CRYPTOCURRENCY, SECURITY
AND MONETARY POLICY

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ABSTRACT

The effect of cryptocurrency security protocol on the medium of exchange properties were studied. Notably, cryptocurrencies make use of cryptography as holders of coins store their currencies such as Bitcoin at an address on a decentralized ledger that is linked to both a private and public key. If the device storing a private key is lost or damaged, the coins will be lost permanently. Consequently, the currency loss creates conditions where private money circulates. It is also likely to impact whether the supply of private money over time would be socially efficient.

Intermediaries can be valuable firms that help individuals avoid uncertainty from holding cryptocurrency by providing safe-keeping of tokens. We consider that there may be different types of intermediaries who offer different services including safe-keeping and providing transactions services. Notably, we find financial intermediation improves the volume of decentralized trade regardless of the type of bank. However, the largest gains occur when banks only provide safe-keeping for tokens that are borrowed. With banks providing safekeeping for all tokens, token loss will be eliminated but a steady-state equilibrium only exists when total token production is capped.

Lastly, in order to study the effect of monetary policy on price movements of cryptocurrency, we use a standard Vector Autoregression (VAR) framework to identify shocks to the stance of monetary policy. We find that positive shocks to the size of the money stock cause the prices of both Bitcoin and Ethereum to increase. From this

perspective, our results do indicate that investors have sought out access to cryptocurrencies as central banks have adopted easy monetary policies since the end of the Great Financial Crisis.

DEDICATION

This thesis is dedicated to everyone who helped me and guided me through the trials and tribulations of creating this manuscript. In particular, my Mom and Dad for their continued support, and to my good friends Robert and Erika.

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CHAPTER 1

INTRODUCTION

In 2008, shortly after the financial crisis, Satoshi Nakomoto introduced Bitcoin as an alternative to the centralized financial system. Bitcoin is a type of digital currency known as a cryptocurrency. Cryptocurrencies make use of a decentralized ledger known as a blockchain. 7,000 different cryptocurrencies have been established since then, but Bitcoin still remains the most popular. Consequently, an academic literature studying cryptocurrency has also emerged. In particular, there has been work considering whether privately created outside monies with Bitcoin's properties could circulate in equilibrium. However, no one has investigated what impact the unique security features of a cryptocurrency has on its ability to function as a medium of exchange.

A unique feature of cryptocurrency is that such forms of money make use of cryptography where coin holders store their bitcoin at a public address on a decentralized ledger that is linked to both a private and public key. As the identity of the individual behind the public address is unknown, public key cryptography is necessary for network verification. The public address is similar to a pen name in that it's a pseudonym. When a transfer request from the address is made to the Bitcoin network, the crypto holder uses the private key to digitally sign the message containing the request. In order to complete the transfer, the Bitcoin network verifies the digital signature with the public key. The verification process illustrates the importance of the security of the private key.

In order to prevent hackers from gaining access to their private keys, crypto owners often store their private key on a hardware wallet offline in "cold storage". The hardware wallet is a type of flash drive that performs the function of a crypto wallet. For example, the Bitcoin wallet is responsible for generating the random private key that is a 256-bit number identified by 64 characters in the range 0-9 or A-F. The wallet then uses the public key, which is mathematically related to the private key, to create the public address. However, if the hardware wallet is lost, damaged or the passcode to the device is forgotten, access to the tokens is permanently lost. The losses are far from trivial. In fact, recent estimates from Chainalysis indicates that as of 2017, nearly 4 million bitcoins alone have been lost which is around one-quarter of the aggregate supply of existing bitcoins.

In the first chapter, we study the impact of permanent coin loss on the equilibrium properties of cryptocurrency in a modern model of monetary exchange. We are able to show how the circulation of private money depends on the shape of the money producer's cost function. Additionally, we show that a stable price equilibrium can exist even if the costs are strictly convex and equal to zero at the origin as new coins need to be produced in order to offset the amount of tokens that are lost over time. Moreover, we discuss how the relationship between the price of coins and the degree of token loss also depends on the shape of the cost function. To be specific, if the cost function is strictly convex, the price of tokens adjusts to the rate of token loss along with the equilibrium money stock. Under linear costs, the money stock entirely absorbs the degree of loss.

Previously, the existence of a stable price steady state was not able to be shown to exist with a privately produced currency unless there was an incentive in place to limit production. Due to the token loss, this is not necessary in our model with tokens being removed from the money supply each period. However, money holders take account of the negative consequences that stem from the potential to lose a portion of their currency. As a result, an efficient return on the currency is not possible without a central authority removing an additional portion of the money supply.

In the next chapter, we consider the impact of market based institutions on the efficiency of cryptocurrency as a medium of exchange. Consequently, the OCC recently extended to banks the same authority over cryptocurrency: "Because national banks are authorized to perform safekeeping and custody services for physical assets, national banks are likewise permitted to provide those same services via electronic means (i.e., custody of cryptocurrency.)" Further, they stress that these services could be extended and intertwined with other important functions of intermediaries such as the extension of credit: "By providing such services, banks can continue to fulfill the intermediation function they have historically played in providing payment, loan, and deposit services."

Consequently, as economists continue to think about the larger impact of cryptocurrency in the global financial system, they cannot ignore the role that they will eventually play in the traditional, deposit-based banking system. That is, economists need to develop frameworks where institutions bank in terms of cryptocurrencies such as bitcoin. In particular, recently crypto credit markets such as Bitfinex and Bitbond have been established in which borrowing and lending in terms of bitcoin has emerged. It is not hard to envision that deposit-based intermediaries which have long played an essential function in terms of the extension of credit will begin to do the same. That is, economists need to develop frameworks in which individuals do not engage in borrowing and lending in outside money issued by sovereign governments but instead by profit-maximizing issuers of digital fiat money. Such loans are not subject to an "inflation tax" where revenues may be redistributed back into the economy — rather they are in terms of assets that are subject to permanent loss due to separation of holders from their private keys. For this reason, central banks also need to begin to consider how borrowing and lending in terms of cryptocurrency may be offered by traditional intermediaries.

In light of these important issues, we seek to answer the following questions. How would the provision of safekeeping services by banks – as has been just recently allowed in the United States – affect the circulation and value of cryptocurrencies? Further, how is financial market activity affected if banks both provide safekeeping and extend credit in terms of cryptocurrency? What is the overall impact of financial intermediation if banks simultaneously provide security, intermediate between borrowers and lenders, and conduct transactions/payments services?

To do so, we introduce microfounded financial intermediaries in an environment with digital money that is subject to permanent loss as in the case of cryptocurrency. We refer to the intermediaries in the framework as "banks." As banks may provide different levels of credit extension, safekeeping, and payments processing, we consider that there are three different types of banks in the model. To begin, we start the analysis with type I banks that only intermediate between borrowers and lenders – they do not provide safekeeping services. Meaning, the borrower is responsible for the security of the cryptocurrency. We proceed to look at type II banks who provide safekeeping over tokens that are borrowed, but not over an individual's other money holdings. Finally, we look at type III banks who process all transactions between buyers and sellers. Consequently, in this setup, token loss would never occur.

Notably, we find financial intermediation improves the volume of decentralized trade regardless of the type of bank. However, the largest gains occur when banks only provide safe-keeping for tokens that are borrowed. With banks providing safekeeping for all tokens, token loss will be eliminated but a steady-state equilibrium only exists when total token production is capped. Now we turn to the last chapter and look at potential causes of cryptocurrency price movements.

As previously mentioned, Satoshi released Bitcoin in the wake of the financial crisis when there was a growing concern and distrust of central banks. That is, there was a growing concern of high levels of inflation as a result of aggressive central bank actions. With interest in cryptocurrency at an all-time high, the objective of this last chapter is to study how central bank activity affects price movements for Bitcoin and Ethereum – the two largest circulating cryptocurrencies. In particular, we look at the impact of aggregate monetary shocks on the price of Bitcoin and Ethereum. To do so, we employ a structural vector autogression (VAR) and identify monetary shocks as othorgonalized innovations to the federal funds rate. However, our focus is on the impact of such shocks on the price of cryptocurrency rather than aggregate real economic variables such as GDP. In particular, we use monthly data from September 2015 to December 2020 to look at the response of Bitcoin and Ethereum to various types of monetary shocks. In particular, we find that positive shocks to the size of the money stock cause the prices of both Bitcoin and Ethereum to increase. From this perspective, our results do indicate that investors have sought out access to cryptocurrencies as central banks have adopted easy monetary policies since the end of the Great Financial Crisis.

CHAPTER 2

CRYPTOCURRENCY AND CURRENCY LOSS

I. INTRODUCTION

In recent years, there has been increased enthusiasm regarding the viability of cryptocurrencies. For example, the value of Bitcoin, the cryptocurrency with the largest circulation, has recently been established near $60,000. In addition, there has been significant appreciation of other cryptocurrencies such as Ethereum and Dogecoin in the past several years. Consequently, an academic literature studying cryptocurrency has also emerged. In particular, the ideas put forward by Hayek (1999) are often invoked when considering whether privately created outside monies could circulate in equilibrium.

In terms of thinking about the viability of cryptocurrency, there has also been increasing awareness of the costs of producing such forms of money. In this sense, the costs of producing digitally issued money can be significantly higher than pieces of paper issued by sovereign governments. Notably, a recent contribution by Fernandez-Villaverde and Sanches (2019, hereafter, FVS) has emphasized how the technologies for producing digital money play a significant role in determining whether private forms of money would exist. In particular, their framework shows that the shape of the cost function determines the relationship between equilibrium prices and the incentive to produce money over time. To be specific, an equilibrium with stable prices can only exist if the cost function is strictly increasing and locally linear around the origin. That is, if the marginal cost of producing tokens is equal to zero, there would be incentives to produce money until it eventually becomes worthless.

Yet, in addition to the relatively unique features of the costs of producing digital money, cryptocurrencies are also embedded with distinctive security features. For example, a special feature of cryptocurrency is that such forms of private outside money are based on cryptography – holders of coins store their currencies such as Bitcoin at an address on a decentralized ledger that is linked to both a private and public key.[1] As the identity of the individual behind the public address is unknown, public key cryptography is necessary for network verification. Similar to a pen name, the public address acts as a pseudonym for the owner of the cryptocurrency. In order to transfer Bitcoin from an address, the crypto holder must use the private key to digitally sign the message containing the request to send Bitcoin.[2] The Bitcoin network completes the transfer once the signature is verified by the public key. The verification process illustrates how possession of the private key allows for control of the corresponding address's funds. Moreover, security of the private key is critical to the use of cryptocurrency.

Interestingly, Kahn et al. (2020) study the degree of attention that is required for security of a private key. In particular, crypto owners store their private key in what is known as a wallet – wallets can be broadly placed in two categories. First, a hot wallet stores the private keys on a device that is connected to the internet. Holding the private keys in hot storage provides more convenience due to easier access, but this accessibility

[1] The private key is known as the signing key and is only known by the address owner. In contrast, the public key used for verification is made available to the Bitcoin network.

[2] A good analogy comes from the 80's film "The Scarlet Pimpernel". While his deeds are known, the identity of the hero is unknown to the public. In order to verify his communications, he signs his letters with a wax insignia shaped like a scarlet pimpernel.

comes with a cost. Due to a connection to a network, the private keys are always at risk from an external web-based attack. In order to prevent this, crypto owners often use a second type of wallet. That is, Bitcoin owners store their private key on a hardware wallet offline in "cold storage".[3] However, this comes with the downside of requiring extra effort by the coin holder. Moreover, if the hardware wallet is lost, damaged or the passcode to the device is forgotten, access to the tokens is permanently lost.[4] The losses are far from trivial. In fact, recent estimates from Chainalysis[5] indicates that as of 2017, nearly 4 million bitcoins alone have been lost which is around one-quarter of the aggregate supply of existing bitcoins.[6] This aspect of many cryptocurrencies is typically omitted from models such as FSV and Martin and Schreft (2006) who study whether privately issued outside money would circulate over time. Consequently, failing to account for permanent coin loss may fundamentally alter the conditions where private money circulates. It is also likely to impact whether the supply of private money over time would be socially efficient.

In light of this gap in the existing literature, the objective of this paper is to study how exogenous token loss affects the conditions where privately produced money circulates. To do so, we include the potential to lose tokens in the model of FVS. As in FVS, we are able

[3]The hardware wallet is a type of flash drive that performs the function of a crypto wallet. For example, the Bitcoin wallet is responsible for generating the random private key that is a 256-bit number identified by 64 characters in the range 0-9 or A-F. The wallet then uses the public key, which is mathematically related to the private key, to create the public address.

[4]Due to the pseudoanomyity and the decentralized nature of crypto networks, it is not possible to go to a trusted third party and retrieve your private key or gain access to your money with an ID.

[5]Chainalysis is a crypto forensics company based in New York that will often assist Interpol and U.S. law agencies in tracking down cyber criminals. For a discussion on their relevance to current events, see "The rise of crypto laundries: how criminals cash out of bitcoin." (Murphy, Financial Times, May 28, 2021).

[6]For example, the CEO of the crypto exchange QuadrigaCX died and took with him any knowledge of the private keys to $190 million in several different types of cryptocurrency. In addition, Stefan Thomas, a German programmer based in San Francisco, forgot the password that unlocked his small hard drive, called the IronKey, that would allow him to access 7,002 bitcoin, which was worth $220 million. For additional discussion, see "Lost Passwords Lock Millionaires Out of Their Bitcoin." (Popper, New York Times, January 12, 2021).

to show how the circulation of private money depends on the shape of the cost function. Yet, in stark contrast to FVS, we show that a stable price equilibrium can exist even if the costs are strictly convex and equal to zero at the origin as new coins need to be produced in order to offset the amount of tokens that are lost over time. Moreover, we discuss how the relationship between the price of coins and the degree of token loss also depends on the shape of the cost function. To be specific, if the cost function is strictly convex, the price of tokens adjusts to the rate of token loss along with the equilibrium money stock. Under linear costs, the money stock entirely absorbs the degree of loss.

The remainder of the paper proceeds as follows. In the next section, we construct a model to show the effects of currency loss. In order to replicate the issue of crypto holders losing access to their private key, we introduce the potential to lose tokens in the model. In this context, we show that a stable-price equilibrium can exist with no cap on production of tokens. Next, we examine whether a stable price equilbrium can exist if there is a cap on token production. We proceed to conduct welfare analysis and then look to examine conditions where government money can displace privately produced outside money that is susceptible to token loss.

II. MODEL

There is a [0,1] set of infinitely-lived consumers who consume and produce goods at different points in discrete time. Further, individuals meet in both a centralized market (CM) and a decentralized market (DM). Consumers also have the ability to transform labor into a perishable good known as the CM good through the use of a linear technology. In addition to consumers, following FVS, there are also entrepreneurs (miners) who create tokens. In the first subperiod, both types of agents interact in the CM and consume the CM good.

In the second sub-period, consumers meet in pairs in the DM but the entrepreneurs are idle. During this time, consumers produce differentiated goods which are also divisible. Due to their specialized tastes and production capabilities, it will never be the case that two consumers would desire what the other produces. This friction creates a role for a medium of exchange. First, a consumer has the ability to produce a good for the other individual with probability $\sigma \in (0, 1/2)$ and will therefore be a seller. Second, there is also the probability $\sigma \in (0, 1/2)$ that he will be a buyer in a given match. Lastly, with probability $(1 - 2\sigma)$, neither individual would desire to trade.

By comparison, entrepreneurs have access to a technology that can create tokens. These tokens can come in either physical or digital form. Counterfeiting is not an issue with the tokens as they can be verified publicly at zero cost. Entrepreneurs create identical tokens, and bear the same costs of production $c : \mathbb{R}_+ \to \mathbb{R}_+$ for minting tokens. We refer to the entrepreneurs as miners when the tokens created are cryptocurrency. In the case of miners, the cost comes from the miners solving a complex computational problem that requires input. For example, the cryptocurrency will have a protocol that determines the amount of work required to solve the problem. To capture these costs, the cost function is strictly increasing and weakly convex: $c'(\Delta) > 0, c''(\Delta) \geq 0$, and $c(0) = 0$. The entrepreneurs will create these tokens to maximize profits which enable trade between buyers and sellers in the DM.

Let $x_t \in \mathbb{R}$ denote the consumer's net consumption of the CM good, and $q_t \in \mathbb{R}_+$ denote consumption of the DM good. In addition, $n_t \in \mathbb{R}_+$ reflects the effort to produce the DM good. Thus, an individual consumer's preferences are denoted by

$$U(x_t, q_t) = x_t + u(q_t) - w(n_t).$$

Note that $u : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, increasing, and strictly concave: $u'(\cdot) > 0$, $u''(\cdot) < 0$, $u'(0) = \infty$, and $u(0) = 0$. Also, $w : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, increasing, and weakly convex: $w'(\cdot) > 0$, $w''(\cdot) \geq 0$, and $w(0) = 0$. With regards to $u(q)$ and $w(q)$, one of the inequalities of the second derivative must be strict. All consumers have the same discount factor $\beta \in (0, 1)$ between dates t and $t + 1$.

By comparison, $x_t^e \in \mathbb{R}_+$ denotes an entrepreneur's consumption of the CM good, and $\Delta_t \in \mathbb{R}_+$ is the production of tokens. The preferences of an entrepreneur are denoted by

$$U^e(x_t^e, \Delta_t) = x_t^e - c(\Delta_t)$$

Meetings in the DM are anonymous, precluding any chance of credit. This friction, along with the double coincidence of wants problem, makes a medium of exchange necessary to facilitate trade. Therefore, although a token is intrinsically worthless, it obtains its value from use as a medium of exchange.

Since the entrepreneurs' only concern is maximizing profits, their behavior can be predicted by other agents. This allows agents to form expectations about the future money supply and use this to determine the relative price level.

CONSUMERS

The value function for the consumer in the CM is denoted by $W(M^c_{t-1}, t)$. The agent will start period t with $M^c_{t-1} \in \mathbb{R}_+$ in his portfolio. Further, $V(M^c_t, t)$ denotes the consumer's value function in the DM. The Bellman equation for the consumer in the CM is

$$W\big(M^c_{t-1}, t\big) = \max_{x_t, M^c_t} \big[x_t + V(M^c_t, t)\big]$$

subject to:

$$\phi_t \cdot M^c_t + x_t = \phi_t \cdot M^c_{t-1}$$

The budget constraint rearranges to:

$$x_t = \phi_t \cdot M^c_{t-1} - \phi_t \cdot M^c_t$$

Substituting into the Bellman Equation:

$$W\big(M^c_{t-1}, t\big) = \max_{M^c_t} \big\{\phi_t \cdot M^c_{t-1} - \phi_t \cdot M^c_t + V(M^c_t, t)\big\}$$

Next, $W(0, t)$ below represents the value function for the consumer at the end of the first subperiod. The agent will choose between the disutility from producing goods and acquiring tokens in the CM versus the value of tokens for use in the DM:

$$W(0, t) = \max_{M^c_t} \big[- \phi_t \cdot M^c_t + V\big(M^c_t, t\big)\big] \tag{1}$$

In contrast to FVS, we introduce the potential for consumers to lose tokens in the model. For example, this could be due to an individual losing access to their private key. The loss of the private key would occur in the DM before they match up with another consumer. Once in the DM, the tokens will be lost with a probability of $\psi \in (0,1)$. We also let $\overline{q}$ represent the anticipated amount of the consumption good to be traded in a match in the DM. In addition, $\overline{d}$ denotes the transfer of tokens to a seller.

Below we present the value function for a consumer at the beginning of the DM. First, with probability ψ, the individual loses his tokens. Further, with probability σ, the individual would like to be a buyer but does not have any tokens to give up in exchange. As a result, consumption will not occur and the individual will return to the CM in the following period. Moreover, with probability $\sigma\psi$, the consumer would have the ability to produce a good for a buyer but the buyer has also lost his tokens. In either scenario, the consumer will return to the CM empty-handed.

$$
\begin{aligned}
V(M_t^c,t) = \psi\{ & \underbrace{\sigma(u(0)+\beta W(0,t+1))}_{match \quad w/ \quad seller} + \underbrace{\sigma\psi[-w(0)+\beta W(0+0,t+1)]}_{potential \quad buyer \quad lost \quad currency} + \\
& \underbrace{\sigma(1-\psi)[-w(\overline{q})+\beta W(0+\overline{d},t+1)}_{successful \quad match \quad w/ \quad buyer} + \underbrace{(1-2\sigma)\beta W(0,t+1)}_{no \quad match} \} \\
& +(1-\psi)\{ \underbrace{\sigma[u(q)+\beta W(M_t^c-d)]}_{match \quad w/ \quad seller} + \underbrace{\sigma\psi[-w(0)+\beta W(M_t^c+0)]}_{potential \; buyer \; lost \; currency} \\
& \underbrace{\sigma(1-\psi)[-w(\overline{q})+\beta W(M_t^c+\overline{d})]}_{successful \quad match \quad w/ \quad buyer} + \underbrace{(1-2\sigma)\beta W(M_t^c,t+1)}_{no \quad match} \}
\end{aligned}
\tag{2}
$$

However, with probability $\sigma(1-\psi)$, the consumer produces a good for a buyer that has not lost their tokens. The consumer will take the tokens given in exchange for the DM goods produced into the next period. Lastly, with probability (1-2σ), the consumer will not be a buyer or seller in the DM. Thus, the consumer will not consume a DM good and would go to the next period without any tokens.

Alternatively, with a probability of $(1 - \psi)$, the consumer does not lose his tokens. Conditional on his money holdings, with a probability of σ, the individual is a buyer and transfers his tokens over to the seller in exchange for the seller's DM good. As a consequence, consumption of the DM good occurs and the individual will not take any tokens into the next period. In addition, with probability $\sigma\psi$, the consumer is capable of producing a good for a buyer but the buyer has lost his tokens. As a result, the consumer will then enter the next period with only his tokens in hand.

With probability $\sigma(1 - \psi)$, the consumer produces a good for a buyer in exchange for tokens. This results in the individual bringing his and the buyer's tokens into the following period. Lastly, with probability $(1 - 2\sigma)$, the consumer will not be a buyer or seller. As a result, consumption does not take place and the individual goes into the next period with his tokens.

Simplifying instances where no trade occurs produces the equation below. Notably, in situations where the individual loses his tokens, we can combine all the instances where no trade occurs. This results in zero utility for the consumer, and such possibilities in (2) drop out entirely. With a probabiliy of $(1 - \sigma(1 - \psi))$, no trade occurs in the DM. Therefore, no consumption occurs and the consumer enters the next period with no tokens.

$$V(M_t^c,t)=\psi\{\underbrace{\sigma(1-\psi)[-w(\overline{q})+\beta W(0+\overline{d},t+1)]}_{successful\quad match\quad w/\quad buyer}+\underbrace{(1-\sigma(1-\psi))\beta W(0,t+1)}_{no\quad money\quad taken\quad to\quad next\quad period}\}+ \tag{3}$$

$$(1-\psi)\{\underbrace{\sigma[u(q)+\beta W(M_t^c-d)]}_{match\quad w/\quad seller}+$$

$$\underbrace{\sigma(1-\psi)[-w(\overline{q})+\beta W(M_t^c+\overline{d},t+1)]}_{successful\quad match\quad w/\quad buyer}+\underbrace{(1-\sigma(1+(1-\psi)))\beta W(M_t^c,t+1)}_{no\quad consumption,\quad takes\quad money\quad to\quad next\quad period}\}$$

Thus, (2) simplifies to (3) above which only includes scenarios in which one of the consumers does not lose their tokens and trade occurs. Further, with the same probability of $\sigma(1-\psi)$, the consumer produces a good for the buyer and receives tokens from the buyer. As a result, the individual enters the next period with the buyer's tokens.

By comparison, we only eliminate the meetings where the potential buyer has no currency if the consumer does not lose his tokens. In this particular case, no production occurs and the consumer does not gain any tokens. Thus, he will head into the next period with his tokens only with a probability of (1-$\sigma(1+(1-\psi))$). Identically before simplification, with probability $\sigma(1-\psi)$, the consumer produces a good for a buyer in exchange for tokens. Additionally, with a probability of σ, the individual buys the DM good from a seller in exchange for his money holdings. Further, if $\psi=0$, one obtains the standard probability of $(1-2\sigma)$.

We use Nash bargaining to determine the terms of trade between the two consumers in the DM. We let $\theta\in[0,1]$ be the bargaining power of the consumer when he acts as a buyer. The terms of trade $(q,d)\in\mathbb{R}_+^2$ in the DM are determined by solving

$$\max_{(q,d)\in\mathbb{R}^2_+} \big[u(q) - \beta \cdot \phi_{t+1} \cdot d\big]^{\theta} \big[- w(q) + \beta \cdot \phi_{t+1} \cdot d\big]^{1-\theta} \tag{4}$$

subject to the participation constraint of each agent.

Next, we have the buyer's participation constraint:

$$u(q) - \beta \cdot \phi_{t+1} \cdot d \geq 0$$

And, the seller's constraint:

$$-w(q) + \beta \cdot \phi_{t+1} \cdot d \geq 0,$$

Finally, we have the buyer's liquidity constraint:

$$d \leq M_t^c,$$

The amount of production in the DM satisfies:

$$u'(q) = w'(q) \tag{5}$$

where q^* satisfies $u'(q^*) = w'(q^*)$, so that there is a socially efficient level of production in DM if the liquidity constraint does not bind. Next, we find d when the liquidity constraint is not binding:

$$(1-\theta)u(q^*) + \theta w(q^*) = \beta\phi_{t+1} \cdot d \tag{6}$$

That is, the discounted real value of the total tokens to be exchanged is a weighted average of the buyer's utility and the seller's disutility. If the buyer has more bargaining power, then the value of the tokens depends more on the disutility of the seller since they incur a loss of utility in production.

In summary, if the liquidity constraint does not bind, bargaining power does not affect the amount of production in the DM – it only affects the cost in terms of the value of the tokens exchanged.

Next, we can use the bargaining solution to express the value of money exchanged. The amount of production of the DM good will allow for the expression of money traded in terms of the buyer's utility and the seller's effort. $m : \mathbb{R}_+ \to \mathbb{R}_+$ represents the real value of money holdings. Set $m(q)$ equal to $\beta \cdot \phi_{t+1} \cdot d$, and you have:

$$m(q) \equiv \frac{\theta w(q)u'(q) + (1-\theta)u(q)w'(q)}{\theta u'(q) + (1-\theta)w'(q)}$$

We will let $\beta \cdot \phi \cdot M_0$ represent the real value of the consumer's portfolio in the steady state. When the liquidity constraint is binding, we will have $\beta \cdot \phi_{t+1} \cdot d_t = \beta \cdot \phi_{t+1} \cdot M_t^c$:

$$\beta\phi_{t+1}M_t^c = \frac{\theta w(q)u'(q) + (1-\theta)u(q)w'(q)}{\theta u'(q) + (1-\theta)w'(q)}$$

Thus, q can be obtained from the inverse function:

$$q = m^{-1}(\beta\phi_{t+1}M_t^c)$$

For the general case, the solution to the bargaining problem is given by:

$$q(M_t^c, t) = \begin{cases} m^{-1}(\beta \cdot \phi_{t+1} \cdot M_t^c) & if \\ \phi_{t+1} \cdot M_t^c < \beta^{-1}[\theta w(q^*) + (1-\theta)u(q^*)] & \\ & \\ q* & if \\ \phi_{t+1} \cdot M_t^c \geq \beta^{-1}[\theta w(q^*) + (1-\theta)u(q^*)] & \end{cases}$$

and:

$$\phi_{t+1} \cdot d(M_t^c, t) = \begin{cases} \phi_{t+1} \cdot M_t^c & if \\ \phi_{t+1} \cdot M_t^c < \beta^{-1}[\theta w(q*) + (1-\theta)u(q*)] & \\ & \\ \beta^{-1}[\theta w(q^*) + (1-\theta)u(q^*)] & if \\ \phi_{t+1} \cdot M_t^c \geq \beta^{-1}[\theta w(q^*) + (1-\theta)u(q^*)] & \end{cases}$$

SPECIAL CASE. In order to simplify our analysis, consider a case where the buyer has all the bargaining power $(\theta = 1)$:

$$w(q^*) = \beta\phi_{t+1} \cdot d$$

Because the buyer has all of the bargaining power, he extracts all of the surplus from the transaction in the meeting. The buyer will only transfer enough tokens to just cover the effort from the seller.

In the case when d is binding:

$$\phi_{t+1} \cdot d = \phi_{t+1} M_t^c$$

If the buyer has full bargaining power:

$$m(q) \equiv w(q)$$

and so

$$w(q) = \beta \phi_{t+1} M_t^c$$

Upon solving to get the quantity of the DM good produced in terms of the real value of tokens:

$$\beta \phi_{t+1} M_t^c = w^{-1}(\beta \phi_{t+1} M_t^c) \tag{7}$$

We let q represent the amount of the DM good produced for the buyer. The seller will receive no surplus in this transaction. The seller's effort will be equivalent to the real value of tokens offered. Thus, the amount of the DM good produced by the seller is increasing in the real value of a token until the buyer's liquidity constraint no longer binds.

In the case where the buyer has all the bargaining power ($\theta = 1$), the solution to the bargaining problem for the general cost function becomes:

$$q(M_t^c, t) = \begin{cases} w^{-1}(\beta \cdot \phi_{t+1} \cdot M_t^c) & if \\ \phi_{t+1} \cdot M_t^c < \beta^{-1} w(q^*) & \\ & \\ q^* & if \\ \phi_{t+1} \cdot M_t^c \geq \beta^{-1} w(q^*) & \end{cases}$$

and

$$\phi_{t+1} \cdot d(M_t^c, t) = \begin{cases} \phi_{t+1} \cdot M_t^c & if \\ \phi_{t+1} \cdot M_t^c < \beta^{-1} w(q^*) & \\ & \\ \beta^{-1} w(q^*) & if \\ \phi_{t+1} \cdot M_t^c \geq \beta^{-1} w(q^*) & \end{cases}$$

The solution to the bargaining problem characterizes real expenditures in the DM, $\phi_{t+1} \cdot M_t^c$, as a function of the real value of buyer's portfolio.

PORTFOLIO CHOICE

For the portfolio choice the producer will only need to consider the disutility from producing the CM good to acquire money balances, and the gains from trade when the tokens are not lost. We also include from the value function all the possibilities for the agent in the event his tokens are not lost. This will give the agent all the information he needs to determine the optimal amount of tokens to carry into the DM.

$$-\phi_t M_t^c+(1-\psi)\Big\{\sigma[u(q)+\beta W(M_t^c-d(M_t^c,t))]+\sigma[(1-\psi)(-w(\overline{q})+\beta W(M_t^c+\overline{d}(M_t^c,t+1),t+1)$$

$$+\psi(-w(0)+\beta W(M_t^c+0,t+1))]+(1-2\sigma)\beta W(M_t^c,t+1)\Big\}$$

We can substitute $\beta\phi_{t+1}M_t$ for $W(M_t^c, t+1)$ and simplify:

$$\underbrace{-\phi_t M_t^c}_{value\ \ foregone\ \ in\ \ CM} \quad + \quad \underbrace{\sigma(1-\psi)[u(q)-\beta\phi_{t+1}d]}_{(utility\ \ -\ \ money\ \ transfer)\ \ w/\ \ match} \quad + \quad \underbrace{(1-\psi)\beta\phi_{t+1}M_t^c}_{value\ \ taken\ \ to\ \ next\ \ period} \tag{8}$$

The consumer will maximize utility based on equation (8). First, the agent considers the disutility required to carry tokens out of the CM. Next, the agent takes into account the meetings with agents who produce a good for him. This will only occur if the consumer does not lose his tokens. Thus, with a probability of $\sigma(1-\psi)$ the consumer will use his tokens to purchase goods from another agent. Lastly, the consumer will consider the value of tokens taken into the CM in the next period. This only occurs if the consumer does not lose his tokens with a probability of $(1-\psi)$.

The first-order condition for money balances is:

$$-\phi_t+(1-\psi)\Big\{\sigma(u'(q)\frac{\partial q_t}{\partial M_t^c}-\beta\phi_{t+1})+\beta\phi_{t+1}\Big\}=0$$

We let $A = \phi_{t+1} \cdot M_t^c$ and substitute into the first-order condition along with $\frac{\partial q(M_t^c,t)}{\partial M_t^c} = \frac{\beta\phi_{t+1}}{m'(q(M_t^c,t))}$ to get:

$$\phi_t = \beta\phi_{t+1}\Big(\sigma(1-\psi)\frac{u'(m^{-1}(\beta A))}{m'(m^{-1}(\beta A))} + (1-\sigma)(1-\psi)\Big)$$

If the buyer's liquidity constraint no longer binds, $\sigma(1-\psi)[u(q) - \beta\phi_{t+1}d]$ remains constant with any change in M_t^c. The first-order condition for money balances will now become:

$$\phi_t = \beta\phi_{t+1}(1-\psi)$$

Thus, the optimal choice satisfies:

$$\phi_t = \beta\phi_{t+1}L_\theta(\phi_{t+1} \cdot M_t^c).$$

We can define $\gamma_{t+1} = \frac{\phi_{t+1}}{\phi_t}$ and express in terms of the rate of return

$$1 = \beta\gamma_{t+1}L_\theta(\gamma_{t+1}\phi_t M_t^c)$$

along with the transversality condition

$$\lim_{t\to\infty} \beta^t \cdot \phi_t \cdot M_t^c$$

where $L_\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is given by:

$$L_\theta(A) = \begin{cases} \sigma(1-\psi)\frac{u'(m^{-1}(\beta A))}{m'(m^{-1}(\beta A))} + (1-\sigma)(1-\psi) & if\ A < \beta^{-1}\left(\theta w(q^*) + (1-\theta)u(q^*)\right) \\ 1-\psi & if\ A \geq \beta^{-1}\left(\theta w(q^*) + (1-\theta)u(q^*)\right) \end{cases}$$

This function represents the marginal benefit of taking additional tokens into the DM. The term $\sigma(1 - \psi)\frac{u'(m^{-1}(\beta A))}{m'(m^{-1}(\beta A))}$ represents the utility from consuming the DM good produced by an agent in the DM. This will occur if the liquidity constraint does not bind. If it does bind, no more DM goods will be purchased with an increase in tokens brought to the DM. The only value the additional tokens will have is in the CM next period. This will of course be affected by the probability of loss of currency (ψ). The greater the chance of currency loss, the lower the benefit of taking tokens in the DM. Without the currency, the agent will not have the ability to purchase a DM good and will lose any potential value of holding tokens in the next period.

We will now solve for the portfolio problem for our previous special case. In turn, this allows us to solve for a special case with functional forms. We substitute $\frac{u'(m^{-1}(\beta A))}{m'(m^{-1}(\beta A))} = \frac{u'(w^{-1}(\beta A))}{w'(w^{-1}(\beta A))}$ and $L_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is given by:

$$L_1(A) = \begin{cases} \sigma(1-\psi)\frac{u'(w^{-1}(\beta A))}{w'(w^{-1}(\beta A))} + (1-\sigma)(1-\psi) & if\ A < \beta^{-1}w(q^*) \\ 1-\psi & if\ A \geq \beta^{-1}w(q^*) \end{cases}$$

We continue with the assumption the buyer has all the bargaining power. In addition, let's say the utility from consumption can be defined as $u(q) = 2q^{1/2}$, and $w(q) = q$ represents the cost of producing DM goods. We solve and the special case with functional forms $L_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is given by:

$$L_1(\phi_{t+1}M_t^c) = \begin{cases} \sigma(1-\psi)\frac{1}{\sqrt{\beta\phi_{t+1}M_t^c}} + (1-\sigma)(1-\psi) & if\ \phi_{t+1}M_t^c < \beta^{-1}w(q^*) \\ 1-\psi & if\ \phi_{t+1}M_t^c \geq \beta^{-1}w(q^*) \end{cases}$$

With our functional forms, it becomes obvious that the liquidity benefit from carrying money into the DM decreases in real money holdings.

ENTREPRENEUR'S PROBLEM

We now turn our focus to the supply of tokens in the economy. With no central bank, entrepreneurs determine the quantity of money in the economy. For the entrepreneur's problem, we first denote $M_t \in \mathbb{R}_+$ as the per capita supply of currency in period t. Next, we denote $\Delta_t \in \mathbb{R}$ as the entrepreneur's net circulation of newly minted tokens in period t (mined if new cryptocurrency). Along with the probability of currency loss ψ, we can now denote the law of motion for the total amount of tokens at the beginning of period t by

$$M_t = \Delta_t + M_{t-1} - \psi M_{t-1},$$

The term ψM_{t-1} represent the currency loss from the DM. We denote the entrepreneur's money holdings with $M_t^e \in \mathbb{R}_+$. As long as $\Delta_t \geq 0$, then the the entrepreneur's budget constraint can be shown as:

$$x_t + \phi_t M_t^e = \phi_t \Delta_t + \phi_t M_{t-1}^e \quad \forall t \geq 0$$

Recall that the entrepreneur only consumes the CM good and will not participate in the DM. So unless $\beta(1-\psi)\phi_{t+1} \geq \phi_t$, the entrepreneur will choose not to hold currency across time periods. Consequently, $M^e_t = M^e_{t+1} = 0$, and the budget constraint can be rewritten as

$$x_t = \phi_t \Delta_t$$

The entrepreneur will consume an amount of the CM good equal to the amount of new tokens created. Since $x_t \geq 0$, our previous assumption $\Delta_t \geq 0$ must be true.

Entrepreneurs take the price of tokens as given. We have $\Delta^*_t \in \mathbb{R}_+$ as the solution to the profit maximization problem of an entrepreneur is given by

$$\Delta^*_t \in arg \max_{\Delta \in \mathbb{R}_+} \big[\phi_t \Delta_t - c(\Delta_t)\big].$$

The amount of new token creation will adjust accordingly to any price change of tokens. Therefore, the solution to the entrepreneur's profit maximization problem implies:

$$M_t = \Delta^*_t + (1-\psi)M_{t-1} \quad \forall t \geq 0.$$

Importantly, there are two factors that determine the change of the money stock from one period to the next. In the process of maximizing profits, the entrepreneurs will add to the money stock. In addition, any tokens misplaced will be lost permanently from the money stock each period. The equation highlights an important distinction between currency loss and theft. In particular, theft will cause the agents to lose their tokens, but it will not cause a decrease in the money supply.[7] Next, we look at equilibrium conditions for the economy.

III. EQUILIBRIUM

The market-clearing condition for money is

$$M_t = M_t^c + M_t^e, \quad \forall t$$

As shown earlier, $M_t^e = 0$ and the market-clearing condition becomes

$$M_t = M_t^c$$

We now move to define a monetary equilibrium in our framework.

DEFINITION 1: *A perfect-foresight monetary equilibrium is an array* $\{M_t, M_t^c, \Delta_t^*, \phi_t\}_{t=0}^{\infty}$ *satisfying the following conditions at all dates* $t \geq 0$.

5 CONDITIONS FOR EQUILIBRIUM

i. Money Demand for DM

$$\phi_t = \beta \phi_{t+1} L_\theta(\phi_{t+1} \cdot M_t^c) \tag{9}$$

ii. Transversality Condition

$$\lim_{t \to \infty} \beta^t (\phi_t \cdot M_t^c) = 0 \tag{10}$$

[7] He et al. (2008) introduce theft in model to highlight the security properties of banking. Thieves steal money from the honest agents in order to spend money in the following CM. While this affects the value of money and production, the stock of money remains unchanged.

iii. Profit Maximization for Entrepreneur

$$\Delta_t^* \in arg \max_{\Delta_t \in \mathbb{R}_+} \left[\phi_t \Delta_t - c(\Delta_t)\right] \tag{11}$$

iv. Law of Motion for Currency Creation

$$M_t = \Delta_t^* + (1-\psi)M_{t-1}, \quad \forall t \geq 0. \tag{12}$$

v. Money Market Clearing Condition

$$M_t = M_t^c. \tag{13}$$

Our analysis begins with entrepreneurs having a strictly convex cost function. The first proposition shows conditions with price stability:

PROPOSITION 1: *Suppose that* $\psi > 0$ *and buyers have all the bargaining power,* $(\theta = 1)$*. In addition, consider the following functions for utility from consumption,* $u(q) = 2q^{\frac{1}{2}}$*, the cost of producing DM goods* $w(q) = q$*, and the costs of producing new tokens* $c(\Delta) = \Delta^2$*. Under these conditions, there exists a steady state price* $\overline{\phi} = \frac{\sigma(1-\psi)\sqrt{2\beta\psi}}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$*. At the steady-state price, entrepreneurs will maintain a steady state level of tokens* $\overline{M} = \frac{\sqrt{\beta}\sigma(1-\psi)}{\sqrt{2\psi}(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$*. The steady-state price of tokens is decreasing in the rate of currency loss, and increasing in the probability of a successful trade. Finally, the steady-state money supply is decreasing in the rate of currency loss, and increasing in the likelihood of a successful trade.*

The results show that a steady-state monetary equilibrium with a stable price is possible with a strictly convex cost function for entrepreneurs, $c(\Delta) = \Delta^2$. The proposition also shows the effect of currency loss on production of new tokens in the steady-state. In particular, the steady-state money supply $\overline{M}$ is decreasing in the rate of currency loss ψ. The increase in loss of tokens will directly lower the money supply. In addition, the steady-state price of tokens is decreasing in the rate of currency loss. As tokens get their value from their purchasing power in the decentralized market, consumers make purchases less often in the decentralized market when the rate of currency loss is higher. As a result, the tokens will be worth less to the consumer. Moreover, this also indirectly lowers the steady-state money supply since entrepreneurs would have reduced incentives to produce additional tokens if the price of tokens is lower.

In addition, the proposition shows the effect of a higher matching rate in the decentralized market. Notably, the steady-state price $\overline{\phi}$ is increasing in the probability of a successful trade (σ) in the decentralized market. As mentioned in the previous paragraph, tokens derive their value from use in trade for goods in the decentralized market. As a result, the tokens gain value due to the increased chance of trade. Moreover, this leads to the steady-state money supply $\overline{M}$ increasing in successful trades. The higher price provides additional incentives for entrepreneurs to produce new tokens.

In light of the results presented in our Propostion 1, we would like to compare how our results stand in relationship to previous work. First, in stark contrast to our analysis, FVS argue that a stable price equilibrium does not generally exist when the costs of producing new tokens is strictly convex. In particular, the marginal cost of producing tokens is zero at production of zero tokens. Thus, entrepreneurs will continue to produce new tokens as long as the price of tokens is not zero — in turn, consumers expect entrepreneurs to produce new tokens until prices are driven to zero.

In fact, in order to generate a stable price equilibrium, FVS require that a hard cap is placed on token production at a future date. Since consumers believe that token production will eventually cease, a stable price equilibrium is possible. In contrast, we show that such caps are not required — as token loss occurs over time, new tokens need to be produced so that *the supply of tokens will remain constant & prices are stable.* In comparison to FVS where the price of tokens converges monotonically to zero, the introduction of token loss leads to the currency converging to a positive price. As a result, this helps the cryptocurrency to avoid self-fulfilling inflationary episodes.

Martin & Schreft were also able to establish the existence of an equilibrium for a privately-issued currency. The authors use a mechanism that is similar to the one used by FVS. In their model, Martin & Schreft instill agents with a belief that any money minted after a specified threshold number will be worthless. Therefore, once the number of units of money has reached the threshold, agents will have no incentive to mint new currency. In both models, agents have reason to believe that money creation will not be unbounded. As a result, conditions for an equilibrium do exist.

Our results have particular importance for cryptocurrencies without an algorithm that places a limit on total token production. Notably, Dogecoin, Monero, and Ethereum are all cryptocurrencies without limitations on total production. Consequently, token loss increases the potential for stability of cryptocurrencies. Moreover, the results also pertain to cryptocurrency with a cap at a finite date. Instead of a date in the future, the cryptocurrency would automatically converge to a stable price.

We now look to establish that a steady-state is also consistent with a linear cost function for entrepreneurs.

PROPOSITION 2: *Suppose consumers have the same utility functions from the previous proposition but* $c(\Delta) = c_0\Delta$ *represents the cost of producing tokens for entrepreneurs. Then there exists a steady-state price* $\overline{\phi} = c_0$ *and steady-state money supply* $\overline{M} = \frac{1}{c_0}\frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$.

The steady-state price when the entrepreneur has a linear cost function does not adjust with changes in the rate of currency loss. Therefore, the steady-state money supply is more sensitive to a change in the rate of currency loss with a lower c_0. In equilibrium, a higher priced cryptocurrency will have a higher cost c_0 to produce than a lower priced cryptocurrency. As a result of the higher price, it takes less of a decrease in token supply to meet the lower demand for real money balances caused by the increase in currency loss.

Consequently at lower prices, the steady-state money supply of a cryptocurrency with a linear cost function for entrepreneurs is more sensitive to a change in the rate of currency loss than a cryptocurrency with a strictly convex cost function. However, at higher prices the cryptocurrency with a strictly convex cost function requires larger adjustments to maintain the steady-state level of the money supply than a cryptocurrency with a linear cost function.

FINITE SUPPLY OF TOKENS

Up until this point, miners face no constraint on minting new currency. However, in the case of most cryptocurrencies, the protocol sets an upper limit on token creation. For example, due to the Bitcoin algorithm, mining of new Bitcoins will stop in the year 2140 after 21 million have been created. Similar to Bitcoin, we consider a cap on the amount of cryptocurrency that can be mined at each date. In addition, miners cannot create any new tokens past a certain date. Let $\overline{\Delta}_t \in \mathbb{R}_+$ denote the date-t cap on cryptocurrency.

The entrepreneur's profit maximization problem can be shown as

$$\Delta_t^* \in arg \max_{0 \leq \Delta \leq \overline{\Delta}_t} [\phi_t \Delta - c(\Delta)].$$

Now, with the new constraints, we prove that a steady-state does not exist regardless of the cost function.

PROPOSITION 3 *Suppose there is a set of caps* $\{\overline{\Delta}_{t=0}^{\infty}\}$ *on token creation. These caps are such that* $\overline{\Delta}_t > 0$ *at dates* $0 \leq t \leq T$ *and* $\overline{\Delta}_t = 0$ *at all subsequent dates* $t \geq T+1$*, given a finite date* $T > 0$*. A stable price steady-state does not exist under this set of conditions as long as* $\psi > 0$*.*

In contrast to FVS, a stable price equilibrium is not possible when cryptocurrency production is capped. This would preclude a stable price for a lot of cryptocurrencies. Bitcoin, Litecoin and Bitcoin cash all have a hard cap as part of their protocol. Once tokens are no longer created, the token loss would introduce deflationary pressure that makes a stable price impossible. That is, price stability would only be possible if token loss $\psi = 0$. While there is a lot of effort being put into developing technology to better secure private keys, there will always be a possibility of losing access to tokens.

This highlights a big issues for money provided privately. During some periods a deflationary path for money may be appropriate, but other times it could be extremely harmful. For instance, during the financial crisis in 2008, the increase in demand for money called for an increase in growth of the money supply. A shrinking money supply would have extended the time necessary for the economy to recover.

COROLLARY 1. *If the government issues money, a stable price equilibrium is possible for a cryptocurrency with a cap on total production.*

With a central bank money, it is possible to get the same results as the economy without currency loss. The government could declare their digital currency must be accepted on par with the cryptocurrency. Define $\Delta^g \in \mathbb{R}_+$ as the real value of government token production and $M^{+g} \in \mathbb{R}_+$ as the real value of private plus government money. With a government digital money susceptible to loss, the central bank can follow a simple rule to maintain a stable price. The rule for the the government can be defined by

$$\Delta_t^g = \omega M_{t-1}^{+g}, \quad \forall t > T$$

with the money growth rate $\omega = \psi$. This would allow the number of tokens to remain constant, and the proposition would be valid.

However, a different rule will need to be followed for a government issued digital currency held directly with the central bank. Token holders would lose their anonymity, but would no longer deal with the issue of token loss. With no token loss with government money's, the growth rate of money would no longer be one for one with the rate of token loss. Starting with the time period after the cap, the growth rate can be defined by

$$\Sigma_{t=1}^{\infty} \omega_t = \Sigma_{t=1}^{\infty} \psi \cdot (1 - \psi)^{T+t-1}.$$

As the government fraction of the total money stock grows, the percentage of tokens that needs replacing shrinks.

IV. WELFARE PROPERTIES

Following the welfare analysis in FVS, there is a strictly positive amount of tokens in the initial period. This is done to allow for record-keeping without having the government engage in the costly process of minting tokens. A solution to the social planner's problem is characterized by the surplus maximizing quantity q^* in the DM. With the ability to implement lump-sum taxes in the CM, Rocheteau (2012) shows that a social planner can achieve the efficient level of production in the DM by removing tokens in the DM. After the initial date, the economy will be in the first-best allocation since there will be no more costs for token production.

In order to simplify the analysis, we continue to impose that buyers are able to make take-it-or-leave-it offers to sellers. This will eliminate the inefficiencies associated with the hold-up problem, and allow the focus to be on issues related to token loss.

Recall the definition of money demand from the proof of proposition 1

$$z(\gamma) \equiv \frac{1}{\gamma} L_1^{-1}\left(\frac{1}{\beta\gamma}\right).$$

Real money demand is defined as a function of the real return on currency. Due to consumer preferences, $\frac{u'(q_t)}{w'(q_t)}$ is strictly decreasing in q_t. We assume preferences that imply the demand for real balances decreases in the inflation rate. As a result, $z_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing.

In this equilibrium, the quantity traded in the DM $\hat{q}$ satisfies

$$\sigma(1-\psi)\frac{u'(\hat{q_{cl}})}{w'(\hat{q_{cl}})}+1-\sigma(1-\psi)-\psi=\frac{1}{\beta},$$

We know production with the stable price equilibrium is below the socially efficient quantity of $q*$. A comparison can now be made to an environment without token loss. Here, we can contrast production in the DM to the FVS model with a cap on total production. In order to represent the FVS model, assume $\psi = 0$ due to the lack of token loss.

PROPOSITION 4: *Suppose we have 2 economies with an equilibrium with a stable price. Now, suppose for both economies buyers have the same bargaining power, $(\theta = \hat{\theta})$. In addition for both economies, consider the following functions for utility from consumption, $u(q) = \hat{u(q)}$, the cost of producing DM goods $w(q) = \hat{w(q)}$, and the costs of producing new tokens $c(\Delta) = \hat{c(\Delta)}$. One economy has currency loss while the other has none. If we let $\sigma = \hat{\sigma}$ and $\beta = \hat{\beta}$ for both economies and $\psi > 0$ for the economy that has currency loss, the DM good production with a stable price equilibrium is less than an economy without currency loss, $\hat{q_{cl}} < \hat{q}$.*

With a digital currency suffering token loss, there will be less allocation of the DM good in comparison to the FVS model. The intuition behind this is self-evident. Agents take into account the possiblity of losing money in the DM and as a result are less willing to carry money into the DM. This lowers the tokens worth in exchange and as a store of value. In addition, there will be less tokens available for trade due to agents misplacing tokens. Both of these factors lead to decrease in production in the decentralized market.

Furthermore, the result illustrates the impact on welfare from this feature of a private digital currency. Next, we look at another feature of digital currency that impacts this result.

CAP ON EACH PERIOD'S TOKEN PRODUCTION

As mentioned previously, Bitcoin and most cryptocurrencies have an algorithm that places an upper bound on total token creation. In addition, the algorithm adjusts mining difficulty to maintain a determinant trend in the long run. In the short run, once the algorithm adjusts, the production of tokens is guided by profit and loss. In order to emulate this, we will introduce a cap on each period's production $\overline{\Delta}_t = \alpha M_t$ for all $t \geq 0$.

In equilibrium, a necessary condition for efficiency is to have the real rate of return on money equal to the rate of time preference. This usually ensures that there is no opportunity cost of holding money balances in the DM. Since this involves a strictly positive real return on money in equilibrium, it cannot normally be achieved with purely private token creation. However, with an algorithm limiting each periods token production, we can show a positive real return for private currency that suffers loss in a steady state.

PROPOSITION 5: *Suppose that* $\psi > 0$ *and there is a set of caps* $\{\overline{\Delta}_{t=0}^{\infty}\}$ *on token creation. These caps are such that* $\overline{\Delta}_t \in (0, \psi\overline{M})$ *for all* $t \geq 0$*. Then there exists a stationary monetary equilibrium with a strictly positive real return on money.*

With a positive real return on currency, the price of tokens will rise to a level that induces entrepreneurs to produce more tokens than are lost. At this point, it becomes unsustainable for tokens to continue to increase in value permanently. The maximum demand for money will be met and tokens must be removed to enable market clearing. In order to prevent this, a cap must be in place to prevent entrepreneurs from increasing the level of token production. Moreover, the cap must be set to a level lower than the amount of tokens misplaced each period. Otherwise, the economy converges to a stable price steady state.

In comparison to FVS, a purely private arrangement can provide a positive real return on money. This holds for FVS even after a finite date where currency production no longer occurs. Without currency loss, the money stock will remain constant at this point. However, while currency loss does make a positive return possible, this is not enough to get efficient production in the DM. Normally, a currency with a rate of return equal to time preference provides efficiency in the DM. However, there will still be a cost to holding money with $\gamma = \frac{1}{\beta}$. The potential to lose currency makes it costly to hold money at this rate of return. Recall the money demand function

$$\phi_t = \beta\phi_{t+1}\Big(\sigma(1-\psi)\frac{u'(m^{-1}(\beta A))}{w'(m^{-1}(\beta A))} + (1-\sigma)(1-\psi)\Big)$$

To get efficient production we must have $u^{'}(q^*) = w^{'}(q^*)$. This will give us

$$\phi_t = \beta\phi_{t+1}\Big(\sigma(1-\psi)\frac{1}{1} + (1-\sigma)(1-\psi)\Big)$$

$$\phi_t = \beta\phi_{t+1}(1-\psi)$$

$$\frac{1}{\beta(1-\psi)} = \frac{\phi_{t+1}}{\phi_t} = \gamma_{t+1}.$$

Thus, efficient production in the DM is only possible with a return $\gamma_{t+1} = \frac{1}{\beta(1-\psi)}$. This illustrates how the Friedman rule, $\gamma = \frac{1}{\beta}$, will not obtain the efficient allocation. Similar to results in He et al., the consumer takes into account the potential to lose access to their currency. As a result, the return on currency needs to compensate for that loss in order to achieve the optimal allocation.

PROPOSITION 6: *There is no purely private arrangement that can deliver optimal production in the DM.*

Similar to FVS, entrepreneurs would need to take tokens out of circulation for a steady state with efficient production to be possible. This is not possible since it is not consistent with profit seeking behavior. Even after the upper bound on token creation has been reached, a private currency cannot maximize the surplus in the DM. A state actor needs to step in and remove tokens from circulation.

Let's say the government implements a lump sum tax $\tau \geq 0$ in the CM on consumers. Due to linear costs in the CM, the tax will be neutral and not lead to distributional effects. With the government taxing money out of the money supply, there will be an increase in the return of money. With an increase in the real value of tokens, entrepreneurs have an incentive to create more tokens each period. However, with a cap on token production, entrepreneurs will face an upper bound on token production.

PROPOSITION 7: *With a cap on token production each period $\overline{\Delta} = \alpha M$, there can be an efficient return with a tax $\tau = (\alpha + (1-\beta)(1-\psi))M_t$.*

In this case, a shrinking money supply that achieves an efficient return is still socially wastful. The economy would still incur welfare loss each period in the CM. The loss comes from the cost of producing tokens as compared to a central bank creating money at almost no cost. The welfare loss from Bitcoin mining bears a resemblance to the extraction of gold under the gold standard. Milton Friedman (1951) estimated the resource costs associated with running a gold standard were around one and half percent of national income. This would have been about half of the increase in income each year. [8] Current estimates of Bitcoin mining put the cost around $1.2 billion annually. Even the upper bound on token creation from Bitcoin's algorithm cannot address this issue. After all the tokens are created, the miner's activity is still necessary for the network to remain active. Instead of a reward of new Bitcoin and transaction fees, the miners receive transaction fees only.

Another factor to consider is the negative externality from Bitcoin mining energy usage. Vries (2018) estimates the Bitcoin network consumes around 2.55 gigawatts of electricity, and could eventually consume 7.67 gigawatts. This would put Bitcoin's energy use in the range of developed nations such as Austria (8.2 gigawatts).[9] Currently, the limited number of transactions on the network make this a poor substitute for most fiat regimes.

V. MONETARY POLICY

We now examine the proper role for monetary policy in the presence of privately-issued currencies. We now extend to the government the ability to introduce its own brand of currency. Recall, that an efficient equilibrium is not possible with an economy with a purely private arrangement. The authors investigate if is possible to achieve a socially optimal outcome with the introduction of government money and monetary policy.

[8] Allan Meltzer (1983) did a more recent estimate with the current income money ratio and it was lower. Yet, it would still amount to 16% of the increase in income per year.

[9] In an attempt to put this in perspective, Saad Imran (2018) makes the point that Bitcoin mining uses around 2.22% of the electricity of the world's data centers.

The government creates a brand of tokens that is referred to as currency g, and we will refer to tokens created by entrepreneurs as currency p. The government budget constraint becomes

$$\phi_t^g \Delta_t^g + \tau_t = c(\Delta_t^g), \tag{14}$$

where $\tau_t \in \mathbb{R}$ is the real value of lump-sum taxes, $\phi_t^g \in \mathbb{R}_+$ is the real value of government-issued currency, and $\Delta_t^g \in \mathbb{R}$ is the amount of the government brand issued at date t.

Two features set the government issuer apart from its private counterpart. First, the ability to tax consumers and withdraw money from circulation. We demonstrate this in corollary 1 and proposition 8. We will now focus on the other property that differentiates government tokens. That is, similar to the second part of corollary 1, the government could issue a digital currency that is held directly at the Federal Reserve. This type of currency would not suffer from token loss that occurs from a decentralized ledger. Thus, the law of motion for government money is given by

$$\overline{M}_t^g = \Delta_t^{*,g} + \overline{M}_{t-1}^g \tag{15}$$

at all dates, given an initial condition $M_{-1}^g \in \mathbb{R}_+$.

DEFINITION 2. *A perfect-foresight monetary equilibrium is an array* $\{M_t, M_t^c, \phi_t, \phi_t^g, \Delta_t^*, \Delta_t^{*,g}, \tau_t\}_{t=0}^{\infty}$ *satisfying (9)-(15) at all dates* $t \geq 0$.

Unlike FVS, consumers will consider the private money inferior due to token loss. For an equilibrium, the government money would have a lower rate of return then the private money in the economy. We can define the government return on money $\gamma_{t+1} = \frac{\phi^g_{t+1}}{\phi^g_t}$ and find equilibrium conditions for it to be valued.

PROPOSITION 8: *Suppose* $\psi = 0$ *for money issued by the government* M^g*. The condition* $\gamma^p_t = \frac{\gamma^g_t}{(1-\psi)}$ *must hold in order for both currencies to be valued in equilibrium.*

Agents will take into account the possibility of losing private tokens in their decision for money holdings. As a result, the return on the private digital currency must compensate for this possibility. With $\gamma^g_t = (1-\psi)\gamma^p_t$, the government digital currency allows for the same level of consumption in the DM. This leads us to the following corollary.

COROLLARY 2. *With a government rule that ensures* $\gamma^g_t > (1-\psi)\gamma^p_t$*, there is no equilibrium where agents value private money.*

The government can drive out private money by guaranteeing a return lower than private money. Agents will consider the fact government money is not susceptible to loss and adjust accordingly. As long as government compensates for the expected token loss, agents will prefer to hold the government issued tokens. In FVS, agents do not naturally discriminate between government and private money. As a result, the government must peg the real value of government money to a high enough level to drive out private money. Entrepreneurs will not shrink their supply of currency to match the positive return of government money. This will drive private money out as agents prefer the money with a higher return. In contrast, with the addition of token loss to private currency, agents will naturally discriminate and prefer government money as long as the return compensates for token loss.

FVS point out how a private currency can help discipline the central bank by ensuring that it provides a good money. With the addition of token loss, the government will have more room for discretion. As the private money will be discriminated against due to the token loss arising from its security properties. This does allow the government more flexibility in dealing with economic downturns, but still provides an incentive to keep inflation under control. However, there is a caveat to go along with this result. Namely, we ignore the benefits of anonymity that a private digital currency can provide.

CONCLUSION

In recent years, there has been increased enthusiasm regarding the viability of cryptocurrencies. Consequently, an academic literature studying cryptocurrency has also emerged. In terms of thinking about the viability of cryptocurrency, there has also been increasing awareness of the costs of producing such forms of money. In this sense, the costs of producing digitally issued money can be significantly higher than pieces of paper issued by sovereign governments. Yet, in addition to the relatively unique features of the costs of producing digital money, cryptocurrencies are also embedded with unique security features. Notably, a unique feature of cryptocurrency is that such forms of private outside money are based on cryptography — holders of coins store their currencies such as Bitcoin at an address on a decentralized ledger that is linked to both a private and public key. If a private key is lost or damaged, the coins are permanently missing. This aspect of many cryptocurrencies is typically omitted from previous that study whether privately issued outside money would circulate over time. Consequently, failing to account for permanent coin loss may fundamentally alter the conditions where private money circulates. It is also likely to impact whether the supply of private money over time would be socially efficient. In light of this gap in the existing literature, the objective of this paper is to study how exogenous token loss affects the conditions where privately produced money circulates. In particular, we discuss how the relationship between the price of coins and the degree of token loss also depends on the shape of the cost function.

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CHAPTER 3

CRYPTO, SECURITY, AND FINANCIAL INTERMEDIATION

I. INTRODUCTION

Satashi Nakamoto created the cryptocurrency known as bitcoin in 2008. Since then, many other cryptocurrencies have been introduced and have been circulating in the global financial system. Notably, recent estimates of the total market capitalization of the over 7,000 different cryptocurrencies are almost $350 billion. In particular, the total capitalization of bitcoin is about $200 billion followed by ethereum at around $40 billion.[10]

Why have cryptocurrencies become increasingly more accepted? Some value the privacy that they derive from conducting transactions in cryptocurrency. However, many also value that alternatives to the centralized financial system are important as central banks continue printing money and devaluing fiat monies issued by sovereign governments. That is, one of the initial motivations for establishing cryptocurrencies such as bitcoin was to avoid inflation risk coming out of the recent Great Financial Crisis. This has further been magnified due to central bank responses to the current public health crisis across the globe. Consequently, some economists argue that physical cash could eventually be eliminated. In this way, cryptocurrency could prove valuable as a means to conduct private trustless transactions.[11]

[10] Data from https://coinmarketcap.com.
[11] Kahn et al. (2005) demonstrate the importance of anonymity in dealing with identity theft from transactions. A consumer using cash or a digital replacement is not at risk of financial loss associated with identity theft.

A unique feature of cryptocurrency is that such forms of money make use of cryptography where coin holders store their bitcoin at a public address on a decentralized ledger that is linked to both a private and public key.[12] As the identity of the individual behind the public address is unknown, public key cryptography is necessary for network verification. The public address is similar to a pen name in that it's a pseudonym. When a transfer request from the address is made to the Bitcoin network, the crypto holder uses the private key to digitally sign the message containing the request. In order to complete the transfer, the Bitcoin network verifies the digital signature with the public key. The verification process illustrates the importance of the security of the private key. The person in possession of the private key is in control of the funds at the address.

In order to prevent hackers from gaining access to their private keys, crypto owners often store their private key on a hardware wallet offline in "cold storage". The hardware wallet is a type of flash drive that performs the function of a crypto wallet. For example, the Bitcoin wallet is responsible for generating the random private key that is a 256-bit number identified by 64 characters in the range 0-9 or A-F. The wallet then uses the public key, which is mathematically related to the private key, to create the public address. However, if the hardware wallet is lost, damaged or the passcode to the device is forgotten, access to the tokens is permanently lost.[13] The losses are far from trivial. In fact, recent estimates from Chainalysis[14] indicates that as of 2017, nearly 4 million bitcoins

[12] The private key is known as the signing key and is only known by the address owner. In contrast, the public key used for verification is made available to the Bitcoin network.

[13] Due to the pseudoanomyity and the decentralized nature of crypto networks, it is not possible to go to a trusted third party and retrieve your private key or gain access to your money with an ID.

[14] Chainalysis is a crypto forensics company based in New York that will often assist Interpol and U.S. law agencies in tracking down cyber criminals. For a discussion on their relevance to current events, see "The rise of crypto laundries: how criminals cash out of bitcoin." (Murphy, Financial Times, May 28, 2021).

alone have been lost which is around one-quarter of the aggregate supply of existing bitcoins.[15]

Yet, as shown by He et al. (2005, 2008), one of the essential roles of banks in the financial system is to provide safekeeping of different assets.[16] Consequently, the OCC recently extended to banks the same authority over cryptocurrency: "Because national banks are authorized to perform safekeeping and custody services for physical assets, national banks are likewise permitted to provide those same services via electronic means (i.e., custody of cryptocurrency.)" Further, they stress that these services could be extended and intertwined with other important functions of intermediaries such as the extension of credit: "By providing such services, banks can continue to fulfill the intermediation function they have historically played in providing payment, loan, and deposit services."[17]

Consequently, as economists continue to think about the larger impact of cryptocurrency in the global financial system, they cannot ignore the role that they will eventually play in the traditional, deposit-based banking system. That is, economists need to develop frameworks where institutions bank in terms of cryptocurrencies such as bitcoin. In particular, recently crypto credit markets such as Bitfinex and Bitbond have been established in which borrowing and lending in terms of bitcoin has emerged. It is not hard to envision that deposit-based intermediaries which have long played an essential

[15]For additional discussion, see "Lost Passwords Lock Millionaires Out of Their Bitcoin." (Popper, New York Times, January 12, 2021).

[16]In contrast to He et al., we study a framework where the money supply is produced by profit-maximizing agents. Thus, it is endogenous rather than following simple exogenous growth rate. Further, the money stock is subject to permanent loss over time.

[17]Office of the Comptroller of the Currency (Interpretative Letter #1170: July 2020). For additional discussion, see "Banks can now hold Bitcoin: Behind the OCC's big decision and why it matters." (Roberts, Fortune, July 22, 2020).

function in terms of the extension of credit will begin to do the same. That is, economists need to develop frameworks in which individuals do not engage in borrowing and lending in outside money issued by sovereign governments but instead by profit-maximizing issuers of digital fiat money. Such loans are not subject to an “inflation tax” where revenues may be redistributed back into the economy — rather they are in terms of assets that are subject to permanent loss due to separation of holders from their private keys. For this reason, central banks also need to begin to consider how borrowing and lending in terms of cryptocurrency may be offered by traditional intermediaries.

In light of these important issues, we seek to answer the following questions. How would the provision of safekeeping services by banks – as has been just recently allowed in the United States – affect the circulation and value of cryptocurrencies? Further, how is financial market activity affected if banks both provide safekeeping and extend credit in terms of cryptocurrency? What is the overall impact of financial intermediation if banks simultaneously provide security, intermediate between borrowers and lenders, and conduct transactions/payments services?

To do so, we extend the framework of Berentsen, Camera, and Waller (2007) with microfounded financial intermediaries by replacing central bank fiat currency with digital money that is subject to permanent loss as in the case of cryptocurrency. In particular, in order to consider the circulation of cryptocurrency as in Fernandez-Villaverde & Sanches (2019), we introduce tokens that are produced by profit-maximizing issuers of money rather than outside money issued by a sovereign government. Yet, in contrast to Fernandez-Villaverde & Sanches, we consider that a percentage of the stock of tokens will be lost over time in the absence of safe-keeping services.

We refer to the intermediaries in the framework as "banks." As in Berentsen, Camera, and Waller (BCW), banks have access to a record keeping technology that keeps track of the financial histories of agents. As banks may provide different levels of credit extension, safekeeping, and payments processing, we consider that there are three different types of banks in the model.[18] To begin, we start the analysis in a way that is close to BCW in that type I banks only intermediate between borrowers and lenders – they do not provide safekeeping services. However, in contrast to BCW, the borrower is responsible for the security of the cryptocurrency. We proceed to look at type II banks who provide safekeeping over tokens that are borrowed, but not over an individual's other money holdings. Finally, we look at type III banks who process all transactions between buyers and sellers. Consequently, in this setup, token loss would never occur. It is this part of our analysis that most closely matches up with BCW. Yet, we also show that a steady-state would not exist unless there is a hard cap on token production.

The complications from establishing private keys and providing security over cryptocurrency has also been studied in other work but in slightly different contexts. For example, Kahn et al (2020) delve deeper into security protocols and look at various tradeoffs in dealing with theft of the private key. On the one hand, lowering the number of addresses that hold funds attacts hackers because each address is worth more. To avoid this possibility, holders can increase password strength. However, the password is susceptible to theft each time it is used during a transaction.[19] As a result, customers need

[18]As discussed by BCW, credit functions like illiquid bonds in Kocherlakota (2003). Yet, in our framework, we look at the transfer of money produced by profit-maximizing agents.

[19]Kahn & Roberds (2008) investigate identity theft and find two types of equilibria can exist that are relevant for the circulation of cryptocurrency. To begin, one equilibrium involves "existing account fraud" which includes theft of an existing card. The second type of equilibrium involves "new account fraud" which means that the data is stolen and used to create a new credit card account. Cryptocurrency users

to follow a strict security protocol to ensure thieves do not gain access to their password. Moreover, a difficult password and protocol require effort from the customer. With the bank and customer sharing liability, the bank faces a moral hazard problem. That is, the bank cannot control the customers actions and the customer will not fully internalize the costs of theft and underprovide the level of care in regards to security. Consequently, there is an important role for the government to regulate security protocols. Nevertheless, our framework diverges by looking at microfoundations for monetary exchange and financial intermediaries. As a result, we focus our attention on the issues related to loss of currency on the money supply. Not withstanding, Kahn et al raise an important issue that has relavance to our paper. That is, the ability to scale with the technology used leaves the holder of the currencies susceptible to losses not possible with physical cash.[20]

The remainder of the paper is organized as follows. Section II describes the benchmark physical environment. Section III looks at a symmetric equilibrium where banks can enforce financial contracts at zero cost. In particular, it looks at economic activity separately under type I banks, type II banks, and type III banks. Section IV extends the analysis to look at different outcomes in the absence of enforcement. Section V offers concluding comments.

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do not have to worry about the latter more costly type of fraud. However, existing account fraud is a significant issue. If an individual's "identity" is stolen by obtaining a holder's private key, they are limited to spending the balance at the address. Consequently, an account used frequently for transactions should be maintained with a low balance. Further, accounts should be maintained in separate wallets to avoid potential contagion resulting from a wallet's stolen password.

[20] For example, from "Lost Passwords Lock Millionaires Out of Their Bitcoin." Stefan Thomas, a German programmer based in San Francisco, forgot the password that unlocked his small hard drive, called the IronKey, that would allow him to access 7,002 bitcoin, which was worth $220 million. This amount of cash would be too bulky to carry around and lose at one time.

II. ENVIRONMENT

There is a [0,1] set of infinitely-lived consumers who consume and produce goods at different points in discrete time. Consumers first meet in a decentralized market (First Market or DM) and then in a centralized market (Second Market or CM). In addition to consumers, following Fernandez-Villaverde & Sanches (2019), there are also entrepreneurs (miners) who create tokens. Hereafter, we simply refer to their paper as 'FS'. In the second market, both types of agents interact and consume.

In the first sub-period, consumers meet in pairs but entrepreneurs are idle. Consumers receive a preference shock to whether they can produce or consume at the beginning of the period. This friction along with the absence of record keeping creates a role for a medium of exchange. In contrast to BCW, profit driven entrepreneurs provide the medium of exchange, not central banks. A consumer can produce but not consume with a probability of n and can consume but not produce with a probability $1 - n$. We will call a consumer who wants to consume a buyer and a consumer who can produce a seller. Buyers get utility $u(q_b)$ from q_b consumption in the first market where $u'(q_b) > 0$, $u''(q_b) < 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$. Sellers incur utility loss $c_1(q_s)$ from production of q_s units of goods in the first market such that $c_1'(q_s) \geq 0$ and $c_1''(q_s) \geq 0$.

In addition to consumers, banks are also active. In particular, banks compete to offer their services in a perfectly competitive environment. In the DM, after the preference shock, sellers have no need for their money holdings in the DM and buyers would like to increase their money holdings for purchases. To attract deposits from sellers, banks will offer interest on deposits and loan out this money to buyers in exchange for interest on the loan. In the CM, banks and consumers will settle their debt. To start, banks will be given a technology to force repayment at zero cost. In the last section, banks will no longer have access to this technology and can only deter default by excluding consumers from the financial system.

By comparison, entrepreneurs have access to an additional technology that creates tokens. These tokens come in digital form.[21] Counterfeiting is not an issue with the tokens as they can be verified publicly at zero cost. Entrepreneurs create identical tokens and bear the same costs of production in the second sub-period $c_2 : \mathbb{R}_+ \to \mathbb{R}_+$ for minting tokens. We refer to the entrepreneurs as miners since the tokens created are cryptocurrency. In the case of miners, the cost comes from the miners solving a complex computational problem that requires input. For example, the cryptocurrency will have a protocol that determines the amount of work required to solve the problem. We denote the cost function of producing tokes as $c_2(\Delta)$ where Δ denotes the number of tokens produced since the process takes place in the second sub-period. To capture these costs, the cost function is strictly increasing and weakly convex: $c_2'(\Delta) > 0$, $c_2''(\Delta) \geq 0$, and $c_2(0) = 0$. Entrepreneurs will create these tokens in exchange for the centralized market good produced by consumers. The entrepreneur's profit motive determines the supply of tokens which enable trade between buyers and sellers in the decentralized market.

By comparison, x^e denotes an entrepreneur's consumption of the good produced in the second market, and Δ is the production of tokens. Entrepreneurs are only active in the second market with their preferences denoted by

[21] While some people may use paper wallets that resemble paper bills, this is only to store the private key and address of their bitcoins – miners play no part in producing them. Paper wallets are only one way among many of personally safekeeping a private key.

$$U^e(x^e, \Delta) = x^e - c_2(\Delta)$$

For the most part, meetings in the first market are anonymous, precluding any chance of credit. This friction, along with the fact that goods are perishable, makes a medium of exchange necessary to facilitate trade. Although a token is intrinsically worthless, it obtains its value from use as a medium of exchange. Since entrepreneurs' only motivation is to maximize profits, their behavior can be predicted by other agents. This allows agents to form expectations about the future money supply and use this to determine the relative price level over time.

In the second market all consumers consume and produce with utility $U(x)$ from x consumption, $U'(x) > 0, U'(0) = \infty, U'(+\infty) = 0$ and $U''(x) \leq 0$. In addition, consumers have the ability to transform labor into a perishable good through the use of a linear technology. The consumers either consume the good or use it to exchange with entrepreneurs and consumers for tokens. By introducing differences in preferences among consumers across both sub-periods, money holdings are degenerate at the beginning of a period.[22] The discount factor across dates is $\beta \in (0, 1)$.

BANKS, RECORD KEEPING AND STORAGE

Following BCW, financial intermediation occurs through perfectly competitive firms who accept nominal deposits and make nominal loans. In contrast to BCW, banks facilitate borrowing and lending in terms of digital tokens. Moreover, the unique security features of cryptocurrency that lead to token loss gives rise to the need of provision of

[22] In this manner, our framework follows Lagos and Wright (2005).

security. However, with banks providing security, the consumer will lose the privacy of his transactions that comes with anonymity. Due to these potential concerns, the banking system around cryptocurrency could develop in a variety of ways. In order to address this, we will consider "banks" that provide three different levels of security in our analysis. Consequently, we can analyze the benefits to allocation and welfare due to the different functions of the banking system.

All three types of banks make use of a technology at zero cost that allows record keeping of financial histories of individuals. In particular, all banks will have a record of anyone who defaults on their one-period loan. Due to linear production costs in the CM, consumers have no need to smooth out the repayment of loans to the bank. Similar to BCW, banks do not have a record of trading histories in the first market. However, in our framework, type II and type III banks will have a partial record of trading history due to processing transactions for consumers. Similar to BCW, there is no inside money issued that will influence the money supply. Moreover, in contrast to models with theft such as Kahn et al, He et al, and Sanches & Williamson (2010), the security features from banks have a direct influence on the supply of money. Specifically, money holdings are not shuffled from honest agents to thieves as in prior work, but lost from the money supply permanently.

As mentioned previously, consumers receive a preference shock upon entering the first market. After the preference shock, consumers will have the opportunity to conduct business with banks. No longer in need of a medium of exchange, sellers will deposit their digital tokens with banks. Buyers looking to supplement their money holdings can borrow the digital tokens deposited at the banks. After dealing with the banks, buyers and sellers will meet up to trade. The timing described will be the same in each of the different bank type's environment. However, with each change of bank type, there will be variations in provision of security of the money balances.

There will be three different cases concerning the banking setup. The first case will only have type I banks available. In this scenario, the bank will transfer the digital tokens to the borrower, but does not provide safekeeping of tokens. As a result, the tokens will be susceptible to currency loss since the borrower is responsible for the security of the tokens. In addition, the seller will be shielded from token loss as the buyer is liable for any borrowed tokens lost in the DM. From this perspective, our framework differs in a significant way from BCW. Some users of digital currency may prefer this arrangement due to the anonymity that can still be retained.[23] Moreover, we can analyze the benefits specifically from financial intermediation in terms of a digital currency subject to token loss.

The additional scenarios allow us to analyze the benefits of banks providing safekeeping. The second case will look at only type II banks. In this scenario, banks hold onto borrowed tokens to provide safekeeping for the buyer and will transfer tokens to the seller during the meetup between buyer and seller. The buyer retains tokens brought into the DM and supplements the transaction with the tokens left after token loss. In this case, the banks ensure that tokens borrowed from the bank will not be susceptible to token loss.

[23]The European Union's fifth Anti Money Laundering Directive (AMLD5) calls for the need for strict identity verification measures of customers participating in the crypto exchanges. Even outside the European Union, exchanges abide by the know your customer laws. Coinbase, the largest Crypto exchange, will go so far as not to process any transaction to LocalBitcoins. LocalBitcoins is an exchange that does not ask for an ID and does not track transactions. In addition to the Know Your Customer (KYC) compliance, it calls for stronger Anti Money Laundering (AML) measures. Specifically, it calls for exchanges to monitor for any suspicious transactions made by exchanges. These type of rules would certainly be enforced with banks involved with crypto transactions.

However, the buyer will lose anonymity with purchases made by the bank, but retain privacy with any remaining purchases. Lastly, in the third case, type III banks provide safekeeping for *both* buyers and sellers as they are able to process transactions in the DM between buyers and sellers. That is, a buyer will deposit all of his tokens with the type III bank at the beginning of the first sub-period. In this manner, token loss will be eliminated, but a buyer loses all anonymity with transactions processed by banks in the first market. Under most conditions for cryptocurrency, the elimination of token loss precludes the existence of steady state. We will begin by looking at type I banks.

WELFARE

We now solve for the planner's first best allocation. The expected steady state lifetime utility of a consumer at the beginning of the period is given by:

$$(1-\beta)\mathcal{W} = (1-n)u(q_b) - nc_1(q_s) + U(x) - x, \tag{16}$$

where q_b is consumption and q_s is production in the first market. Agents are treated symmetrically by a planner who maximizes (1) subject to the feasibility constraint

$$(1-n)q_b = nq_s.$$

Optimal consumption of the good in the second market will satisfy

$$U'(x^*) = 1.$$

In addition, optimal consumption and production of the good in the first market are such that

$$u'(q_b) = c_1'(\frac{1-n}{n}q_b).$$

We let $q^* \equiv q_b^* = \frac{n}{(1-n)}q_s^*$ to yield

$$u'(q_b^*) = c_1'\left(\frac{1-n}{n}q^*\right).$$

The following is a description of one time period in chronological order. To begin, the first market starts with consumers observing their production and consumption shocks. Again, consumers become either a buyer with a probability $(1 - n)$ or seller with a probability n. After the shock, the banking sector opens and agents can borrow or deposit money. The banking sector then closes and trade takes place. This leads into the second market where they also trade goods, but in addition settle financial claims with the bank.

Let ϕ be the real price of money in the second market. All consumers and entrepreneurs follow identical strategies and real allocations are constant over time in a stationary and symmetrical equilibrium. We define M as the average stock of money held by consumers. In contrast, m denotes an individual's holdings of money balances. In a stationary equilibrium, real money balances at the end of the period are time-invariant

$$\phi M = \phi_{+1}M_{+1}.$$

Defining γ as the growth rate of money along with time invariance in the steady-state:

$$\frac{\phi}{\phi_{+1}} = \frac{P_{+1}}{P} = \frac{M_{+1}}{M} = \gamma.$$

Let $V(m)$ denote the expected value from trading in market 1 with nominal money balances m at time t. In addition, $W(m,l,d)$ denotes the value function of a consumer entering the second market with m units of money, l loans, and d deposits at time t.

A CONSUMER'S PROBLEM IN THE SECOND MARKET

Following BCW, in the second market, consumers produce h goods and consume x, repay any loans, redeem deposits, and adjust their money balances. Entrepreneurs produce tokens for the Second Market good. A consumer who borrows l units of money in the first market will repay $(1+i)l$ units of money, with i as the nominal interest rate on the loan. A consumer who deposits d units of money will receive $(1+i_d)d$ units of money, with i_d as the nominal interest rate on deposits.

In the second market, the consumer's problem is:

$$W(m,l,d) = \max_{x,h,m_{+1}} \big[U(x) - h + \beta V_{+1}(m_{+1})\big]$$

$$s.t. \quad x + \phi m_{+1} = h + \phi m + \phi(1+i_d)d - \phi(1+i)l, \tag{17}$$

where m_{+1} is the money taken into period $t+1$. Upon substituting for h, the consumer's problem becomes

$$W(m,l,d) = \phi\big[m - (1+i)l + (1+i_d)d\big] + \max_{x,m_{+1}} \big[U(x) - x - \phi m_{+1} + \beta V_{+1}(m_{+1})\big]$$

The first order condition for the second market good is

$$U'(x) = 1.$$

Notably, the consumer's wealth does not influence the optimal choice for x. This is a result of the consumer's quasi-linear preferences. Moreover, the consumption of the second market good is always the same. The consumer will adjust his production of the second market good to ensure money is leftover to bring into the first market. The consumer's demand for tokens satisfies:

$$\phi = \beta V'_{+1}(m_{+1})$$

where $V'_{+1}(m_{+1})$ is the marginal value of an additional unit of money taken into period $t+1$. The price of a token is a function of its store of value and the goods it can buy in the following period's first market.

The consumer's decision to borrow or loan money in the first market depends on the interest costs of repayment (or the return on deposits) offered by intermediaries. In addition, the seller must consider the benefit of the tokens in the next period when deciding how much of the first market good to produce. In particular, the benefit of producing goods to acquire tokens is reflected in the following envelope condition:

$$W_m(m, l, d) = \phi.$$

The value of a marginal increase in tokens held is reflected by the decrease in the disutility of labor in the second market by being able to sell tokens.

When choosing how much to borrow, the consumer faces the marginal cost of paying back the loan in the following second market. The envelope condition for loans is

$$W_l(m, l, d) = -\phi(1 + i).$$

That is, they will incur $\phi(1 + i)$ units of disutility as they will need to work more to repay their loans. Finally, the envelope condition for deposits is

$$W_d(m, l, d) = \phi(1 + i_d).$$

The benefit of depositing funds comes from the labor savings in the second market.

A CONSUMER'S PROBLEM IN THE FIRST MARKET

We will first look at activity in the presence of type I banks. As a benchmark, though type I banks can enforce loan contracts, they cannot settle payments between the buyer and seller. This leaves the consumer in charge of keeping track of the private key for all tokens. Consequently, consumers are subject to token loss with a probability ψ. Also, p denotes the nominal price of goods. In order to determine p, we will use competitive pricing instead of bargaining.[24] Buyers will never deposit funds and sellers will never take out loans, $l_s = d_b = 0$.

[24] Competitive pricing was previously used in Lagos & Rocheteau (2005), Berensten (2005) and Aruoba (2006).

An agent with m units of money at the beginning of the first market has expected lifetime utility

$$V(m) = (1-n)[u(q_b) + W((1-\psi)m + (1-\psi)l - pq_b, l)] +$$

$$n[-c_1(q_s) + W((1-\psi)(m-d) + pq_s, d)], \tag{18}$$

where pq_b is the amount of money spent as a buyer and pq_s the money received by a seller. If a buyer, money will be brought into the second market if what is retained of the loan and money brought into the first period exceeds pq_b. However, this will not ever be the case since the buyer decides on the loan amount in the first market. The buyer will not borrow more money then he plans on spending due to the interest costs of the loan. For the seller, he will hold on to all of the money received in the first market for producing goods for a buyer. Again, the key difference compared to BCW is that we study an endogenously produced money supply where such tokens are subject to loss. This also differentiates our work from FS.

SELLERS' DECISIONS

A seller will deposit money immediately upon receiving his preference shock in the first market. The bank will pay this back plus the interest on deposits in the following second market. If the seller does not deposit a portion of his money it will be subject to loss. Thus, in this case, he can only expect to retain $(1-\psi)(m-d)$ units of tokens at the end of the first market. Once the seller deposits the money with a bank he is no longer responsible for the tokens in the current sub-period.

The problem of a seller is

$$\max_{q_s,d}[-c_1(q_s) + W((1-\psi)(m-d) + pq_s, d)]$$

$$s.t. \quad d \leq m$$

The seller has two decisions to make at this point. First, he needs to determine the quantity of the first market good to produce and how much money to deposit. The seller's first order condition for q_s is

$$-c_1'(q_s) + W_m((1-\psi)(m-d) + pq_s, d))(p) = 0.$$

In choosing how much to produce, the seller looks at tradeoffs over disutility to be incurred in the DM and CM. From the envelope condition for money balances:

$$c_1'(q_s) = p\phi.$$

The amount sellers produce is independent of m and d due to linear preferences. The seller trades off the level of disutility between production in the DM and CM.

Next, we take a look at the seller's decision of how much money to deposit with banks. The seller's deposit decision becomes

$$\mathscr{L} \max_{d} = -c_1(q_s) + W((1-\psi)(m-d) + pq_s, d) - \lambda_d(d-m)$$

where λ_d is the Lagrange multiplier on the deposit constraint. The first order condition with respect to d is:

$$-(1-\psi)W_m + W_d - \lambda_d = 0$$

Substitute the envelope conditions $W_m = \phi$ and $W_d = \phi(1+i_d)$ to obtain

$$\lambda_d = \phi(i_d + \psi)$$

With $i_d > -\psi$ the constraint binds and $d = m$. Notably, in contrast to BCW, the seller would be willing to tolerate negative interest rates as long as banks offer some sort of protection against token loss. Consequently, under this condition, sellers will deposit all of their money balances at the bank.

BUYERS' DECISIONS

After the consumer receives his shock to become a buyer, he must determine the size of the loan to take out from the bank. Since type I banks do not have a safe-keeping function, the buyer will not deposit any of his cash balances. Moreover, the tokens from the loan are susceptible to loss.

If an agent is a buyer in the first market, his problem becomes

$$\max_{q_b, l}[u(q_b) + W((1-\psi)m + (1-\psi)l - pq_b, l)]$$

$$s.t. \quad pq_b \leq (1-\psi)(m+l), \quad l \leq \bar{l},$$

In contrast to BCW, the amount borrowed along with money brought into the market are susceptible to currency loss. The buyer can only spend a portion of money he brings into the market, $(1-\psi)m$, plus a fraction of what they borrow, $(1-\psi)l$.

We want to emphasize here that token loss is different than inflation. First with inflation, there is the potential for seignorage revenues to be rebated back into the economy. Second, only buyers who do not deposit funds are susceptible to token loss.

The buyer also faces a constraint on the loan size that is bounded above by $\bar{l}$. The buyer takes the constraint as given but it is determined endogenously. The buyer's problem becomes

$$\mathscr{L} \max_{q_b,l} = u(q_b) + W((1-\psi)m + (1-\psi)l - pq_b, l) - \lambda(pq_b - (1-\psi)(m+l)) - \lambda_l(l-\bar{l})$$

where λ is the multiplier on the buyer's cash constraint and λ_l on the borrowing constraint.

We will look at the buyer's choice of quantity of the first market good q_b first. The buyer's first order condition becomes

$$u'(q_b) = pW_m + p\lambda.$$

The buyer's desired amount of consumption depends on the cost of working to repay the loan and whether the liquidity constraint binds. Take $W_m = \phi$ and $\frac{c_1'(q_s)}{\phi} = p$ and substitute

$$u'(q_b) = c_1'(q_s)\Big(1 + \frac{\lambda}{\phi}\Big)$$

After the substitutions, we compare the buyer's gain in value from an additional unit of consumption versus the seller's disutility from producing the additional amount of goods. Notably, the cash constraint drives a wedge between the buyer's and seller's marginal returns. Thus, this inefficiency drives down the value of trade between the buyer and the seller.

The buyer's first order condition for l is

$$(1-\psi)W_m + (1-\psi)\lambda = -W_l + \lambda_l$$

Each unit of funding provides the buyer with more cash that he values at rate $(1-\psi)W_m$. It also helps relieve the constraint on money holdings which he values at rate $(1-\psi)\lambda$. However, the buyer will be held back by his debt brought into the second market.

From the previous envelope conditions: $W_m = \phi$ and $W_l = -\phi(1+i)$

$$\lambda = \frac{\phi(i+\psi) + \lambda_l}{(1-\psi)}$$

If $\lambda = 0$, then $u'(q^*) = c_1'(q^*)$ and trades are efficient. However, for this to occur, deflation must be at a rate $\gamma = \beta$ and currency loss must be zero and the loan constraint must not bind. Upon substituting for λ, the quantity traded in the first market satisfies:

$$\frac{u'(q_b)}{c_1'(q_s)} = 1 + \frac{(i+\psi)}{(1-\psi)} + \frac{\lambda_l}{\phi(1-\psi)}$$

If the borrowing constraint does not bind,

$$\frac{u'(q_b)}{c_1'(q_s)} = 1 + \frac{(i+\psi)}{(1-\psi)} = \frac{1+i}{1-\psi}$$

The buyer borrows up to the point where the marginal benefit of borrowing equals the marginal cost. In comparison to BCW, he will not borrow as much. He then spends all his money and consumes $q_b = (1 - \psi)(m + l)/p$. Here, we also see the costs of currency loss. Additionally, trades are inefficient as long as $i > -\psi$.

As a benchmark, suppose $\psi = 0$ as in BCW. Then, we get the usual type of inefficiency which comes from the need to wait to receive deposit returns by the seller. However, the problem is magnified here because the borrower will lose some of the tokens obtained.

If the borrowing constraint binds then $\lambda_l > 0$ and

$$u'(q_b) > \frac{1+i}{1-\psi} \cdot c_1'(q_s)$$

The value of an extra unit of a loan exceeds its marginal cost. Consequently, a borrower may be willing to pay more than the prevailing loan rate, but banks may be worried about default. The interest rate may not rise to clear the market and so credit rationing might occur depending on the ability of banks to enforce loan contracts. So, the borrower borrows $\bar{l}$, spends all of his money and consumes $q_b = (1 - \psi)(m + \bar{l})/p$.

All the buyers enter the period with the same amount of money and therefore face the same problem. The same is true for the seller and so q_b and q_s is the same for all buyers and sellers respectively.

$$q_s = \frac{1-n}{n} q_b$$

A BANK'S PROBLEM

As was previously stated, we begin by focusing on type I banks. For such banks, the only function they serve is to intermediate between borrowers and lenders. Banks accept nominal deposits and pay out the nominal interest rate i_d. They also make nominal loans l at a nominal rate i. There are no operating costs for banks. In addition, the credit market and deposit market are perfectly competitive. Therefore, all participants take interest rates as given. Finally, banks are not subject to reserve requirements.

The representative bank solves the following problem per borrower:

$$\max_{l}(i - i_d)l$$

$$s.t. \quad l \leq \bar{l}, \quad u(q_b) - (1+i)l\phi \geq \Gamma,$$

where Γ is the reservation value of the borrower. The reservation value represents the surplus a borrower would receive from taking out a loan from another bank.

We consider two different settings concerning enforcement of loan contracts. First, banks can force repayment at no cost. Consequently $\bar{l} = \infty$ as the borrowing constraint would never bind. In a second case, the banks cannot force repayment, but any borrower who fails to pay will be shut out of the banking sector. Later on, we derive conditions to ensure voluntary repayment to determine $\bar{l}$.

We can represent the bank's problem as

$$\mathscr{L} \max_{l} = (i - i_d)l - \lambda_L(l - \bar{l}) + \lambda_\Gamma(u(q_b) - (1+i)l\phi - \Gamma)$$

where λ_L is the Lagrange multiplier on the constraint on a bank's lending. Also, λ_Γ is the Lagrange multiplier on the participation constraint of the borrower. The bank's first order condition for how much to loan becomes:

$$i - i_d - \lambda_L + \lambda_\Gamma \Big[u'(q_b) \frac{dq_b}{dl} - (1+i)\phi \Big] = 0$$

As described in BCW, with $i - i_d > 0$ the bank will loan as much as possible. Therefore, $u'(q_b)\frac{dq_b}{dl} = (1+i)\phi$ and it will be the case that $\lambda_\Gamma > 0$. However, competition will drive interest rates to parity: $i = i_d$.

From $q_b = (1-\psi)(m+l)/p$

$$\frac{dq_b}{dl} = \frac{(1-\psi)}{p} \Rightarrow \frac{dl}{dq_b} = \frac{p}{(1-\psi)}$$

As p increases, it costs more to obtain goods in the DM and the buyer will choose to borrow more funds. Since $p\phi = c_1'(q_s)$

$$\frac{dq_b}{dl} = \frac{(1-\psi)\phi}{c_1'(q_s)}$$

Thus, a bank's decision on how much to lend becomes

$$i - i_d - \lambda_L + \lambda_\Gamma \Big[u'(q_b) \frac{(1-\psi)\phi}{c_1'(q_s)} - (1+i)\phi \Big] = 0$$

Since the bank will receive no profits due to free entry:

$$\phi\lambda_\Gamma\Big[\frac{(1-\psi)u'(q_b)}{c_1'(q_s)} - (1+i)\Big] = \lambda_L$$

$$\frac{(1-\psi)u'(q_b)}{c_1'(q_s)} = 1 + i + \frac{\lambda_L}{\phi\lambda_\Gamma}$$

With $\bar{l} = \infty$, $\lambda_L = 0$ and the loan amount satisfies $\frac{u'(q_b)}{c_1'(q_s)} = \frac{(1+i)}{(1-\psi)}$. This is the same amount that satisfies the buyer's desired demand since the loan constraint does not bind. However, if enforcement of loan contracts is not costless or banks are unable simply to enforce contracts, then the lending constraint may bind so that $\lambda_L > 0$. In this case, trade in the DM will be even more inefficient with $\frac{u'(q_b)}{c_1'(q_s)} > \frac{(1+i)}{(1-\psi)}$. In comparison to BCW, the degree of inefficiency is stronger because of the rate of token loss.

MONEY SUPPLY AND THE ENTREPRENEUR'S PROBLEM

With no central bank, entrepreneurs determine the quantity of money in the economy. For the entrepreneur's problem, we first denote $M \in \mathbb{R}_+$ as the per capita supply of currency in the current period and $M_{-1} \in \mathbb{R}_+$ as the supply of currency in the previous period. Next, we denote $\Delta \in \mathbb{R}_+$ as the entrepreneur's net circulation of newly minted tokens in the current period (or mined).

The evolution of the money supply depends on the rate of currency loss. Along with the probability of currency loss ψ, the law of motion for tokens depends on the type of bank available and consumer behavior. We continue with the case where only type I banks are active.

With type I banks, we can denote the law of motion for the total amount of tokens at the beginning of the DM in the current period by

$$M = \Delta + (1 - \psi)M_{-1}.$$

with ψM_{-1} representing the currency loss from the First Market. Consumers are responsible for the security of tokens in the money supply.

We denote the entrepreneur's money holdings with $M^e \in \mathbb{R}_+$. If we assume that $\Delta \geq 0$, then the the entrepreneur's budget constraint can be shown as

$$x + \phi M^e = \phi\Delta + \phi M^e_{-1} \quad \forall t \geq 0$$

The entrepreneur will not participate in the First Market. Further, unless $\beta(1 - \psi)\phi \geq \phi_{-1}$, the entrepreneur will choose not to hold currency across time periods. Thus, $M^e_{-1} = M^e = 0$, and the budget constraint can be rewritten as

$$x = \phi\Delta.$$

The entrepreneur will consume an amount of the CM good equal to the amount of new tokens created. Since $x \geq 0$, then $\Delta \geq 0$.

Entrepreneurs take the price of tokens as given. We denote $\Delta^* \in \mathbb{R}_+$ as the solution to the profit maximization problem of an entrepreneur:

$$\Delta^* \in arg \max_{\Delta \in \mathbb{R}_+} \left[\phi\Delta - c_2(\Delta)\right].$$

The amount of new token creation will adjust accordingly to any price change in tokens. Therefore, the solution to the entrepreneur's profit maximization problem satisfies:

$$\phi = c_2'(\Delta^*).$$

The entrepreneur will create additional tokens until the disutility from the work needed to create an additional token is equivalent to the real value of the token. For each period, the optimal amount of tokens for the entrepreneur to produce is $\Delta^* = (c_2')^{-1}(\phi)$.

MARGINAL VALUE OF MONEY

In choosing the amount of money balances to acquire in the CM, consumers need to consider the probability of being a buyer $(1-n)$ and being a seller n in the following DM. A buyer values their tokens for the first market good they can purchase. Also, if a buyer, they need to consider the amount of money that can be borrowed from banks. This will reduce the amount of tokens they carry into the first market. If a seller, the tokens derive worth from the store of value function. However, they also take the probability of token loss into account.

Thus, the marginal value of money starts with:

$$V'(m) = (1-n)\left[u'(q_b)\frac{\partial q_b}{\partial m} + W_m\Big((1-\psi) - p\frac{\partial q_b}{\partial m} + (1-\psi)\frac{\partial l}{\partial m}\Big) + W_l\frac{\partial l}{\partial m}\right]$$

$$+n\left[-c_1'(q_s)\frac{\partial q_s}{\partial m} + W_m\left((1-\psi) + p\frac{\partial q_s}{\partial m} - (1-\psi)\frac{\partial d}{\partial m}\right) + W_d\frac{\partial d}{\partial m}\right]$$

Money brought into the DM by a seller will not affect his production of the DM good.

Therefore, we can substitute $\partial q_s/\partial m = 0$ for the consumer's decision on money holdings. Also, with $i_d > -\psi$, a seller will deposit all money holdings and we need to substitute $\partial d/\partial m = 1$. Lastly, use the envelope conditions for cash, loans and deposits to get

$$V'(m) = (1-n)\left[u'(q_b)\frac{\partial q_b}{\partial m} + \phi\Big((1-\psi) - p\frac{\partial q_b}{\partial m} + (1-\psi)\frac{\partial l}{\partial m}\Big) - \phi(1+i)\frac{\partial l}{\partial m}\right] + n\phi(1+i_d)$$

If a buyer in the first market, the consumer must weigh the additional utility provided from consumption of the first market good bought with an additional unit of money. Additionally, the buyer considers the gain in utility from borrowing less funds from banks. If a seller, an additional unit of money will reduce his labor in the second market. In particular, this will depend on the interest rate paid by banks on deposits.

All of the tokens held by a buyer will be spent in the first market. Consequently, $p(\partial q_b/\partial m) = (1-\psi) + (1-\psi)\partial l/\partial m$ and the first order condition becomes:

$$V'(m) = (1-n)\left[u'(q_b)\frac{\partial q_b}{\partial m} - \phi(1+i)\frac{\partial l}{\partial m}\right] + n\phi(1+i_d)$$

As a buyer, the net role of money demand is to purchase more goods and reduce borrowing costs. Now we can substitute $\frac{\partial l}{\partial m} = \frac{p}{(1-\psi)}\frac{\partial q_b}{\partial m} - 1$

$$V'(m) = (1-n)\left[\frac{\partial q_b}{\partial m}(u'(q_b) - \frac{\phi(1+i)p}{(1-\psi)}) + \phi(1+i)\right] + n\phi(1+i_d)$$

Take the equilibrium condition from the credit market $\frac{u'(q_b)}{c'(q_s)} = \frac{(1+i)}{(1-\psi)}$ and substitute

$c_1'(q_s) = p\phi$. Now we can substitute $u'(q_b) = p\phi\frac{(1+i)}{(1-\psi)}$

$$V'(m) = (1-n)\frac{(1-\psi)u'(q_b)}{p} + n\phi(1+i_d) \tag{19}$$

which is similar to BCW, but the marginal value of tokens is lower in our setup due to token loss. In regards to the first term, an extra unit of money allows consumers to buy more goods depending on the rate of token loss and prices in the DM. The second term reflects the store of value which is enhanced by depositing funds with banks. These funds are not susceptible to token loss because buyers are responsible for all tokens that are borrowed from banks. In a symmetric equilibrium $q_s = \frac{1-n}{n}q_b$. If $m < m^*$ then $0 < q_b < q^*$ and so $\partial q_b/\partial m > 0$. Thus, $V''(m) < 0$. When $m \geq m^*$ then $q_b = q^*$ implying $\partial q_b/\partial m = 0$. Thus, $V''(m) = 0$. Hence, $V(m)$ is a concave function.

From $c_1'(q_s) = p\phi$ in (4)

$$V'(m) = \phi\left[(1-n)\frac{(1-\psi)u'(q_b)}{c'(q_s)} + n(1+i_d)\right] \tag{20}$$

Equation (5) reflects the dual role of money. When a buyer, he receives $(1-\psi)u'(q_b)/c_1'(q_s)$ due to the medium of exchange role. If the agent is a seller, he lends money and receives $1+i_d$ from the return on deposits. In particular, the extent of token loss reduces the value of money.

III. EQUILIBRIUM WITH ENFORCEMENT

First, as a benchmark, we consider that all types of banks can enforce repayment at no cost. As default is not possible, there are no borrowing constraints for agents. However, banks cannot dictate terms of trade between private agents. Banks in this case will post interest rates on loans that will clear the market in equilibrium. Also, there will be no cap on total token production.

In comparison to FS, we establish that a monetary equilibrium with price stability exists if the costs of producing new tokens are strictly convex where $c_2'(0) = 0$. To do so, we find the nominal interest rate that will clear the market. From $\phi = \beta V_{+1}'(m_{+1})$, substitute into (5) to get:

$$\frac{\phi_{-1}}{\beta} = \phi\left[(1-n)\frac{(1-\psi)u'(q_b)}{c_1'(q_s)} + n(1+i_d)\right]$$

Now use $q_s = \frac{1-n}{n}q_b$ and $\frac{\phi_{-1}}{\phi} = \gamma$ and subtract both sides by the real value of a token ϕ:

$$\frac{\gamma-\beta}{\beta} = (1-n)\left[\frac{(1-\psi)u'(q_b)}{c_1'(\frac{1-n}{n}q_b)} - 1\right] + ni_d \tag{21}$$

The right-hand side measures the value of bringing a marginal unit of money into the first market. The first term is the net benefit (marginal benefit minus marginal cost) of spending the unit of money on goods when a buyer and the second term is the net benefit from depositing a unit of idle balances when a seller. The first term is lowered by the potential to lose currency in the first market.

Since banks can enforce repayment, they will be unconstrained and $\bar{l} = \infty$. Consequently, $\lambda_l = 0$ and $\frac{u'(q_b)}{c_1'(q_s)} = \frac{1+i}{(1-\psi)}$ in (6) yields

$$\frac{\gamma - \beta}{\beta} = (1-n)i + ni_d$$

The first term now represents the interest saving from borrowing one less unit of money when a buyer.

Zero profits for banks implies $i = i_d$:

$$\frac{\gamma - \beta}{\beta} = i \tag{22}$$

In terms of q_b:

$$\frac{\gamma - \beta}{\beta} = \frac{(1-\psi)u'(q_b)}{c_1'(\frac{1-n}{n}q_b)} - 1 \tag{23}$$

DEFINITION 1. When repayment of loans can be enforced, a monetary equilibrium with credit is an interest rate i satisfying (7) and a quantity q_b satisfying (8).

Of note, the rate of token loss plays no part in the market clearing interest rate. However, the rate of currency loss will impact the quantity of trade in the DM. As the rate of token loss increases, consumers will retain less of their money holdings for trade. As a result, consumers will demand less tokens and consumption levels fall in the DM.

Next, we determine the steady state stock of tokens for a given price $\overline{\phi}$. We consider three different initial values for tokens: $\phi = \overline{\phi}$, $\phi > \overline{\phi}$ and $\phi < \overline{\phi}$.

LEMMA 1: *Suppose that the cost of producing tokens is $c_2(\Delta) = \Delta^2$ and there is no cap on token production. Under these conditions, entrepeneurs after n periods will produce just* *enough currency to replace what is lost each period* $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$*. The steady-state money supply is* $\overline{M} = \frac{1}{2\psi}\overline{\phi}$.

In summary, a stable price is the only possibility when there is no cap on total token production. By comparison, in FS, there is always an incentive for entrepreneurs to produce additional tokens and the supply of money will grow unbounded without token loss if the cost of producing tokens is strictly convex.[25] However, in our framework, entrepreneurs must have sufficient incentive to continually produce tokens over time to make up for token loss each period. Next, we establish a steady state price of tokens, and use this to determine the equilibrium money supply. In order to do this, we assume functional forms for the cost function for sellers and utility function for buyers.

PROPOSITION 1: *Suppose that the cost of producing tokens is* $c_2(\Delta) = \Delta^2$ *and there is no cap on their production. In addition, consider the following functions for utility from consumption,* $u(q) = 2q^{\frac{1}{2}}$ *and the cost of producing DM goods* $c_1(q) = q$*. Under these conditions, the steady state money stock is* $\overline{M} = \sqrt{\frac{1}{2\psi}\beta^2(1-\psi)n}$*. Moreover, the value of tokens in the centralized market is* $\overline{\phi} = \sqrt{2\psi\beta^2(1-\psi)n}$ *and the price level in the decentralized market is* $\overline{p} = \frac{(1-n)}{n\sqrt{2\psi\beta^2(1-\psi)n}}$*. In order to clear the market, banks must pay* $\overline{i} = \frac{1-\beta}{\beta}$*. Equilibrium consumption in the DM is* $\overline{q_b} = \left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2$*. It is decreasing in the rate of token loss and is increasing in* β *and* n*.*

Note that both the steady-state value of a unit of money and the price of goods are non-monotonic in the rate of token loss. If $\psi \in (0, 1/2]$, the steady state price $\overline{\phi}$ is increasing in ψ. This occurs as entrepreneurs require a higher price to replace the

[25]In order to achieve a stationary money supply, the FS environment needs a cap on total token production similar to Bitcoin. In Martin and Shreft (2006), agents are given what is known as L-beliefs. That is, agents do not value currency minted after a specified threshold. This will ensure agents stop minting currency at some point and the money supply is bounded.

increased amount of token loss. However, the steady state price decreases from $\psi \in [1/2, 1)$. At extreme rates of currency loss, consumers have little incentive to carry tokens into the first market and the price of tokens must be lower.

With $\psi > 0$, there is less production in the DM. The currency loss decreases production in two ways. First, consumers will not be able to use all money brought into the DM. Second, the possibility of losing tokens discourages consumers from bringing money into the DM. Next, we find the equilibrium condition with no intermediation and compare with our results.

We can compare this allocation to a setting without financial intermediation by setting $i_d = -\psi$ and solving for the quantity consumed in the absence of financial intermediation, $\tilde{q}_b$. The equilibrium condition without intermediaries can be represented by

$$\frac{1-\beta}{\beta} = (1-n)\left[\frac{(1-\psi)u'(\tilde{q}_b)}{c_1'(\frac{1-n}{n}\tilde{q}_b)} - 1\right] - n\psi \tag{24}$$

Of note, currency has a lower store of value in the presence of token loss. The following lemma highlights the importance of the diminished store of value to our result.

LEMMA 2: *The quantity of consumption in the DM in the presence of intermediaries is higher than in an economy without intermediaries as long as* $1 > \beta(1-\psi)$. *In other words, as long as* $\psi > 0$.

By Lemma 2, it's unambiguous that credit increases the production of the good in the DM. As consumers discount the future less, the difference in production levels of the DM good decreases. However, in comparison to BCW, consumers cannot completely self-insure at $\beta = 1$. That is, currency loss will discourage consumers from carrying enough money into the DM to fully self-insure. In this manner, the contribution of credit in the economy depends on the extent of token loss.

Importantly, we show financial intermediaries without safekeeping improve the allocation in the face of currency loss. In contrast, banks in He et al. (2005, 2008) offer demand deposits which are less likely to be stolen than physical cash. Here, type I banks shield the seller from token loss, but this is due to the responsibility of security being transferred to borrowers. No service or inside money is being provided to reduce the rate of token loss, but intermediation still offsets some of the loss of production due to token loss. Consequently, due to concerns of anonymity, institutions with similar properties to type I banks could play an important role in increasing the circulation of cryptocurrency.

Now let's compare the equilibrium with credit and currency loss to the model in FS. In particular, we compare to the FS case with a cap on total production that leads to a stable price equilibrium. However, to get a better comparison, we will replace bargaining in the FS model with competitive pricing. For the stable price, substitute $\gamma = 1$ and $\psi = 0$ since there is no currency loss.

LEMMA 3: *If* $\psi < \frac{n(1-\beta)}{1-n\beta}$, *production of the DM good is greater than the framework of FS with competitive pricing under a constant money supply.*

Similar to the content of Lemma 2, Lemma 3 provides additional insights regarding the contribution of financial intermediaries. Notably, in FS, consumers would be less willing to carry tokens into the DM because as a seller the tokens would be idle. In this way, less consumption will occur in the DM because the buyer's liquidity constraint binds at a lower level of money holdings. At lower rates of token loss, the gains from being able to deposit tokens outweigh the potential loss of tokens for spending. Further, as the number of sellers increases, the gains from holding money would be higher and individuals would tolerate somewhat higher rates of token loss. Next, we include the CM to find hours worked and make a comparison to BCW.

For consumers entering the CM, the expected amount of work is

$$\overline{h} = x^* + \frac{\psi(1-n)}{(1-\psi)} c_1'(\overline{q}_s)\overline{q}_b \tag{25}$$

In comparison to BCW, where $\overline{h} = x^*$, there will be more hours worked in the centralized market due to token loss. $(1-n)$ represents the portion of consumers who are buyers and $\frac{\psi}{(1-\psi)} c_1'(q_s)q_b$ represents the extra work done to make up for the money and loans lost. Next, we calculate hours worked without intermediaries to get

$$\tilde{h} = x^* + \frac{\psi}{(1-\psi)} c_1'(q_s)\tilde{q}_b \tag{26}$$

We get the following corollary from comparing hours worked with and without intermediaries.

COROLLARY 1. *Suppose that* $1 > \beta > \frac{1-\sqrt{1-n}}{n(1-\psi)}$. *Under this condition, financial intermediation leads to less hours worked in the CM and improves welfare in all cases.*

In comparing (11) to (10), it is easy to see that without banks a larger share of the money spent in the DM will need to be recovered. In contrast to the model with no credit, buyers are unable to borrow funds so the money brought into the market by sellers is idle. Sellers will not be shielded from currency loss, and will have to work to recover the tokens lost in the DM.

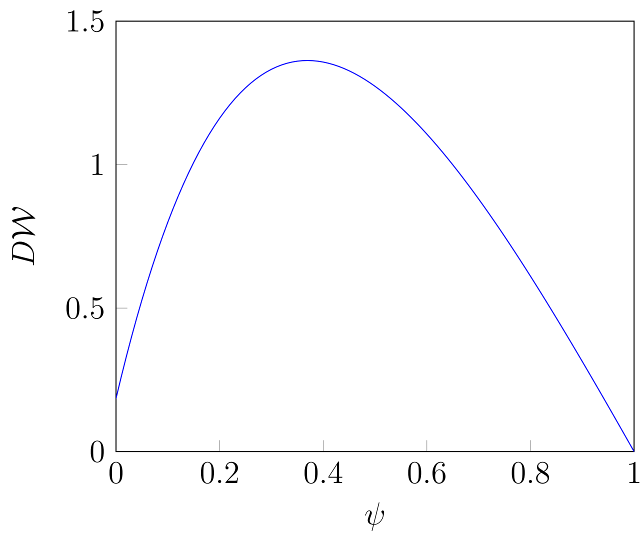

Figure 1: Difference in Welfare - Type I Banks

However, consumers carry more tokens into the DM at higher rates of token loss with financial intermediaries. As a result, consumers work more hours despite needing to recover a smaller portion of their tokens.

Notably, in figure 1, we can examine the contribution of financial interemediation in relationship to the extent of token loss, ψ. To do so, we assign a discount factor $\beta = 0.95$ and portion of sellers $n = 0.4$. As one can observe, the difference in welfare is positive and is increasing in the rate of token loss at lower rates of token loss. In particular, the availability of credit ensures that tokens are not idle in the DM. Consequently, with an increase in DM production, consumers see an increase in welfare even when more hours are worked in the CM. However, the difference in welfare starts to lower at higher rates of token loss. This is due to a dramatic drop in economic activity in the DM due to the high likelihood of a consumer losing his tokens. The maximum contribution of intermediation to welfare takes place when the probability of token loss is near 40%.

Next, we examine the source of the improvement in welfare. Does the buyer or seller or both benefit from the availability of intermediaries? Similar to BCW, we demonstrate that all the gains in welfare from intermediation result from the benefit it provides for a seller.

PROPOSITION 2. *The gain in welfare from financial intermediation is due to the fact that it allows payment of interest to depositors and shields the seller from token loss.*

As in BCW, the proof shows that in equilibrium agents are indifferent between borrowing and financing their consumption. Yet, in contrast to BCW, financial intermediaries play an additional role of shielding sellers from token loss. With borrowers able to use the seller's currency, there is a reduction in the total amount of currency needed for similar levels of consumption in the DM. By shielding sellers from token loss, banks reduce the aggregate amount of tokens that need to be replaced in the CM. Instead of *all* consumers working to replace lost tokens, it is now only the portion of consumers who were buyers doing so. This is why at lower rates of token loss, if currency loss increases, the difference in welfare increases more in favor of the market with credit.

CAP ON TOKEN PRODUCTION EACH PERIOD

In this section we will assume that there is a cap on token production each period. We will denote the token production each period as αM_t. Under certain circumstances, this will allow for a deflationary equilibrium with the amount of tokens decreasing each period.

LEMMA 4: *Suppose there is a cap on token production $\hat{\Delta}_t = \alpha M_t$. If the cap is lower than the amount of tokens lost each period $\alpha M_t < \psi M_t$, the supply of tokens will decrease at a rate $\gamma = 1 - (\psi - \alpha)$.*

To summarize, there are two possibilities when there is a cap on token production each period. First, with $\hat{\Delta}_t = \alpha M_t \geq \psi M_t$, entrepreneurs will eventually produce just enough tokens to replace the currency lost. Second, with $\hat{\Delta}_t = \alpha M_t < \psi M_t$, entrepreneurs will not be able to produce as many tokens as they would like. This leads to a decrease in the money supply each period. By comparison, in the framework of FS, a cap greater than zero on each period's token creation would not be enough to prevent the supply of money from growing unbounded. Without token loss in the FS framework, the supply of tokens still grows unchecked but at a slower rate. Next, we find the equilibrium quantity of consumption in the DM.

From our previous solution for equilibrium quantity $q = \left[\frac{\beta(1-\psi)n}{\gamma(1-n)}\right]^2$ and γ from Lemma 4:

$$q_b = \left[\frac{\beta(1-\psi)n}{[1-(\psi-\alpha)](1-n)}\right]^2$$

In each succesive period, ϕ will increase at a rate $\frac{1}{1-(\psi-\alpha)}$ and the money supply will decrease at a rate $1-(\psi-\alpha)$. In comparison to the case with no cap, consumers will carry a larger real value of tokens into the DM. As a result, the quantity of consumption of the DM good will be higher.

Next, we look at the nominal interest rate. Substitute $\gamma = 1-(\psi-\alpha)$ into equation (7)

$$\frac{1-(\psi-\alpha)-\beta}{\beta} = i$$

In terms of q_b, substitute $\gamma = 1-(\psi-\alpha)$ into equation (8)

$$\frac{1-(\psi-\alpha)-\beta}{\beta} = \frac{(1-\psi)u'(q_b)}{c'(\frac{1-n}{n}q_b)} - 1$$

The cap on token production each period leads to a lower nominal interest rate. In fact, the buyer will be willing to borrow more due to the lower cost of loans. The seller will receive a lower return on deposits, but will benefit from the increased value of the tokens the following period. This entices consumers to carry a larger real value of tokens to the DM and leads to greater consumption of the DM good. This behavior is also responsible for lower interest rates on loans.

Note that it is possible to have negative interest rates with a cap with a high rate of currency loss and a low cap. This occurs if $\psi > 1 + \alpha - \beta$. Sellers will accept negative interest rates on their deposits in order to avoid token loss. We now turn to analyzing the benefits of safekeeping from financial intermediaries.

TYPE II BANKS

In this section, type II banks will be available to consumers in the DM. In contrast to type I banks, these banks are in charge of the private key for tokens that are borrowed. Tokens will not be subject to loss as they are held by the bank until the requested transfer for the borrower during exchange in the DM occurs. However, tokens brought in the DM by buyers will be subject to token loss prior to their transactions. From this perspective, the role of intermediaries is more important because of their actions in the economy's payments system and the additional safekeeping function.

The consumer's problem in the second market will be the same as when type I banks are available. However, for the first market, we need to take into account the fact that the loan will not be susceptible to loss because these tokens are protected by banks who process payments on behalf of buyers. An agent with m units of money at the beginning of the first market has expected lifetime utility

$$V(m) = (1-n)[u(q_b) + W((1-\psi)m + l - pq_b, l)]$$

$$n[-c_1(q_s) + W((1-\psi)(m-d) + pq_s, d)], \tag{27}$$

where the buyer does not occur any loss from his loan. There is no need to look at the seller's decision since the rest of the problem is identical to our previous case.

BUYERS' DECISIONS

After the consumer receives his shock to become a buyer, he must determine the size of the loan to take out from the bank. Type II banks have a safe-keeping function, but the buyer may still lose some of his tokens prior to exchange in the DM. If an agent is a buyer in the first market, his problem becomes

$$\max_{q_b, l} [u(q_b) + W((1-\psi)m + l - pq_b, l)]$$

$$s.t. \quad pq_b \leq (1-\psi)m + l, \quad l \leq \bar{l},$$

The buyer also faces a constraint on the loan size that is bounded above by $\bar{l}$. The buyer's first order condition for the quantity of the first market good q_b is the same as our case with only type I banks.

$$u'(q_b) = c_1'(q_s)\Big(1 + \frac{\lambda}{\phi}\Big)$$

Since all of the loan is retained for spending, the buyer's choice for a loan will be different. Without token loss, the loan will buy more of the DM good than the case with type I banks. The buyer's first order condition for l is

$$W_m + \lambda = -W_l + \lambda_l$$

With the addition of type II banks, a buyer will value each unit of loans as cash at a higher rate W_m rather than $(1 - \psi)W_m$. Similarly, the buyer values the relief to the constraint on money holdings at a higher rate λ rather than $(1 - \psi)\lambda$. Both of these results are due to the bank's safekeeping. This will lead to the buyer taking out a larger size loan.

From the previous envelope conditions: $W_m = \phi$ and $W_l = -\phi(1 + i)$

$$\lambda = \phi i + \lambda_l$$

Upon substituting for λ, the quantity traded in the first market satisfies:

$$\frac{u'(q_b)}{c_1'(q_s)} = 1 + i + \frac{\lambda_l}{\phi}$$

In contrast to the setting with only type I banks, the rate of currency loss will play no direct role in the loan decision. That is, through type II banks, the buyer will take out a larger loan at similar interest rates. From this perspective, the additional safe-keeping function of such intermediaries promotes credit market activity. If the borrowing constraint does not bind,

$$\frac{u'(q_b)}{c_1'(q_s)} = 1 + i$$

In this case the buyer will borrow up to the point where the marginal benefit of borrowing equals the marginal cost. He then spends all his money and consumes $q_b = ((1-\psi)m + l)/p$. However, trades are inefficient as long as $i > 0$. Unlike the case with type I banks, interest rates will never be negative.

A BANK'S PROBLEM

We move our focus to the type II banks. For such banks, they play an additional function other than to intermediate between borrowers and lenders. They also provide safe-keeping and process payments for borrowers. However, we assume this is costless and will not affect the bank's problem.

Since loans are not susceptible to token loss, the loan amount satisfies $\frac{u'(q_b)}{c_1'(q_s)} = 1+i$ when repayment of loans is costless to enforce. Also, if loans are not enforceable, the borrowing constraint will bind and the loan amount satisfies $\frac{u'(q_b)}{c_1'(q_s)} > 1+i$. In comparison to our previous case, the degree of inefficiency is not as severe due to the absence of token loss from borrowed funds.

MONEY SUPPLY AND THE ENTREPRENEUR'S PROBLEM

With type II banks, we can denote the law of motion for the total amount of tokens at the beginning of the DM in the current period by

$$M = \Delta + (1 - (1-n)\psi)M_{-1}.$$

In contrast to an economy with only type I banks, only the tokens brought into the DM by buyers will be subject to loss $(1-n)M_{-1}$. That is, buyers are only responsible for the security of the tokens carried into the DM. Type II banks will shield the buyer's borrowed funds from token loss until the time of transaction. With banks providing safekeeping for the seller's deposited funds, the aggregate rate of token loss will decrease as the number of sellers increases. Everything else will be identical to the previous case.

MARGINAL VALUE OF MONEY

We now look at the role that the safekeeping function provided to borrowers has on activity in the DM. Yet, the condition regarding the marginal value of tokens brought in the DM is the same as an economy with only type I banks. Identical to an economy with type I banks, the seller does not suffer token loss for the money deposited. However, as we show below, the consumer will borrow a larger amount for a given interest rate because of safekeeping provided by banks.

We consider the effect of safekeeping of deposits on nominal interest rates. Zero profits for the banks implies $i = i_d$ therefore:

$$i = \frac{1 - \beta(1 - (1-n)\psi)}{\beta(1 - (1-n)\psi)} \tag{28}$$

The interest rate will be higher with type II banks that are active. In terms of q_b:

$$\frac{1-\beta}{\beta} = \frac{(1 - (1-n)\psi)u'(q_b)}{c'(\frac{1-n}{n}q_b)} - 1 \tag{29}$$

The quantity of the DM good that satisifies the above condition is higher than the case with only type I banks. With the same condition for the marginal value of money, why does a consumer bring more money into the DM? At a given interest rate, the buyer would prefer to borrow all the money for purchases from banks. However, the market would not clear under these conditions. Notably, from condition (13), we see that the interest rate is higher than with type I banks. That is, the buyer is willing to pay a higher rate on a loan due to the safekeeping function of type II banks. As the rate of currency loss increases, the value of the safe-keeping function of the banks increases. Therefore, the rate buyers pay on a loan is increasing in the rate of token loss. Next, competition between banks drives the interest rate on deposits to the same as the higher rate on loans. In turn, the higher interest rate paid on deposits entices consumers to bring a higher real value of money needed to meet condition (14) as they may be sellers who earn returns on their deposits with banks. Next, we determine the steady state stock of tokens for a given price $\overline{\phi}$.

LEMMA 5: *Suppose that the cost of producing tokens is $c_2(\Delta) = \Delta^2$ and there is no cap on token production. Under these conditions, entrepeneurs after n periods will produce just enough currency to replace what is lost each period $\Delta^*_{+n} = \overline{\Delta}^* = (1-n)\psi\overline{M}$. The steady-state money supply is $\overline{M} = \frac{\overline{\phi}}{2(1-n)\psi}$ which is higher than an economy with type I banks.*

The extent of the increase in money supply depends on the proportion of sellers in the economy. The structure of the proof closely follows that of Lemma 1. The only difference is the substitution of new aggregate rate of currency loss $(1 - n)\psi$ which is lower than under type I banks. In order to see how activity in the DM is affected, we can use our equilibrium conditions to establish the steady-state values.

PROPOSITION 3: *Suppose that the cost of producing tokens is $c_2(\Delta) = \Delta^2$ and there is no cap on their production. In addition, consider the following functions for utility from consumption, $u(q) = 2q^{\frac{1}{2}}$ and the cost of producing DM goods $c_1(q) = q$. Under these conditions, the steady state money stock is $\overline{M} = \sqrt{\frac{\beta^2(1-(1-n)\psi)n}{2(1-n)\psi}}$. Moreover, the value of* *tokens in the centralized market is* $\overline{\phi} = \sqrt{2(1-n)\psi\beta^2(1-(1-n)\psi)n}$ *and the price level in the decentralized market is* $\overline{p} = \frac{1-n}{n\overline{\phi}} = \frac{(1-n)}{n\sqrt{2(1-n)\psi\beta^2(1-(1-n)\psi)n}}$*. The real value of tokens* $(\overline{\phi})$ *increases with the rate of token loss as long as* $\psi < \frac{1}{2-n}$*. For larger values of* ψ*, the real value of tokens falls. In order to clear the market, banks must pay* $\overline{i} = \frac{1-\beta(1-(1-n)\psi)}{\beta(1-(1-n)\psi)}$*. Equilibrium consumption in the DM is* $\overline{q} = \left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2$*. Similar to consumption with type I banks, it is decreasing in the rate of token loss and is increasing in* β *and* n*. However, in contrast to an economy with type I banks, the decrease in consumption from an increase in the rate of currency loss is dampened.*

The availability of type II banks increases the steady state money supply. With banks safekeeping and processing payments from borrowed tokens, consumers are willing to bring more tokens into the DM. Along with the decrease of tokens vulnerable to loss in the DM, the larger quantity of tokens leads to greater consumption in the DM. Next, we compare the change in the price of tokens.

In order to explain the price behavior of tokens in the CM, we look at the relationship between the rate of token loss and the real price of tokens. If $\psi \in (0, 1/2(1-n)]$, the steady state price $\overline{\phi}$ is increasing in ψ. In comparison, with only type I banks, the price of tokens begins to fall at $\psi = 1/2$. Because of the safekeeping function of type II banks, consumers can tolerate a higher rate of token loss from their individual money holdings $(\overline{m})$. As the number of sellers increases, less of the money supply is susceptible to the higher rate of token loss. Therefore, consumers can purchase more of the DM good with tokens and will place more value on them. As a result, the economy with type II banks will have a larger money supply. In this manner, although a smaller percentage of tokens will be lost each period, there will be more tokens lost and a higher price on tokens is needed as an incentive to entrepreneurs to replace the lost currency. Now, we will contrast the results with type I banks. The comparison will highlight the benefits of banks providing safekeeping of the seller's tokens. First, we compare the allocation to a setting with type I banks.

LEMMA 6: *The quantity of consumption in the DM in the presence of type II intermediaries is higher than in an economy without intermediaries and an economy with type I intermediaries.*

With type II banks available, the quantity of consumption in the DM sees a larger increase than with the addition of type I banks. Consumers bring more tokens to the DM in pursuit of the higher return on deposits. Banks are able to pay out a higher rate due to the buyers' willingness to pay a higher interest rate on loans. Buyers are willing to pay a higher interest rate due to the elimination of token loss on loans. In addition to the increase in money brought into the DM, buyers retain a higher portion of tokens due to the safekeeping provided by the banks.

Further, we also have the following Lemma which looks at the quantity of trade in the DM:

LEMMA 7: *If* $\psi < \frac{n(1-\beta)}{(1-n)(1-n\beta)}$, *production of the DM good is greater than the framework of FS with competitive pricing.*

Due to safekeeping of the seller's tokens, the condition is satisfied with a higher rate of token loss than Lemma 3. As previously mentioned, the seller's tokens remain idle in the FS framework. In comparison, not only is the seller shielded from token loss in our model, he receives a higher return on deposits due to loans no longer being subject to loss. As a result, the increase in store of value for sellers outweighs the potential loss of tokens for buyers at much higher rates of token loss. Consequently, circulation of tokens will be greater in these instances as consumers are able to withstand much higher rates of currency loss. Moreover, as the number of sellers increases, the consumer's toleration for higher rates increases.

Finally, we include the CM to find hours worked and make a comparison to type I banks to get the following corollary:

COROLLARY 2: *Suppose that* $\psi < \frac{1}{2-n}$. *Under this condition, there will be less hours worked in economies with type II banks. In addition, type II banks increase the consumer's welfare.*

As a result of safekeeping, consumers work to replace a smaller portion of tokens carried in the DM with type II banks. However, increasing rates of token loss depress the quantity of trade with type I banks more quickly. Consequently, with type II banks,

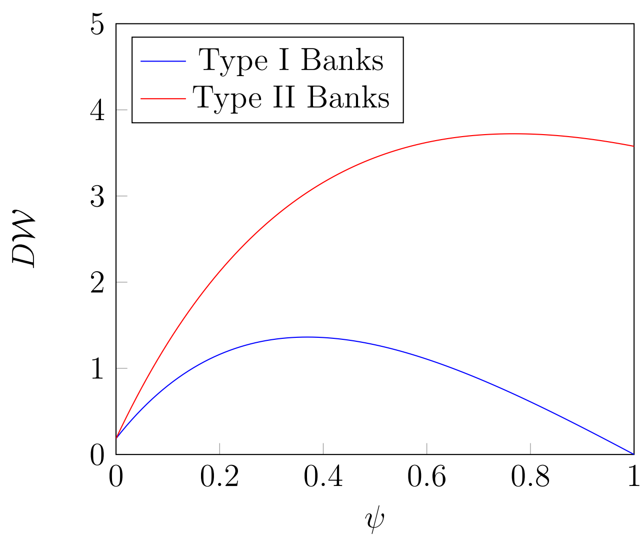

Figure 2: Difference in Welfare - Type II Banks

consumers carry a much larger sum of currency into the DM as tokens are lost more frequently. As a result of larger money holdings in the DM, the real value of tokens misplaced is higher and leads to more work in the CM. However, consumers still see an increase in welfare due to the higher levels of production in the DM. We can use the parameters from our previous examples to illustrate this.

As figure 2 illustrates, the difference in welfare gains continues to grow at high rates of token loss. Even though there is more work in the CM for type II banks, the gains from a higher level of production in the DM offset this and more. For type II banks, the increase in welfare is positive and is increasing in the rate of currency loss until a maximum around 70%. In contrast to type I banks, type II banks promote trade at the highest levels of token loss and raise welfare for consumers. Thus, it is clear that the additional safe-keeping function has an important role in the financial system. Moreover, the importance of this role increases as token loss becomes more frequent. Next, we examine how the welfare gains from intermediation with type II banks are distributed across depositors and borrowers.

PROPOSITION 4. *Due to the additional safekeeping functions of type II, a seller receives a larger return on deposits and a larger gain in welfare than when financial intermediation is provided by type I banks.*

In comparison to the economy with type I banks, interest rates and lending are higher in economies where type II banks are active. Yet, the welfare gains occur exclusively among depositors who receive higher interest rates on loans. As the rate of token loss increases, the buyer is willing to pay a higher interest rate because of the safekeeping function provided by banks on tokens that are borrowed. In turn, interest rates on loans increase to the point where the buyer is indifferent between taking out a loan and acquiring enough tokens to obtain self-insurance. Thus, similar to BCW and an economy with type I banks, a consumer will see no benefit as a buyer from taking out a loan. In contrast, as a result of the premium paid to type II banks, the seller will see a larger gain in welfare.

CAP ON TOKEN PRODUCTION EACH PERIOD

In this section we will assume that there is a cap on token production each period. We will denote the token production each period as αM_t. Under certain circumstances, this will allow for a deflationary equilibrium with the amount of tokens decreasing each period.

LEMMA 8: *Suppose there is a cap on token production $\hat{\Delta}_t = \alpha M_t$. If the cap is lower than the amount of tokens lost each period $\alpha M_t < (1-n)\psi M_t$, the supply of tokens will decrease at a rate $\gamma = 1 - ((1-n)\psi - \alpha)$.*

In our setting with type II banks, the range of caps that lead to deflation will decrease. In order for $\hat{\Delta}_t = \alpha M_t > (1-n)\psi M_t$, the cap needs to be lower in comparison to an economy with only type I banks. If $\hat{\Delta}_t = \alpha M_t > (1-n)\psi M_t$, the net decrease in the money supply is smaller with type II banks available. With less deflation, the buyer will consume less of the DM good.

From our previous solution for equilibrium quantity $q = \left[\frac{\beta(1-(1-n)\psi)n}{\gamma(1-n)}\right]^2$ and γ from Lemma 8:

$$q_b = \left[\frac{\beta(1-(1-n)\psi)n}{[1-((1-n)\psi-\alpha)](1-n)}\right]^2$$

In each successive period, ϕ will increase at a rate $\frac{1}{1-((1-n)\psi-\alpha)}$ and the money supply will decrease at a rate $1-((1-n)\psi-\alpha)$. In comparison to the case with no cap, consumers will carry a larger real value of tokens into the DM. As a result, the quantity of consumption of the DM good will be higher.

Next, we look at the nominal interest rate. Substitute $\gamma = 1 - ((1-n)\psi - \alpha)$ into equation (7)

$$i = \frac{1 - ((1-n)\psi - \alpha) - \beta(1 - (1-n)\psi)}{\beta(1 - (1-n)\psi)}$$

In terms of q_b, substitute $\gamma = 1 - ((1-n)\psi - \alpha)$ into equation (8)

$$\frac{1 - ((1-n)\psi - \alpha) - \beta}{\beta} = \frac{(1 - (1-n)\psi)u'(q_b)}{c'(\frac{1-n}{n}q_b)} - 1$$

The cap on token production each period leads to a lower nominal interest rate. The buyer will be willing to borrow more due to the lower cost of loans. The seller will receive a lower return on deposits, but will benefit from the increased value of the tokens the following period. This entices consumers to bring more tokens to the DM and leads to greater consumption of the DM good.

TYPE III BANKS

In contrast to the previous section, both buyers and sellers deposit all their money holdings with banks. Here, banks provide safe-keeping for the entire money supply in the DM and process all payments. With banks keeping track of the private key for tokens, the money supply will not suffer any token loss. As a result, the model in this section shares some of the equilibrium results from BCW. However, our model has some unique results regarding welfare. For example, in contrast to BCW and our first two cases, the addition of intermediaries does not alter the amount of work in the CM. Moreover, the improvement in welfare does not increase with the rate of token loss. Also, in addition to a seller seeing an increase in welfare from interest on deposits, a buyer will benefit from taking out a loan.

CAP ON TOTAL TOKEN PRODUCTION

By establishing zero token loss, type III banks limit the potential equilibria in the economy. With $c_2'(0) = 0$, entrepreneurs have an incentive to create tokens as long as the price is greater than zero. This will lead to an unbounded money supply and an equilibrium with runaway inflation. If a cap on token production each period $\hat{\Delta}_t = \alpha M_t > 0$ is introduced, the money supply will grow at a slower pace, but it will still be unbounded. Moreover, the economy will still be in an equilibrium with runaway inflation. Similar to FS, the unbounded money supply restricts the existence of a steady state. That is, a steady state only exist when token production at time period T is capped at zero every period $\hat{\Delta}_{t \geq T} = 0$. With a cap on total production, the money supply becomes constant and there is a stable price steady state[26]. In contrast to our earlier results, deflation is no longer possible. As deflation is no longer a possibility, an efficient allocation is not possible even with the elimination of currency loss. If a cap is placed on total token production, we get the following corollary for type III banks.

COROLLARY 3. *With $\Delta = 0$, the introduction of type III banks improves the allocation in the DM which improves welfare.*

The nominal interest rate and quantity of consumption will be identical to BCW with a growth rate of the money stock $\gamma = 1 - \psi$. With respect to BCW, consumers in the CM will work an identical number of hours $h = x^*$. Moreover, this is even the case for consumers in the absence of banks. Surprisingly, with the cap set at zero, consumers will

[26]While Bitcoin has a cap of 21 million bitcoins, cryptocurrencies such as Ethereum and Monero have no cap on total token production.

receive the same level of welfare improvement regardless of the rate of token loss. If $\gamma = 1 - \psi$, the resulting rise in the value of tokens offsets any token loss. We will demonstrate with a numerical example that the increase in welfare does not change with rate of token loss in the economy.

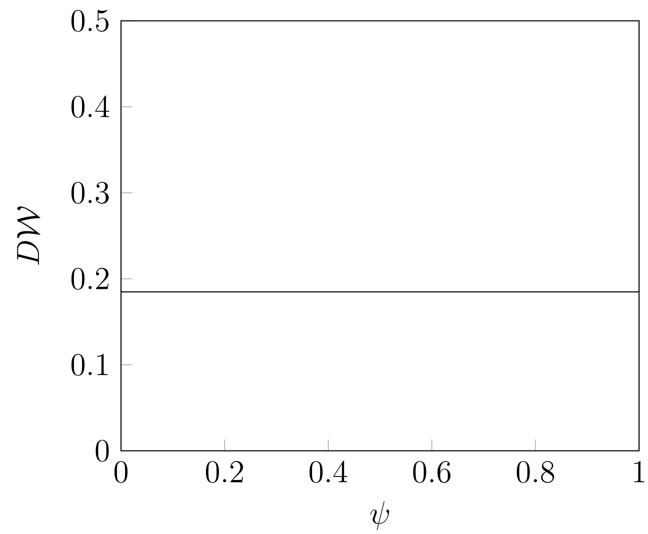

Figure 3: Difference in Welfare - Type III Banks

For the graph above illustrating the difference in welfare, we use the same parameter values as in our previous numerical examples. The difference in welfare is positive but constant with a change in the rate of currency loss. As the rate of token loss increases, the money stock will shrink by a corresponding percentage. Consequently, the increase in the value of tokens ensures the level of production remains the same in the DM. Moreover, with no new tokens created, consumers will not have to work extra in the CM to replace any lost money holdings. The importance of credit is not in eliminating token loss. Rather, similar to BCW, it is in ensuring that balances are never idle and paying out an interest on deposits.

Next, we examine the source of the improvement in welfare from type III banks. In this part of our analysis, we look at a stable price equilibrium due to banks eliminating token loss. A consumer outside of the banking system will suffer token loss without the benefit of deflation. Does the seller or buyer benefit? Or both? We demonstrate that a consumer will benefit from loans when a buyer and the seller benefits from interest received from his deposits.

PROPOSITION 5. *In addition to allowing payments of interest to depositors and shielding the seller from token loss, financial intermediation increases welfare for a buyer by eliminating token loss.*

If a consumer self-insures for consumption shocks instead of borrowing, the expected extra hours worked will be

$$\frac{\psi n(1-n)}{(1-\beta)(1-\psi)}c_1'(q_s)q_b.$$

If a consumer is a buyer, he will expect to work extra every period without access to a loan $\Sigma_{t=0}^{\infty}\beta^t(1-n) = \frac{1-n}{1-\beta}$. In particular, he will replace the portion of the extra money brought in to replace the loan $\frac{\psi n}{(1-\psi)}c'(q_s)q_b$.

Unlike the case where buyers do not deposit their funds with type II banks, the consumer benefits from taking out a loan. Identical to the case with type II banks, the buyer is shielded from any token loss on loans. However, in contrast to our previous case, the buyer will not have to pay a premium on the loan for this benefit. With the consumer knowing his tokens brought into the DM are always protected from loss, he will not require a higher interest on deposits to bring more tokens into the DM. Thus, a buyer will not have to pay a higher interest rate on loans and will work less if he borrows.

PROPOSITION 6. *With token production capped at zero, an increase in the rate of token loss has no effect on welfare improvements from financial intermediation and safekeeping of tokens.*

With a cap on total production, the level of welfare is independent of the rate of token loss and the type of bank that is active in the financial system. With type I banks, the money stock will shrink at a rate $\gamma = 1 - \psi$. Similar to an economy with type III banks, any increase in the rate of token loss leads to corresponding increase in the real value of tokens. Further, the amount of hours worked in the CM will also be independent of the rate of loss. Consequently, the welfare gains from intermediation will not depend on ψ. The same principles hold true for type II banks as well. Thus, for all three types of banks, the rate of token loss has no effect on DM production and welfare.

The same principle holds true with an inter-bank comparison. For instance, type III banks eliminate token loss, but consumers receive no inflation as a result. On the other hand, consumers receive no protection from currency loss with type I banks, but have a higher return on tokens. Any benefits from safekeeping by type III banks will be offset by the decrease in the return to the currency. The same holds true when a comparison is made between any of the banks. Therefore, with $\Delta = 0$, consumers will have identical consumption in the DM and the same level of welfare regardless of the type of bank available.

Now we can use proposition 5, lemma 6 and corollary 2 to get the following corollary.

COROLLARY 4. *The largest welfare gain occurs with the introduction of type II banks.* As shown by proposition 5, all three bank types provide the same welfare benefits with $\Delta = 0$. In all other cases, type II banks provide the largest welfare benefits. With a portion of tokens susceptible to loss, a steady state exists in the absence of a cap on total token production. The safekeeping of a portion of tokens provided by type II banks facilitates an increase in the circulation of tokens. This is a likely scenario to play out with cryptocurrency in the future. The security of cryptocurrency requires a high level of knowledge and effort. As a result, individuals will outsource security to institutions that specialize in this area of expertise. However, there will still be some individuals wary of using the banking system, and others would like to have privacy with a small portion of their money holdings. Not dissimilar to our current situation, where the use of physical cash is still popular.

IV. EQUILIBRIUM WITHOUT ENFORCEMENT

In the previous section, banks could force repayment of loans cost free. However, with the type of uncertainty surrounding regulation of cryptocurrency, it is also useful to look at an economy where this is not possible. Consequently, agents may now have an incentive to default in market 2 on a loan obtained in market 1. The only means for banks to offset the short-run benefit from defaulting on a loan is exclusion from the banking system. This occurs because banks will share the agent's repayment history with other banks leading to exclusion from any other banks upon default. For credit to exist with no enforcement, the cost of being shut out of the banking system must be greater than repayment of the loan. With this being the only available punishment for banks, the real borrowing constraint $\phi\bar{l}$ will become finite. Notably, the borrowing constraint is endogenous in which we derive conditions to ensure voluntary repayment. As the buyer exhausts all of his money balances in the first market, his expected utility from entering the second market with no money can be shown by

$$W(m) = U(x^*) - h_b + \beta V_{+1}$$

where h_b represents the buyer's production in the second market when repaying the loan.

Now consider when he does not repay the loan. The defaulting borrower will have more time for leisure in the second market due to not having to repay the loan, but will be forever excluded from the banking sector after the default occurs. Consequently, no bank will loan money to him. Further, if he deposits money the bank will confiscate the debt owed. With a hat denoting the deviator's optimal choice, the buyer's expected discounted utility becomes

$$\hat{W}(m) = U(\hat{x}) - \hat{h}_b + \beta \hat{V}_{+1}(\hat{m}_{+1}).$$

Similar to BCW, the growth rate of the money supply γ affects the value of being in the banking system $W(m)$ and the value of being excluded $\hat{W}(m)$. In addition, the rate of token loss plays a part as well.

In order for an equilibrium with money and credit to exist, a consumer must receive more utility with access to the banking system than from being excluded. Consequently, we must have $W(m) \geq \hat{W}(m)$. In turn, the borrowing constraint $\phi\bar{l}$ must satisfy

$$W(m) = \hat{W}(m).$$

In a constrained equilibrium, consumers borrowing constraint binds $\phi l = \phi\bar{l}$ and the banks ration loans. If the consumer's constraint does not bind $\phi l < \phi\bar{l}$, the result will be an unconstrained equilibrium with similar conditions to our previous sections.

TYPE I BANKS:

LEMMA 9. *With no cap on token production, the real borrowing constraint* $\phi\bar{l}$ *satisfies*

$$\phi\bar{l} = \frac{\beta}{(1+i)(1-\beta)}\left\{\Psi(q_b,\hat{q}_b) + c_1'(q_s)\Big(\frac{1-\beta}{\beta(1-\psi)}\Big)\left[\hat{q}_b - (1-n)q_b\right]\right\}, \tag{30}$$

where

$$\Psi(q_b,\hat{q}_b) = (1-n)[u(q_b) - u(\hat{q}_b)] + \frac{c_1'(q_s)}{(1-\psi)}[(1-n(1-\psi))\hat{q}_b - (1-n)q_b] \geq 0.$$

With $i > -\psi$, banks lend out all of their deposits in equilibrium so real lending satisfies

$$l = \frac{n}{1-n}M$$

With $\phi\bar{l}$ always positive, an equilibrium with credit always exists. Next, we define conditions where there is no constraint on lending from banks.

DEFINITION 2. A monetary equilibrium with unconstrained credit is a triple $(q_b, \hat{q}_b, i)$ satisfying

$$\frac{\gamma-\beta}{\beta} = (1-n)\Big[\frac{(1-\psi)u'(q_b)}{c_1'(q_s)} - 1\Big] + ni, \tag{31}$$

$$\frac{\gamma-\beta}{\beta} = (1-n)\Big[\frac{(1-\psi)u'(\hat{q}_b)}{c_1'(\hat{q}_s)} - 1\Big] - n\psi, \tag{32}$$

$$\frac{(1-\psi)u'(q_b)}{c_1'(q_s)} = 1+i \tag{33}$$

such that $0 < \phi l = nc_1'(q_s)q_b < \phi\bar{l}$, where $q_s = \frac{1-n}{n}q_b$. The above conditions will be satisfied when banks meet the demand for loans from buyers in the DM. The bank can loan the full amount without concern for default due to the high costs of being excluded from the banking system. However, under certain conditions, banks will not loan enough tokens to buyers and the borrowing constraint will bind due to default risk. Next, we define conditions where this takes place.

DEFINITION 3. A monetary equilibrium with constrained credit is a triple $(\overline{\overline{q_b}}, \hat{q}_b, \bar{\bar{i}})$ satisfying (16), (17), (18) where $nc'(\overline{\overline{q_s}})\overline{\overline{q_b}} = \phi\bar{l}$ and $\overline{\overline{q_s}} = \frac{1-n}{n}\overline{\overline{q_b}}$.

In a constrained equilibrium, the banks will indirectly ration the amount of loans. Instead of offering the interest rate on deposits from an unconstrained equilibrium, intermediaries will offer lower interest rates to borrowers and returns on deposits to sellers. The smaller return on deposits lowers the amount of tokens consumers bring into the DM. With a reduced amount of tokens in the DM, banks have less money to loan out in the constrained credit equilibrium. However, since interest rates are lower, banks do not need to worry about default and they can safely lend out all of the tokens deposited. In contrast to BCW, financial intermediation close to the Friedman rule does not cause a breakdown of financial intermediation. With positive rates of token loss, a consumer still has an incentive to repay loans.

In an unconstrained equilibrium, $c_1'(q_s)nq_b = \phi l < \phi\bar{l}$. Since $i = (1-\beta)/\beta$ in an unconstrained equilibrium from (15) we have

$$(1-\beta)(1+i)c_1'(q_s)nq_b < \beta\Psi(q_b, \hat{q}_b) + \beta + c_1'(q_s)\left(\frac{1-\beta}{\beta(1-\psi)}\right)\left[\hat{q}_b - (1-n)q_b\right] \tag{34}$$

Define

$$g(i,\beta,\psi) = (1-\beta)(1+i)c_1'(q_s)nq_b$$
$$f(i,\beta,\psi) = \beta\Psi(q_b,\hat{q}_b) + \beta + c_1'(q_s)\Big(\frac{1-\beta}{\beta(1-\psi)}\Big)\big[\hat{q}_b - (1-n)q_b\big]$$

We will let $\Delta(i,\beta,\psi) = f(i,\beta,\psi) - g(i,\beta,\psi)$. When Δ is positive it will be beneficial for a consumer to default. In comparison to BCW where only two variables determine whether a consumer will deviate, the system becomes much more complex with the addition of token loss. Moreover, it becomes too difficult to prove analytically the values of β and ψ where a constrained equilibrium exists. Instead, we use numerical examples in order to demonstrate the effects of token loss on the potential for default. For the table at the top of the next page, we will use the functional forms and parameter values from our previous numerical examples:

ψ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
q_b	0.4011	0.3249	0.2567	0.1965	0.1444	0.1003	0.0642	0.0361
$\hat{q}_b$	0.3756	0.2701	0.1908	0.1313	0.0872	0.0550	0.0321	0.0165
$f(i,\beta,\gamma)$	0.0127	0.0103	0.0081	0.0062	0.0046	0.0032	0.0020	0.0011
$g(i,\beta,\gamma)$	0.0116	0.0286	0.0398	0.0461	0.0483	0.0468	0.0422	0.0349
$\Delta(i,\beta,\gamma)$	0.0010	−0.0184	−0.0317	−0.0399	−0.0437	−0.0436	−0.0402	−0.0338
p	0.00	5.8844	4.4133	3.8522	3.6034	3.5306	3.6034	3.8522
ϕ	0.00	0.2549	0.3399	0.3894	0.4163	0.4248	0.4163	0.3894
i	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526

Table 1: Effects of currency loss w/o enforcement w/ type I banks

Based upon the values of delta, there will be an unconstrained equilibrium as long as $\psi > 0.0045$. As the rate of currency loss rises, the importance of having access to the banking system increases. If a consumer that has deviated is a seller, the cost of having tokens idle increases as the rate of currency loss rises.

Next, we look at the implications of a change in the discount rate on default. The discount rate plays an important role in a consumer's outlook on the future without access to the financial system. In order to illustrate this, we hold the rate of currency loss constant $\psi = 0.1$ and adjust β.

β	0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25
q_b	0.3249	0.2601	0.1521	0.1659	0.1089	0.0729	0.0441	0.0225
$\hat{q}_b$	0.2701	0.1944	0.1367	0.0933	0.0608	0.0373	0.0207	0.0098
$f(i,\beta,\gamma)$	0.0103	0.0275	0.0405	0.0491	0.05346	0.0534	0.0491	0.0405
$g(i,\beta,\gamma)$	0.0286	0.0352	0.0360	0.0329	0.0274	0.0207	0.0138	0.0076
$\Delta(i,\beta,\gamma)$	-0.0184	-0.0076	0.0044	0.0161	0.02603	0.0327	0.0353	0.0328
p	5.8844	6.5766	7.4536	8.6002	10.1639	12.4226	15.9719	22.3606
ϕ	0.2549	0.2281	0.2013	0.1744	0.1476	0.1207	0.0939	0.0671
i	0.0526	0.1765	0.3333	0.5384	0.8182	1.2222	1.8571	3.0

Table 2: Effects of discount rate w/o enforcement w/ type I banks

There is an unconstrained equilibrium as long as $\beta > 0.78$. However, as the consumers discount the future more, the immediate benefit of defaulting increases. Consequently, as consumers discount the future more, the banks will have to lower interest rates to counter the additional incentive to default. Next, we look at the equilibrium with no enforcement with type II banks.

TYPE II BANKS:

LEMMA 10. *With no cap on token production, the real borrowing constraint $\phi\bar{l}$ satisfies*

$$\phi\bar{l} = \frac{\beta}{(1+i)(1-\beta)}\left\{\Psi(q_b,\hat{q}_b) + u'(q_b)\frac{(1-(1-n)\psi)\hat{q}_b}{1-\psi} - u'(q_b)(1-n)q_b]\right\}, \tag{35}$$

where

$$\Psi(q_b,\hat{q}_b) = (1-n)[u(q_b) - u(\hat{q}_b)] - nc_1'(q_s)\hat{q}_b \geq 0.$$

In any equilibrium with $i > 0$, banks lend out all of their deposits so real lending satisfies

$$l = \frac{n}{1-n} M$$

We use the parameter values and functional forms of the previous example for the following table.

ψ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
q_b	0.4011	0.3544	0.3106	0.2697	0.2316	0.1965	0.1642	0.1349
$\hat{q}_b$	0.3756	0.2701	0.1907	0.1313	0.0872	0.0550	0.0321	0.0165
$f(i,\beta,\gamma)$	0.0127	0.0119	0.0111	0.0103	0.0096	0.0088	0.0081	0.0073
$g(i,\beta,\gamma)$	0.0116	0.0430	0.0687	0.0894	0.1060	0.1189	0.1288	0.1360
$\Delta(i,\beta,\gamma)$	0.0010	-0.0311	-0.0575	-0.0790	-0.0963	-0.1101	-0.1207	-0.1286
p	0.00	7.4333	5.4324	4.5949	4.1334	3.8522	3.6777	3.5761
ϕ	0.00	0.2018	0.2761	0.3264	0.3629	0.3894	0.4078	0.4193
i	0.0526	0.1198	0.1961	0.2837	0.3850	0.5036	0.6447	0.8148

Table 3: Effects of currency loss w/o enforcement w/ type II banks

In comparison to an economy with access to only type I banks, the consumer will stop defaulting at a lower rate of token loss $\psi < 0.0028$. Also, in contrast to the previous section, the cost of deviating increases at a higher rate as the rate of token loss rises. Not only does a seller lose more tokens that are idle if he has deviated, he also loses out on receiving the higher rate of interest paid on deposits. Banks are able to pay this out due to a premium that a buyer is willing to pay on loans due to the safekeeping of tokens. Due to the loss of access to the bank's safekeeping, a deviator will carry less tokens into the DM than a non-deviator. Thus, there will be a large divergence between q_b and $\hat{q}_b$ at high rates of token loss.

Next, we hold the rate of currency loss constant at $\psi = 0.1$ and adjust β. There will be an unconstrained equilibrium with $\beta > 0.76$. In comparison to type I banks, there is only

β	0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25
q_b	0.3544	0.2837	0.2209	0.1659	0.1188	0.0795	0.0481	0.0245
$\hat{q}_b$	0.2701	0.1944	0.1368	0.0933	0.0608	0.0373	0.0207	0.0098
$f(i,\beta,\gamma)$	0.0119	0.0319	0.0470	0.0570	0.06204	0.0620	0.0570	0.0470
$g(i,\beta,\gamma)$	0.0430	0.0467	0.0451	0.0397	0.0322	0.0239	0.0158	0.0086
$\Delta(i,\beta,\gamma)$	−0.03117	−0.0148	0.0019	0.0173	0.0297	0.0381	0.0412	0.0383
p	7.433	8.3079	9.4155	10.8641	12.8393	15.6925	20.1762	28.2466
ϕ	0.2018	0.1805	0.1593	0.1380	0.1168	0.0956	0.0743	0.0531
i	0.1198	0.2515	0.4184	0.6366	0.9342	1.3640	2.0395	3.2553

Table 4: Effects of discount rate w/o enforcement w/ type II banks

a small effect with the elimination of token loss on loaned funds. Next, we look at the case with type III banks.

TYPE III BANKS

LEMMA 11. *With a cap on tokens $\hat{\Delta} = 0$, the real borrowing constraint $\phi\bar{l}$ satisfies*

$$\phi\bar{l} = \frac{\beta}{(1+i)(1-\beta)}\left\{(1-n)\Psi(q_b,\hat{q}_b) + u'(q_b)(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b]\right\},$$

where

$$\Psi(q_b,\hat{q}_b) = (1-n)[u(q_b) - u(\hat{q}_b)] - nc_1'(q_s)\hat{q}_b \geq 0.$$

In any equilibrium with $i > 0$, banks lend out all of their deposits so real lending satisfies

$$\phi l = \frac{n}{1-n}\phi M$$

To guarantee repayment in a constrained equilibrium banks charge a nominal loan rate, $\bar{i}$, that is below the market clearing rate.

We will use the same functional forms and parameter values for the table below. With a rate of token loss $\psi < 0.0019$, there will be a constrained equilibrium. In comparison to

ψ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
q_b	0.4011	0.4011	0.4011	0.4011	0.4011	0.4011	0.4011	0.4011
$\hat{q}_b$	0.3756	0.2701	0.1907	0.1313	0.0872	0.0550	0.0321	0.0165
$f(i,\beta,\gamma)$	0.0127	0.0127	0.0127	0.0127	0.0127	0.0127	0.0127	0.0127
$g(i,\beta,\gamma)$	0.0116	0.0647	0.1120	0.1544	0.1926	0.2273	0.2588	0.2876
$\Delta(i,\beta,\gamma)$	0.0010	−0.0520	−0.0993	−0.1417	−0.1799	−0.2146	−0.2462	−0.2749
p	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
ϕ	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
i	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526

Table 5: Effects of currency loss w/o enforcement w/ type III banks

an economy with type I and type II banks, it requires a much lower rate of token loss to discourage default on loans. With banks eliminating all token loss, the rate of token loss has no effect on consumers with access to financial intermediaries. This makes it extremely costly to default even for consumers with a high time preference. We demonstrate this in the next table where we hold the rate of currency loss constant at 0.1.

β	0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25
q_b	0.4011	0.3211	0.25	0.1877	0.1344	0.09	0.0544	0.0277
$\hat{q}_b$	0.2701	0.1944	0.1368	0.0933	0.0608	0.0373	0.0207	0.0098
$f(i,\beta,\gamma)$	0.0127	0.034	0.05	0.0606	0.066	0.066	0.0606	0.05
$g(i,\beta,\gamma)$	0.0647	0.0641	0.0585	0.0498	0.0395	0.0288	0.0187	0.0101
$\Delta(i,\beta,\gamma)$	−0.0520	−0.0301	−0.0085	0.0108	0.0264	0.0372	0.0419	0.0398
p	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
ϕ	n/a	n/a	n/a	n/a	n/a	n/a	n/a	n/a
i	0.0526	0.1765	0.3333	0.5384	0.8182	1.2222	1.8571	3.0

Table 6: Effects of discount rate w/o enforcement w/ type III banks

There is an unconstrained equilibrium with a discount factor as low as $\beta = 0.70$. Even with a population that discounts the future a lot and a lower rate of currency loss, it is too costly in the future to lose access to this type of financial intermediary. Banks can lend to individuals with a high time preference without threat of default.

V. CONCLUSION

In this paper, we evaluate how financial intermediaries can play an important role in economic activity by providing safekeeping of cryptocurrency. As banks may provide different levels of credit extension, safekeeping, and payments processing, we consider that there are three different types of banks in the model. To begin, we start the analysis in a way that is close to BCW in that type I banks only intermediate between borrowers and lenders – they do not provide safekeeping services. However, in contrast to BCW, the borrower is responsible for the security of the cryptocurrency. We proceed to look at type II banks who provide safekeeping over tokens that are borrowed, but not over an individual's other money holdings. Finally, we look at type III banks who process all transactions between buyers and sellers.

Even without safekeeping to reduce token loss, banks can play a more important role than in the BCW model. In particular, since banks help insure that no tokens are idle, there will be less tokens needed for trade in the DM. This is valuable in our framework as there will be less tokens lost each period. Next, we show the impact of safekeeping of tokens by banks. This will further prevent token loss and reduce the costs that are associated with producing tokens which is valuable due to the high costs associated with the mining of bitcoin.[27] In addition, type III banks eliminate all token loss. Notably, once token loss is eliminated, it is only possible to have a steady state with a cap on token creation.

Finally, we also look at activity in the absence of enforcement on repayment of loans. Similar to BCW, a consumer will not want to be shut out of the banking sector due to the

[27] For example, it is well known that bitcoin mining uses electricity heavily.

interest paid on deposits. Moreover, in contrast to BCW, if a deviator is cut off from credit, the individual will bear an even higher cost due to the loss of safekeeping. Consequently, in most circumstances, consumers will want to avoid default.

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CHAPTER 4

MONETARY SHOCKS AND THE VALUE OF BITCOIN

I. INTRODUCTION

In 2008, Satoshi Nakamoto introduced Bitcoin as an alternative to the current centralized monetary system. Notably, in November of the same year, the Federal Reserve began to initiate a number of monetary programs in an effort to mitigate the adverse effects of the Great Financial Crisis on the U.S. economy. To begin, the Federal Reserve began purchasing mortgage-backed securities and other debt obligations of Fannie Mae, Freddie Mac, and Ginnie Mae. Furthermore, in the following month, the Federal Reserve exhausted use of its primary tool – the target for the federal funds rate – by moving to a range for the target from 0 to $\frac{1}{4}$ percent. How would the Federal Reserve conduct policy in the absence of its conventional tool? In response, they announced that they would principally be supporting the U.S. economy through the size of its balance sheet. Consequently, the monetary base in the United States increased to unprecedented levels over the following years.

Such actions led to fears that the inflation rates from the 1970s and early 1980s would re-emerge and provided motivation for early Bitcoin enthusiasts. Though the worries of high inflation rates have not yet been confirmed, Bitcoin has seen a dramatic rise in price to its current levels at around $60,000. In part, the initial rationale for holding Bitcoin has resurfaced as the Federal Reserve has again relied on its balance sheet and printing money since the early stages of panic in the financial system during the COVID-19 public health crisis.

Furthermore, over 7,000 different cryptocurrencies have been established since the introduction of Bitcoin by Satoshi. The most popular competitors to Bitcoin have set themselves apart by adding additional features. For example, Monero and Z-Cash offer further layers of encryption to provide more privacy and security. There are also different "stable coins" that peg their value to the exchange rate of the dollar or other sovereign currencies. On the other hand, some cryptocurrencies such as DogeCoin have gained acceptance simply because of their popularity as a meme. However, for the last several years, Ethereum has been Bitcoin's biggest competitor with a current price around $4,000.

With interest in cryptocurrency at an all-time high, the objective of this paper is to study how central bank activity affects price movements for Bitcoin and Ethereum – the two largest circulating cryptocurrencies. In particular, we look at the impact of aggregate monetary shocks on the price of Bitcoin and Ethereum. To do so, we follow Christiano et al. (1996) by employing a structural vector autogression (VAR) and identify monetary shocks as othorgonalized innovations to the federal funds rate. However, our focus is on the impact of such shocks on the price of cryptocurrency rather than aggregate real economic variables such as GDP. In particular, we use monthly data from September 2015 to December 2020 to look at the response of Bitcoin and Ethereum to various types of monetary shocks.

II. LITERATURE REVIEW

In response to the increasing enthusiasm regarding Bitcoin and other cryptocurrencies, an academic literature has started to emerge which studies the role of cryptocurrencies in the financial system and its value over time. To begin, we focus on initial theoretical contributions which focus on the role of such currencies as a medium of exchange.

Notably, Sanches and Fernandez-Villaverde (2019) find that it is possible for cryptocurrencies to exist and can avoid losses over time from inflation. In particular, this can take place if such tokens have a cap on total production. However, cryptocurrencies such as Ethereum and Monero do not have algorithms placing a limit on total token production.

A big selling point for Bitcoin is that individuals can conduct transactions by allowing users to remain pseudonymous. In order to provide pseudonymity, Bitcoin and other cryptocurrencies make use of digital cryptography where private keys are necessary for transactions. However, if the private key is misplaced, the tokens are permanently lost. Glenn and Reed (2021) introduce these properties into a version of the Sanches and Fernandez-Villaverde model and demonstrate that a stable price equilibrium can exist even if there is not a cap on production. Thus, it is possible for a cryptocurrency like Ethereum to avoid over-issuance because new tokens will need to be created over time in order to maintain the supply of tokens and avoid deflation. However, they show that the degree of token loss lowers its value and hinders its ability to promote exchange.

There have also been some empirical contributions in recent years. For example, Ali et al. (2020) focus on the store of value role of cryptocurrency. In particular, they find that Bitcoin is not a good hedge against the stock market as its value appears to be positively correlated with stock market activity. Yet, Dyhrberg (2016) finds that Bitcoin can be used alongside gold to hedge against the stock market. In addition, Demir et al. (2018) point out that Bitcoin can be used as a hedge against uncertainty over economic policy. However, our paper is the first to adopt a rigorous time-series methodology to look at how monetary policy shocks affect the price of Bitcoin and Ethereum.

III. DATA

We use monthly data from August 2015 to December 2020. The start date was chosen so that we could include Ethereum in our analysis which did not go live until July 2015. The main source for obtaining accurate cryptocurrency data comes from crypto exchanges such as Coinbase, Kraken, and Binance. As discussed by Alexander and Dakos (2019), a significant amount of papers on cryptocurrencies use non-traded price data from websites such as Cryptocompare (CC), Coinmarketcap (CM), and Coingecko (CG) instead. They rank both coins and exchanges by trading volume and market capitalisation. However, this presents an issue since a lot of the exchanges inflate their numbers artificially in order to attract more traders. New exchanges often have a 'zero fee' structure which turns transaction fees around by actually rewarding market makers for placing limit orders with the exchange's own coin. This actively encourages volume inflation because market makers earn coins through wash trading, which is still legitimate in these unregulated exchanges. In addition, the price data coming from the wash trading is not reflective of genuine trades and should be disregarded. Thus, using volume data and in our case price data that includes these exchanges would be erroneous.

Instead, we choose to take our data from exchanges with verifiable trading volume. By "verifiable trading volume," we mean reported volumes that we are confident do not include any fake trades and other manipulations that boost the exchanges' trade volume. To ensure that junk data is excluded, we use price data from Coindesk. Coindesk gathers data from the eight leading cryptocurrency exchanges.[28] In order to find a monthly price, we took the average of the daily closing price of both Bitcoin and Ethereum.

As mentioned previously, Bitcoin has a very volatile price history. The price of Bitcoin was around $250 in August of 2015. However, the price had been as high as $1,154 in November 2013, but fell sharply due to rumors of mismanagement and insufficient security practices at the then biggest crypto exchange Mt. Gox. Shortly thereafter in Feb 2014, Mt. Gox was once again the center of controversy with an announcement of a supposed hack and declaration of bankruptcy. After the exchange lost around 850,000 bitcoins, the price of the digital currency fell from $850 to $580.

The supply of Bitcoin has also contributed to its sharp rise in price over time. As mentioned previously, Satoshi programmed Bitcoin to have a finite quantity of production. The Bitcoins are introduced into the network as a reward to provide incentive for individuals to use their computer's computational power. The Bitcoin reward for Miners decreases by 50% every 4 years. In 2016, Miners saw their reward halve from 25 to 12.5 bitcoin. The halving of the Bitcoin reward doubles the stock-to-flow ratio and usually precedes large increases in the cryptocurrency's price. Along with the halving, interest from institutional investors put upward pressure on the price of Bitcoin in 2017. The

[28] Several studies from reputable research desks have been conducted to find the exchanges with verifiable trade volume, most notably from Bitwise Asset Management, Digital Asset Research and The Block. Bitfinex, Coinbase, Kraken, Bitstamp, bitFlyer, Gemini, itBit, and Poloniex were considered trustworthy by all three research desks.

dramatic increase in price from $772 in January 2017 to a new high later that year of $18,984 on December 18th can be seen in Figure 1. In part due to Bitcoin futures trading at the Chicago Mercantile Exchange (CME) and the Cboe options exchange (CBOE), the price crashed back down to $6332 by the start of February 2018. Yet, over the next two years, the price of Bitcoin was fairly stable. However, in March 2020, the price rose back to $12,300 but then slowly fell back to $5,500. The fall of the price of Bitcoin in March coincided with a flight to the dollar due to the initial panic from the coronavirus.

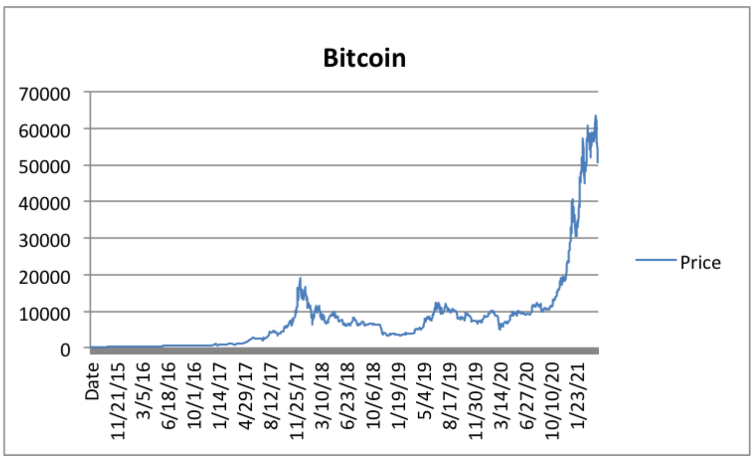

Figure 4: Bitcoin Price History

With the payment of $1,200 stimulus checks to most Americans in response to the health crisis, investors put upward pressure on the price again. In addition, the price got a boost from anticipation of the next halving. In turn, the price of Bitcoin rose to just under $10,000 around the date of the halving on May 11, 2020. Moreover, on March 15th, the Federal Reserve lowered the target for federal funds rate to zero once again. In addition, the FOMC announced a new round of quantitative easing. There was an immediate purchase of $80 billion dollars of assets and plans to purchase at least $700 billion over the coming months. Consequently, the Federal Reserve's balance sheet rose from $4 trillion to over $7 trillion in a matter of months. As a result of the aggressive response from central banks in response to Covid, investors showed a renewed concern for inflation with regards to the U.S. dollar. In turn, many market observers began speaking approvingly of the potential of cryptocurrencies to develop into a store of value to hedge against inflation from increased government spending during the pandemic. Further, around the time Paypal announced its crypo checkout service in October, the price of Bitcoin has steadily rose from $10,500 to its current price of $60,000.

Finally, other periods of economic instability appear to lead to positive price changes in Bitcoin. For example, there was an increase in Bitcoin usage in Venezuela after the country began to experience hyperinflation. More recently, Nigerians have been using the cryptocurrency to work around the country's conglomeration of exchange rates that hurt its citizens in regards to remittance payments. Also, there was an uptick in the price of Bitcoin after announcements from the Chinese government for capital controls.

In addition to Bitcoin, Ethereum is a popular cryptocurrency that is second only to Bitcoin in market cap size. In particular, in 2013, Ethereum founder Vitalik Buterin released the white paper. In August of 2015, Buterin's team launched the (ETH) platform with each coin trading at $2. In 2016, Ethereum's price rose from $0.9 at the beginning of the year to an all-time high of $20.5 in mid-June. However, a vulnerability found inside the contract for "The DAO"[29], a type of decentralized autonomous organization, was exploited by a hacker on June 17, 2016 to drain approximately 3.6 million ETH from the

[29] The DAO was an early decentralized autonomous organization (DAO) intended to act as an investor-directed venture capital firm. The project raised $150 million USD worth of ether (ETH) and was one of the earliest crowdfunding efforts and high-profile projects built on the Ethereum blockchain.

fund. The attacker would single-handedly have owned around 4.4% of the total supply of ETH if no change was made. Since this was a smart contract, the instructions from the code that were written from the contract would automatically be executed. Due to the severity of the attack, a bailout was agreed upon and a controversial proposal (EIP 779) was drawn up. The proposal would allow everyone to withdraw their ether from the DAO contract. On July 20th, a majority of mining power supported a fork which implemented this change, while a smaller community decided to split off and rename the old chain to Ethereum Classic.

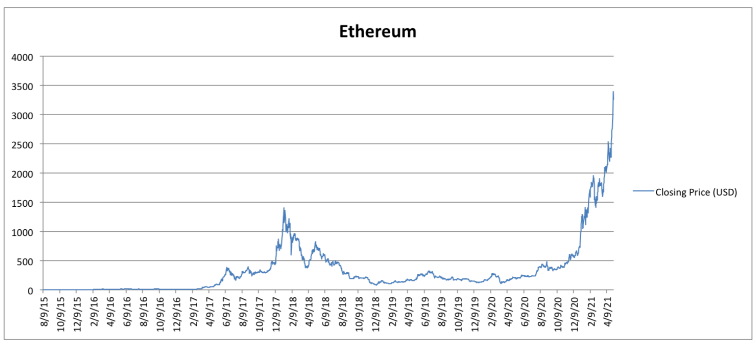

Figure 5: Ethereum Price History

These events led to the price dropping rapidly and closing out the year at $8. In 2017, Ethereum really gained in popularity and closed the year around $800. As can be seen in Figure 2, Ethereum reached a then high of $1,360 on January 13th, 2018 – about a month later than Bitcoin's high. Bitcoin's price dropped a month earlier due to the ability to short with futures trading. Ethereum was not part of the futures market until February 2021. The price crashed back down and dropped to $400 in April. It saw a sharp increase to $750 but slowly fell the rest of the year, reaching a low of $80 in December. Similar to

Bitcoin, there was a lull in price changes over the next couple of years. Yet, in March of 2020, it fell to $123 due to the flight to the dollar. Nevertheless, due to the strong public policy response to the public health crisis, Ethereum rose to over $400 in September and finished the year at $800 and has risen sharply all year to its current level around $3,000. It has been increasing more steadily than Bitcoin with the explanation that its inclusion in futures trading has led to the faster rise in price.

Finally, in terms of our data on the stance of U.S. monetary policy, the federal funds rate was at the zero lower bound in the first three and last two months of our sample. As a result, the Federal Reserve adopted unconventional policy tools such as large-scale asset purchases (quantitative easing) and forward guidance to influence long-term interest rates. In order to deal with the zero lower bound, we use the "shadow policy rate" estimated by Wu and Xia (2016).[30]

IV. METHODOLOGY

In terms of our empirical methodology, we follow Christiano et al. (1996) by constructing a VAR model to analyze the impact of shocks on macroeconomic variables and the prices of Bitcoin and Ethereum. The authors use a modelling strategy that involves choosing variables for real output, nominal effects, and monetary policy. We expand on this strategy by including variables for cryptocurrency.[31]

[30] The shadow rates are updated monthly at the Atlanta Federal Reserve and are available at www.atlantafed.org/cqer/research/wu-xia-shadow-federal-funds-rate.aspx

[31] In comparison to our framework, Biais et al. (2021) look at how the price of Bitcoin responds to changes in transaction fees. Notably, in their analysis, they assume that investors in each period know the inflation rate going into the following period. However, our focus is much different in that we look at how unanticipated monetary shocks affect the price of Bitcoin and Ethereum. Liu and Tsyvinski (2020) also provide another interesting contribution but their work looks at the role of network effects driven by user adoption that affect the returns to various cryptocurrencies. Again, our work is distinct as we look at how exogenous unanticipated changes in monetary policy affect the returns to different cryptocurrencies.

The analysis begins with our basic extension of the Christiano et al. structure. The endogenous variables in Z_t are specified as

$$Z_t : [Y_t, EMP_t, P_t, PCM_t, NTR_t, FFR_t, M1_t, ETHP_t, BTC_t]$$

with the vector including the log levels of industrial production (Y), payroll employment (EMP), core-CPI (P), crude materials (PCM), the U.S. money stock (M1), the price of Bitcoin (BTC), and the price of Ethereum (ETH). The ratio of non-borrowed reserves to total reserves (NTR) is included along with the effective Federal funds rate (FFR).

The standard form VAR is

$$Z_t = \mu + \Sigma_{i=1}^{k} A_i Z_{t-i} + e_t \tag{36}$$

where μ is a vector of constants and trend terms, A_i is a parameter matrix, e_t is a vector of serially uncorrelated error terms, and k is the number of lags. We impose a lag length of 4 for our model. Rewriting (1) as:

$$\mathbf{A}(L)Z_t = e_t \tag{37}$$

and inverting matrix $\mathbf{A}(L)$ we obtain the reduced-form equation:

$$Z_t = \mathbf{A}(L)^{-1} e_t \tag{38}$$

We assume the error terms are related to the underlying shocks, ϵ_t, by

$$e_t = \mathbf{C}\epsilon_t \tag{39}$$

After imposing a Choleski factorization, we estimate $\mathbf{C}$ and the parameter matrix $\mathbf{A_i}$ by applying ordinary least squares equation by equation to (1) and (3). Therefore, the dynamic effects of the price of Bitcoin and Ethereum in our aggregate model are obtained from

$$Z_t = \mathbf{B}(L)\epsilon_t \tag{40}$$

where $\mathbf{B}(L) = \mathbf{A}(L)^{-1}\mathbf{C}$. : Although we use a Choleski factorizaton for identification purposes, it should be noted that the system is very robust to different orderings. We assume that cryptocurrency prices are orthogonal to the contemporaneous information set of the real economy, nominal effects, and monetary policy.

V. RESULTS

We begin by summarizing the effects of a positive innovation from the FFR on the price of Bitcoin and Ethereum. We see the change in the price of Bitcoin reaches its high at around 1% after 2 months and then declines until it is negative after 9 months have passed. The effect of the innovation follows a similar pattern with Ethereum – there is a sharp increase in the price until it peaks with a 3% increase at 3 months. It then trends down until it is negative after 8 months. Our results may seem counter-intutive but it might also be that the FFR shock is a proxy for economic performance that is not captured through industrial production or other variables in the VAR. A possible explanation could be that the economy is performing well when there is a positive shock from the FFR. With the economy performing well, investors may be more willing to make investments in the cryptocurrency market. Notably, the effect of output shocks on the prices of both cryptocurrencies follows the same pattern as as an innovation from the FFR.

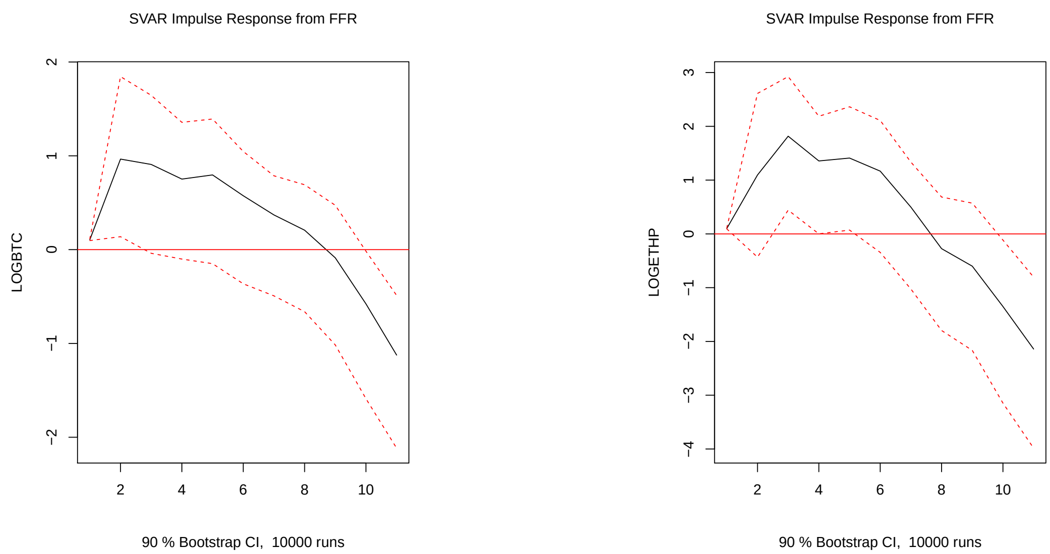

Figure 6: Federal Funds Rate Shock

Now, we take a look at the effect of a shock to the size of M1 on both Ethereum and Bitcoin. Similar to our previous results, they both exhibit the same pattern. The prices of both increase as expected with a positive shock to M1, and then after 10 months they experience a drop in price.

Bitcoin sees a peak in its change in price of 1% which can last for as long as 4 months. For Ethereum, it peaks a little bit later. It sees its largest percent rise of 1.75% after 3 months, but goes negative 1 month sooner than Bitcoin at 9 months. Thus, our results do suggest that investors may seek out access to cryptocurrencies as central banks continue to

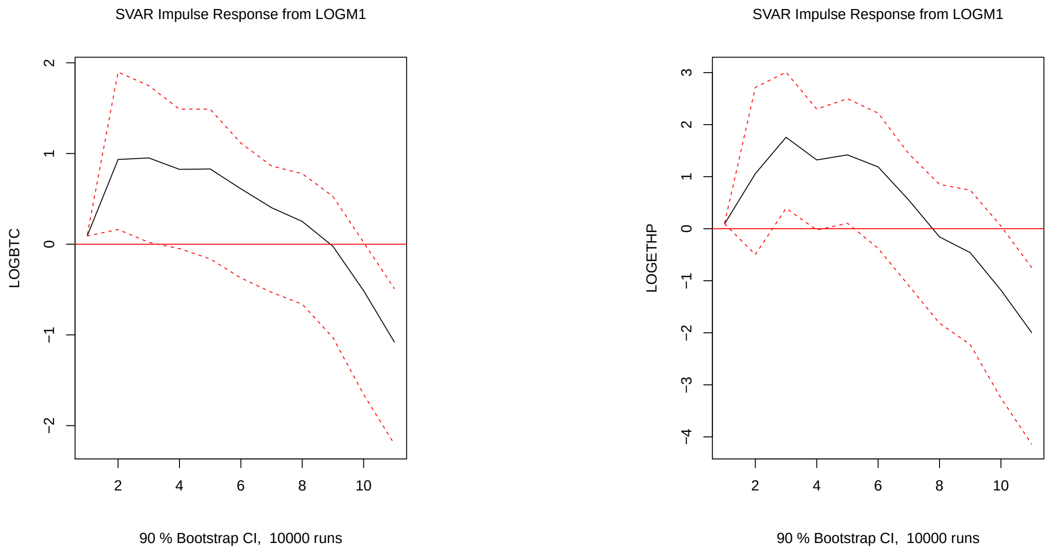

Figure 7: M1 Shock

print large amounts of money.

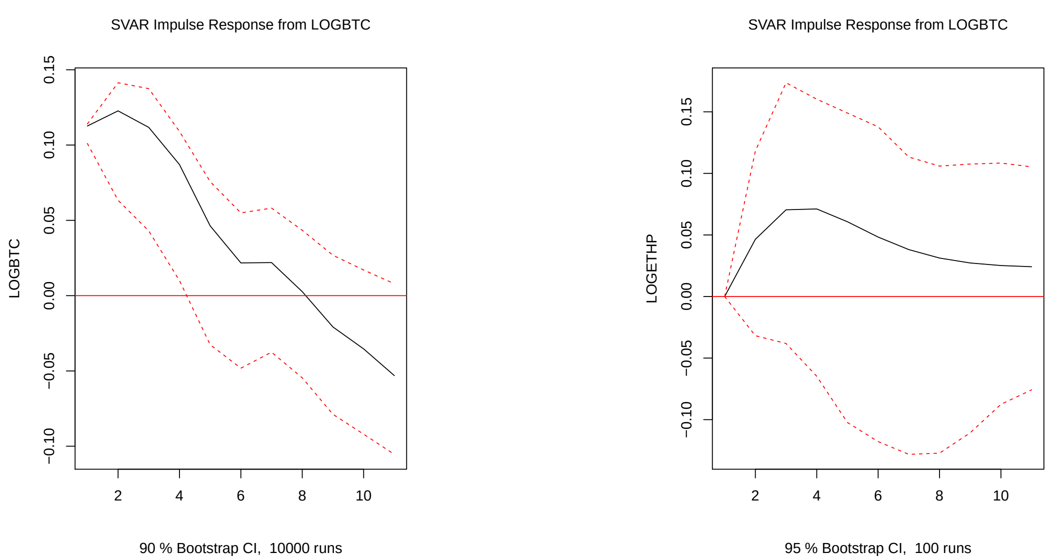

Figure 8: Bitcoin Price Shocks

We turn next to evaluate the impact of price shocks to either cryptocurrency. While a shock to the price of Bitcoin does not seem to impact the value of Ethereum, shocks to

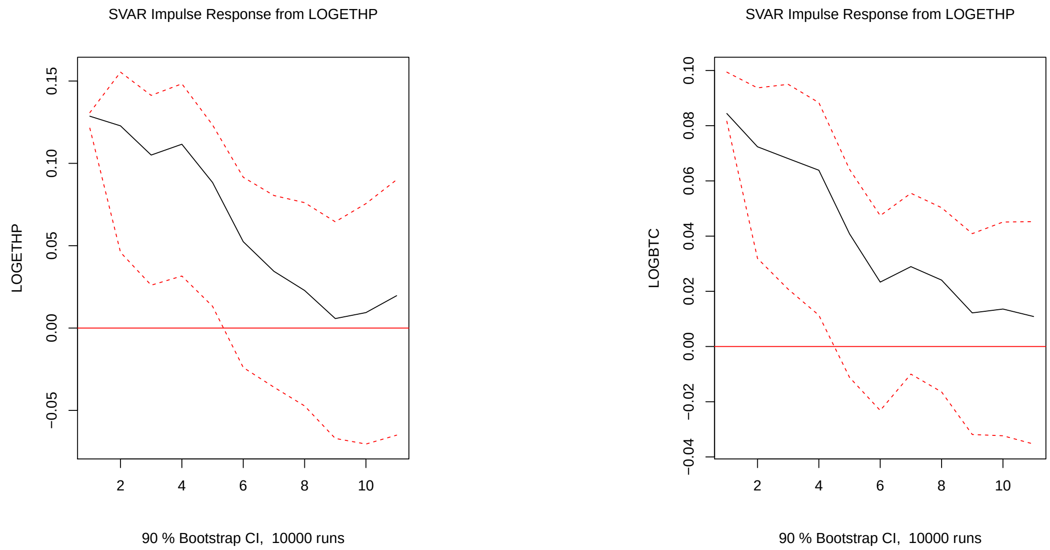

Figure 9: Ethereum Price Shocks

Ethereum prices do affect the price of Bitcoin. From this perspective, shocks to the price of cryptocurrencies other than Bitcoin may reflect an increase in the desire of investors to acquire cryptocurrencies as a whole. However, shocks to the price of Bitcoin may not have the same impact as Bitcoin has the largest circulation of all cryptocurrencies.

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CHAPTER 5

OVERALL CONCLUSION

In conclusion, the chapters answer some important questions in regards to the viability of Satoshi's claim. That is, cryptocurrency can serve as an alternative to our current centralized financial system. A stable price steady state exists due to the potential for a non trivial amount of permanent token loss. The currency loss ensures that the money supply is bounded and does not lead to a self fulfilling hyper-inflationary equilibrium. However, cryptocurrency as currently constituted is limited in its ability to respond to changes in money demand. Moreover, cryptocurrency is unable to provide an efficient return without intervention from a centralized authority.

Additionally, banks can play an important role in widespread adoption of crytocurrency as a medium of exchange. Financial intermediation can increase the volume of trade and the willingness of households to hold cryptocurrency as money. With a portion of tokens subject to loss, it is important that money holdings are not idle and put to a productive use. Banks can also provide safekeeping and greatly reduce the rate of currency loss. This will have the duel effect of increasing the willingness of individuals to hold cryptocurrency as money, and increase economic activity by allowing a larger amount of money available for trade. The largest gains occur when banks only provide safe-keeping for a portion of the currency. When banks provide safekeeping for all tokens, token loss will be eliminated but a steady-state is only possible with token creation capped at zero each period. This has particular relevance for cryptocurrencies with no cap on total token creation, such as Ethereum, Monero and Elon Musk's favorite meme coin Dogecoin. The result highlights one of the limitations of using a cryptocurrency as a medium of exchange.

Lastly, there is evidence that aggressive central bank activity leads to more interest in using cryptocurrency as a medium of exchange. If high rates of inflation continue, more people may turn to holding cryptocurrency. As a result, the widespread use of cryptocurrency becomes more likely in spite of its limitations as a currency.

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APPENDIX A

PROOF OF RESULTS OF NASH BARGAINING IN A DM MEETUP:

Next, we have the FOC for q :

$$\theta\big[u(q)-\beta\cdot\phi_{t+1}\cdot d\big]^{\theta-1}\big[u'(q)\big]\big[-w(q)+\beta\cdot\phi_{t+1}\cdot d\big]^{1-\theta}+$$

$$\big[u(q)-\beta\cdot\phi_{t+1}\cdot d\big]^{\theta}\big(1-\theta\big)\big[-w(q)+\beta\cdot\phi^{1}_{t+1}\cdot d\big]^{-\theta}(-w'(q))=0,$$

$$\theta\big[u(q)-\beta\cdot\phi_{t+1}\cdot d\big]^{\theta-1}\big[u'(q)\big]\big[-w(q)+\beta\cdot\phi_{t+1}\cdot d\big]^{1-\theta}=$$

$$\big[u(q)-\beta\cdot\phi_{t+1}\cdot d\big]^{\theta}\big(1-\theta\big)\big[-w(q)+\beta\cdot\phi_{t+1}\cdot d\big]^{-\theta}(w'(q))$$

$$\theta\big[u(q)-\beta\cdot\phi_{t+1}\cdot d\big]^{-1}\cdot u'(q)\cdot\big[-w(q)+\beta\cdot\phi_{t+1}\cdot d\big]=(1-\theta)w'(q)$$

$$\theta u'(q)\big[-w(q)+\beta\cdot\phi_{t+1}\cdot d\big]=(1-\theta)w'(q)\big[u(q)-\beta\cdot\phi_{t+1}\cdot d\big] \quad (A.1)$$

Next, we determine the first-order condition for d :

$$\theta\big[u(q) - \beta \cdot \phi_{t+1} \cdot d\big]^{\theta-1}\big[-\beta\phi_{t+1}\big]\big[-w(q) + \beta \cdot \phi_{t+1} \cdot d\big]^{1-\theta} +$$

$$\big[u(q) - \beta \cdot \phi_{t+1} \cdot d\big]^{\theta}\big(1-\theta\big)\big[-w(q) + \beta \cdot \phi_{t+1} \cdot d\big]^{-\theta} \cdot (\beta\phi_{t+1}) = 0,$$

$$\theta\big[u(q) - \beta \cdot \phi_{t+1} \cdot d\big]^{-1} \cdot \big[-w(q) + \beta \cdot \phi_{t+1} \cdot d\big] = (1-\theta)$$

$$\theta\big[-w(q) + \beta \cdot \phi_{t+1} \cdot d\big] = (1-\theta)\big[u(q) - \beta \cdot \phi_{t+1} \cdot d\big] \tag{A.2}$$

$$[-w(q) + \beta \cdot \phi_{t+1} \cdot d] = \frac{(1-\theta)}{\theta}[u(q) - \beta \cdot \phi_{t+1} \cdot d] \tag{A.3}$$

We can now substitute from (18) into (16) to find q^* when the liquidity constraint is not binding:

$$\theta u'(q)[-w(q) + \beta \cdot \phi_{t+1} \cdot d] = (1-\theta)w'(q)[u(q) - \beta \cdot \phi_{t+1} \cdot d]$$

$$\theta u'(q)\frac{(1-\theta)}{\theta}\,[u(q) - \beta \cdot \phi_{t+1} \cdot d] = (1-\theta)w'(q)\,[u(q) - \beta \cdot \phi_{t+1} \cdot d]$$

$$u'(q) = w'(q) \tag{A.4}$$

$q^* \in \mathbb{R}_+$ satisfies $u'(q^*) = w'(q^*)$, so that there is a socially efficient level of production in DM if the liquidity constraint does not bind.

Next, substitute $q_t = q^*$ into (18) to find d when the liquidity constraint is not binding:

$$\theta\left[-w(q^*) + \beta \cdot \phi_{t+1} \cdot d\right] = (1-\theta)\left[u(q^*) - \beta \cdot \phi_{t+1} \cdot d\right]$$

$$(1-\theta)u(q^*) - (1-\theta)\left[\beta \cdot \phi_{t+1} \cdot d\right] = -\theta w(q^*) + \theta\left[\beta \cdot \phi_{t+1} \cdot d\right]$$

$$(1-\theta)u(q^*) + \theta w(q^*) = \beta\phi_{t+1} \cdot d \tag{A.5}$$

■

PROOF OF REAL MONEY HOLDINGS:

$$-\theta u'(q)w(q) + \theta u'(q)\beta \cdot \phi_{t+1} \cdot d = (1-\theta)w'(q)u(q) - (1-\theta)w'(q)\beta \cdot \phi_{t+1} \cdot d$$

$$\theta u'(q)\beta \cdot \phi_{t+1} \cdot d + (1-\theta)w'(q)\beta \cdot \phi_{t+1} \cdot d =$$

$$(1-\theta)w'(q)u(q) + \theta u'(q)w(q)$$

$$\beta \cdot \phi_{t+1} \cdot d = \frac{\theta w(q)u'(q) + (1-\theta)u(q)w'(q)}{\theta u'(q) + (1-\theta)w'(q)} \tag{A.6}$$

■

PROOF OF SPECIAL CASE. Set $\theta = 1$ when d is not binding

$$(1-1)u(q^*) + 1 \cdot w(q^*) = \beta\phi_{t+1} \cdot d$$

$$w(q^*) = \beta\phi_{t+1} \cdot d$$

In the case when d is binding:

$$\phi_{t+1} \cdot d = \phi_{t+1} M_t^c$$

If the buyer has full bargaining power:

$$m(q) \equiv \frac{(1)w(q)u'(q) + (1-1)u(q)w'(q)}{(1)u'(q) + (1-1)w'(q)}$$

simplify

$$m(q) \equiv \frac{w(q)u'(q)}{u'(q)}$$

simplify

$$m(q) \equiv w(q)$$

and so

$$w(q) = \beta\phi_{t+1} M_t^c$$

Upon solving to get the quantity of the DM good produced in terms of the real value of tokens:

$$\beta\phi_{t+1}M_t^c = w^{-1}(\beta\phi_{t+1}M_t^c)$$

■

PROOF OF CONSUMER'S PORTFOLIO CHOICE:

$$-\phi_t M_t^c + (1-\psi)\Big\{\sigma[u(q) + \beta\phi_{t+1}M_t^c - \beta\phi_{t+1}d] + \sigma[(1-\psi)(-w(\overline{q}) + \beta\phi_{t+1}M_t^c + \beta\phi_{t+1}\overline{M_t^c}) + \psi\beta\phi_{t+1}M_t^c]$$

$$+(1-2\sigma)\beta\phi_{t+1}M_t^c\Big\}$$

Now, we can sum up all the terms with $\phi_{t+1}M_t^c$ inside the $(1-\psi)$ bracket.

$$\sigma[\beta\phi_{t+1}M_t^c] + \sigma[(1-\psi)(\beta\phi_{t+1}M_t^c) + \psi\beta\phi_{t+1}M_t^c] + (1-2\sigma)\beta\phi_{t+1}M_t^c$$

$$\sigma\beta\phi_{t+1}M_t^c + \sigma[\beta\phi_{t+1}M_t^c] + (1-2\sigma)\beta\phi_{t+1}M_t^c$$

$$\beta\phi_{t+1}M_t^c$$

Simplify further by eliminating the work function and tokens received. Remember, we are only concerned with optimizing the amount of tokens for a purchase in the DM.

$$-\phi_t M_t^c + (1-\psi)\Big\{\sigma[u(q) - \beta\phi_{t+1}d] + \beta\phi_{t+1}M_t^c\Big\}$$

Proof of Money Demand: The first-order condition for money balances is:

$$-\phi_t + (1-\psi)\Big\{\sigma(u'(q)\frac{\partial q_t}{\partial M_t^c} - \beta\phi_{t+1}) + \beta\phi_{t+1}\Big\} = 0$$

To solve for $\frac{\partial q(M_t^c,t)}{\partial M_t^c}$, remember that $m\left(q(M_t^c,t)\right) = \beta \cdot \phi_{t+1} \cdot M_t^c$

Upon taking the derivative with respect to M_t^c:

$$m'\left(q(M_t^c,t)\right)\frac{\partial q(M_t^c,t)}{\partial M_t^c} = \beta\phi_{t+1}$$

$$\frac{\partial q(M_t^c,t)}{\partial M_t^c} = \frac{\beta\phi_{t+1}}{m'\left(q(M_t^c,t)\right)}$$

We let $A = \phi_{t+1} \cdot M_t^c$ and substitute into the first-order condition:

$$\phi_t = \sigma(1-\psi)u'\left(m^{-1}(\beta A)\right)\frac{\beta\phi_{t+1}}{m'\left(q(M_t^c,t)\right)} - \sigma(1-\psi)\beta\phi_{t+1} + (1-\psi)\beta\phi_{t+1}$$

$$\phi_t = \beta\phi_{t+1}\Big(\sigma(1-\psi)\frac{u'(m^{-1}(\beta A))}{m'(m^{-1}(\beta A))} - \sigma(1-\psi) + (1-\psi)\Big)$$

$$\phi_t = \beta\phi_{t+1}\Big(\sigma(1-\psi)\frac{u'(m^{-1}(\beta A))}{m'(m^{-1}(\beta A))} + (1-\sigma)(1-\psi)\Big)$$

If the buyer's liquidity constraint no longer binds, $\sigma(1-\psi)[u(q) - \beta\phi_{t+1}d]$ remains constant with any change in M_t^c. The first-order condition for money balances will now become:

$$-\phi_t + \Big(1-\psi\Big)\beta\phi_{t+1} = 0$$

$$-\phi_t + \beta\phi_{t+1}\Big(1 - \psi\Big) = 0$$

$$\phi_t = \beta\phi_{t+1}\Big(1 - \psi\Big)$$

■

PROOF OF MONEY DEMAND FOR SPECIAL CASE:

Let's say the utility from consumption can be defined as $u(q) = 2q^{1/2}$, and $w(q) = q$ represents the cost of producing DM goods. Therefore, if the buyer has full bargaining power:

$$w(q) = q = \beta\phi_{t+1}M_t^c$$

Solve for the inverse to get the quantity of the DM good produced in terms of the real value of tokens:

$$q = \beta\phi_{t+1}M_t^c = w^{-1}(\beta\phi_{t+1}M_t^c)$$

and so

$$w(w^{-1}(\beta\phi_{t+1}M_t^c)) = \beta\phi_{t+1}M_t^c$$

Now, we use $u(q) = 2q^{1/2}$ to get

$$u(w^{-1}(\beta\phi_{t+1}M_t^c)) = 2(\beta\phi_{t+1}M_t^c)^{1/2}$$

Now, we solve for $\frac{u'(w^{-1}(\beta A))}{w'(w^{-1}(\beta A))}$

$$\frac{u'(w^{-1}(\beta A))}{w'(w^{-1}(\beta A))} = \frac{\beta \phi_{t+1}}{\beta \phi_{t+1} \sqrt{\beta \phi_{t+1} M_t^c}}$$

simplify

$$\frac{u'(w^{-1}(\beta A))}{w'(w^{-1}(\beta A))} = \frac{1}{\sqrt{\beta \phi_{t+1} M_t^c}}$$

now substitute into L_1 to get

$$L_1(\phi_{t+1} M_t^c) = \sigma(1-\psi) \frac{1}{\sqrt{\beta \phi_{t+1} M_t^c}} + (1-\sigma)(1-\psi)$$

We can also use the first-order condition for the portfolio choice:

$$\phi_t = \sigma(1-\psi) u'(w^{-1}(\beta \phi_{t+1} M_t^c)) + \beta \phi_{t+1} \Big((1-\sigma)(1-\psi) \Big)$$

Substituting $w^{-1}(\beta \phi_{t+1} M_t^c) = q = \beta \phi_{t+1} M_t^c$, and solving for $u'(q)$.

$$u(q) = \frac{q^{\frac{1}{2}}}{\frac{1}{2}} = \frac{[\beta \phi_{t+1} M_t^c]^{\frac{1}{2}}}{\frac{1}{2}}$$

$$u'(w^{-1}(\beta \phi_{t+1} M_t^c)) = \frac{1}{2} \frac{\beta \phi_{t+1} [\beta \phi_{t+1} M_t^c]^{-\frac{1}{2}}}{\frac{1}{2}}$$

$$\phi_t = \sigma(1-\psi) \frac{\beta \phi_{t+1}}{[\beta \phi_{t+1} M_t^c]^{\frac{1}{2}}} + \beta \phi_{t+1} \Big((1-\sigma)(1-\psi) \Big)$$

The first-order condition for portfolio problem becomes

$$\phi_t = \beta\phi_{t+1}\Big(\frac{\sigma(1-\psi)}{[\beta\phi_{t+1}M_t^c]^{\frac{1}{2}}} + (1-\sigma)(1-\psi)\Big)$$

giving us the same condition for L_1

$$L_1(\phi_{t+1}M_t^c) = \sigma(1-\psi)\frac{1}{\sqrt{\beta\phi_{t+1}M_t^c}} + (1-\sigma)(1-\psi)$$

■

PROOF OF PROPOSITION 1:

First, by condition iii, the first order condition to the entrepreneur's problem in the steady-state is:

$$\overline{\phi} = c'(\Delta^*)$$

Thus, token creation each period will satisfy $\Delta^* = (c')^{-1}(\overline{\phi})$. In turn, we have the following law of motion $\overline{M} = (c')^{-1}(\overline{\phi}) + (1-\psi)\overline{M}$. Moreover, it is easy to see that $\overline{M} = \frac{1}{\psi}(c')^{-1}(\overline{\phi})$ in the steady state.

Under the above cost function, the entrepreneur will create $\frac{1}{2}\overline{\phi}$ tokens each period. Given that token creation is $\frac{1}{2}\overline{\phi}$ each period, we can easily show the steady-state level of the money supply to be $\overline{M} = \frac{1}{2\psi}\overline{\phi}$.

Next, define the value of tokens in the steady-state as $\overline{b} \equiv \overline{\phi M}$:

$$1 = \beta\overline{\gamma}L_1\left(\overline{\gamma}\overline{b}\right)$$

Rearrange to get a consumer's money demand

$$\bar{b} = \frac{1}{\bar{\gamma}} L_1^{-1}(\frac{1}{\beta\overline{\gamma}})$$

define as $z_1(\gamma)$

$$\bar{b} = \frac{1}{\bar{\gamma}} L_1^{-1}(\frac{1}{\beta\overline{\gamma}}) \equiv z_1(\overline{\gamma}).$$

Because $\overline{\gamma} = 1$, the above equation becomes

$$\bar{b} = L_1^{-1}(\frac{1}{\beta}) \equiv z_1(1).$$

Now, impose the steady state on the first-order condition for portfolio problem

$$\overline{\phi} = \beta\overline{\phi}\Big(\frac{\sigma(1-\psi)}{[\beta\bar{b}]^{\frac{1}{2}}} + (1-\sigma)(1-\psi)\Big)$$

$$1 = \beta\Big(\frac{\sigma(1-\psi)}{[\beta\bar{b}]^{\frac{1}{2}}} + (1-\sigma)(1-\psi)\Big)$$

$$\frac{1}{\beta} = L_1 = \frac{\sigma(1-\psi)}{[\beta\bar{b}]^{\frac{1}{2}}} + (1-\sigma)(1-\psi)$$

Now solve for $\bar{b}$ to get real money demand:

$$\bar{b} = L_1^{-1}(\frac{1}{\beta}) = \frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$$

The real money demand in the steady-state can be shown by:

$$z_1(1) = \frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$$

To allow a stable price, the real money demand must satisfy $\bar{b} < \beta^{-1}w(q^*)$. Otherwise, the rate of return on currency must be greater than 1. The liquidity constraint no longer binds when the real value of money holding becomes

$$\frac{u'(q*)}{m'(q*)} = 1 = \frac{1}{\sqrt{\beta \cdot \overline{\phi} \cdot \overline{M}}} \rightarrow \sqrt{\beta \cdot \overline{\phi} \cdot \overline{M}} = 1 \rightarrow \overline{\phi} \cdot \overline{M} = \frac{1}{\beta}$$

In order to show $\bar{b} < \beta^{-1}w(q^*)$, we take $\frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2} < \frac{1}{\beta}$ and multiply by β on both sides

$$\frac{\beta^2(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2} < 1$$

$$\beta^2(\sigma(1-\psi))^2 < (1-\beta(1-\psi)+\beta\sigma(1-\psi))^2$$

$$\beta(\sigma(1-\psi)) < 1-\beta(1-\psi)+\beta\sigma(1-\psi)$$

$$0 < 1-\beta(1-\psi)$$

Since $\beta < 1$, it will be the case that $0 < 1-\beta(1-\psi)$.

The market clearing condition in the steady-state implies

$$\overline{\phi M} = z_1(1)$$

Setting the real value of money in the steady-state equal to real money demand, we solve for the price of tokens needed in the steady-state.

$$z_1(1) = \frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2} = \frac{1}{2\psi}\overline{\phi}^2 = \overline{\phi M}$$

$$\frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2} = \frac{1}{2\psi}\overline{\phi}^2$$

$$\frac{2\psi\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2} = \overline{\phi}^2$$

The price of tokens in the steady-state becomes:

$$\overline{\phi} = \frac{\sigma(1-\psi)\sqrt{2\beta\psi}}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$$

To see what effect having an increase in likelihood of a trade in the DM on the steady-state price, take the partial derivative of $\overline{\phi}$ with respect to σ:

$$\frac{\partial\overline{\phi}}{\partial\sigma} = \frac{\sqrt{2\beta\psi}(1-\psi)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))} + \frac{\sqrt{2\beta\psi}\sigma(1-\psi)(-1)\beta(1-\psi)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$$

$$\frac{\partial\overline{\phi}}{\partial\sigma} = \underbrace{\frac{\sqrt{2\beta\psi}(1-\psi)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}}_{\oplus} - \underbrace{\frac{\sqrt{2\beta\psi}\beta\sigma(1-\psi)^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}}_{\oplus}$$

Assume first term is greater than second term

$$\frac{\sqrt{2\beta\psi}(1-\psi)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))} > \frac{\sqrt{2\beta\psi}\beta\sigma(1-\psi)^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$$

$$\frac{1}{1} > \frac{\beta\sigma(1-\psi)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$$

$$1-\beta(1-\psi)+\beta\sigma(1-\psi) > \beta\sigma(1-\psi)$$

$$1-\beta(1-\psi) > 0$$

Since $\beta \in (0,1)$, we know the above is true. Thus, $\frac{\partial\overline{\phi}}{\partial\sigma} > 0$.

To see the effect of currency loss on the steady-state price, take the partial derivative of $\overline{\phi}$ with respect to ψ:

$$\frac{\partial\overline{\phi}}{\partial\psi} = \frac{2\beta\psi}{2\sqrt{2\beta\psi}}\frac{\sigma(1-\psi)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$$

$$+\sqrt{2\beta\psi}\Big(\frac{-\sigma}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))} + \frac{\sigma(1-\psi)(-1)(\beta-\beta\sigma)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}\Big)$$

$$\frac{\partial\overline{\phi}}{\partial\psi} = \underbrace{\frac{\beta\psi\sigma(1-\psi)}{\sqrt{2\beta\psi}(1-\beta(1-\psi)+\beta\sigma(1-\psi))}}_{\oplus} - \underbrace{\frac{\sigma\sqrt{2\beta\psi}}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}}_{\oplus} - \underbrace{\frac{\sqrt{2\psi}\beta^{\frac{3}{2}}\sigma(1-\psi)(1-\sigma)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}}_{\oplus}$$

Assume first term smaller than second plus the third term:

$$\frac{\beta\psi\sigma(1-\psi)}{\sqrt{2\beta\psi}(1-\beta(1-\psi)+\beta\sigma(1-\psi))} < \frac{\sigma\sqrt{2\beta\psi}}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))} + \frac{\sqrt{2\psi}\beta^{\frac{3}{2}}\sigma(1-\psi)(1-\sigma)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$$

simplify

$$\frac{\beta\psi(1-\psi)}{\sqrt{2\beta\psi}} < \sqrt{2\beta\psi} + \frac{\sqrt{2\psi}\beta^{\frac{3}{2}}(1-\psi)(1-\sigma)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$$

We know the last term is positive so if we show 2nd term larger than 1st term will still hold true

$$\frac{\sqrt{\beta\psi}(1-\psi)}{\sqrt{2}} < \sqrt{2\beta\psi}$$

$$\frac{\sqrt{\psi}(1-\psi)}{\sqrt{2}} < \sqrt{2\psi}$$

$$\sqrt{\psi}(1-\psi) < 2\sqrt{\psi}$$

$$(1-\psi) < 2$$

Which gives us $\frac{\partial\overline{\phi}}{\partial\psi} < 0$.

Substitute $\overline{\phi} = \frac{\sigma(1-\psi)\sqrt{2\beta\psi}}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$ into the steady-state money supply to get

$$\overline{M} = \frac{1}{2\psi}\overline{\phi} = \frac{1}{2\psi}\cdot\frac{\sigma(1-\psi)\sqrt{2\beta\psi}}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))} = \frac{\sqrt{\beta}\sigma(1-\psi)}{\sqrt{2\psi}(1-\beta(1-\psi)+\beta\sigma(1-\psi))}$$

To find the effect of an increase in currency loss on the steady-state level of tokens, take the partial derivative of $\overline{M}$ with respect to ψ:

$$\frac{\partial\overline{M}}{\partial\psi} = -\frac{\overline{\phi}}{2\psi^2} + \frac{1}{2\psi}\frac{\partial\overline{\phi}}{\partial\psi} = -\frac{\overline{\phi}}{2\psi^2} + \ominus$$

Thus, $\frac{\partial\overline{M}}{\partial\psi} < 0$.

In order to find the effect of successful trades on the steady-state money supply, take the partial derivative of $\overline{M}$ with respect to σ:

$$\frac{\partial \overline{M}}{\partial \sigma} = \frac{1}{2\psi}\Big(\frac{\partial \overline{\phi}}{\partial \sigma}\Big) = \frac{1}{2\psi}\Big(\oplus\Big)$$

Therefore, $\frac{\partial \overline{M}}{\partial \sigma} < 0$. ■

PROOF OF PROPOSITION 2:

First by condition (iii) in the steady-state equilibrium would imply that the first order condition to the entrepreneur's problem in the steady-state be represented by:

$$\overline{\phi} = c_0$$

The amount of tokens produced at this price is indeterminate. With entrepreneurs producing tokens in the range $\Delta^* \in [0, \infty]$ each period. In the steady-state condition (v) must hold:

$$\overline{\phi M} = z_1(1)$$

From the previous proof, we substitute $z_1(1) = \frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$ and for the steady-state price $\overline{\phi} = c_0$ to get

$$c_0\overline{M} = \frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$$

Which will give the steady-state supply of tokens

$$\overline{M} = \frac{1}{c_0}\frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$$

Next, by condition iv we get the law of motion for tokens represented by

$$\overline{M} = \Delta^* + (1-\psi)\overline{M}$$

It is easy to show that $\Delta^* = \psi\overline{M} = \frac{\psi}{c_0}\frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}$.

To find the effect of an increase in currency loss on the steady-state level of tokens, take the partial derivative of $\overline{M}$ with respect to ψ:

$$\frac{\partial\overline{M}}{\partial\psi} = \frac{\partial}{\partial\psi}\left\{\frac{1}{c_0}\frac{\beta(\sigma(1-\psi))^2}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2}\right\}$$

$$\frac{\partial\overline{M}}{\partial\psi} = \frac{1}{c_0}\left\{\frac{2\beta(\sigma(1-\psi))(-\beta\sigma)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^2} + \frac{\beta(\sigma(1-\psi))^2(-2)(\beta-\beta\sigma)}{(1-\beta(1-\psi)+\beta\sigma(1-\psi))^3}\right\}$$

$$\frac{\partial\overline{M}}{\partial\psi} = \frac{-2\beta^2\sigma^2(1-\psi)(1-\beta(1-\psi)+\beta\sigma(1-\psi)) - 2\beta^2\sigma^2(1-\sigma)(1-\psi)}{c_0(1-\beta(1-\psi)+\beta\sigma(1-\psi))^3}$$

$$\frac{\partial\overline{M}}{\partial\psi} = \frac{\overbrace{-2\beta^2\sigma^2(1-\psi)}^{\ominus}\overbrace{(1-\beta(1-\psi)+\beta\sigma(1-\psi)+(1-\sigma)(1-\psi))}^{\oplus}}{\underbrace{c_0(1-\beta(1-\psi)+\beta\sigma(1-\psi))^3}_{\oplus}}$$

Which gives us $\frac{\partial\overline{M}}{\partial\psi} < 0$.

A change in the rate of currency loss will have no effect on the steady-state price, $\frac{\partial\overline{\phi}}{\partial\psi} = 0$. Set Ω, then compare the $\frac{\partial\overline{M}}{\partial\psi}$ from proposition 1 to proposition 2.

$$\frac{\partial\overline{M}^{p2}}{\partial\psi} = \frac{-2\beta^2\sigma^2(1-\psi)(\Omega+(1-\sigma)(1-\psi))}{c_0\Omega^3}$$

$$\frac{\partial \overline{M}^{p1}}{\partial \psi} = \frac{-\sqrt{\frac{\beta}{2}}\frac{\sigma}{\psi^{\frac{3}{2}}}(1-\psi)}{\Omega} + \frac{\sqrt{\frac{\beta\psi}{2}}\sigma(1-\psi)}{\Omega} - \frac{\sigma\sqrt{2\beta\psi}}{\Omega} - \frac{\sqrt{2\psi}\beta^{\frac{3}{2}}\sigma(1-\psi)(1-\sigma)}{\Omega^2}$$

$$\frac{\partial \overline{M}^{p1}}{\partial \psi} = \frac{-\sqrt{\frac{\beta}{2}}\frac{\sigma}{\psi^{\frac{3}{2}}}(1-\psi)\Omega^2}{\Omega^3} + \frac{\sqrt{\frac{\beta\psi}{2}}\sigma(1-\psi)\Omega^2}{\Omega^3} - \frac{\sigma\sqrt{2\beta\psi}\Omega^2}{\Omega^3} - \frac{\sqrt{2\psi}\beta^{\frac{3}{2}}\sigma(1-\psi)(1-\sigma)\Omega}{\Omega^3}$$

Multiply through by Ω^3 and combine terms and assume $\frac{\partial \overline{M}^{p2}}{\partial \psi} < \frac{\partial \overline{M}^{p1}}{\partial \psi}$

$$\frac{-2}{c_0}\beta^2\sigma^2(1-\psi)(\Omega+(1-\sigma)(1-\psi)) < \Omega^2\Big(-\sqrt{\frac{\beta}{2}}\sigma(1-\psi)(\frac{1}{\psi^{\frac{3}{2}}}-\psi^{\frac{1}{2}})-\sigma\sqrt{2\beta\psi}\Big)-\sqrt{2\psi}\beta^{\frac{3}{2}}\sigma(1-\psi)(1-\sigma)\Omega$$

$$\frac{-2\beta^2\sigma^2(1-\psi)(\Omega+(1-\sigma)(1-\psi))}{\Omega^2\Big(-\sqrt{\frac{\beta}{2}}\sigma(1-\psi)(\frac{1}{\psi^{\frac{3}{2}}}-\psi^{\frac{1}{2}})-\sigma\sqrt{2\beta\psi}\Big)-\sqrt{2\psi}\beta^{\frac{3}{2}}\sigma(1-\psi)(1-\sigma)\Omega} > c_0$$

As c_0 gets larger the adjustment to a change in ψ to maintain a steady-state is smaller.

■

PROOF OF PROPOSITION 3:

Consider a set of caps with the property that $\overline{\Delta}_t > 0$ at dates $0 \le t \le T$ and $\overline{\Delta}_t = 0$ at all subsequent dates $t \ge T+1$, given a finite date $T > 0$. Set $\phi_t = \overline{\phi}$ at all dates $t \ge T+1$, with the constant $\overline{\phi} > 0$ satisfying

$$1 = \beta L_1(\overline{\phi}\Sigma_{\tau=0}^{T}\overline{\Delta}_\tau).$$

This comes from

$$\phi_t = \beta\phi_{t+1}L_1(\phi_{t+1}M_t)$$

$\phi_t = \phi_{t+1} = \overline{\phi}$ because $t \geq T+1$. So you have

$$\overline{\phi} = \beta\overline{\phi}L_1(\overline{\phi}M_t)$$

cancel out $\overline{\phi}$ on both sides

$$1 = \beta L_1(\overline{\phi}M_t)$$

Since no more tokens are created at this point, at all future dates $\overline{\phi}\Sigma_{\tau=0}^{T}\overline{\Delta}_\tau{=}M_t$

$$1 = \beta L_1(\overline{\phi}\Sigma_{\tau=0}^{T}\overline{\Delta}_\tau).$$

The above equation is not possible with currency loss. Let's assume that we have price stability and all the tokens have been created. We will now introduce currency loss for each period into the above equation.

$$1 = \beta L_1(\overline{\phi}\Sigma_{m=T+j}^{\infty}(1-\psi)_m\Sigma_{\tau=0}^{T}\overline{\Delta}_\tau).$$

One of two things needs to happen for this equation to remain true. Entrepreneurs would need to create more tokens or the price would need to increase. The first possibility is not possible because token creation has been capped. The second causes a contradiction. It would no longer be the case that $\phi_t = \phi_{t+1}$. Therefore, a sequence with price stability does not exist with currency loss and a cap on total token creation. ■

PROOF OF PROPOSITION 4:

$$\frac{1}{\beta} = \sigma(1-\psi)\frac{u'(\hat{q_{cl}})}{w'(\hat{q_{cl}})} + 1 - \sigma(1-\psi) - \psi = \sigma\frac{u'(\hat{q})}{w'(\hat{q})} + 1 - \sigma = \frac{1}{\beta}$$

$$\sigma(1-\psi)\frac{u'(\hat{q_{cl}})}{w'(\hat{q_{cl}})} + 1 - \sigma(1-\psi) - \psi = \sigma\frac{u'(\hat{q})}{w'(\hat{q})} + 1 - \sigma$$

$$\sigma(1-\psi)\frac{u'(\hat{q_{cl}})}{w'(\hat{q_{cl}})} + \sigma\psi - \psi = \sigma\frac{u'(\hat{q})}{w'(\hat{q})}$$

As you can see $\frac{u'(\hat{q_{cl}})}{w'(\hat{q_{cl}})} > \frac{u'(\hat{q})}{w'(\hat{q})}$ and since $\frac{u'(q)}{w'(q)}$ is decreasing in q then $\hat{q_{cl}} < \hat{q}$. ■

PROOF OF PROPOSTION 5:

Suppose $\phi < \overline{\phi}$, then token production is less than what is needed for a stable price $\Delta^* < \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. There will be less new tokens created than tokens lost from the money supply in the previous period $\Delta^* < \psi M_{-1}$. Since token production will be less than the amount of tokens lost it will be the case that $\gamma < 1$ and thus deflation is observed. The price of tokens will be greater than the previous period $\phi_{+1} > \phi$. This will lead to more tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} > \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with deflation in this economy with no cap.

With a cap $\overline{\Delta}^* = \alpha\overline{M} < \psi\overline{M}$, the number of tokens will decrease in the steady state and allow for a positive real return on currency. ■

PROOF OF PROPOSITION 6:

In equilibrium the real value of tokens must remain the same. In order to have a steady state with $\overline{\gamma} = \frac{1}{\beta(1-\psi)}$, the amount of tokens must evolve according to

$$\phi_{t+1} M_{t+1} = \frac{1}{\beta(1-\psi)} M_{t+1} = 1 \cdot M_t = \phi_t M_t$$

Now substitute $\phi_{t+1} = \frac{1}{\beta(1-\psi)}, \phi_t = 1$ and $M_{t+1} = \beta(1-\psi) M_t$ into $\phi_{t+1} M_{t+1} = \phi_{t+1} \Delta_t^* + \phi_{t+1}(1-\psi) M_t$ to get

$$\beta(1-\psi) M_t = \Delta_{t+1}^* + (1-\psi) M_t$$

$$(\beta(1-\psi) - (1-\psi)) M_t = \Delta_{t+1}^*$$

$$(\beta - 1)(1-\psi) M_t = \Delta_{t+1}^*$$

$$\underbrace{(\beta - 1)}_{(-)} \underbrace{(1-\psi)}_{(+)} \overline{M} = \overline{\Delta}^* < 0$$

■

PROOF OF PROPOSITION 7:

Similar to the proof of proposition 6, the amount of tokens must evolve according to

$$\phi_{t+1} M_{t+1} = \frac{1}{\beta(1-\psi)} M_{t+1} = 1 \cdot M_t = \phi_t M_t$$

Now substitute $\phi_{t+1} = \frac{1}{\beta(1-\psi)}, \phi_t = 1$ and $M_{t+1} = \beta(1-\psi) M_t$ into $\phi_{t+1} M_{t+1} = \phi_{t+1} \Delta_t^* + \phi_{t+1}(1-\psi) M_t$ to get

$$\beta(1-\psi) M_t = \Delta_{t+1}^* + (1-\psi) M_t$$

Now, we place a cap on token production $\overline{\Delta} = \alpha M$ and there is a tax τ each period.

$$\beta(1-\psi)M_t = \alpha M_{t+1} + (1-\psi)M_t - \tau$$

$$\tau = (\alpha + (1-\psi) - \beta(1-\psi))M_t$$

$$\tau = (\alpha + (1-\beta)(1-\psi))M_t > 0$$

■

PROOF OF PROPOSITION 8:

We must have $q^g = q^p$ in order for agents to hold both currencies. The returns must yield the same amount of production in the DM. Recall the money demand function for tokens with currency loss.

$$\phi_t^p = \beta\phi_{t+1}^p\Big(\sigma(1-\psi)\frac{u'(q^p)}{w'(q^p)} + (1-\sigma)(1-\psi)\Big)$$

$$\frac{1}{\gamma_{t+1}^p} = \frac{\phi_t^p}{\phi_{t+1}^p} = \beta\Big(\sigma(1-\psi)\frac{u'(q^p)}{w'(q^p)} + (1-\sigma)(1-\psi)\Big)$$

Now if you divide both sides by $(1-\psi)$ you get

$$\frac{1}{(1-\psi)\gamma_{t+1}^p} = \beta\Big(\sigma\frac{u'(q^p)}{w'(q^p)} + (1-\sigma)\Big)$$

Since $q^p = q^g$ we have

$$\frac{1}{(1-\psi)\gamma_{t+1}^p} = \beta\Big(\sigma \frac{u'(q^p)}{w'(q^p)} + (1-\sigma)\Big) = \beta\Big(\sigma \frac{u'(q^g)}{w'(q^g)} + (1-\sigma)\Big) = \frac{1}{\gamma_{t+1}^g}$$

which gives us $\gamma_{t+1}^g = (1-\psi)\gamma_{t+1}^p$. ■

APPENDIX B

PROOF OF LEMMA 1:

First, suppose that $\phi = \overline{\phi}$, then token production will be equal to what is needed for a stable price $\Delta^* = \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production is equal to the amount of tokens lost it will be the case that $\gamma = 1$ and thus the price level will be constant over time. The price of tokens will be the same as the previous period $\phi_{+1} = \phi$. This will lead to the same amount of tokens being created as the previous period $\Delta^*_{+1} = \Delta^*$.

This process will repeat itself ensuring token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. Thus, $\gamma = 1$ for the remaining periods making a steady state possible.

Next, assume $\phi > \overline{\phi}$, then token production will greater than what is needed for a stable price $\Delta^* > \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production will be greater than the amount of tokens lost it will be the case that $\gamma > 1$ and thus inflation occurs. Over time, the price of tokens will be less than the previous period $\phi_{+1} < \phi$. This will lead to less tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} < \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with inflation in this economy.

Lastly, suppose $\phi < \overline{\phi}$, then token production is less than what is needed for a stable price $\Delta^* < \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. There will be less new tokens created than tokens lost from the money supply in the previous period $\Delta^* < \psi M_{-1}$. Since token production will be less than the amount of tokens lost it will be the case that $\gamma < 1$ and thus deflation is observed. The price of tokens will be greater than the previous period $\phi_{+1} > \phi$. This will lead to more tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} > \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with deflation in this economy. ■

PROOF OF PROPOSITION 1:

From Lemma 1 the steady-state money supply is $\overline{M} = \frac{1}{2\psi}\overline{\phi}$. We use equation (8) to determine the quantity of production in the decentralized market.

$$\frac{\gamma - \beta}{\beta} = \frac{(1-\psi)u'(q_b)}{c'(\frac{1-n}{n}q_b)} - 1$$

With our functional forms:

$$\frac{\gamma - \beta}{\beta} = \frac{(1-\psi)q^{-1/2}}{\frac{1-n}{n}} - 1$$

$$\frac{\gamma}{\beta} = \frac{(1-\psi)q^{-1/2}}{\frac{1-n}{n}}$$

$$q^{1/2} = \frac{\beta(1-\psi)n}{\gamma(1-n)}$$

$$q = \left[\frac{\beta(1-\psi)n}{\gamma(1-n)}\right]^2$$

Substitute $\gamma = 1$

$$q = \left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2$$

From the expression for the amount of trade in the DM, we can determine the equilibrium price. We begin with the buyer's liquidity constraint

$$pq_b = (1-\psi)(m+l).$$

From the buyer's perspective, we consider the amount of loans that they are able to receive. First, note that every period, the portion of consumers who become buyers n deposit their 'm' units of money at banks. Thus, $n \cdot m$ represents the total amount of credit funding available to all sellers. This amount of funding will be distributed across the portion of the population that are buyers $(1-n)$.

$$(1-n)l = nm$$

$$l = \frac{n}{(1-n)}m$$

Therefore, we have:

$$pq_b = (1-\psi)(m + \frac{n}{1-n}m)$$

$$pq_b = (1-\psi)m(\frac{1-n+n}{1-n})$$

$$pq_b = \frac{(1-\psi)m}{1-n} \Rightarrow (1-n)pq_b = (1-\psi)m$$

Now rearrange $(1-n)pq_b = (1-\psi)m$ to get

$$(1-n)q_b = \frac{(1-\psi)m}{p}$$

Since $p\phi = c'(\frac{1-n}{n}q_b) = \frac{1-n}{n}$ then $p = \frac{1-n}{n\phi}$. Now substitute to get

$$(1-n)q_b = \frac{(1-\psi)m}{\frac{1-n}{n\phi}}$$

rearrange

$$(1-n)q_b = (1-\psi)\frac{n\phi}{1-n}m$$

Substitute $m = \overline{M}$ and $q_b = \left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2$

$$(1-n)\left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2 = (1-\psi)\frac{n\phi}{1-n}\overline{M}$$

Now substitute the steady state money supply, $\overline{M} = \frac{1}{2\psi}\overline{\phi}$

$$(1-n)\left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2 = (1-\psi)\frac{n\phi}{1-n}\frac{1}{2\psi}\overline{\phi}$$

simplify and switch sides

$$\frac{1}{2\psi}\overline{\phi}^2(1-\psi) = (\beta(1-\psi))^2 n$$

$$\overline{\phi}^2 = \frac{(\beta(1-\psi))^2 n}{\frac{1}{2\psi}(1-\psi)}$$

$$\overline{\phi}^2 = 2\psi\beta^2(1-\psi)n$$

$$\overline{\phi} = \sqrt{2\psi\beta^2(1-\psi)n}$$

The price level in the decentralized market

$$p = \frac{1-n}{n\overline{\phi}} = \frac{(1-n)}{n\sqrt{2\psi\beta^2(1-\psi)n}}$$

■

PROOF OF LEMMA 2:

To show $\tilde{q}_b < q_b$ as long as $i_d \geq -\psi$, we use our functional forms:

$$\frac{1-\beta}{\beta} = (1-n)\left[\frac{(1-\psi)\tilde{q}_b^{-1/2}}{\frac{1-n}{n}} - 1\right] - n\psi \tag{B.1}$$

$$\frac{1}{\beta} = (1-\psi)n\tilde{q}_b^{-1/2} + n - n(\psi) \tag{B.2}$$

$$\frac{1}{\beta} - n(1-\psi) = (1-\psi)n\tilde{q}_b^{-1/2} \tag{B.3}$$

$$\frac{1-\beta n(1-\psi)}{\beta} = (1-\psi)n\tilde{q}_b^{-1/2} \tag{B.4}$$

$$\tilde{q}_b^{1/2} = \frac{\beta n(1-\psi)}{1-\beta n(1-\psi)}$$

$$\tilde{q}_b = \left[\frac{\beta n(1-\psi)}{1-\beta n(1-\psi)}\right]^2$$

Now show $q_b > \tilde{q}_b$

$$q_b = \left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2 > \left[\frac{\beta n(1-\psi)}{1-\beta n(1-\psi)}\right]^2 = \tilde{q}_b$$

$$\frac{\beta(1-\psi)n}{(1-n)} > \frac{\beta n(1-\psi)}{1-\beta n(1-\psi)}$$

$$\frac{1}{(1-n)} > \frac{1}{1-\beta n(1-\psi)}$$

$$1-\beta n(1-\psi) > 1-n$$

$$-\beta n(1-\psi) > -n$$

$$1 > \beta(1-\psi)$$

PROOF OF LEMMA 3:

$$\frac{1-\beta}{\beta} = (1-n)\left[\frac{\ddot{q}_b^{-1/2}}{\frac{1-n}{n}} - 1\right] - n(0)$$

$$\frac{1-\beta}{\beta} = n\ddot{q}_b^{-1/2} - (1-n)$$

$$\frac{1}{\beta} = n\ddot{q}_b^{-1/2} + n$$

$$\frac{1-n\beta}{\beta} = n\ddot{q}_b^{-1/2}$$

$$\ddot{q}_b^{1/2} = \frac{n\beta}{1-n\beta}$$

$$\ddot{q}_b = \left[\frac{n\beta}{1-n\beta}\right]^2$$

Now find conditions where the quantity produced in our model is greater than FS.

$$\left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2 > \left[\frac{n\beta}{1-n\beta}\right]^2$$

$$\frac{\beta(1-\psi)n}{(1-n)} > \frac{n\beta}{1-n\beta}$$

$$\frac{(1-\psi)}{(1-n)} > \frac{1}{1-n\beta}$$

$$(1-\psi)(1-n\beta) > 1-n$$

$$-\psi(1-n\beta) > 1-n-1+n\beta$$

$$-\psi(1-n\beta) > -n+n\beta$$

$$\frac{n-n\beta}{1-n\beta} > \psi$$

■

PROOF OF HOURS WORKED IN CM TYPE I BANKS:

Money holdings are heterogeneous due to trade shocks and financial transactions from the first market. Thus, if we set $m = M_{-1}$, money holdings for buyers will be 0, and $\frac{1}{n}(1-\psi)M_{-1}$ for sellers.

Take the inverse of $U'^{*}) = 1$ to get $x^* = U'^{-1}(1)$. The buyer's production in the second market can be derived as follows:

$$h_b = x^* + \phi\{m_{+1} + (1+i)l\}$$

The buyer must produce enough of the CM good to consume x^*, pay off the loan from the bank $(1 + i)l$ and also procure enough money for the DM next period. In equilibrium, $m_{+1} = M = M_{-1} - \psi M_{-1} + \Delta_{-1}$:

$$h_b = x^* + \phi\{M_{-1} - \psi M_{-1} + \Delta_{-1} + (1+i)l\}$$

Since $\Delta = \psi M_{-1}$ in a stable price equilibrium, simplify to get:

$$h_b = x^* + \phi\{M_{-1} + (1+i)l\}$$

Take the buyer's liquidity constraint $pq_b = (1 - \psi)(M_{-1} + l)$ and substitute $p\phi = c'(q_s)$. Now substitute to get

$$h_b = x^* + \frac{c'(q_s)q_b}{(1-\psi)} + \phi\{il\} \tag{B.5}$$

Since $(1 - n)l = nm$, we can substitute into the liquidity constraint to get $p\phi q_b = (1 - \psi)\phi(\frac{(1-n)}{n}l + l) = (1-\psi)\phi\frac{l}{n} \Rightarrow \phi l = \frac{n}{(1-\psi)}p\phi q_b$. Use $p\phi = c'(q_s)$ to get $\phi l = \frac{n}{(1-\psi)}c'(q_s)q_b$ and substitute into the equation:

$$h_b = x^* + \frac{c'(q_s)q_b}{(1-\psi)} + \frac{in}{(1-\psi)}c'(q_s)q_b \tag{B.6}$$

So, with token loss taken into account, the buyer recovers the production cost of the DM good and tokens lost in the first market and also pay back the interest on the loan.

The seller's production is

$$h_s = x^* + \phi\{m_{+1} - [pq_s + M_{-1} + i_d d]\}$$

The seller will work enough hours to get the efficient amount of the good x^* and money to take into the DM next period less the real value of the good they sold in the DM pq_s and the money deposited plus interest $[M_{-1} + i_d d]$.

Use the fact that in equilibrium $m_{+1} = M = M_{-1} + \Delta_{-1} - \psi_1 M_{-1}$:

$$h_s = x^* + \phi\{M_{-1} + \Delta_{-1} - \psi_1 M_{-1} - [pq_s + M_{-1} + i_d d]\}$$

Since $\Delta = \psi M_{-1}$ in a stable price equilibrium, simplify to get:

$$h_s = x^* - \phi\{pq_s + i_d d\}$$

We can now substitute $c'(q_s) = p\phi$ to get:

$$h_s = x^* - c'(q_s)q_s - \phi i_d d$$

The expected hours worked h satisfies:

$$h = (1-n)h_b + nh_s = x^* + \frac{(1-n)(1+in)}{(1-\psi)}c'(q_s)q_b - nc'(q_s)q_s - n\phi i_d d$$

since in equilibrium $q_b = \frac{n}{1-n}q_s$ and $i(1-n)l = i_d nd$.

$$h = (1-n)h_b + nh_s = x^* + \frac{(1-n)(1+in)}{(1-\psi)}c'(q_s)q_b - (1-n)c^{'}(q_s)q_b - i(1-n)l$$

substitute $l = \frac{n}{(1-\psi)}c'(q_s)q_b$

$$h = (1-n)h_b + nh_s = x^* + \frac{(1-n)(1+in)}{(1-\psi)}c'(q_s)q_b - \frac{(1-n)(1-\psi)}{(1-\psi)}c^{'}(q_s)q_b - \frac{ni(1-n)}{(1-\psi)}c'(q_s)q_b$$

$$h = x^* + \frac{(1-n)(1+in-1+\psi-in)}{(1-\psi)}c'(q_s)q_b$$

$$h = x^* + \frac{(1-n)\psi}{(1-\psi)}c'(q_s)q_b$$

■

HOURS WORKED WITHOUT BANKS:

Take the buyer's liquidity constraint $p\tilde{q}_b = (1-\psi)(M_{-1})$ and substitute $p\phi = c'(\tilde{q}_s)$. Now substitute to get

$$\tilde{h}_b = x^* + \frac{c'(q_s)\tilde{q}_b}{(1-\psi)}$$

So the buyer recovers the production cost of the DM good and tokens lost in the first market.

The seller's production is

$$\tilde{h}_s = x^* + \phi\{m_{+1} - [p\tilde{q}_s + (1-\psi)M_{-1}]\}$$

The seller will work enough hours to get the efficient amount of the good x^* and money to take into the DM next period less the real value of the good they sold in the DM $p\tilde{q}_s$ and their money brought into the market.

$$\tilde{h}_s = x^* - \phi\{p\tilde{q}_s\} + \frac{c'(\tilde{q}_s)\tilde{q}_b}{(1-\psi)} - \frac{(1-\psi)c'(q_s)\tilde{q}_b}{(1-\psi)}$$

We can now substitute $c'(\tilde{q}_s) = p\phi$ to get:

$$\tilde{h}_s = x^* - c'(\tilde{q}_s)\tilde{q}_s + \frac{\psi}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b$$

The expected hours worked h satisfies:

$$\tilde{h} = (1-n)h_b + nh_s = x^* + \frac{(1-n)}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b - nc'(\tilde{q}_s)\tilde{q}_s + \frac{n\psi}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b$$

$$\tilde{h} = (1-n)h_b + nh_s = x^* + \frac{(1-n)}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b - \frac{(1-n)(1-\psi)}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b + \frac{n\psi}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b$$

simplify

$$\tilde{h} = (1-n)h_b + nh_s = x^* + \frac{\psi(1-n)}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b + \frac{n\psi}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b$$

simplify

$$\tilde{h} = x^* + \frac{\psi}{(1-\psi)}c'(\tilde{q}_s)\tilde{q}_b \qquad \blacksquare$$

PROOF FOR HOURS WORKED TO BE LESS WITH INTERMEDIARIES:

$$\tilde{h} - h = x^* + \frac{\psi}{(1-\psi)} c'(\tilde{q}_s)\tilde{q}_b - x^* - \frac{\psi(1-n)}{(1-\psi)} c'(q_s) q_b > 0$$

$$\tilde{h} - h = \frac{\psi}{(1-\psi)} \frac{(1-n)}{n} \left[\frac{\beta n(1-\psi)}{1-\beta n(1-\psi)} \right]^2 - \frac{\psi(1-n)}{(1-\psi)} \frac{(1-n)}{n} \left[\frac{\beta(1-\psi)n}{(1-n)} \right]^2 > 0$$

multiply through by $\frac{(1-\psi)n}{\psi(1-n)}$

$$\left[\frac{\beta n(1-\psi)}{1-\beta n(1-\psi)} \right]^2 - (1-n) \left[\frac{\beta(1-\psi)n}{(1-n)} \right]^2 > 0$$

$$\left[\frac{\beta n(1-\psi)}{1-\beta n(1-\psi)} \right]^2 > (1-n) \left[\frac{\beta(1-\psi)n}{(1-n)} \right]^2$$

simplify

$$\frac{1}{(1-\beta n(1-\psi))^2} > \frac{1}{(1-n)}$$

simplify

$$1 - n > (1 - \beta n(1-\psi))^2$$

take the square root of both sides

$$\sqrt{1-n} > 1 - \beta n(1-\psi)$$

simplify

$$\beta n(1-\psi) > 1 - \sqrt{1-n}$$

simplify

$$\beta > \frac{1-\sqrt{1-n}}{n(1-\psi)}$$

■

PROOF OF PROPOSITION 2:

We will assume that at some point in time t an agent decides to not borrow ever again but continue to deposit at the beginning of the second market. It is optimal for him to buy the same quantity since the optimal choice still satisfies (8). His money balance will then become $m^{nb}_{+1} = m_{+1} + l_{+1}$. An agent who decides to never borrow has to carry around a higher money balance. This saves him from paying interest on the loans in the future. This means consumption and production in market 1 are not affected. The difference in lifetime payoff comes from hours worked.

Coming into market 2 he will need to work extra hours to hold extra money to make up for the loan l_{+1}, so the hours worked becomes:

$$h^{nb}_b = x^* + \phi\{m_{+1} + l_{+1} + (1+i)l\}$$

$$= x^* + \frac{(1+in)c'(q_s)q_b}{(1-\psi)} + \phi l_{+1},$$

where as if he sold in the previous market

$$h^{nb}_s = x^* + \phi\{m_{+1} + l_{+1} - [pq_s + M_{-1} + i_d d]\}$$

$$= x^* - c'(q_s)q_s + \phi l_{+1} - \phi id.$$

The expected hours worked will now satisfy

$$h^{nb} = (1-n)h_b^{nb} + nh_s^{nb} = x^* + \frac{\psi(1-n)}{(1-\psi)}c'(q_s)q_b + \phi l_{+1}.$$

Consequently, from $h = (1-n)h_b + nh_s = x^* + \frac{\psi(1-n)}{(1-\psi)}c'(q_s)q_b$ the additional hours worked are

$$\overline{h} - h = \phi l_{+1} = \frac{\gamma n}{(1-\psi)}c'(q_s)q_b > 0 \tag{B.7}$$

Since we showed earlier $\phi l = \frac{n}{(1-\psi)}c'(q_s)q_b$ and since $l_{+1} = \gamma l$ in a steady state. With $\gamma = 1$, we get $\phi l_{+1} = \frac{n}{(1-\psi)}c'(q_s)q_b$ and so the extra money to compensate for the loan amount will be the same as the amount shown earlier. The loan amount will need to be higher with inflation and lower with deflation.

Now we look at the hours worked in some future period 2. He will not have a loan to pay back $(1+i)l$ and so hours worked becomes:

$$\hat{h}_b = x^* + \phi\{m_{+1} + l_{+1}\}$$

Since $\Delta = \psi M_{-1}$ we substitute $c'(q_s)q_b = \phi(1-\psi)(m+l)$ to get

$$\hat{h}_b = x^* + \frac{c'(q_s)q_b}{(1-\psi)} - \phi l + \phi l_{+1}$$

We can now substitute $\phi l_{+1} - \phi l = \frac{n}{(1-\psi)}c'(q_s)q_b(\gamma - 1) = \frac{n}{(1-\psi)}c'(q_s)q_b(1-1) = 0$ to get

$$\hat{h}_b = x^* + \frac{c'(q_s)q_b}{(1-\psi)}$$

is the hours worked for a buyer in market 1. If he sold in market 1 it becomes

$$\hat{h}_s = x^* + \phi\{m_{+1} + l_{+1} - [pq_s + (1+i)d^{nb}]\}$$

substituting $c'(q_s) = \phi p$

$$\hat{h}_s = x^* - c'(q_s)q_s + \phi\{m_{+1} + l_{+1} - [(1+i)d^{nb}]\}$$

now substitute $m_{+1} = M = M_{-1} - \psi M_{-1} + \Delta_{-1}$

$$\hat{h}_s = x^* - c'(q_s)q_s + \phi\{M_{-1} - \psi M_{-1} + \Delta_{-1} + l_{+1} - [(1+i)d^{nb}]\}$$

since $\psi M_{-1} = \Delta_{-1}$

$$\hat{h}_s = x^* - c'(q_s)q_s + \phi\{M_{-1} + l_{+1} - [(1+i)d^{nb}]\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{c'(q_s)q_b}{(1-\psi)} - \phi\Big\{l - l_{+1} + (1+i)d^{nb}\Big\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{c'(q_s)q_b}{(1-\psi)} + \frac{n}{(1-\psi)}c'(q_s)q_b(\gamma - 1) - \phi\Big\{(1+i)d^{nb}\Big\}$$

since $\gamma = 1$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{c'(q_s)q_b}{(1-\psi)} - \phi\Big\{(1+i)d^{nb}\Big\}$$

since you self insure for consumption shocks $d^{nb} = m + l$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{c'(q_s)q_b}{(1-\psi)} - \phi\Big\{(1+i)(m+l)\Big\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{c'(q_s)q_b}{(1-\psi)} - (1+i)\frac{c'(q_s)q_b}{(1-\psi)}$$

$$\hat{h}_s = x^* - c'(q_s)q_s - i\frac{c'(q_s)q_b}{(1-\psi)}$$

We can now substitute $i = \frac{1-\beta}{\beta}$ to get

$$\hat{h}_s = x^* - c'(q_s)q_s - \frac{1-\beta}{\beta}\frac{c'(q_s)q_b}{(1-\psi)}$$

The expected hours will satisfy:

$$\hat{h} = (1-n)\hat{h}_b + n\hat{h}_s = (1-n)\big[x^* + \frac{c'(q_s)q_b}{(1-\psi)}\big]$$

$$+n\big[x^* - c'(q_s)q_s - \frac{1-\beta}{\beta}\frac{c'(q_s)q_b}{(1-\psi)}\big]$$

First add x^* terms:

$$(1-n+n)x^* = x^*$$

Now we can substitute $q_s = \frac{(1-n)}{n}q_b$ into $c'(q_s)q_s$ to get

$$c'(q_s)\frac{(1-n)}{n}q_b$$

Now we can add $c'(q_s)q_b$ terms:

$$\left\{(1-n)\Big(\frac{1}{1-\psi}\Big) - n\Big(\frac{(1-n)}{n} + \frac{(1-\beta)}{\beta(1-\psi)}\Big)\right\}c'(q_s)q_b$$

$$\left\{\frac{(1-n)}{(1-\psi)} - \frac{(1-n)(1-\psi)}{(1-\psi)} - \frac{n(1-\beta)}{\beta(1-\psi)}\right\}c'(q_s)q_b$$

$$\frac{\psi(1-n)}{(1-\psi)}c'(q_s)q_b - \frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b$$

Now combine x^* and $c'(q_s)q_b$ terms to get:

$$\hat{h} = x^* + \frac{\psi(1-n)}{(1-\psi)}c'(q_s)q_b - \frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b$$

The expected gain from this strategy is

$$\hat{h} - h = x^* + \frac{\psi(1-n)}{(1-\psi)}c'(q_s)q_b - \frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b - x^* - \frac{\psi(1-n)}{(1-\psi)}c'(q_s)q_b$$

$$\hat{h} - h = -\frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b$$

The expected gain in the future from this strategy becomes:

$$h^{nb} - h + \Sigma_{t=1}^{\infty}\beta^t(\hat{h} - h)$$

Sum of an infinite geometric series $\Sigma_{t=1}^{\infty}\beta^t = \frac{\beta}{1-\beta}$

$$h^{nb} - h + \frac{\beta}{1-\beta}(\hat{h} - h)$$

Now substitute $h^{nb} - h = \frac{n}{(1-\psi)} c'(q_s) q_b$ and $\hat{h} - h = -\frac{n(1-\beta)}{\beta(1-\psi)} c'(q_s) q_b$

$$\frac{n}{(1-\psi)} c'(q_s) q_b + \frac{\beta}{1-\beta}\Big(- \frac{n(1-\beta)}{\beta(1-\psi)}\Big) c'(q_s) q_b$$

simplify

$$\frac{n}{(1-\psi)} c'(q_s) q_b - \frac{n}{(1-\psi)} c'(q_s) q_b$$

Which gives us

$$\Big\{0\Big\} c^{'}(q_s) q_b = 0$$

So agents will be indifferent to borrowing with a stable price and saving the money themselves.

PROOF OF LEMMA 4: If $\hat{\Delta}_t = \alpha M_t < \psi M_t$, there will be a decrease in tokens each period. As a consequence, the price of tokens will increase each period $\phi_{t+1} > \phi_t$, but token production will stay the same due to the cap. Consequently, it is possible for a steady state with deflation to exist.

In order to find γ, we take the law of motion for the supply of tokens and substitute $\hat{\Delta}_t = \alpha M_{t-1}$.

$$M_t = \alpha M_{t-1} + (1-\psi) M_{t-1}$$

$$M_t = (1 - (\psi - \alpha)) M_{t-1}$$

$$\frac{M_t}{M_{t-1}} = 1 - (\psi - \alpha)$$

Which gives us $\gamma = \frac{M_t}{M_{t-1}} = 1 - (\psi - \alpha)$. ■

PROOF OF TYPE II BANK'S PROBLEM:

The representative bank solves the following problem per borrower:

$$\max_{l} (i - i_d) l$$

$$s.t. \quad l \leq \bar{l}, \quad u(q_b) - (1+i) l \phi \geq \Gamma.$$

where Γ is the reservation value of the borrower. The reservation value is the borrower's surplus from receiving a loan at another bank.

We consider two different settings concerning enforcement of loan contracts. First, banks can force repayment at no cost. Consequently $\bar{l} = \infty$ as the borrowing constraint would never bind. In a second case, the banks cannot force repayment, but any borrower who fails to pay will be shut out of the banking sector. Later on, we derive conditions to ensure voluntary repayment to determine $\bar{l}$.

We can represent the bank's problem as

$$\mathscr{L} \max_{l} = (i - i_d) l - \lambda_L (l - \bar{l}) + \lambda_\Gamma (u(q_b) - (1+i) l \phi - \Gamma)$$

where λ_L is the Lagrange multiplier on the constraint on a bank's lending. Also, λ_Γ is the Lagrange multiplier on the participation constraint of the borrower. The bank's first order condition for how much to loan becomes:

$$i - i_d - \lambda_L + \lambda_\Gamma \left[u'(q_b) \frac{dq_b}{dl} - (1+i) \phi \right] = 0$$

As described in BCW, with $i - i_d > 0$ the bank will loan as much as possible. Therefore, $u'(q_b)\frac{dq_b}{dl} = (1+i)\phi$ and it will be the case that $\lambda_\Gamma > 0$. However, competition will drive interest rates to parity $i = i_d$. The bank's problem matches our previous case up to this point. Now we must consider the fact that the loan does not suffer token loss.

From $q_b = ((1-\psi)m + l)/p$

$$\frac{dq_b}{dl} = \frac{1}{p} \Rightarrow \frac{dl}{dq_b} = p$$

As p increases, it costs more to obtain goods in the DM and the buyer will choose to borrow more funds. Since $p\phi = c'(q_s)$

$$\frac{dq_b}{dl} = \frac{\phi}{c'(q_s)}$$

Thus, a bank's decision on how much to lend becomes

$$i - i_d - \lambda_L + \lambda_\Gamma \Big[u'(q_b)\frac{\phi}{c'(q_s)} - (1+i)\phi \Big] = 0$$

Since the bank will receive no profits due to free entry

$$\phi\lambda_\Gamma \Big[\frac{u'(q_b)}{c'(q_s)} - (1+i) \Big] = \lambda_L$$

$$\frac{u'(q_b)}{c'(q_s)} - (1+i) = \frac{\lambda_L}{\phi\lambda_\Gamma}$$

$$\frac{u'(q_b)}{c'(q_s)} = 1 + i + \frac{\lambda_L}{\phi\lambda_\Gamma}$$

■

PROOF OF MARGINAL VALUE OF MONEY TYPE II BANKS:

Thus, the marginal value of money starts with:

$$V'(m) = (1-n)\left[u'(q_b)\frac{\partial q_b}{\partial m} + W_m\Big((1-\psi) - p\frac{\partial q_b}{\partial m} + \frac{\partial l}{\partial m}\Big) + W_l\frac{\partial l}{\partial m}\right]$$

$$+n\left[-c'(q_s)\frac{\partial q_s}{\partial m} + W_m\left((1-\psi) + p\frac{\partial q_s}{\partial m} - (1-\psi)\frac{\partial d}{\partial m}\right) + W_d\frac{\partial d}{\partial m}\right]$$

Money brought into the DM by a seller will not affect his production of the DM good. Therefore, we can substitute $\partial q_s/\partial m = 0$ for the consumer's decision on money holdings. Also, with $i_d > -\psi$, a seller will deposit all money holdings and we need to substitute $\partial d/\partial m = 1$. Lastly, use the envelope conditions for cash, loans and deposits to get

$$V'(m) = (1-n)\left[u'(q_b)\frac{\partial q_b}{\partial m} + \phi\Big((1-\psi) - p\frac{\partial q_b}{\partial m} + \frac{\partial l}{\partial m}\Big) - \phi(1+i)\frac{\partial l}{\partial m}\right] + n\phi(1+i_d)$$

If a buyer in the first market, the consumer must weigh the additional utility provided from consumption of the first market good bought with an additional unit of money. Additionally, the buyer considers the gain in utility from borrowing less funds from banks. If a seller, an additional unit of money will reduce his labor in the second market. In particular, this will depend on the interest rate paid by banks on deposits.

All of the tokens held by a buyer will be spent in the first market. Consequently, $p(\partial q_b/\partial m) = (1-\psi) + \partial l/\partial m$ and the first order condition becomes:

$$V'(m) = (1-n)\left[u'(q_b)\frac{\partial q_b}{\partial m} - \phi(1+i)\frac{\partial l}{\partial m}\right] + n\phi(1+i_d)$$

As a buyer, the net role of money demand is to purchase more goods and reduce borrowing costs. Now we can substitute $\frac{\partial l}{\partial m} = p\frac{\partial q_b}{\partial m} - (1-\psi)$

$$V'(m) = (1-n)\left[u'(q_b)\frac{\partial q_b}{\partial m} - \phi(1+i)\left[p\frac{\partial q_b}{\partial m} - (1-\psi)\right]\right] + n\phi(1+i_d)$$

$$V'(m) = (1-n)\left[\frac{\partial q_b}{\partial m}(u'(q_b) - \phi(1+i)p) + (1-\psi)\phi(1+i)\right] + n\phi(1+i_d)$$

Take the equilibrium condition from the credit market $\frac{u'(q_b)}{c'(q_s)} = 1+i$ and substitute $c'(q_s) = p\phi$. Now we can substitute $u'(q_b) = p\phi(1+i)$

$$V'(m) = (1-n)\left[\frac{\partial q_b}{\partial m}(0) + \frac{(1-\psi)u'(q_b)}{p}\right] + n\phi(1+i_d)$$

$$V'(m) = (1-n)\frac{(1-\psi)u'(q_b)}{p} + n\phi(1+i_d) \tag{B.8}$$

From $c'(q_s) = p\phi$ in (4)

$$V'(m) = \phi\left[(1-n)\frac{(1-\psi)u'(q_b)}{c'(q_s)} + n(1+i_d)\right]$$

■

PROOF FOR NOMINAL INTEREST RATE WITH TYPE II BANKS:

Identical to the first case, we get the following for bringing an additional unit of money.

$$\frac{\gamma - \beta}{\beta} = (1-n)\left[\frac{(1-\psi)u'(q_b)}{c'(\frac{1-n}{n}q_b)} - 1\right] + ni_d \tag{B.9}$$

A difference emerges when we substitute to find the nominal interest rate. Substitute $\lambda_l = 0$ and $\frac{u'(q_b)}{c'(q_s)} = 1 + i$ in (22) yields

$$\frac{\gamma - \beta}{\beta} = (1-n)\left[(1-\psi)(1+i) - 1\right] + ni_d$$

$$\frac{\gamma - \beta}{\beta} = (1-n)((1-\psi)i - \psi) + ni_d$$

The first term now represents the interest saving from borrowing one less unit of money when a buyer. The banks can charge a higher rate due to this.■

PROOF OF LEMMA 5:

First, suppose that $\phi = \overline{\phi}$, then token production will be equal to what is needed for a stable price $\Delta^* = \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production is equal to the amount of tokens lost it will be the case that $\gamma = 1$ and thus the price level will be constant over time. The price of tokens will be the same as the previous period $\phi_{+1} = \phi$. This will lead to the same amount of tokens being created as the previous period $\Delta^*_{+1} = \Delta^*$.

This process will repeat itself ensuring token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = (1-n)\psi\overline{M}$. Thus, $\gamma = 1$ for the remaining periods making a steady state possible.

Next, assume $\phi > \overline{\phi}$, then token production will greater than what is needed for a stable price $\Delta^* > \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. Since token production will be greater than the amount of tokens lost it will be the case that $\gamma > 1$ and thus inflation occurs. Over time, the price of tokens will be less than the previous period $\phi_{+1} < \phi$. This will lead to less tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} < \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = (1-n)\psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with inflation in this economy.

Lastly, suppose $\phi < \overline{\phi}$, then token production is less than what is needed for a stable price $\Delta^* < \overline{\Delta}^* = \frac{1}{2}\overline{\phi}$. There will be less new tokens created than tokens lost from the money supply in the previous period $\Delta^* < (1-n)\psi M_{-1}$. Since token production will be less than the amount of tokens lost it will be the case that $\gamma < 1$ and thus deflation is observed. The price of tokens will be greater than the previous period $\phi_{+1} > \phi$. This will lead to more tokens being created than the previous period since token production is increasing in price $\Delta^*_{+1} > \Delta^*$.

This process will continue for n periods until $\phi_{+n} = \overline{\phi}$. Token production for each period equals token loss $\Delta^*_{+n} = \overline{\Delta}^* = \psi\overline{M}$. At this point $\gamma = 1$ for the remaining periods. Thus, it is not possible to have a steady state with deflation in this economy. ■

PROOF OF PROPOSITION 3:

From Lemma 1 the steady-state money supply is $\overline{M} = \frac{1}{2(1-n)\psi}\overline{\phi}$. We use equation (8) to determine the quantity of production in the decentralized market.

$$\frac{\gamma - \beta}{\beta} = \frac{(1-(1-n)\psi)u'(q_b)}{c'(\frac{1-n}{n}q_b)} - 1$$

With our functional forms:

$$\frac{\gamma - \beta}{\beta} = \frac{(1-(1-n)\psi)q^{-1/2}}{\frac{1-n}{n}} - 1$$

$$\frac{\gamma}{\beta} = \frac{(1-(1-n)\psi)q^{-1/2}}{\frac{1-n}{n}}$$

$$q^{1/2} = \frac{\beta(1-(1-n)\psi)n}{\gamma(1-n)}$$

$$q = \left[\frac{\beta(1-(1-n)\psi)n}{\gamma(1-n)}\right]^2$$

Substitute $\gamma = 1$

$$q_b = \left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2$$

From the expression for the amount of trade in the DM, we can determine the equilibrium price. We begin with the buyer's liquidity constraint

$$pq_b = (1-\psi)m + l.$$

From the buyer's perspective, we consider the amount of loans that they are able to receive. First, note that every period, the portion of consumers who become buyers n

deposit their 'm' units of money at banks. Thus, $n \cdot m$ represents the total amount of credit funding available to all sellers. This amount of funding will be distributed across the portion of the population that are buyers $(1-n)$.

$$(1-n)l = nm$$

$$l = \frac{n}{(1-n)}m$$

Therefore, we have:

$$pq_b = (1-\psi)m + \frac{n}{1-n}m$$

$$pq_b = (\frac{(1-\psi)(1-n)+n}{1-n})m$$

$$pq_b = \left(\frac{1-(1-n)\psi}{1-n}\right)m$$

Since $p\phi = c'(\frac{1-n}{n}q_b) = \frac{1-n}{n}$

$$p = \frac{1-n}{n\phi}$$

Now rearrange $(1-n)pq_b = (1-(1-n)\psi)m$ to get

$$(1-n)q_b = \frac{(1-(1-n)\psi)m}{p}$$

$$(1-n)q_b = \frac{(1-(1-n)\psi)m}{\frac{1-n}{n\phi}}$$

$$(1-n)q_b = (1-(1-n)\psi)\frac{n\phi}{1-n}m$$

Substitute $m = \overline{M}$ and $q_b = \left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2$

$$(1-n)\left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2 = (1-(1-n)\psi)\frac{n\phi}{1-n}\overline{M}$$

Now substitute the steady state money supply, $\overline{M} = \frac{1}{2(1-n)\psi}\overline{\phi}$

$$(1-n)\left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2 = (1-(1-n)\psi)\frac{n\phi}{1-n}\frac{1}{2(1-n)\psi}\overline{\phi}$$

simplify and switch sides

$$\frac{1}{2(1-n)\psi}\overline{\phi}^2(1-(1-n)\psi) = (\beta(1-(1-n)\psi))^2 n$$

$$\overline{\phi}^2 = \frac{(\beta(1-(1-n)\psi))^2 n}{\frac{1}{2(1-n)\psi}(1-(1-n)\psi)}$$

$$\overline{\phi}^2 = 2(1-n)\psi\beta^2(1-(1-n)\psi)n$$

$$\overline{\phi} = \sqrt{2(1-n)\psi\beta^2(1-(1-n)\psi)n}$$

The price level in the decentralized market

$$p = \frac{1-n}{n\overline{\phi}} = \frac{(1-n)}{n\sqrt{2(1-n)\psi\beta^2(1-(1-n)\psi)n}}$$

■

PROOF OF LEMMA 6:

Compare $q_b > \tilde{q}_b$

$$q_b = \left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2 > \left[\frac{\beta n(1-\psi)}{1-\beta n(1-\psi)}\right]^2 = \tilde{q}_b$$

$$\frac{\beta(1-(1-n)\psi)n}{(1-n)} > \frac{\beta n(1-\psi)}{1-\beta n(1-\psi)}$$

$$\frac{(1-(1-n)\psi)}{(1-n)} > \frac{(1-\psi)}{1-\beta n(1-\psi)}$$

$$(1-(1-n)\psi)(1-\beta n(1-\psi)) > (1-n)(1-\psi)$$

$$1+(1-n)\psi\beta(1-\psi) > \beta(1-\psi)$$

PROOF OF LEMMA 7:

Find conditions where the quantity produced in our model is greater than FS.

$$\left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2 > \left[\frac{n\beta}{1-n\beta}\right]^2$$

$$\frac{\beta(1-(1-n)\psi)n}{(1-n)} > \frac{n\beta}{1-n\beta}$$

$$\frac{(1-(1-n)\psi)}{(1-n)} > \frac{1}{1-n\beta}$$

$$-(1-n)\psi(1-n\beta) > 1-n-1+n\beta$$

$$\frac{n-n\beta}{(1-n)(1-n\beta)} > \psi$$

PROOF OF HOURS WORKED IN CM TYPE II BANKS:

Money holdings are heterogeneous due to trade shocks and financial transactions from the first market. Thus, if we set $m = M_{-1}$, money holdings for buyers will be 0, and $\frac{1}{n}(1 - \psi)M_{-1}$ for sellers.

Take the inverse of $U'^{*}) = 1$ to get $x^{*} = U'^{-1}(1)$. The buyer's production in the second market can be derived as follows:

$$h_b = x^{*} + \phi\{m_{+1} + (1+i)l\}$$

The buyer must produce enough of the CM good to consume x^{*}, pay off the loan from the bank $(1 + i)l$ and also procure enough money for the DM next period. In equilibrium, $m_{+1} = M = M_{-1} - (1-n)\psi M_{-1} + \Delta_{-1}$:

$$h_b = x^{*} + \phi\{M_{-1} - (1-n)\psi M_{-1} + \Delta_{-1} + (1+i)l\}$$

Since $\Delta = (1-n)\psi M_{-1}$ in a stable price equilibrium, simplify to get:

$$h_b = x^{*} + \phi\{M_{-1} + (1+i)l\}$$

Take the buyer's liquidity constraint $pq_b = (1-\psi)M_{-1} + l$ and substitute $p\phi = c'(q_s)$. Now substitute to get

$$h_b = x^{*} + c'(q_s)q_b + \psi M_{-1} + \phi\{il\} \tag{B.10}$$

Since $(1 - n)l = nm$, we can substitute into the liquidity constraint to get $pq_b = \left(\frac{1-(1-n)\psi}{1-n}\right)m \Rightarrow m = \frac{1-n}{(1-(1-n)\psi)}pq_b$ and $pq_b = \frac{1-(1-n)\psi}{n}l$. Use $p\phi = c'(q_s)$ to get $\phi m = \frac{1-n}{(1-(1-n)\psi)}c'(q_s)q_b$ and $\phi l = \frac{n}{(1-(1-n)\psi)}c'(q_s)q_b$ and substitute into the equation:

$$h_b = x^* + c'(q_s)q_b + \frac{\psi(1-n)}{(1-(1-n)\psi)}c'(q_s)q_b + \frac{in}{(1-(1-n)\psi)}c'(q_s)q_b \tag{B.11}$$

simplify

$$h_b = x^* + c'(q_s)q_b + \frac{\psi(1-n)+in}{(1-(1-n)\psi)}c'(q_s)q_b \tag{B.12}$$

So the buyer recovers the production cost of the DM good and tokens lost in the first market and also pay back the interest on the loan.

The seller's production is

$$h_s = x^* + \phi\{m_{+1} - [pq_s + M_{-1} + i_d d]\}$$

The seller will work enough hours to get the efficient amount of the good x^* and money to take into the DM next period less the real value of the good they sold in the DM pq_s and the money deposited plus interest $[M_{-1} + i_d d]$.

Use the fact that in equilibrium $m_{+1} = M = M_{-1} + \Delta_{-1} - (1-n)\psi M_{-1}$:

$$h_s = x^* + \phi\{M_{-1} + \Delta_{-1} - (1-n)\psi M_{-1} - [pq_s + M_{-1} + i_d d]\}$$

Since $\Delta = (1-n)\psi M_{-1}$ in a stable price equilibrium, simplify to get:

$$h_s = x^* - \phi\{pq_s + i_d d\}$$

We can now substitute $c'(q_s) = p\phi$ to get:

$$h_s = x^* - c'(q_s)q_s - \phi i_d d$$

The expected hours worked h satisfies:

$$h = (1-n)h_b + nh_s = x^* + (1-n)c'(q_s)q_b + \frac{(1-n)(\psi(1-n)+in)}{(1-(1-n)\psi)}c'(q_s)q_s - nc'(q_s)q_s - n\phi i_d d$$

since in equilibrium $q_b = \frac{n}{1-n}q_s$ and $i(1-n)l = i_d nd$.

$$h = (1-n)h_b + nh_s = x^* + (1-n)c'(q_s)q_b + \frac{(1-n)(\psi(1-n)+in)}{(1-(1-n)\psi)}c'(q_s)q_b - (1-n)c'(q_s)q_b - i(1-n)l$$

substitute $l = \frac{n}{(1-\psi)}c'(q_s)q_b$ and simplify

$$h = (1-n)h_b + nh_s = x^* + \frac{(1-n)(\psi(1-n)+in)}{(1-(1-n)\psi)}c'(q_s)q_b - \frac{ni(1-n)}{(1-(1-n)\psi)}c'(q_s)q_b$$

$$h = (1-n)h_b + nh_s = x^* + \frac{(1-n)(\psi)(1-n)}{(1-(1-n)\psi)}c'(q_s)q_b$$

$$h = x^* + \frac{\psi(1-n)^2}{(1-(1-n)\psi)}c'(q_s)q_b$$

■

PROOF OF COROLLARY 2:

We compare hours worked with type II banks to the hours worked in the case with type I banks

$$h_{II} = x^* + \frac{\psi(1-n)^2}{(1-(1-n)\psi)} c'(q_s)q_b < x^* + \frac{\psi(1-n)}{(1-\psi)} c'(q_s)q_b = h_I$$

$$\frac{\psi(1-n)^2}{(1-(1-n)\psi)} \frac{(1-n)}{n} \left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2 < \frac{\psi(1-n)}{(1-\psi)} \frac{(1-n)}{n} \left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2$$

$$\frac{(1-n)}{(1-(1-n)\psi)} \left[\frac{\beta(1-(1-n)\psi)n}{(1-n)}\right]^2 < \frac{1}{(1-\psi)} \left[\frac{\beta(1-\psi)n}{(1-n)}\right]^2$$

$$\frac{(1-n)}{(1-(1-n)\psi)} (1-(1-n)\psi)^2 < \frac{1}{(1-\psi)} (1-\psi)^2$$

$$(1-n)(1-(1-n)\psi) < 1-\psi$$

$$1-(1-n)\psi - n + n(1-n)\psi < 1-\psi$$

$$-\psi + n\psi - n + n\psi - n^2\psi < -\psi$$

$$2n\psi - n^2\psi < n$$

$$\psi(2n - n^2) < n$$

$$\psi < \frac{n}{(2n-n^2)}$$

$$\psi < \frac{1}{2-n}$$

■

PROOF OF PROPOSITION 4:

We will assume that at some point in time t an agent decides to not borrow ever again but continue to deposit at the beginning of the second market. It is optimal for him to buy the same quantity since the optimal choice still satisfies (8). His money balance will then become $m^{nb}_{+1} = m_{+1} + l_{+1}$. An agent who decides to never borrow has to carry around a higher money balance. This saves him from paying interest on the loans in the future. This means consumption and production in market 1 are not affected. The difference in lifetime payoff comes from hours worked.

Coming into market 2 he will need to work extra hours to hold extra money to make up for the loan l_{+1}, so the hours worked becomes:

$$h^{nb}_b = x^* + \phi\{m_{+1} + l_{+1} + (1+i)l\}$$

$$= x^* + c'(q_s)q_b + \frac{\psi(1-n) + in}{(1-(1-n)\psi)}c'(q_s)q_b + \phi l_{+1},$$

where as if he sold in the previous market

$$h^{nb}_s = x^* + \phi\{m_{+1} + l_{+1} - [pq_s + M_{-1} + i_d d]\}$$

$$= x^* - c'(q_s)q_s + \phi l_{+1} - \phi id.$$

The expected hours worked will now satisfy

$$h^{nb} = (1-n)h^{nb}_b + nh^{nb}_s = x^* + \frac{\psi(1-n)^2}{(1-(1-n)\psi)}c'(q_s)q_b + \phi l_{+1}.$$

Consequently, from $h = (1-n)h_b + nh_s = x^* + \frac{\psi(1-n)^2}{(1-(1-n)\psi)}c'(q_s)q_b$ the additional hours worked are

$$\overline{h} - h = \phi l_{+1} = \frac{\gamma n}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b > 0 \tag{B.13}$$

Since we showed earlier $\phi l = \frac{n}{(1-(1-n)\psi)}c'(q_s)q_b$ and since $(1-\psi)l_{+1} = \gamma l$ in a steady state. Since the tokens are held as cash, we have to factor this in. With $\gamma = 1$, we get $\phi l_{+1} = \frac{n}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b$ and so the extra money to compensate for the loan amount will be the same as the amount shown earlier. The loan amount will need to be higher with inflation and lower with deflation.

Now we look at the hours worked in some future period 2. He will not have a loan to pay back $(1+i)l$ and so hours worked becomes:

$$\hat{h}_b = x^* + \phi\{m_{+1} + l_{+1}\}$$

We substitute $m = \frac{1-n}{1-(1-n)\psi}pq_b$ and $\phi l_{+1} = \frac{n}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b$ to get

$$\hat{h}_b = x^* + \frac{(1-\psi)(1-n) - n}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b$$

simplify

$$\hat{h}_b = x^* + \frac{1-(1-n)\psi}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b$$

is the hours worked for a buyer in market 1. If he sold in market 1 it becomes

$$\hat{h}_s = x^* + \phi\{m_{+1} + l_{+1} - [pq_s + (1+i)d^{nb}]\}$$

substituting $c'(q_s) = \phi p$

$$\hat{h}_s = x^* - c'(q_s)q_s + \phi\{m_{+1} + l_{+1} - [(1+i)d^{nb}]\}$$

now substitute $m_{+1} = M = M_{-1} - \psi M_{-1} + \Delta_{-1}$

$$\hat{h}_s = x^* - c'(q_s)q_s + \phi\{M_{-1} - \psi M_{-1} + \Delta_{-1} + l_{+1} - [(1+i)d^{nb}]\}$$

since $\psi M_{-1} = \Delta_{-1}$

$$\hat{h}_s = x^* - c'(q_s)q_s + \phi\{M_{-1} + l_{+1} - [(1+i)d^{nb}]\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{1-(1-n)\psi}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b - \phi\Big\{(1+i)d^{nb}\Big\}$$

since you self insure for consumption shocks $d^{nb} = m + l$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{1-(1-n)\psi}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b - \phi\Big\{(1+i)(m+l)\Big\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + \frac{1-(1-n)\psi}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b - (1+i)\frac{1-(1-n)\psi}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b$$

$$\hat{h}_s = x^* - c'(q_s)q_s - \frac{i(1-(1-n)\psi)}{(1-\psi)(1-(1-n)\psi)}c'(q_s)q_b$$

Take $\frac{1-\beta}{\beta} = i - (1-n)\psi(1+i)$ and solve for i to get

$$\frac{1-\beta}{\beta} = i - (1-n)\psi - (1-n)\psi i$$

$$i(1-(1-n)\psi) = \frac{1-\beta(1-(1-n)\psi)}{\beta}$$

$$i = \frac{1-\beta(1-(1-n)\psi)}{\beta(1-(1-n)\psi)}$$

$$\hat{h}_s = x^* - c'(q_s)q_s - \frac{1-\beta(1-(1-n)\psi)}{\beta(1-(1-n)\psi)} \frac{(1-(1-n)\psi)}{(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

$$\hat{h}_s = x^* - c'(q_s)q_s - \frac{1-\beta(1-(1-n)\psi)}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

The expected hours will satisfy:

$$\hat{h} = (1-n)\hat{h}_b + n\hat{h}_s = (1-n)\big[x^* + \frac{1-(1-n)\psi}{(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b\big]$$

$$+n\big[x^* - c'(q_s)q_s - \frac{1-\beta(1-(1-n)\psi)}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b\big]$$

First add x^* terms:

$$(1-n+n)x^* = x^*$$

Now we can substitute $q_s = \frac{(1-n)}{n} q_b$ into $c'(q_s)q_s$ to get

$$c'(q_s)\frac{(1-n)}{n} q_b$$

Now we can add $c'(q_s)q_b$ terms:

$$\Bigg\{(1-n)\Big(\frac{1-(1-n)\psi}{(1-\psi)(1-(1-n)\psi)}\Big) - n\Big(\frac{(1-n)}{n} + \frac{1-\beta(1-(1-n)\psi)}{\beta(1-\psi)(1-(1-n)\psi))}\Big)\Bigg\} c'(q_s)q_b$$

$$\Bigg\{\frac{(1-n)(1-(1-n)\psi)}{(1-\psi)(1-(1-n)\psi)} - \frac{(1-n)(1-\psi)(1-(1-n)\psi)}{(1-\psi)(1-(1-n)\psi)} - \frac{n(1-\beta(1-(1-n)\psi))}{\beta(1-\psi)(1-(1-n)\psi)}\Bigg\} c'(q_s)q_b$$

$$\left\{\frac{\psi(1-n)(1-(1-n)\psi)}{(1-\psi)(1-(1-n)\psi)} - \frac{n(1-\beta(1-(1-n)\psi))}{\beta(1-\psi)(1-(1-n)\psi)}\right\} c'(q_s)q_b$$

$$\frac{\beta\psi(1-n)(1-(1-n)\psi) - n(1-\beta(1-(1-n)\psi))}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

Now combine x^* and $c'(q_s)q_b$ terms to get:

$$\hat{h} = x^* + \frac{\beta\psi(1-n)(1-(1-n)\psi) - n(1-\beta(1-(1-n)\psi))}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

The expected gain from this strategy is

$$\hat{h}-h = x^* + \frac{\beta\psi(1-n)(1-(1-n)\psi) - n(1-\beta(1-(1-n)\psi))}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b - x^* - \frac{\psi(1-n)^2}{(1-(1-n)\psi)} c'(q_s)q_b$$

$$\hat{h}-h = \frac{\beta\psi(1-n)(1-(1-n)\psi) - n(1-\beta(1-(1-n)\psi))}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b - \frac{\beta(1-\psi)\psi(1-n)^2}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

$$\hat{h}-h = \frac{\beta\psi(1-n) - \beta\psi^2(1-n)^2 - n + n\beta(1-(1-n)\psi) - \beta\psi(1-n)^2 + \beta\psi^2(1-n)^2}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

$$\hat{h}-h = \frac{\beta\psi(1-n) - n + n\beta(1-(1-n)\psi) - \beta\psi(1-n)^2}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

$$\hat{h}-h = \frac{\beta\psi - \beta\psi n - n + n\beta - n\beta\psi + n^2\beta\psi - \beta\psi + 2n\beta\psi - \beta\psi n^2}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

$$\hat{h}-h = \frac{-n(1-\beta)}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s)q_b$$

$$\overline{h} - h + \Sigma_{t=1}^{\infty}\beta^t(\hat{h}-h)$$

Sum of an infinite geometric series $\Sigma_{t=1}^{\infty}\beta^t = \frac{\beta}{1-\beta}$

$$\overline{h} - h + \frac{\beta}{1-\beta}(\hat{h}-h)$$

Now substitute $\overline{h} - h = \frac{n}{(1-\psi)(1-(1-n)\psi)} c'(q_s) q_b$ and $\hat{h} - h = \frac{-n(1-\beta)}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s) q_b c'(q_s) q_b$

$$\frac{n}{(1-\psi)(1-(1-n)\psi)} c'(q_s) q_b + \frac{\beta}{1-\beta} \Big(\frac{-n(1-\beta)}{\beta(1-\psi)(1-(1-n)\psi)} c'(q_s) q_b \Big) c'(q_s) q_b$$

simplify

$$\frac{n}{(1-\psi)(1-(1-n)\psi)} c'(q_s) q_b + \frac{-n}{(1-\psi)(1-(1-n)\psi)} c'(q_s) q_b$$

Which gives us

$$\frac{0}{(1-\psi)(1-(1-n)\psi)} c'(q_s) q_b = 0$$

■

PROOF OF LEMMA 8:

If $\hat{\Delta}_t = \alpha M_t < (1-n)\psi M_t$, there will be a decrease in tokens each period. As a consequence, the price of tokens will increase each period $\phi_{t+1} > \phi_t$, but token production will stay the same due to the cap. Consequently, it is possible for a steady state with deflation to exist.

In order to find γ, we take the law of motion for the supply of tokens and substitute $\hat{\Delta}_t = \alpha M_{t-1}$.

$$M_t = \alpha M_{t-1} + (1-(1-n)\psi) M_{t-1}$$

$$M_t = (1 - ((1-n)\psi - \alpha)) M_{t-1}$$

$$\frac{M_t}{M_{t-1}} = 1 - ((1-n)\psi - \alpha)$$

Which gives us $\gamma = \frac{M_t}{M_{t-1}} = 1 - (\psi - \alpha)$. ■

PROOF OF DM PRODUCTION W/ CAP $\alpha = 0$ AND NO BANKING:

$$\frac{\gamma - \beta}{\beta} = (1-n)\left[\frac{(1-\psi)\tilde{q}_{\alpha,b}^{-1/2}}{\frac{1-n}{n}} - 1\right] - n\psi$$

substitute $\gamma = 1 - \psi + \alpha$

$$\frac{1 - \psi + \alpha - \beta}{\beta} = (1-n)\left[\frac{(1-\psi)\tilde{q}_{\alpha,b}^{-1/2}}{\frac{1-n}{n}} - 1\right] - n\psi$$

substitute $\alpha = 0$

$$\frac{1 - \psi + 0 - \beta}{\beta} = (1-n)\left[\frac{(1-\psi)\tilde{q}_{\alpha,b}^{-1/2}}{\frac{1-n}{n}} - 1\right] - n\psi$$

$$\frac{1-\psi}{\beta} = (1-\psi)n\tilde{q}_{\alpha,b}^{-1/2} + n - n(\psi)$$

$$\frac{1-\psi}{\beta} - n(1-\psi) = (1-\psi)n\tilde{q}_{\alpha,b}^{-1/2}$$

$$\frac{1 - \psi - \beta n(1-\psi)}{\beta} = (1-\psi)n\tilde{q}_{\alpha,b}^{-1/2}$$

$$\tilde{q}_{\alpha,b}^{1/2} = \frac{\beta n(1-\psi)}{1 - \psi - \beta n(1-\psi)}$$

$$\tilde{q}_{\alpha,b} = \left[\frac{\beta n(1-\psi)}{1 - \psi - \beta n(1-\psi)}\right]^2$$

$$\tilde{q}_{\alpha,b} = \left[\frac{\beta n(1-\psi)}{(1-\beta n)(1-\psi)}\right]^2$$

$$\tilde{q}_{\alpha,b} = \left[\frac{\beta n}{1-\beta n}\right]^2$$

PROOF OF CONSUMPTION W/ TYPE III BANKING & CAP AT 0:

$$\tilde{q}_{\alpha,b} = \left[\frac{\beta n}{1-\beta n}\right]^2 < \left[\frac{\beta n}{1-n}\right]^2 = q_b$$

$$\frac{\beta n}{1-\beta n} < \frac{\beta n}{1-n}$$

$$1 - n < 1 - \beta n$$

$$\beta < 1$$

HOURS WORKED WITHOUT BANKS AND A CAP $\alpha = 0$:

A buyer's production will be

$$\tilde{h}_b^{\alpha} = x^* + m_{+1}$$

With a cap on production, money holdings will evolve $m_{+1} = (1 - \psi + \alpha)m$

$$\tilde{h}_b^{\alpha} = x^* + (1 - \psi + \alpha)m$$

A cap on production will be set a $\alpha = 0$

$$\tilde{h}_b^{\alpha} = x^* + (1 - \psi)m$$

Take the buyer's liquidity constraint $pq_b = (1 - \psi)(M_{-1})$ and substitute $p\phi = c'(q_s)$.
Now substitute to get

$$\tilde{h}_b^\alpha = x^* + \frac{(1-\psi)c'(q_s)\tilde{q}_b}{(1-\psi)} = x^* + c'(q_s)\tilde{q}_b$$

So the buyer recovers the production cost of the DM good but not the tokens lost in the first market due to deflation from the cap.

The seller's production is

$$\tilde{h}_s^\alpha = x^* + \phi\{m_{+1} - [pq_s + (1-\psi)m]\}$$

The seller will work enough hours to get the efficient amount of the good x^* and money to take into the DM next period less the real value of the good they sold in the DM pq_s and their money retained that they brought into the market.

$$\tilde{h}_s^\alpha = x^* + \phi(1-\psi)m - \phi\{pq_s\} - \phi(1-\psi)m$$

The seller loses tokens brought into the dm, but due to deflation it will be enough for purchases in the DM next period.

$$\tilde{h}_s^\alpha = x^* - \phi pq_s$$

We can now substitute $c'(q_s) = p\phi$ to get:

$$\tilde{h}_s^\alpha = x^* - c'(q_s)q_s$$

The expected hours worked h satisfies:

$$\tilde{h}^{\alpha} = (1-n)h_b + nh_s = x^* + (1-n)c'(q_s)\tilde{q}_b - nc'(q_s)q_s$$

substitute $q_s = \frac{1-n}{n}q_b$

$$\tilde{h}^{\alpha} = (1-n)h_b + nh_s = x^* + (1-n)c'(q_s)\tilde{q}_b - nc'(q_s)\frac{1-n}{n}q_b$$

simplify

$$\tilde{h}^{\alpha} = (1-n)h_b + nh_s = x^* + (1-n)c'(q_s)\tilde{q}_b - (1-n)c'(q_s)q_b$$

simplify

$$\tilde{h}^{\alpha} = x^*$$

■

PROOF OF PROPOSITION 6:

We will assume that at some point in time t an agent decides to not borrow ever again but continue to deposit at the beginning of the second market. It is optimal for him to buy the same quantity since the optimal choice still satisfies (8). His money balance will then become $m_{+1}^{nb} = m_{+1} + l_{+1}$. An agent who decides to never borrow has to carry around a higher money balance. This saves him from paying interest on the loans in the future. This means consumption and production in market 1 are not affected. The difference in lifetime payoff comes from hours worked.

Coming into market 2 he will need to work extra hours to hold extra money to make up for the loan l_{+1}, so the hours worked becomes:

$$h_b^{nb} = x^* + \phi\{m_{+1} + l_{+1} + (1+i)l\}$$

$$= x^* + \phi(m_{+1} + l) + \phi(il) + \phi l_{+1}$$

$$= x^* + c'(q_s)q_b + inc'(q_s)q_b + \phi l_{+1}$$

where as if he sold in the previous market

$$h_s^{nb} = x^* + \phi\{m_{+1} + l_{+1} - [pq_s + M_{-1} + i_d d]\}$$

$$= x^* - c'(q_s)q_s + \phi l_{+1} - \phi i_d d.$$

The expected hours worked will now satisfy

$$h^{nb} = (1-n)h_b^{nb} + nh_s^{nb} = x^* + \phi l_{+1}.$$

Consequently, from $h = (1-n)h_b + nh_s = x^*$ the additional hours worked are

$$h^{nb} - h = \phi l_{+1} = \frac{\gamma nc'(q_s)q_b}{(1-\psi)} > 0 \tag{B.14}$$

Since we showed earlier $\phi l = nc'(q_s)q_b$ and since $l_{+1} = \frac{\gamma l}{(1-\psi)}$ in a steady state. With $\gamma = 1$, we get $\phi l_{+1} = \frac{nc'(q_s)q_b}{(1-\psi)}$ and so the extra money to compensate for the loan amount will depend on currency loss. The higher the rate of currency loss, the more labor needed to self insure to maintain same level of consumption in the first market. The loan amount will need to be higher with inflation and lower with deflation.

Now we look at the hours worked in some future period 2. He will not have a loan to pay back $(1+i)l$ and so hours worked becomes:

$$\hat{h}_b = x^* + \phi\{m_{+1} + l_{+1}\}$$

We substitute $c'(q_s)q_b = \phi(M_{-1} + l)$ to get

$$\hat{h}_b = x^* + c'(q_s)q_b - \phi l + \phi l_{+1}$$

We can now substitute $\phi l_{+1} - \phi l = \frac{n}{(1-\psi)}c'(q_s)q_b - nc'(q_s)q_b = \frac{n}{(1-\psi)}c'(q_s)q_b - \frac{n(1-\psi)}{(1-\psi)}c'(q_s)q_b = \frac{n\psi}{(1-\psi)}c'(q_s)q_b$ to get

$$\hat{h}_b = x^* + c'(q_s)q_b + \frac{n\psi c'(q_s)q_b}{(1-\psi)}$$

is the hours worked for a buyer in market 1. If he sold in market 1 it becomes

$$\hat{h}_s = x^* + \phi\{m_{+1} + l_{+1} - [pq_s + (1+i)d^{nb}]\}$$

substituting $c'(q_s) = \phi p$

$$\hat{h}_s = x^* - c'(q_s)q_s + \phi\{m_{+1} + l_{+1} - [(1+i)d^{nb}]\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + c'(q_s)q_b - \phi\Big\{l - l_{+1} + (1+i)d^{nb}\Big\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + c'(q_s)q_b + \frac{n\psi}{(1-\psi)}c'(q_s)q_b - \phi\Big\{(1+i)d^{nb}\Big\}$$

since you self insure for consumption shocks $d^{nb} = m + l$

$$\hat{h}_s = x^* - c'(q_s)q_s + c'(q_s)q_b + \frac{n\psi c'(q_s)q_b}{(1-\psi)} - \phi\Big\{(1+i)(m+l_{+1})\Big\}$$

$$\hat{h}_s = x^* - c'(q_s)q_s + c'(q_s)q_b + \frac{n\psi c'(q_s)q_b}{(1-\psi)} - (1+i)c'(q_s)q_b - (1+i)\frac{n\psi c'(q_s)q_b}{(1-\psi)}$$

$$\hat{h}_s = x^* - c'(q_s)q_s - \frac{i(1-\psi+n\psi)}{(1-\psi)}c'(q_s)q_b$$

We can now substitute $i = \frac{1-\beta}{\beta}$ to get

$$\hat{h}_s = x^* - c'(q_s)q_s - \frac{1-\beta}{\beta}\frac{(1-\psi+n\psi)}{(1-\psi)}c'(q_s)q_b$$

The expected hours will satisfy:

$$\hat{h} = (1-n)\hat{h}_b + n\hat{h}_s = (1-n)\big[x^* + c'(q_s)q_b + \frac{n\psi c'(q_s)q_b}{(1-\psi)}\big]$$

$$+n\big[x^* - c'(q_s)q_s - \frac{1-\beta}{\beta}\frac{(1-\psi+n\psi)}{(1-\psi)}c'(q_s)q_b\big]$$

First add x^* terms:

$$(1-n+n)x^* = x^*$$

Now we can substitute $q_s = \frac{(1-n)}{n}q_b$ into $c'(q_s)q_s$ to get

$$c'(q_s)\frac{(1-n)}{n}q_b$$

Now we can add $c'(q_s)q_b$ terms:

$$\left\{(1-n)\Big(\frac{1-\psi+n\psi}{1-\psi}\Big)-n\Big(\frac{(1-n)}{n}+\frac{1-\beta}{\beta}\frac{(1-\psi+n\psi)}{(1-\psi)}\Big)\right\}c'(q_s)q_b$$

$$\left\{\frac{(1-n)(1-\psi+n\psi)}{(1-\psi)}-\frac{(1-n)(1-\psi)}{(1-\psi)}-\frac{n(1-\beta)(1-\psi+n\psi)}{\beta(1-\psi)}\right\}c'(q_s)q_b$$

$$\left\{\frac{(1-n)(n\psi)}{(1-\psi)}-\frac{n(1-\beta)(1-\psi+n\psi)}{\beta(1-\psi)}\right\}c'(q_s)q_b$$

$$\left\{\frac{(1-n)(n\psi)}{(1-\psi)}-\frac{n(1-\beta)(1-\psi+n\psi)}{\beta(1-\psi)}\right\}c'(q_s)q_b$$

$$\left\{\frac{\beta(1-n)(n\psi)}{\beta(1-\psi)}-\frac{n(1-\beta)(1-\psi+n\psi)}{\beta(1-\psi)}\right\}c'(q_s)q_b$$

$$\left\{\frac{\beta n\psi-\beta n^2\psi}{\beta(1-\psi)}-\frac{n-n\psi+n^2\psi-n\beta+n\beta\psi-\beta n^2\psi}{\beta(1-\psi)}\right\}c'(q_s)q_b$$

$$\left\{\frac{\beta n-n+n\psi-n^2\psi}{\beta(1-\psi)}\right\}c'(q_s)q_b$$

Now combine x^* and $c'(q_s)q_b$ terms to get:

$$\hat{h}=x^*+\frac{\psi n(1-n)}{\beta(1-\psi)}c'(q_s)q_b-\frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b$$

The expected gain from this strategy is

$$\hat{h}-h=x^*+\frac{\psi n(1-n)}{\beta(1-\psi)}c'(q_s)q_b-\frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b-x^*$$

$$\hat{h}-h=\frac{\psi n(1-n)}{\beta(1-\psi)}c'(q_s)q_b-\frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b$$

The expected gain in the future from this strategy becomes:

$$h^{nb} - h + \Sigma_{t=1}^{\infty}\beta^t(\hat{h} - h)$$

Sum of an infinite geometric series $\Sigma_{t=1}^{\infty}\beta^t = \frac{\beta}{1-\beta}$

$$h^{nb} - h + \frac{\beta}{1-\beta}(\hat{h} - h)$$

Now substitute $h^{nb} - h = \frac{n}{(1-\psi)}c'(q_s)q_b$ and $\hat{h} - h = \frac{\psi n(1-n)}{\beta(1-\psi)}c'(q_s)q_b - \frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b$

$$\frac{n}{(1-\psi)}c'(q_s)q_b + \frac{\beta}{1-\beta}\Big(\frac{\psi n(1-n)}{\beta(1-\psi)}c'(q_s)q_b - \frac{n(1-\beta)}{\beta(1-\psi)}c'(q_s)q_b\Big)$$

simplify

$$\frac{n}{(1-\psi)}c'(q_s)q_b + \frac{\psi n(1-n)}{(1-\beta)(1-\psi)}c'(q_s)q_b - \frac{n(1-\beta)}{(1-\beta)(1-\psi)}c'(q_s)q_b$$

simplify

$$\frac{n}{(1-\psi)}c'(q_s)q_b + \frac{\psi n(1-n)}{(1-\beta)(1-\psi)}c'(q_s)q_b - \frac{n}{(1-\psi)}c'(q_s)q_b$$

Which gives us

$$\frac{\psi n(1-n)}{(1-\beta)(1-\psi)}c'(q_s)q_b$$

PROOF OF PROPOSITION 5:

First show production the same with all three banks with cap regardless of rate of token loss. Recall $q_b^I = \left[\frac{\beta(1-\psi)n}{[1-(\psi-\alpha)](1-n)}\right]^2$, $q_b^{II} = \left[\frac{\beta(1-(1-n)\psi)n}{[1-((1-n)\psi-\alpha)](1-n)}\right]^2$, $q_b^{III} = \left[\frac{\beta n}{(1-n)}\right]^2$

Set $\alpha = 0$ and show $q_b^I = q_B^{II}$

$$q_b^I = \left[\frac{\beta(1-\psi)n}{[1-(\psi-0)](1-n)}\right]^2 = \left[\frac{\beta(1-(1-n)\psi)n}{[1-((1-n)\psi-0)](1-n)}\right]^2 = q_b^{II}$$

simplify

$$q_b^I = \left[\frac{\beta(1-\psi)n}{[1-\psi](1-n)}\right]^2 = \left[\frac{\beta(1-(1-n)\psi)n}{[1-(1-n)\psi](1-n)}\right]^2 = q_b^{II}$$

$$q_b^I = \left[\frac{\beta n}{(1-n)}\right]^2 = \left[\frac{\beta n}{(1-n)}\right]^2 = q_b^{II}$$

Thus, $q_b^I = q_b^{II} = q_b^{III}$ and ψ does not affect production in the DM. Thus, the additional safekeeping functions have no production benefits.

Now we need to show that financial intermediation brings no additional benefit as token loss increases. As shown previously, production with no banks is given by

$$q_b^{nb} = \left[\frac{\beta n}{1-\beta n}\right]^2$$

The rate at which agents discount the future affects increases in production from intermediation, but the rate of token loss has no effect.

Now, we just need to show that the hours worked is not affected by the rate of token loss. First, show the hours worked with no token creation $\alpha = 0$ and type I banks.

PROOF OF HOURS WORKED IN CM TYPE I BANKS:

Money holdings are heterogeneous due to trade shocks and financial transactions from the first market. Thus, if we set $m = M_{-1}$, money holdings for buyers will be 0, and $\frac{1}{n}(1-\psi)M_{-1}$ for sellers.

Take the inverse of $U'(x^*) = 1$ to get $x^* = U'^{-1}(1)$. The buyer's production in the second market can be derived as follows:

$$h_b = x^* + \phi\{m_{+1} + (1+i)l\}$$

The buyer must produce enough of the CM good to consume x^*, pay off the loan from the bank $(1+i)l$ and also procure enough money for the DM next period. In equilibrium, $m_{+1} = M = M_{-1} - \psi M_{-1} + 0$, since $\Delta = 0$:

$$h_b = x^* + \phi\{M_{-1} - \psi M_{-1} + 0 + (1+i)l\}$$

simplify to get:

$$h_b = x^* + \phi\{(1-\psi)m + (1+i)l\}$$

Take the buyer's liquidity constraint $pq_b = (1-\psi)(m + \frac{n}{1-n}m)$ and substitute $p\phi = c'(q_s)$. Now substitute to get

$$h_b = x^* + \frac{(1-\psi)(1-n)c'(q_s)q_b}{(1-\psi)} + \phi\{il\} \tag{B.15}$$

Since $(1-n)l = nm$, we can substitute into the liquidity constraint to get $p\phi q_b = (1-\psi)\phi(\frac{(1-n)}{n}l + l) = (1-\psi)\phi\frac{l}{n} \Rightarrow \phi l = \frac{n}{(1-\psi)}p\phi q_b$. Use $p\phi = c'(q_s)$ to get $\phi l = \frac{n}{(1-\psi)}c'(q_s)q_b$ and substitute into the equation:

$$h_b = x^* + (1-n)c'(q_s)q_b + \frac{(1+i)n}{(1-\psi)}c'(q_s)q_b \tag{B.16}$$

Since there is no token creation, the buyer recovers the production cost of the DM good, but not tokens lost in the first market. In addition, he also pay back the interest on the loan.

The seller's production is

$$h_s = x^* + \phi\{m_{+1} - [pq_s + M_{-1} + i_d d]\}$$

The seller will work enough hours to get the efficient amount of the good x^* and money to take into the DM next period less the real value of the good they sold in the DM pq_s and the money deposited plus interest $[M_{-1} + i_d d]$.

Use the fact that in equilibrium $m_{+1} = M = M_{-1} + \Delta_{-1} - \psi_1 M_{-1}$:

$$h_s = x^* + \phi\{M_{-1} + \Delta_{-1} - \psi_1 M_{-1} - [pq_s + M_{-1} + i_d d]\}$$

Since $\Delta = 0$ due to cap:

$$h_s = x^* - \phi\{\psi M_{-1} + pq_s + i_d d\}$$

We can now substitute $c'(q_s) = p\phi$ to get:

$$h_s = x^* - \frac{\psi(1-n)}{(1-\psi)} c'(q_s) q_b - c'(q_s) q_s - \phi i_d d$$

The expected hours worked h satisfies:

$$h = (1-n)h_b + nh_s = x^* + (1-n)(1-n)c'(q_b)q_b + \frac{(1-n)(1+i)n}{(1-\psi)}c'(q_s)q_b - \frac{\psi(1-n)n}{(1-\psi)}c'(q_s)q_b$$

since in equilibrium $q_b = \frac{n}{1-n}q_s$ and $i(1-n)l = i_d nd$.

$$h = x^* + (1-n)(1-n)c'(q_b)q_b + \frac{(1-n)(1+i)n}{(1-\psi)}c'(q_s)q_b - \frac{\psi(1-n)n}{(1-\psi)}c'(q_s)q_b - (1-n)c^{'}(q_s)q_b - i(1-n)l$$

substitute $l = \frac{n}{(1-\psi)}c'(q_s)q_b$ and simplify

$$h = x^* - n(1-n)c'(q_b)q_b + \frac{(1-n)(1+i)n}{(1-\psi)}c'(q_s)q_b - \frac{\psi(1-n)n}{(1-\psi)}c'(q_s)q_b - \frac{ni(1-n)}{(1-\psi)}c'(q_s)q_b$$

$$h = x^* + \frac{(1-n)n(-(1-\psi)+(1+i)-\psi-i)}{(1-\psi}c'(q_s)q_b$$

$$h = x^* + \frac{0}{(1-\psi)}c'(q_s)q_b$$

$$h = x^*$$

The rate of currency loss plays no role in hours worked with a cap $\Delta = 0$ and Type I Banks in use.

Now we show work with type II banks is not dependent on rate of currency loss with a cap on token production $\Delta = 0$.

PROOF OF HOURS WORKED IN CM TYPE II BANKS:

Money holdings are heterogeneous due to trade shocks and financial transactions from the first market. Thus, if we set $m = M_{-1}$, money holdings for buyers will be 0, and $\frac{1}{n}(1 - (1-n)\psi)M_{-1}$ for sellers.

Take the inverse of $U'^{*}) = 1$ to get $x^* = U'^{-1}(1)$. The buyer's production in the second market can be derived as follows:

$$h_b = x^* + \phi\{m_{+1} + (1+i)l\}$$

The buyer must produce enough of the CM good to consume x^*, pay off the loan from the bank $(1+i)l$ and also procure enough money for the DM next period. In equilibrium, $m_{+1} = M = M_{-1} - (1-n)\psi M_{-1} + \Delta_{-1}$:

$$h_b = x^* + \phi\{M_{-1} - (1-n)\psi M_{-1} + \Delta_{-1} + (1+i)l\}$$

Since $\Delta = 0$ with a cap at zero on token production, simplify to get:

$$h_b = x^* + \phi\{(1 - (1-n)\psi)M_{-1} + (1+i)l\}$$

Since $(1 - n)l = nm$, we can substitute into the liquidity constraint to get $pq_b = \left(\frac{1-(1-n)\psi}{1-n}\right)m \Rightarrow m = \frac{1-n}{(1-(1-n)\psi)}pq_b$ and $pq_b = \frac{1-(1-n)\psi}{n}l$. Use $p\phi = c'(q_s)$ to get $\phi m = \frac{1-n}{(1-(1-n)\psi)}c'(q_s)q_b$ and $\phi l = \frac{n}{(1-(1-n)\psi)}c'(q_s)q_b$ and substitute into the equation:

$$h_b = x^* + \frac{(1-(1-n)\psi)(1-n)}{(1-(1-n)\psi)}c'(q_s)q_b + \frac{(1+i)n}{(1-(1-n)\psi)}c'(q_s)q_b \tag{B.17}$$

simplify

$$h_b = x^* + (1-n)c'(q_s)q_b + \frac{(1+i)n}{(1-(1-n)\psi)}c'(q_s)q_b \tag{B.18}$$

So the buyer recovers the buyers portion of money for the production cost of the DM good and also pays back the principle and interest on the loan. Due to the cap on token production, he does not recover the lost tokens from his initial money holdings.

The seller's production is

$$h_s = x^* + \phi\{m_{+1} - [pq_s + M_{-1} + i_d d]\}$$

The seller will work enough hours to get the efficient amount of the good x^* and money to take into the DM next period less the real value of the good they sold in the DM pq_s and the money deposited plus interest $[M_{-1} + i_d d]$.

Use the fact that in equilibrium $m_{+1} = M = M_{-1} + \Delta_{-1} - (1-n)\psi M_{-1}$:

$$h_s = x^* + \phi\{M_{-1} + \Delta_{-1} - (1-n)\psi M_{-1} - [pq_s + M_{-1} + i_d d]\}$$

Since $\Delta = 0$ with a cap on total token production, simplify to get:

$$h_s = x^* - \phi\{(1-n)\psi M_{-1} + pq_s + i_d d\}$$

We can now substitute $c'(q_s) = p\phi$ to get:

$$h_s = x^* - \frac{(1-n)\psi(1-n)}{(1-(1-n)\psi)}c'(q_s)q_b - c'(q_s)q_s - \phi i_d d$$

The expected hours worked h satisfies:

$$h = (1-n)h_b + nh_s = x^* + (1-n)^2 c'(q_s)q_b + \frac{(1-n)(1+i)n}{(1-(1-n)\psi)}c'(q_s)q_b$$

$$-\frac{n\psi(1-n)^2}{(1-(1-n)\psi)}c'(q_s)q_b - nc'(q_s)q_s - n\phi i_d d$$

since in equilibrium $q_b = \frac{n}{1-n}q_s$ and $i(1-n)l = i_d nd$.

$$h = x^* + (1-n)^2 c'(q_s)q_b + \frac{(1-n)(1+i)n}{(1-(1-n)\psi)}c'(q_s)q_b - \frac{n\psi(1-n)^2}{(1-(1-n)\psi)}c'(q_s)q_b - (1-n)c'(q_s)q_b - i(1-n)l$$

substitute $l = \frac{n}{(1-(1-n)\psi)}c'(q_s)q_b$ and simplify

$$h = x^* - \frac{(n(1-n) - n(1-n)^2\psi)c'(q_s)q_b}{(1-(1-n)\psi)} + \frac{(1-n)(1+i)n}{(1-(1-n)\psi)}c'(q_s)q_b$$

$$-\frac{n\psi(1-n)^2}{(1-(1-n)\psi)}c'(q_s)q_b - \frac{ni(1-n)}{(1-(1-n)\psi)}c'(q_s)q_b$$

simplify

$$h = x^* - \frac{(n(1-n) - n(1-n)^2\psi)c'(q_s)q_b}{(1-(1-n)\psi)} + \frac{(1-n)n}{(1-(1-n)\psi)}c'(q_s)q_b - \frac{n\psi(1-n)^2}{(1-(1-n)\psi)}c'(q_s)q_b$$

simplify to get

$$h = x^*$$

Wiht $h = x^*$, the rate of token loss has no bearing on hours worked with type II banks.

This holds true for hours worked with type III banks and no banking also.

We have shown that the rate of token loss has no effect on hours worked and production in the DM when token creation is capped at zero. We did this by showing it does not effect hours worked with all three types of banks and no banks. In addition, we showed that production is not influenced by token loss in all three types of banks and no banks. Thus, the rate of token loss has no effect on the welfare benefit from introducing banking in an environment with $\Delta = 0$. ■

PROOF OF LEMMA 9:

We will derive the endogenous real borrowing constraint $\phi\bar{l}$. The quantity will be the maximal real loan the borrower will be willing to repay in the second market at given market prices. For buyers entering the second market with no money, who repay their loans, the expected discounted utility in a steady state is

$$W(m) = U(x^{*}) - h_b + \beta V_{+1}(m_{+1}),$$

where h_b is a buyer's production in the second market if he repays his loan. Consider a borrower who borrowed $\bar{l}$ in market 1 and is contemplating defaulting on his loans in market 2. A deviating buyer's expected discounted utility is

$$\hat{W}(m) = U(\hat{x}) - \hat{h}_b + \beta \hat{V}_{+1}(\hat{m}_{+1}),$$

the hat indicates the optimal choice by a deviator. Thus $\phi\bar{l}$ is the value of borrowing such that $W(m) = \hat{W}(m)$ or

$$U(x^{*}) - U(\hat{x}) + \hat{h}_b - h_b + \beta[V_{+1}(m_{+1}) - \hat{V}_{+1}(\hat{m}_{+1})] = 0 \tag{B.19}$$

The continuation payoffs are

$$\hat{V}_{+1}(\hat{m}_{+1}) = (1-\beta)^{-1}[(1-n)u(\hat{q}_b) - nc(\hat{q}_s) + U(x^*) - \hat{h}],$$

$$V_{+1}(m_{+1}) = (1-\beta)^{-1}[(1-n)u(q_b) - nc(q_s) + U(\hat{x}) - h]. \tag{B.20}$$

We now derive $\hat{x}, \hat{q}_b, \hat{q}_s$ and $\hat{h}_b$. In the last market the deviating buyer's program is

$$\hat{W}(\hat{m}) = \max_{\hat{x}, \hat{h}_b, \hat{m}_{+1}} [U(\hat{x}) - \hat{h}_b + \beta \hat{V}_{+1}(\hat{m}_{+1})]$$

$$s.t. \quad \hat{x} + \phi \hat{m}_{+1} = \hat{h}_b + \phi \hat{m}.$$

Substitute the budget constraint into the payoff function

$$\hat{x} + \phi \hat{m}_{+1} - \phi \hat{m} = \hat{h}_b$$

$$\hat{W}(\hat{m}) = \max_{\hat{x}, \hat{h}_b, \hat{m}_{+1}} [U(\hat{x}) - (\hat{x} + \phi \hat{m}_{+1} - \phi \hat{m}) + \beta \hat{V}_{+1}(\hat{m}_{+1})]$$

The choice for consumption of the CM good must satisfy

$$U'(\hat{x}) = 1$$

which is the same for a non-deviator. Therefore, $\hat{x} = x^*$. With quasi-linear preferences, exclusion from the banking system has no effect on the choice of the CM good. However, this is not the case with the decision on money holdings. The decision on money to carry into the DM must satisfy

$$-\phi + \beta \hat{V}_{+1}'(\hat{m}_{+1}) = 0.$$

The marginal value of money will be lower due to no longer receiving interest on deposits when a seller. In addition, the consumer will no longer be able to take out loans and must adjust accordingly.

In market one, we know that a seller who has deviated will still face the same problem when deciding to sell goods. Thus, the solution to his problem will be $c^{'}(\hat{q}_s) = p\phi$. Moreover, the deviator will produce the same amount of goods as non-deviators so $\hat{q}_s = q_s = \frac{1-n}{n} q_b$.

$$U(x^*) - U(\hat{x}) + \hat{h}_b - h_b + \beta[(1-\beta)^{-1}[(1-n)u(q_b) - nc(q_s)$$

$$+ U(\hat{x}) - h] - (1-\beta)^{-1}[(1-n)u(\hat{q}_b) - nc(\hat{q}_s) + U(x^*) - \hat{h}]] = 0 \tag{B.21}$$

We know $U(x^*) = U(\hat{x})$ so we can eliminate these terms

$$h_b - \hat{h}_b = \frac{\beta}{1-\beta}\left[(1-n)u(q_b) - nc(q_s) - h - (1-n)u(\hat{q}_b) + nc(\hat{q}_s) + \hat{h}\right]$$

Note that on the left, $h_b - \hat{h}_b$ represents the gain in the CM if the buyer was to deviate and not pay back his loan.

Since $c(q_s) = c(\hat{q}_s)$, we can eliminate both terms and simplify to get

$$h_b - \hat{h}_b = \frac{\beta}{1-\beta}\left[(1-n)[u(q_b) - u(\hat{q}_b)] + \hat{h} - h\right] \tag{B.22}$$

The right hand side represents what a deviator will lose by being shut out from the banking system for the following periods. Now we derive $h_b - \hat{h}_b$. We first look at a buyer who pays his loans. He will work

$$h_b = x^* + \phi m_{+1} + \phi(1+i)\bar{l}$$

If he defaults on his loans, he works

$$\hat{h}_b = x^* + \phi \hat{m}_{+1}$$

Now subtract the hours worked for a deviator from the non-deviator to get

$$h_b - \hat{h}_b = x^* + \phi m_{+1} + \phi(1+i)\bar{l} - x^* - \phi \hat{m}_{+1}$$

Next, we use the equilibrium condition that a defaulter's money balances must grow at the rate γ so $\hat{m}_{+1} = \hat{m} + \Delta - \psi\hat{m} = \gamma\hat{m}$ and $m_{+1} = m + \Delta - \psi m = \gamma m$.

$$= \phi(1+i)\bar{l} - \phi\gamma(\hat{m} - m). \tag{B.23}$$

If a buyer chooses to default, then he would not have to work as much because he does not have to earn enough money to repay the loan plus interest $\phi(1+i)\bar{l}$. Yet, without access to credit in the future, he will have to work more because he cannot borrow funds to finance consumption in the DM. Further, upon defaulting, the consumer will not get the benefit of interest on deposits because of his exclusion from the banking system.

Once the agent defaults, as a buyer he spends $p\hat{q}_b$ units of money so his hours worked are

$$\hat{h}_b = x^* + \phi\hat{m}_{+1}$$

For a seller we have

$$\hat{h}_s = x^* + \phi\hat{m}_{+1} - \phi((1-\psi)\hat{m} + p\hat{q}_s)$$

A seller who is a deviator will only retain a portion of his tokens and also lose out on the interest on deposits.

$$= x^* + (\gamma - (1-\psi))\phi(\hat{m}) - \phi p\Big(\frac{1-n}{n}\Big)q_b$$

So for a defaulter expected hours worked are $\hat{h} = (1-n)\hat{h}_b + n\hat{h}_s = x^* + (\gamma - n(1-\psi))\phi(\hat{m}) - \phi p(1-n)q_b$. However, if he does not deviate he works $h = x^* + \frac{\psi(1-n)}{(1-\psi)}c'(\overline{q}_s)\overline{q}_b = x^* + \frac{\psi(1-n)}{(1-\psi)}\phi p q_b$ and so

$$\hat{h} - h = (\gamma - n(1-\psi))\phi\hat{m} - \phi p(1-n)q_b - \frac{\psi(1-n)}{(1-\psi)}\phi p q_b$$

$$\hat{h} - h = (\gamma - n(1-\psi))\phi\hat{m} - \phi\frac{p(1-n)}{1-\psi}q_b$$

At higher rates of token loss the deviator will work less hours, but the loss in utility from consumption of the DM good will be substantially higher.

The money spent on the good for a deviator is $p\hat{q}_b = (1-\psi)\hat{m}$

$$\hat{h} - h = (\gamma - n(1-\psi))\phi[\frac{(p\hat{q}_b)}{(1-\psi)}] - \phi\frac{p(1-n)}{1-\psi}q_b$$

$$\hat{h} - h = \frac{\phi p}{(1-\psi)}[(\gamma - n(1-\psi))\hat{q}_b - (1-n)q_b]$$

$$\hat{h} - h = \frac{c'(q_s)}{(1-\psi)}[(\gamma - n(1-\psi))\hat{q}_b - (1-n)q_b] \tag{B.24}$$

Substitute (19) and (20) into (18) to get

$$\phi(1+i)\bar{l} - \phi\gamma[\frac{(p\hat{q}_b)}{(1-\psi)} - ((1-n)\frac{pq_b}{1-\psi})] =$$

$$\frac{\beta}{1-\beta}\left[(1-n)[u(q_b) - u(\hat{q}_b)] + \frac{c'(q_s)}{(1-\psi)}[(\gamma - n(1-\psi))\hat{q}_b - (1-n)q_b]\right]$$

$$\phi(1+i)\bar{l} =$$

$$\frac{\beta}{1-\beta}\left[(1-n)[u(q_b)-u(\hat{q}_b)] + \frac{c'(q_s)}{(1-\psi)}[(\gamma-n(1-\psi))\hat{q}_b-(1-n)q_b]\right] + \phi\gamma[\frac{(p\hat{q}_b)}{(1-\psi)} - ((1-n)\frac{pq_b}{1-\psi})]$$

$$\phi(1+i)\bar{l} =$$

$$\phi(1+i)\bar{l} = \frac{\beta}{1-\beta}\left[(1-n)[u(q_b)-u(\hat{q}_b)] + \frac{c'(q_s)}{(1-\psi)}[(\gamma-n(1-\psi))\hat{q}_b-(1-n)q_b] + \frac{(1-\beta)\phi\gamma(p\hat{q}_b - (1-n)pq_b)}{\beta(1-\psi)}\right]$$

Use $p\phi = c'(q_s)$ and divide both sides by $(1+i)$

$$\phi\bar{l} =$$

$$\frac{\beta}{(1-\beta)(1+i)}\left[(1-n)[u(q_b)-u(\hat{q}_b)] + \frac{c'(q_s)}{(1-\psi)}[(\gamma-n(1-\psi))\hat{q}_b-(1-n)q_b] + \frac{c'(q_s)}{(1-\psi)}\frac{(1-\beta)\gamma(\hat{q}_b - (1-n)q_b)}{\beta}\right]$$

Substitute $\gamma = 1$

$$\phi\bar{l} =$$

$$\frac{\beta}{(1-\beta)(1+i)}\left[(1-n)[u(q_b)-u(\hat{q}_b)] + \frac{c'(q_s)}{(1-\psi)}[(1-n(1-\psi))\hat{q}_b-(1-n)q_b] + \frac{c'(q_s)}{(1-\psi)}\frac{(1-\beta)(\hat{q}_b - (1-n)q_b)}{\beta}\right]$$

let $\Psi(q_b, \hat{q}_b) = (1-n)[u(q_b) - u(\hat{q}_b)] + \frac{c'(q_s)}{(1-\psi)}[(1-n(1-\psi))\hat{q}_b - (1-n)q_b]$ to get

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\left\{\Psi(q_b,\hat{q}_b) + \frac{c'(q_s)}{(1-\psi)}\frac{(1-\beta)(\hat{q}_b-(1-n)q_b)}{\beta}\right\}$$

which matches conditions in Lemma 11. Next, we need $\phi\bar{l} > 0$ in order for a steady-state equilibrium to exist. Since $\frac{\beta}{(1-\beta)(1+i)} > 0$, we only need to show the inside of the brackets is positive to prove this. Substitute $\frac{1-\beta}{\beta} = \frac{(1-\psi)u'(q_b)}{c'(q_s)} - 1$

$$(1-n)[u(q_b)-u(\hat{q}_b)]+\frac{c'(q_s)}{(1-\psi)}[(1-n(1-\psi))\hat{q}_b-(1-n)q_b]+\frac{c'(q_s)}{(1-\psi)}(\frac{(1-\psi)u'(q_b)}{c'(q_s)}-1)(\hat{q}_b-(1-n)q_b)$$

simplify

$$(1-n)[u(q_b)-u(\hat{q}_b)]+\frac{c'(q_s)}{(1-\psi)}[(1-n(1-\psi))\hat{q}_b-(1-n)q_b]+(u'(q_b)-\frac{c'(q_s)}{(1-\psi)})(\hat{q}_b-(1-n)q_b)$$

simplify

$$(1-n)[u(q_b)-u(\hat{q}_b)]+\frac{c'(q_s)}{(1-\psi)}[-n(1-\psi))\hat{q}_b]+u'(q_b)(\hat{q}_b-(1-n)q_b)$$

simplify

$$(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b+u'(q_b)(\hat{q}_b-(1-n)q_b)$$

Since $u'(q_b) > c'(q_s)$, we can substitute $nc'(q_s)\hat{q}_b = nu'(q_s)\hat{q}_b$

$$(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b+u'(q_b)(\hat{q}_b-(1-n)q_b) > (1-n)[u(q_b)-u(\hat{q}_b)]+u'(q_b)((1-n)\hat{q}_b-(1-n)q_b) > 0$$

As the RHS of the inequality is less than the LHS, we only need to show the RHS is positive.

$$(1-n)[u(q_b)-u(\hat{q}_b)]+u'(q_b)((1-n)\hat{q}_b-(1-n)q_b)>0$$

divide through by $(1-n)$

$$[u(q_b)-u(\hat{q}_b)]+u'(q_b)(\hat{q}_b-q_b)>0$$

$$\frac{u(q_b)-u(\hat{q}_b)}{q_b-\hat{q}_b}>u'(q_b)$$

PROOF OF LEMMA 10:

We will derive the endogenous real borrowing constraint $\phi\bar{l}$ with type II banks.

$$U(x^*)-U(\hat{x})+\hat{h}_b-h_b+\beta[(1-\beta)^{-1}[(1-n)u(q_b)-nc(q_s)$$
$$+U(\hat{x})-h]-(1-\beta)^{-1}[(1-n)u(\hat{q}_b)-nc(\hat{q}_s)+U(x^*)-\hat{h}]]=0 \qquad \text{(B.25)}$$

We know $U(x^*)=U(\hat{x})$ so we can eliminate these terms

$$h_b-\hat{h}_b=\frac{\beta}{1-\beta}\left[(1-n)u(q_b)-nc(q_s)-h-(1-n)u(\hat{q}_b)+nc(\hat{q}_s)+\hat{h}\right]$$

We know $c(q_s)=c(\hat{q}_s)$ so we can eliminate both terms and simplify to get

$$h_b-\hat{h}_b=\frac{\beta}{1-\beta}\left[(1-n)[u(q_b)-u(\hat{q}_b)]+\hat{h}-h\right] \qquad \text{(B.26)}$$

Now we derive $h_b - \hat{h_b}$. We first look at a buyer who pays his loans. He will work

$$h_b = x^* + \phi m_{+1} + \phi(1+i)\bar{l}$$

where we use the equilibrium condition $m_{+1} = m + \Delta - (1-n)\psi m = \gamma m$. If he defaults on his loans, he works

$$\hat{h_b} = x^* + \phi \hat{m}_{+1}$$

where we use the equilibrium condition that a defaulter's money balances must grow at the rate γ so $\hat{m}_{+1} = \hat{m} + \Delta - (1-n)\psi\hat{m} = \gamma\hat{m}$. Note that how much he spent in the previous market 1 is the same whether he repays or not. Thus

$$h_b - \hat{h}_b = x^* + \phi m_{+1} + \phi(1+i)\bar{l} - x^* - \phi\hat{m}_{+1}$$

$$= \phi(1+i)\bar{l} - \phi\gamma(\hat{m} - m). \tag{B.27}$$

Deriving $\hat{h} - h$: Once the agent defaults, as a buyer he spends $p\hat{q}_b$ units of money so his hours worked are

$$\hat{h}_b = x^* + \phi\hat{m}_{+1}$$

For a seller we have

$$\hat{h}_s = x^* + \phi\hat{m}_{+1} - \phi((1-\psi)\hat{m} + p\hat{q}_b)$$

$$= x^* + (\gamma - (1-\psi))\phi\hat{m} - \phi p\Big(\frac{1-n}{n}\Big)q_b$$

So for a defaulter expected hours worked are $\hat{h} = (1-n)\hat{h}_b + n\hat{h}_s = x^* + (\gamma - n(1-\psi))\phi(\hat{m}) - \phi p(1-n)q_b$ while if he does not deviate he works $h = x^* + \frac{\psi(1-n)^2 p\phi q_b}{(1-(1-n)\psi)}$ and so

$$\hat{h} - h = (\gamma - n(1-\psi))\phi(\hat{m}) - \phi p(1-n)q_b - \frac{\psi(1-n)^2 p\phi q_b}{(1-(1-n)\psi)}$$

$$\hat{h} - h = (\gamma - n(1-\psi))\phi(\hat{m}) - \phi p q_b\left(\frac{(1-n)}{(1-(1-n)\psi)}\right) \tag{B.28}$$

the money spent on the good for a deviator is $p\hat{q}_b = (1-\psi)\hat{m}$. Now substitute into (52) to get

$$\hat{h} - h = (\gamma - n(1-\psi))\phi\frac{p\hat{q}_b}{(1-\psi)} - \phi p q_b\left(\frac{(1-n)}{(1-(1-n)\psi)}\right) \tag{B.29}$$

Substitute (53) and (51) into (50) to get

$$\phi(1+i)\bar{l} - \phi\gamma(\frac{p\hat{q}_b}{1-\psi} - (1-n)\frac{pq_b}{(1-(1-n)\psi)}) =$$

$$\frac{\beta}{1-\beta}\left[(1-n)[u(q_b) - u(\hat{q}_b)] + (\gamma - n(1-\psi))\phi\frac{p\hat{q}_b}{(1-\psi)} - \phi p q_b\left(\frac{(1-n)}{(1-(1-n)\psi)}\right)\right]$$

substitute $\gamma = 1$

$$\phi(1+i)\bar{l} =$$

$$\frac{\beta}{1-\beta}\left[(1-n)[u(q_b) - u(\hat{q}_b)] + (1 - n(1-\psi))\phi\frac{p\hat{q}_b}{(1-\psi)} - \phi p q_b\left(\frac{(1-n)}{(1-(1-n)\psi)}\right)\right]$$

$$+\phi(\frac{p\hat{q}_b}{1-\psi} - (1-n)\frac{pq_b}{(1-(1-n)\psi)})$$

Use $p\phi = c'(q_s)$

$$\phi(1+i)\bar{l} =$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b)-u(\hat{q}_b)]+c'(q_s)\big(\frac{(1-n(1-\psi))\hat{q}_b}{(1-\psi)}-q_b\frac{(1-n)}{(1-(1-n)\psi)}\big)\big]$$

$$+c'(q_s)(\frac{\hat{q}_b}{1-\psi}-\frac{(1-n)q_b}{(1-(1-n)\psi)})$$

simplify

$$\phi(1+i)\bar{l}=$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b)-u(\hat{q}_b)]+c'(q_s)\big(\frac{(1-n(1-\psi))\hat{q}_b}{(1-\psi)}-q_b\frac{(1-n)}{(1-(1-n)\psi)}\big)$$

$$+\frac{(1-\beta)c'(q_s)(\frac{\hat{q}_b}{1-\psi}-\frac{(1-n)q_b}{(1-(1-n)\psi)})}{\beta}]$$

substitute $\frac{1-\beta}{\beta}=\frac{(1-(1-n)\psi)u'(q_b)}{c'(q_s)}-1$

$$\phi(1+i)\bar{l}=$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b)-u(\hat{q}_b)]+c'(q_s)\big(\frac{(1-n(1-\psi))\hat{q}_b}{(1-\psi)}-q_b\frac{(1-n)}{(1-(1-n)\psi)}\big)$$

$$+(\frac{(1-(1-n)\psi)u'(q_b)}{c'(q_s)}-1)(c'(q_s)(\frac{\hat{q}_b}{1-\psi}-\frac{(1-n)q_b}{(1-(1-n)\psi)}))]$$

simplify

$$\phi(1+i)\bar{l}=$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b)-u(\hat{q}_b)]+c'(q_s)\big(\frac{(1-n(1-\psi))\hat{q}_b}{(1-\psi)}-q_b\frac{(1-n)}{(1-(1-n)\psi)}\big)$$

$$+((1-(1-n)\psi)u'(q_b)-c'(q_s))(\frac{\hat{q}_b}{1-\psi}-\frac{(1-n)q_b}{(1-(1-n)\psi)}))]$$

simplify

$$\phi(1+i)\bar{l}=$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b)-u(\hat{q}_b)]+c'(q_s)\big(\frac{(1-n(1-\psi))\hat{q}_b}{(1-\psi)}-q_b\frac{(1-n)}{(1-(1-n)\psi)}\big)$$

$$+(1-(1-n)\psi)u'(q_b)(\frac{\hat{q}_b}{1-\psi}-\frac{(1-n)q_b}{(1-(1-n)\psi)}))-c'(q_s)(\frac{\hat{q}_b}{1-\psi}-\frac{(1-n)q_b}{(1-(1-n)\psi)}))]$$

simplify

$$\phi(1+i)\bar{l}=$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b)-u(\hat{q}_b)]+c'(q_s)\big(\frac{(-n(1-\psi))\hat{q}_b}{(1-\psi)}\big)$$

$$+(1-(1-n)\psi)u'(q_b)(\frac{\hat{q}_b}{1-\psi}-\frac{(1-n)q_b}{(1-(1-n)\psi)})]$$

simplify

$$\phi(1+i)\bar{l}=$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b$$

$$+u'(q_b)\frac{(1-(1-n)\psi)\hat{q}_b}{1-\psi}-u'(q_b)(1-n)q_b]$$

just focus on the inside of the bracket

$$(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b+u'(q_b)\frac{(1-(1-n)\psi)\hat{q}_b}{1-\psi}-u'(q_b)(1-n)q_b>0$$

since $u'(q_b)\frac{(1-(1-n)\psi)\hat{q}_b}{1-\psi}>u'(q_b)\hat{q}_b$

$$(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b+u'(q_b)\frac{(1-(1-n)\psi)\hat{q}_b}{1-\psi}-u'(q_b)(1-n)q_b>$$

$$(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b+u'(q_b)\hat{q}_b-u'(q_b)(1-n)q_b>0$$

since $nc'(q_s)\hat{q}_b < nu'(q_b)\hat{q}_b$

$$(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b+u'(q_b)\hat{q}_b-u'(q_b)(1-n)q_b > (1-n)[u(q_b)-u(\hat{q}_b)]+u'(q_b)(1-n)\hat{q}_b$$

now divide through by $(1-n)$

$$u(q_b) - u(\hat{q}_b) + u'(q_b)\hat{q}_b - u'(q_b)q_b > 0$$

$$\frac{u(q_b) - u(\hat{q}_b)}{q_b - \hat{q}_b} > u'(q_b)$$

PROOF OF LEMMA 11:

We will derive the endogenous real borrowing constraint $\phi\bar{l}$ with type III banks.

$$U(x^*) - U(\hat{x}) + \hat{h}_b - h_b + \beta[(1-\beta)^{-1}[(1-n)u(q_b) - nc(q_s)$$

$$+ U(\hat{x}) - h] - (1-\beta)^{-1}[(1-n)u(\hat{q}_b) - nc(\hat{q}_s) + U(x^*) - \hat{h}]] = 0 \tag{B.30}$$

We know $U(x^*) = U(\hat{x})$ so we can eliminate these terms

$$h_b - \hat{h}_b = \frac{\beta}{1-\beta}\big[(1-n)u(q_b) - nc(q_s) - h - (1-n)u(\hat{q}_b) + nc(\hat{q}_s) + \hat{h}\big]$$

We know $c(q_s) = c(\hat{q}_s)$ so we can eliminate both terms and simplify to get

$$h_b - \hat{h}_b = \frac{\beta}{1-\beta}\big[(1-n)[u(q_b) - u(\hat{q}_b)] + \hat{h} - h\big] \tag{B.31}$$

Now we derive $h_b - \hat{h}_b$. We first look at a buyer who pays his loans. He will work

$$h_b = x^* + \phi m_{+1} + \phi(1+i)\bar{l} - \phi(m + \bar{l} - pq_b)$$

$$h_b = x^* + \phi m_{+1} + \phi(1+i)\bar{l} - \phi m_{+1} - \phi\bar{l} + \phi pq_b$$

$$= x^* + i\bar{l} + \phi pq_b,$$

where we use the equilibrium condition $m_{+1} = m = \gamma m$. If he defaults on his loans, he works

$$\hat{h}_b = x^* + \phi\hat{m}_{+1}$$

where we use the equilibrium condition that a defaulter's money balances must grow at the rate γ so $\hat{m}_{+1} = \hat{m} = \gamma\hat{m}$. Note that how much he spent in the previous market 1 is the same whether he repays or not. Thus

$$h_b - \hat{h}_b = x^* + \phi(1+i)\bar{l} + \phi m_{+1} - x^* - \phi\hat{m}_{+1}$$

$$= \phi(1+i)\bar{l} - \phi\gamma(\hat{m} - m). \tag{B.32}$$

Deriving $\hat{h} - h$: Once the agent defaults, as a buyer he spends $p\hat{q}_b$ units of money so his hours worked are

$$\hat{h}_b = x^* + \phi\hat{m}_{+1}$$

For a seller we have

$$\hat{h}_s = x^* + \phi\hat{m}_{+1} - \phi p\hat{q}_s$$

$$= x^* + \phi\hat{m}_{+1} - \phi(1-\psi)\hat{m} - \phi p\Big(\frac{1-n}{n}\Big)q_b$$

So for a defaulter expected hours worked are $\hat{h} = (1-n)\hat{h}_b + n\hat{h}_s = x^* - \phi p(\gamma - n(1-\psi))\frac{\hat{q}_b}{1-\psi} - (1-n)\phi p q_b$ while if he does not deviate he works $h = x^*$ and so

$$\hat{h} - h = \phi(\gamma - n(1-\psi))\frac{p\hat{q}_b}{1-\psi} - \phi p(1-n)q_b$$

$$\hat{h} - h = \phi p(\frac{(\gamma - n(1-\psi))\hat{q}_b}{1-\psi} - (1-n)q_b) \tag{B.33}$$

the money spent on the good for a deviator is $p\hat{q}_b = (1-\psi)\hat{m}$ while for a non-deviator it is $pq_b = \frac{1}{1-n}m$. $\gamma = 1$ since a stable price steady-state is the only steady-state possible. Now substitute into (52) to get

$$\phi(1+i)\bar{l} - \phi(\frac{p\hat{q}_b}{1-\psi} - (1-n)pq_b) =$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b) - u(\hat{q}_b)] - c'(q_s)((1-n)q_b - \frac{(1-n(1-\psi))\hat{q}_b}{1-\psi})\big]$$

$$\phi(1+i)\bar{l} =$$

$$\frac{\beta}{1-\beta}\big[(1-n)[u(q_b) - u(\hat{q}_b)] - c'(q_s)((1-n)q_b - \frac{(1-n(1-\psi))\hat{q}_b}{1-\psi})\big] + \frac{(1-\beta)\phi(\frac{p\hat{q}_b}{1-\psi} - (1-n)pq_b)}{(1-\beta)}$$

$$\phi(1+i)\bar{l} = \frac{\beta}{1-\beta}\big[(1-n)[u(q_b) - u(\hat{q}_b)] - c'(q_s)((1-n)q_b - \frac{(1-n(1-\psi))\hat{q}_b}{1-\psi}) + \frac{(1-\beta)\phi(\frac{p\hat{q}_b}{1-\psi} - (1-n)pq_b)}{\beta}\big]$$

Use $p\phi = c'(q_s)$

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\Big[(1-n)[u(q_b) - u(\hat{q}_b)] - c'(q_s)((1-n)q_b$$

$$-\frac{(1-n(1-\psi))\hat{q}_b}{1-\psi}) + \frac{(1-\beta)c'(q_s)(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b)}{\beta}\Bigg]$$

substitute $\frac{1-\beta}{\beta} = \frac{u'(q_b)}{c'(q_s)} - 1$

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\Bigg[(1-n)[u(q_b) - u(\hat{q}_b)] - c'(q_s)((1-n)q_b$$

$$-\frac{(1-n(1-\psi))\hat{q}_b}{1-\psi}) + (\frac{u'(q_b)}{c'(q_s)} - 1)c'(q_s)(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b)\Bigg]$$

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\Bigg[(1-n)[u(q_b) - u(\hat{q}_b)] - c'(q_s)((1-n)q_b$$

$$-\frac{(1-n(1-\psi))\hat{q}_b}{1-\psi}) + (u'(q_b) - c'(q_s))(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b)\Bigg]$$

eliminate $c'(q_s)(1-n)q_b$

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\Bigg[(1-n)[u(q_b)-u(\hat{q}_b)]+c'(q_s)\frac{(1-n(1-\psi))\hat{q}_b}{1-\psi}+u'(q_b)(\frac{\hat{q}_b}{1-\psi}-(1-n)q_b)-c'(q_s)\frac{\hat{q}_b}{1-\psi}\Bigg]$$

eliminate $c'(q_s)\frac{\hat{q}_b}{1-\psi}$

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\Bigg[(1-n)[u(q_b) - u(\hat{q}_b)] - nc'(q_s)\hat{q}_b + u'(q_b)(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b)\Bigg]$$

Only focus on the inside of brackets

$$(1-n)[u(q_b) - u(\hat{q}_b)] - nc'(q_s)\hat{q}_b + u'(q_b)(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b) > 0$$

since $nc'(q_s)\hat{q}_b > nu'(q_b)\hat{q}_b$ and $u'(q_b)\frac{\hat{q}_b}{1-\psi} > u'(q_b)\hat{q}_b$

$$(1-n)[u(q_b)-u(\hat{q}_b)]-nc'(q_s)\hat{q}_b+u'(q_b)(\frac{\hat{q}_b}{1-\psi}-(1-n)q_b) > (1-n)[u(q_b)-u(\hat{q}_b)]+u'(q_b)((1-n)\hat{q}_b-(1-n)q_b)$$

divide by $(1-n)$

$$u(q_b) - u(\hat{q}_b) + u'(q_b)(\hat{q}_b - q_b) > 0$$

Let $\Psi(q_b, \hat{q}_b) = u(q_b) - u(\hat{q}_b) - c'\big(\frac{1-n}{n}q_b\big)(q_b - \frac{\hat{q}_b}{1-\psi}) > 0$

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\left[(1-n)\Psi(q_b, \hat{q}_b) + \frac{(1-\beta)c'(q_s)(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b)}{\beta}\right] \tag{B.34}$$

Where $\Psi(q_b, \hat{q}_b) > 0$ since

$$\frac{u(q_b) - u(\hat{q}_b)}{q_b - \hat{q}_b} > u'(q_b) > c'(q_s) \tag{B.35}$$

for all $\gamma > \beta$.

Substitute $\frac{1-\beta}{\beta} = \frac{u'(q_b)}{c'(q_s)} - 1$

$$\phi\bar{l} = \frac{\beta}{(1-\beta)(1+i)}\left[(1-n)\Psi(q_b, \hat{q}_b) + (\frac{u'(q_b)}{c'(q_s)} - 1)c'(q_s)(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b)\right] \tag{B.36}$$

We need to only focus on the inside of the brackets

$$(1-n)\Psi(q_b, \hat{q}_b) + (u'(q_b) - c'(q_s))(\frac{\hat{q}_b}{1-\psi} - (1-n)q_b) > 0$$

$$(1-n)\big(u(q_b)-u(\hat{q}_b)-c'\big(q_s\big)(q_b-\frac{\hat{q}_b}{1-\psi})\big)+(u'(q_b)-c'(q_s))(\frac{\hat{q}_b}{1-\psi}-(1-n)q_b)>0$$

$$(1-n)(u(q_b)-u(\hat{q}_b))-(1-n)c'(q_s)q_b+(1-n)\frac{c'(q_s)\hat{q}_b}{1-\psi}+\frac{u'(q_b)\hat{q}_b}{1-\psi}-(1-n)u'(q_b)q_b-\frac{c'(q_s)\hat{q}_b}{1-\psi}+c'(q_s)(1-n)q_b>0$$

simplify

$$(1-n)(u(q_b)-u(\hat{q}_b))+(1-n)c'(q_s)\frac{\hat{q}_b}{1-\psi}+u'(q_b)\frac{\hat{q}_b}{1-\psi}-(1-n)u'(q_b)q_b-c'(q_s)\frac{\hat{q}_b}{1-\psi}>0$$

since $u'(q_b)>c'(q_s)$, we can reduce the LHS by multiplying $u'(q_b)\hat{q}_b$ and $c'(q_s)\hat{q}_b$ by $(1-n)$ to get

$$(1-n)(u(q_b)-u(\hat{q}_b))+(1-n)c'(q_s)\frac{\hat{q}_b}{1-\psi}+(1-n)u'(q_b)\frac{\hat{q}_b}{1-\psi}-(1-n)u'(q_b)q_b-(1-n)c'(q_s)\frac{\hat{q}_b}{1-\psi}>0$$

simplify

$$(1-n)(u(q_b)-u(\hat{q}_b))+(1-n)u'(q_b)\frac{\hat{q}_b}{1-\psi}-(1-n)u'(q_b)q_b>0$$

divide through by (1-n)

$$(u(q_b)-u(\hat{q}_b)+u'(q_b)\frac{\hat{q}_b}{1-\psi}-u'(q_b)q_b>0$$

simplify again

$$u(q_b) - u(\hat{q}_b > u'(q_b)q_b - u'(q_b)\frac{\hat{q}_b}{1-\psi}$$

$$\frac{u(q_b) - u(\hat{q}_b)}{q_b - \frac{\hat{q}_b}{1-\psi}} > u'(q_b)$$

■