price
end

```