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M. D. Diko – University of Amsterdam
R. Goedhart – University of Amsterdam
S. Chakraborti – University of Alabama
R. J. M. M. Does – University of Amsterdam
E. K. Epprecht – Pontifical Catholic University of Rio de Janeiro

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Phase II control charts for monitoring dispersion when parameters are estimated

M. D. Diko[a], R. Goedhart[a], S. Chakraborti[b], R. J. M. M. Does[a], and E. K. Epprecht[c]

[a]Department of Operations Management, University of Amsterdam, Amsterdam, The Netherlands; [b]Department of Information Systems, Statistics and Management Science, University of Alabama, Tuscaloosa, Alabama; [c]Department of Industrial Engineering, Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, Brazil

ABSTRACT

Shewhart control charts are among the most popular control charts used to monitor process dispersion. To base these control charts on the assumption of known in-control process parameters is often unrealistic. In practice, estimates are used to construct the control charts and this has substantial consequences for the in-control and out-of-control chart performance. The effects are especially severe when the number of Phase I subgroups used to estimate the unknown process dispersion is small. Typically, recommendations are to use around 30 subgroups of size 5 each.

We derive and tabulate new corrected charting constants that should be used to construct the estimated probability limits of the Phase II Shewhart dispersion (e.g., range and standard deviation) control charts for a given number of Phase I subgroups, subgroup size and nominal in-control average run-length (ICARL). These control limits account for the effects of parameter estimation. Two approaches are used to find the new charting constants, a numerical and an analytic approach, which give similar results. It is seen that the corrected probability limits based charts achieve the desired nominal ICARL performance, but the out-of-control average run-length performance deteriorate when both the size of the shift and the number of Phase I subgroups are small. This is the price one must pay while accounting for the effects of parameter estimation so that the in-control performance is as advertised. An illustration using real-life data is provided along with a summary and recommendations.

KEYWORDS

average run length (ARL); control charts; control limits; Phase II analysis; process dispersion; statistical process control

Introduction and motivation

The two-sided Shewhart R (sample range) and S (sample standard deviation) control charts are widely used to monitor the process dispersion. In practice, the in-control standard deviation value is usually not known. Then these charts are applied with estimated control limits, where the parameter estimates are obtained from Phase I reference data. When applying these charts, it is common to use the 3-sigma limits, given in most textbooks (Montgomery 2013), where a tabulation of the necessary chart constants can be found. The use of the standard 3-sigma limits is justified on the basis that the distribution of the charting statistic is normal or approximately normal. However, the charting statistics of the R and S charts are highly skewed. As a result, the performance of these charts can be quite questionable, particularly for smaller sample sizes typical in practice. To the best of our knowledge the performances of these charts have not been fully examined in the literature. But, these charts are critical in practical SPC applications since they are the most popular dispersion charts that are used in Phase II for keeping the process dispersion under control, before the location charts are constructed (which need an estimate of the process dispersion) and meaningfully interpreted.

To look more closely into the issues we derive the expressions for the unconditional in-control average run-length ($ICARL$) of the R and S charts (Montgomery 2013), which are based on 3-sigma limits and the assumption of a normal distribution, using the conditioning-unconditioning method (Chen 1998 and Chakraborti 2000), and evaluate them using the statistical software R. The evaluations are done for various values of the number of reference subgroups m, subgroup size n $= 5{,}10$, and nominal in-control average run-length ($ICARL_0$) equal to 370. The results are shown in Table 1, where $PD = 100(\frac{ICARL-370}{370})$ denotes the relative percentage difference between the $ICARL$ and the nominal in-control average run-length ($ICARL_0$), which is equal to 370.

CONTACT R. J. M. M. Does r.j.m.m.does@uva.nl Department of Operations Management, University of Amsterdam, Plantage Muidergracht 12, Amsterdam 1018 TV, The Netherlands.

Table 1. The *ICARL* and *PD* values for the estimated 3-sigma limits of the two-sided Phase II dispersion charts; for n = 5,10; $ICARL_0 = 370$, and various values of *m*.

n	m	R chart with σ_0 estimator $\overline{R}/d_2(n)$		S chart with σ_0 estimator $\overline{S}/c_4(n)$		S chart with σ_0 estimator S_p	
		ICARL	PD	ICARL	PD	ICARL	PD
5	5	4665	1161	745	101	668	80
	10	1000	170	570	54	523	41
	20	422	14	423	14	399	8
	30	332	− 10	365	− 2	349	− 6
	50	278	− 25	317	− 14	308	− 1
	100	245	− 34	282	− 24	278	− 25
	500	222	− 40	256	− 31	256	− 31
10	5	710	92	624	68	592	60
	10	478	29	551	49	530	43
	20	343	− 8	469	27	456	23
	30	300	− 19	429	16	421	14
	50	269	− 27	393	6	388	5
	100	248	− 33	363	− 2	361	− 3
	500	233	− 37	339	− 8	339	− 9

Let us consider first the R chart. This chart uses the average range estimator $\overline{R}/d_2(n)$ for the unknown in-control standard deviation (σ_0), where $d_2(n)$ is the unbiasing constant assuming the normal distribution (Montgomery 2013) and $\overline{R}$ is calculated from the m independent Phase I subgroup ranges $R_1, R_2, \ldots, R_m$. For $n = 5$, in Table 1, it can be seen that the *ICARL* values differ substantially from the nominal value, as the absolute *PD* values range from 10 (for $m = 30$) to 1161 (for $m = 5$). Note that a *PD* value greater (or smaller) than zero indicates that the *ICARL* value is greater (or smaller) than the nominal value 370. Both cases are undesirable. It can also be observed that for $m \geq 30$, the *PD* values are negative, which means that increasing the number of reference subgroups m exacerbates the false alarm rate (*FAR*). Similar results are found for the S chart that uses the Phase I estimator $\overline{S}/c_4(n) = \sum_{i=1}^{m} S_i/mc_4(n)$, where $c_4(n)$ is the unbiasing constant assuming the normal distribution (Montgomery 2013) and the S chart that uses the "pooled" estimator $S_p = \sqrt{\sum_{i=1}^{m} S_i^2/m}$, respectively, where $S_1, S_2, \ldots, S_m$ denote the standard deviations of the m Phase I reference subgroups. Note that the "pooled" estimator is not an unbiased estimator of σ_0. We use this estimator because the unbiasing constant $c_4(m(n-1)+1)$ is already 0.9876 when $m, n = 5$, and it gets even closer to 1 as m and or n increase (0.9975 for $m = 25$ and $n = 5$). Therefore, for all practical purposes this constant is indistinguishable from 1 and hence it is sufficient to use the estimator S_p. The reader is also referred to Mahmoud et al. (2010), where S_p and $S_p/c_4(m(n-1)+1)$ are compared and shown to be practically equal in terms of their probability distributions and mean squared error (MSE). For n = 10, in Table 1, the *PD* values are somewhat better than their counterparts for $n = 5$, but they are still unacceptable.

Based on these results, the standard estimated 3-sigma charts for dispersion cannot be recommended to monitor the dispersion in practice. This is an issue for anyone who uses these control limits available in most textbooks (Montgomery 2013). In fact, most of the commercial software seem to use these same (incorrect) limits. An alternative approach is to use probability limits instead of the classical 3-sigma limits (Diko, Chakraborti, and Graham 2016). This mitigates this issue, but does not solve it entirely. Indeed, Montgomery (2013) mentions the use of probability limits and refers to some tables in Grant and Leavenworth (1986), but it is not clear whether or not these probability limits are commonly used in practice. Woodall (2017) advocates the use of probability limits for the dispersion control charts. For a specified nominal *FAR* (denoted by $\alpha = \alpha_0$) such as 0.0027 or $ICARL_0 = 370$, the probability limits may be constructed using the exact distribution of the charting statistic. This will be discussed in more detail later. As an example, Table 2 shows the *ICARL* and the *PD* values for the two sided R and the S charts with the estimated probability limits, for various values of m, $n = 5$, 10 and $ICARL_0 = 370$.

It can be seen that now the PD values range from −29 to 0 and −32 to −1 for $n = 5$ and $n = 10$, respectively, and they approach zero as m increases, as one might expect (Chen 1998). This means that the difference between the *ICARL* values and the nominal $ICARL_0 = 370$ value is not as bad as what was observed

Table 2. The *ICARL* and *PD* values for the two-sided Phase II *R* and *S* charts with estimated probability limits for various values of *m*; $n = 5$, 10; and $ICARL_0 = 370$.

		R chart with σ_0 estimator $\overline{R}/d_2(n)$		S chart with σ_0 estimator $\overline{S}/c_4(n)$		S chart with σ_0 estimator S_p	
n	m	ICARL	PD	ICARL	PD	ICARL	PD
5	5	269	−27	270	−27	264	−29
	10	302	−18	302	−18	298	−19
	20	327	−11	328	−11	325	−12
	30	334	−10	339	−8	332	−10
	50	339	−8	350	−5	337	−9
	100	349	−5	359	−3	348	−6
	500	368	0	368	0	368	0
10	5	252	−32	252	−32	254	−31
	10	289	−22	289	−22	291	−21
	20	319	−14	319	−14	320	−14
	30	332	−10	332	−10	333	−10
	50	345	−7	345	−7	345	−7
	100	357	−4	357	−4	357	−4
	500	367	−1	367	−1	367	−1

in Table 1. It also means that even though the situation has improved over using the 3-sigma limits, unless m is very large, the estimated probability limits may not lead to the desired $ICARL_0$. The other thing to note is that the *PD* values are remarkably similar for all three estimators.

Thus, from a practical point of view, an important problem still persists. If the number of Phase I subgroups at hand m is small to moderate, even the estimated probability limits of the *R* and *S* charts do not quite maintain an advertised nominal in-control average run-length. Hence, for a given nominal $ICARL_0$ and a given amount of Phase I data, this article derives and tabulates new (correct) charting constants, which account for the effects of parameter estimation. We achieve this by setting the *ICARL* expression equal to some specified nominal value $ICARL_0$ and then evaluating the resulting equation for $\alpha = \alpha(m, n)$. The in-control (*IC*) and out-of-control (*OOC*) average run-length performance of the corrected probability limits charts are calculated and compared to the *IC* and *OOC* average run-length performance of the uncorrected probability limits.

This article is organized as follows. We begin by describing the classical (uncorrected) 3-sigma limits and probability limits Shewhart control charts for dispersion. Next, we derive new (corrected) control limits based on a numerical and an analytic method. Next, a data set from Montgomery (2013) is used to illustrate and discuss the differences between the corrected and uncorrected control limits. Following this, we evaluate the OOC behavior of the newly proposed probability limits. Finally, a summary and recommendations are provided in the last section.

Classical model for probability limits for the dispersion control charts

Suppose that m subgroups (samples) each of size n are available after a successful Phase I analysis to estimate the unknown parameters and set up the control limits that are to be used in prospective Phase II monitoring. Suppose that the data are from normal distributions and as before, let $R_1, R_2, \ldots, R_m$ denote the ranges and $S_1, S_2, \ldots, S_m$ denote the standard deviations of the m Phase I subgroups. As noted earlier, the three commonly used estimators of the unknown in-control process standard deviation σ_0 are (i) $\hat{\sigma}_{01} = \overline{R}/d_2(n)$, based on the average range, (ii) $\hat{\sigma}_{02} = \overline{S}/c_4(n)$, based on the average standard deviation, and (iii) $\hat{\sigma}_{03} = S_p$, the pooled estimator.

Thus, using each of the three Phase I estimators above, the three most popular Phase II Shewhart standard deviation charts are (1) the R chart using the charting statistic $T_{i1} = R_i$ with the unbiased estimator $\hat{\sigma}_{01}$, (2) the S chart using the charting statistic $T_{i2} = S_i$ with the unbiased estimator $\hat{\sigma}_{02}$ and (3) the S chart using the charting statistic $T_{i3} = S_i$ with the estimator $\hat{\sigma}_{03}$, respectively. Note that for all three of the charts, we let $i = m + 1, m + 2, \ldots$ to emphasize that these are Phase II charts, where prospective monitoring starts from the $(m + 1)$th sample having collected m Phase I samples. The subscript $j = 1, 2, 3$ is used to distinguish between the 3 charts. For chart j, we also write the unbiased Phase I estimator $\hat{\sigma}_{0j}$ as $\hat{\sigma}_{0j} = w_j/\varepsilon_{0j}$, where w_j is a biased Phase I estimator based on the charting statistic (i.e., $\bar{R}$, $\bar{S}$, and S_p) and ε_{0j} is its corresponding unbiasing constant (with $\varepsilon_{01} = d_2(n)$, $\varepsilon_{02} = c_4(n)$, $\varepsilon_{03} = c_4\ (m(n-1)+1) \approx 1$ shown in Appendix A). Note also that even though Mahmoud et al. (2010) recommended the estimator S_p, we consider all three estimators here for completeness. It also allows us to contrast our results with those that are found in the current literature.

In general, the control limits of the *jth* Phase II Shewhart chart for the process dispersion, with a charting statistic T_{ij}, can be written as

$$\begin{aligned} U\hat{C}L &= U_{n,\alpha,j}w_j \\ \hat{C}L &= w_j \\ L\hat{C}L &= L_{n,\alpha,j}w_j, \end{aligned} \quad [1]$$

where $U_{n,\alpha,j}$ and $L_{n,\alpha,j}$ are charting constants. These charting constants are based on the $100^*\{1-\alpha/2\}th$ and the $100^*\{\alpha/2\}th$ percentiles of the in-control distribution of the Phase II charting statistic T_{ij}, respectively, and are given in Appendix A. Note that for convenience, the constant ε_{0j}, that divides w_j to form the unbiased estimator, is taken to be a part of each of the charting constants U and L.

Probability limits are based on the in-control distribution of the charting statistic T_{ij}. To this end, note that (i) the in-control distribution of $T_{i1} = R_i$ is that of the random variable $W\sigma$, where W is the sample relative range, which has a well-known distribution for a normal population (see for example, Gibbons and Chakraborti, 2010) (ii) the in-control distribution of T_{i2} and T_{i3} are both that of the random variable $\sqrt{\chi^2_{n-1}}/\sqrt{n-1}$, where χ^2_{n-1} is a chi-square variable with *n*-1 degrees of freedom.

In the Introduction, we argued that to overcome some of the issues associated with using the R and S charts with the estimated 3-sigma limits, the R and S charts with the estimated probability limits are recommended. In this case, the charting constants, $U_{n,\alpha,j}$ and $L_{n,\alpha,j}$ are based on the percentiles of the exact distributions of R_i or S_i. For the R chart, the charting constants are given by $U_{n,\alpha,1} = \frac{F_{W_{n,1-\alpha/2}}}{d_2(n)}$ and $L_{n,\alpha,1} = \frac{F_{W_{n,\alpha/2}}}{d_2(n)}$, where $F_{W_{n,1-\alpha/2}}$ and $F_{W_{n,\alpha/2}}$ denote the $100^*\{1-\alpha/2\}th$ and the $100^*\{\alpha/2\}th$ percentiles of the in-control distribution of the sample relative range W, respectively. Given these charting constants plus $w_1 = \overline{R}$, we can find the control limits by substituting them into Ed. [1]. Similarly, the estimated probability limits for the S charts (i.e., S_2 and S_3) are obtained by substituting $\left(w_2 = \overline{S},\ U_{n,\alpha,2} = \frac{\sqrt{\chi^2_{1-\alpha/2,n-1}}}{\sqrt{n-1}\,c_4(n)},\ L_{n,\alpha,2} = \frac{\sqrt{\chi^2_{\alpha/2,n-1}}}{\sqrt{n-1}\,c_4(n)}\right)$ and $\left(w_3 = S_p,\ U_{n,\alpha,3} = \frac{\sqrt{\chi^2_{1-\alpha/2,n-1}}}{\sqrt{n-1}},\ L_{n,\alpha,3} = \frac{\sqrt{\chi^2_{\alpha/2,n-1}}}{\sqrt{n-1}}\right)$ for S_2 and S_3, respectively, in Eq. [1]. However, these charting constants were originally intended for use with the σ_0 known probability limits, and are thus incorporated using the nominal *FAR* $\alpha = \alpha_0$. Furthermore, they only depend on the Phase II charting statistic, and not on the Phase I estimator or the Phase I sample size. Hence, they are *not* the appropriate constants in the case that σ_0 is unknown. Since these charting constants do not depend on m, they do not properly account for the effect of parameter estimation. In the next section we will correct these charting constants and so their control limits.

The *R* and *S* charts with estimated probability limits and corrected for the effects of parameter estimation

To properly account for the effects of parameter estimation while using the Phase II charts, that is, to account for the effects of using m Phase I samples each of size n to estimate the in-control standard deviation σ_0, we propose to use the following probability limits

$$\begin{aligned} U\hat{C}L &= U_{n,\alpha(m,n),j}w_j \\ \hat{C}L &= w_j \\ L\hat{C}L &= L_{n,\alpha(m,n),j}w_j. \end{aligned} \quad [2]$$

Note that the above control limits are similar in form to those in Eq. [1] except that here we denote α as $\alpha(m,n)$ to emphasize that this probability should be a function of both m and n, to make the correct charting constants L and U depend on both of m and n, and thus account for parameter estimation.

In order to find the charting constants, we need to derive an expression for the unconditional in-control average run-length (*ICARL*). This *ICARL* depends on the in-control distributions of both the Phase I estimator (w_j) and the Phase II charting statistic (T_{ij}). In our derivations, we assume that w_j/σ follows a scaled chi-square distribution $\frac{\varepsilon_{0j}a_{0j}\sqrt{X_{0j}}}{\sqrt{b_{0j}}}$, where X_{0j} denotes a chi-square random variable with b_{0j} degrees of freedom. Formulae and or values for the constants a_{0j}, b_{0j} and ε_{0j} are given in Appendix A, and are based on the well-known Patnaik (1950) approximation (see Chen (1998) for the explicit expressions). Note that conditional on the observed value of the Phase I estimator

w_j (or equivalently, on the realization of X_{0j}), the conditional in-control ($\sigma = \sigma_0$) Phase II run-length distribution is geometric. The success probability of this distribution is equal to the conditional false alarm rate (denoted CFAR), which is defined as

$$\begin{aligned} CFAR_j &= 1 - P\left(\hat{LCL} < T_{ij} < \hat{UCL} | \sigma = \sigma_0\right) \\ &= 1 - P\left(L_{n,\alpha,j} w_j < T_{ij} < U_{n,\alpha,j} w_j | \sigma = \sigma_0\right) \\ &= 1 - P\left(L_{n,\alpha,j}\frac{w_j}{\sigma} < \frac{T_{ij}}{\sigma} < U_{n,\alpha,j}\frac{w_j}{\sigma} | \sigma = \sigma_0\right) \\ &= 1 - P\left(L_{n,\alpha,j}\frac{\varepsilon_{0j} a_{0j}\sqrt{X_{0j}}}{\sqrt{b_{0j}}} < \frac{T_{ij}}{\sigma} \right. \\ &\left. < U_{n,\alpha,j}\frac{\varepsilon_{0j} a_{0j}\sqrt{X_{0j}}}{\sqrt{b_{0j}}} | \sigma = \sigma_0\right) \\ &= CFAR_j\left(X_{0j}, m, n, \alpha\right). \end{aligned} \quad [3]$$

Next, using the conditioning-unconditioning method in Chakraborti (2000), where we integrate over all possible values of X_{0j}, the unconditional *ICARL* of the *jth* Phase II Shewhart dispersion chart can be obtained as

$$ICARL_j(m, n, \alpha) = \int_0^\infty \left[CFAR_j(x, m, n, \alpha)\right]^{-1} f_{\chi^2_{b_{0j}}}(x)\, dx, \quad [4]$$

where $f_{\chi^2_{b_{0j}}}$ denotes the probability density function (pdf) of X_{0j}.

We start with the numerical approach, which solves the equation $ICARL_j(m, n, \alpha(m, n)) = ICARL_0$ numerically for $\alpha(m, n)$.

The numerical approach

The numerical approach finds $\alpha(m, n)$ numerically and uses it to correct the uncorrected constants in Montgomery (2013) as follows:

(i) specifies the values of *m*, *n* at hand and the desired nominal $ICARL_0$;
(ii) uses the exact in control distributions of the charting statistics to (1) define the control limits and (2) determine the expressions for the $CFAR_j(X_{0j}, m, n, \alpha(m, n))$ and the $ICARL_j(m, n, \alpha(m, n))$;
(iii) numerically solves the equation $\int_0^\infty [CFAR_j(x, m, n, \alpha(m, n))]^{-1} f_{\chi^2_{b_{0j}}}(x)dx = ICARL_0$ for the corresponding $\alpha(m, n)$ value; and
(iv) uses $\alpha(m, n)$ to correct the uncorrected charting constants in Montgomery (2013).

For example, for the *R* chart, recall that $L_{n,\alpha(m,n),j} = \frac{F_{W_{n,\alpha(m,n)/2}}}{d_2(n)}$ and $U_{n,\alpha(m,n),j} = \frac{F_{W_{n,1-\alpha(m,n)/2}}}{d_2(n)}$. Consequently, $CFAR_1$ becomes

$$\begin{aligned} &CFAR_1(X_{01}, m, n, \alpha(m, n)) \\ &= 1 - P\left(L_{n,\alpha(m,n),1}\frac{w_1}{\sigma} < \frac{R_i}{\sigma} \right. \\ &\left. < U_{n,\alpha(m,n),1}\frac{w_1}{\sigma} | \sigma = \sigma_0\right) \\ &= 1 - F_{W_n}\left(F_{W_{n,1-\alpha(m,n)/2}}\frac{a_{01}\sqrt{X_{01}}}{\sqrt{b_{01}}}\right) \\ &+ F_{W_n}\left(F_{W_{n,\alpha(m,n)/2}}\frac{a_{01}\sqrt{X_{01}}}{\sqrt{b_{01}}}\right), \end{aligned}$$

where F_{W_n} represents the cumulative distribution function (CDF) of the sample relative range. Using this equation to solve

$$\int_0^\infty \left[CFAR_j(x, m, n, \alpha(m, n))\right]^{-1} f_{\chi^2_{b_{0j}}}(x)\, dx = ICARL_0 \quad [5]$$

will result in the required corrected charting constants. For example, with $m = 5$, $n = 5$, and $ICARL_0 = 370$, the value $\alpha(m, n)$ that satisfies the above equation is 0.001949. This value is then used to correct the uncorrected charting constants for the estimated probability limits Phase II Shewhart *R* chart. The corrected charting constants for the Phase II *R* chart with $\hat{\sigma}_{01}$ as Phase I estimator, are given by

$$U_{5,\alpha(5,5),1} = \frac{F_{W_{5,1-\alpha(5,5)/2}}}{d_2(5)} = \frac{F_{W_{5,1-0.001949/2}}}{d_2(5)} = \frac{5.49281}{2.32593} = 2.3616$$

and

$$L_{5,\alpha(5,5),1} = \frac{F_{W_{5,\alpha(5,5)/2}}}{d_2(5)} = \frac{F_{W_{5,0.001949/2}}}{d_2(5)} = \frac{0.36499}{2.32593} = 0.1569,$$

respectively. For other values of *n*, *m* and $ICARL_0$, the values of $\alpha(m, n)$, $U_{n,\alpha(m,n),1}$ and $L_{n,\alpha(m,n),1}$ are given in Table 3. The *R* codes for finding all these values are

Table 3. New corrected charting constants for the Shewhart R and S charts when parameters are estimated from m Phase I subgroups, $m = 5, 10, 20, 25, 30, 50, 100, 300, 500$ and 1000 each of subgroup size $n = 5, 10$ for a nominal in-control $ARL = 370$ and 500.

			Numerical Approach						Analytical Approach					
			$ICARL0 = 370$			$ICARL0 = 500$			$ICARL0 = 370$			$ICARL0 = 500$		
n	m	Chart	α	L	U	α	L	U	α	L	U	α	L	U
5	5	R - $\bar{R}$	0.001949	0.1569	2.3616	0.001433	0.1451	2.4074	0.001686	0.1513	2.3832	0.001229	0.1396	2.4300
		S - S_p	0.001908	0.1489	2.1547	0.001402	0.1377	2.1939	0.001582	0.1420	2.1786	0.001149	0.1309	2.2188
		S - $\bar{S}$	0.001954	0.1593	2.2890	0.001438	0.1474	2.3306	0.001672	0.1532	2.3103	0.001218	0.1414	2.3527
	10	R - $\bar{R}$	0.002194	0.1617	2.3436	0.001615	0.1496	2.3897	0.002091	0.1598	2.3509	0.001534	0.1477	2.3974
		S - S_p	0.002166	0.1538	2.1383	0.001594	0.1422	2.1777	0.002035	0.1513	2.1464	0.001490	0.1398	2.1862
		S - $\bar{S}$	0.002198	0.1642	2.2729	0.001617	0.1519	2.3148	0.002082	0.1619	2.2803	0.001527	0.1497	2.3225
	20	R - $\bar{R}$	0.002384	0.1652	2.3310	0.001757	0.1529	2.3771	0.002350	0.1646	2.3332	0.001730	0.1522	2.3795
		S - S_p	0.002368	0.1573	2.1268	0.001745	0.1455	2.1662	0.002320	0.1565	2.1294	0.001706	0.1447	2.1690
		S - $\bar{S}$	0.002387	0.1677	2.2614	0.001760	0.1552	2.3033	0.002345	0.1669	2.2639	0.001726	0.1544	2.3060
	25	R - $\bar{R}$	0.002434	0.1660	2.3278	0.001795	0.1537	2.3740	0.002410	0.1656	2.3294	0.001775	0.1532	2.3756
		S - S_p	0.002420	0.1581	2.1239	0.001783	0.1463	2.1634	0.002386	0.1576	2.1258	0.001756	0.1458	2.1653
		S - $\bar{S}$	0.002435	0.1685	2.2587	0.001797	0.1560	2.3005	0.002406	0.1680	2.2603	0.001772	0.1554	2.3024
	30	R - $\bar{R}$	0.002469	0.1666	2.3257	0.001821	0.1542	2.3718	0.002452	0.1664	2.3267	0.001807	0.1539	2.3729
		S - S_p	0.002457	0.1587	2.1219	0.001812	0.1469	2.1614	0.002432	0.1583	2.1233	0.001792	0.1465	2.1628
		S - $\bar{S}$	0.002470	0.1691	2.2566	0.001823	0.1566	2.2985	0.002449	0.1687	2.2579	0.001805	0.1562	2.2999
	50	R - $\bar{R}$	0.002549	0.1680	2.3208	0.001883	0.1555	2.3668	0.002543	0.1679	2.3211	0.001877	0.1554	2.3672
		S - S_p	0.002542	0.1601	2.1175	0.001876	0.1482	2.1569	0.002530	0.1599	2.1181	0.001867	0.1481	2.1575
		S - $\bar{S}$	0.002550	0.1705	2.2522	0.001883	0.1579	2.2941	0.002541	0.1703	2.2527	0.001875	0.1577	2.2947
	100	R - $\bar{R}$	0.002619	0.1692	2.3166	0.001936	0.1567	2.3626	0.002618	0.1692	2.3167	0.001935	0.1566	2.3627
		S - S_p	0.002615	0.1613	2.1137	0.001932	0.1494	2.1531	0.002612	0.1612	2.1139	0.001930	0.1493	2.1533
		S - $\bar{S}$	0.002621	0.1717	2.2484	0.001936	0.1590	2.2903	0.002617	0.1716	2.2486	0.001934	0.1589	2.2904
	300	R - $\bar{R}$	0.002674	0.1701	2.3134	0.001978	0.1575	2.3594	0.002673	0.1701	2.3135	0.001977	0.1575	2.3594
		S - S_p	0.002672	0.1622	2.1109	0.001976	0.1502	2.1502	0.002671	0.1622	2.1109	0.001976	0.1502	2.1502
		S - $\bar{S}$	0.002674	0.1726	2.2455	0.001977	0.1598	2.2874	0.002673	0.1725	2.2456	0.001977	0.1598	2.2874
	500	R - $\bar{R}$	0.002685	0.1702	2.3128	0.001987	0.1577	2.3587	0.002685	0.1702	2.3128	0.001985	0.1577	2.3588
		S - S_p	0.002684	0.1624	2.1103	0.001986	0.1504	2.1496	0.002684	0.1624	2.1103	0.001985	0.1504	2.1496
		S - $\bar{S}$	0.002685	0.1728	2.2450	0.001987	0.1600	2.2867	0.002685	0.1727	2.2450	0.001986	0.1600	2.2868
	1000	R - $\bar{R}$	0.002695	0.1704	2.3122	0.001994	0.1578	2.3581	0.002694	0.1704	2.3123	0.001993	0.1578	2.3582
		S - S_p	0.002694	0.1625	2.1098	0.001994	0.1506	2.1491	0.002693	0.1625	2.1099	0.001993	0.1505	2.1491
		S - $\bar{S}$	0.002695	0.1729	2.2445	0.001994	0.1602	2.2863	0.002694	0.1729	2.2445	0.001993	0.1601	2.2863
10	5	R - $\bar{R}$	0.001813	0.3481	1.9514	0.001328	0.3349	1.9840	0.001467	0.3391	1.9737	0.001057	0.3256	2.0075
		S - S_p	0.001812	0.3534	1.7681	0.001328	0.3402	1.7931	0.001383	0.3419	1.7899	0.000992	0.3282	1.8162
		S - $\bar{S}$	0.001831	0.3638	1.8169	0.001342	0.3502	1.8427	0.001424	0.3527	1.8377	0.001024	0.3388	1.8647
	10	R - $\bar{R}$	0.002095	0.3545	1.9360	0.001538	0.3411	1.9687	0.001965	0.3517	1.9428	0.001435	0.3381	1.9759
		S - S_p	0.002095	0.3598	1.7561	0.001538	0.3463	1.7813	0.001921	0.3560	1.7633	0.001400	0.3424	1.7889
		S - $\bar{S}$	0.002107	0.3702	1.8050	0.001547	0.3564	1.8309	0.001942	0.3665	1.8119	0.001416	0.3525	1.8382
	20	R - $\bar{R}$	0.002319	0.3590	1.9251	0.001708	0.3455	1.9577	0.002280	0.3583	1.9270	0.001675	0.3447	1.9597
		S - S_p	0.002320	0.3644	1.7477	0.001708	0.3508	1.7729	0.002256	0.3632	1.7500	0.001656	0.3495	1.7754
		S - $\bar{S}$	0.002327	0.3748	1.7966	0.001713	0.3608	1.8224	0.002267	0.3736	1.7988	0.001664	0.3596	1.8248
	25	R - $\bar{R}$	0.002378	0.3602	1.9224	0.001752	0.3466	1.9550	0.002352	0.3597	1.9236	0.001730	0.3461	1.9563
		S - S_p	0.002377	0.3655	1.7457	0.001751	0.3519	1.7708	0.002333	0.3647	1.7472	0.001715	0.3510	1.7725
		S - $\bar{S}$	0.002384	0.3759	1.7945	0.001756	0.3619	1.8204	0.002341	0.3751	1.7960	0.001721	0.3611	1.8220
	30	R - $\bar{R}$	0.002420	0.3610	1.9206	0.001783	0.3474	1.9531	0.002403	0.3606	1.9213	0.001769	0.3470	1.9540
		S - S_p	0.002420	0.3663	1.7442	0.001783	0.3527	1.7694	0.002387	0.3657	1.7453	0.001756	0.3521	1.7706
		S - $\bar{S}$	0.002425	0.3767	1.7930	0.001787	0.3627	1.8189	0.002394	0.3761	1.7942	0.001762	0.3621	1.8201
	50	R - $\bar{R}$	0.002516	0.3627	1.9164	0.001857	0.3492	1.9488	0.002512	0.3627	1.9166	0.001852	0.3491	1.9491
		S - S_p	0.002516	0.3681	1.7410	0.001857	0.3545	1.7660	0.002502	0.3679	1.7414	0.001845	0.3542	1.7666
		S - $\bar{S}$	0.002520	0.3785	1.7898	0.001860	0.3645	1.8156	0.002506	0.3783	1.7902	0.001848	0.3642	1.8161
	100	R - $\bar{R}$	0.002602	0.3643	1.9127	0.001922	0.3507	1.9452	0.002602	0.3643	1.9127	0.001922	0.3507	1.9452
		S - S_p	0.002602	0.3697	1.7381	0.001921	0.3560	1.7633	0.002597	0.3696	1.7383	0.001918	0.3559	1.7634
		S - $\bar{S}$	0.002603	0.3801	1.7870	0.001924	0.3661	1.8127	0.002599	0.3800	1.7871	0.001920	0.3660	1.8129
	300	R - $\bar{R}$	0.002667	0.3654	1.9101	0.001973	0.3518	1.9424	0.002668	0.3654	1.9100	0.001973	0.3518	1.9424
		S - S_p	0.002668	0.3708	1.7361	0.001972	0.3571	1.7611	0.002666	0.3708	1.7361	0.001971	0.3571	1.7611
		S - $\bar{S}$	0.002668	0.3812	1.7849	0.001973	0.3672	1.8106	0.002667	0.3812	1.7849	0.001972	0.3672	1.8106
	500	R - $\bar{R}$	0.002681	0.3657	1.9095	0.001983	0.3521	1.9418	0.002681	0.3657	1.9095	0.001983	0.3521	1.9418
		S - S_p	0.002681	0.3710	1.7356	0.001983	0.3574	1.7607	0.002680	0.3710	1.7356	0.001983	0.3574	1.7607
		S - $\bar{S}$	0.002681	0.3815	1.7844	0.001984	0.3675	1.8101	0.002681	0.3815	1.7844	0.001983	0.3674	1.8102
	1000	R - $\bar{R}$	0.002692	0.3659	1.9090	0.001993	0.3523	1.9414	0.002692	0.3659	1.9091	0.001992	0.3522	1.9414
		S - S_p	0.002692	0.3712	1.7353	0.001992	0.3576	1.7603	0.002691	0.3712	1.7353	0.001991	0.3576	1.7603
		S - $\bar{S}$	0.002692	0.3817	1.7841	0.001992	0.3676	1.8098	0.002692	0.3817	1.7841	0.001991	0.3676	1.8098

given in Appendix B as an example for the other codes used in this article.

Similarly, recall that for $j = 2$ we have $L_{n,\alpha(m,n),2} = \frac{\sqrt{\chi^2_{\alpha(m,n)/2,n-1}}}{c_4(n)\sqrt{n-1}}$ and $U_{n,\alpha(m,n),2} = \frac{\sqrt{\chi^2_{1-\alpha(m,n)/2,n-1}}}{c_4(n)\sqrt{n-1}}$, and that $\frac{S}{\sigma} \sim \frac{\sqrt{\chi^2_{n-1}}}{\sqrt{n-1}}$. This can be used to calculate $CFAR_2$ as

$$
\begin{aligned}
&CFAR_2\left(X_{02}, m, n, \alpha\left(m, n\right)\right) \\
&= 1 - P\left(L_{n,\alpha(m,n),2}\frac{w_2}{\sigma} < \frac{S_i}{\sigma}\right. \\
&\left. < U_{n,\alpha(m,n),2}\frac{w_2}{\sigma}|\sigma = \sigma_0\right) \\
&= 1 - F_{\chi^2_{n-1}}\left(\chi^2_{1-\alpha(m,n)/2,n-1}\frac{a_{02}^2 X_{02}}{b_{02}}\right) \\
&+ F_{\chi^2_{n-1}}\left(\chi^2_{\alpha(m,n)/2,n-1}\frac{a_{02}^2 X_{02}}{b_{02}}\right)
\end{aligned}
$$

which can in turn be used to determine the required charting constants. For example, for $j = 2$, $m = 5$, $n = 5$, and $ICARL_0 = 370$, the value of $\alpha(m, n)$ that satisfies Eq. [5] is 0.001954. Thus, the corrected charting constants for the Phase II S chart with $\hat{\sigma}_{02}$ as Phase I estimator, are calculated as

$$
\begin{aligned}
U_{5,\alpha(5,5),2} &= \frac{\sqrt{\chi^2_{1-\alpha(5,5)/2,n-1}}}{c_4(n)\sqrt{n-1}} = \frac{\sqrt{\chi^2_{1-0.001954/2,5-1}}}{0.9400\sqrt{5-1}} \\
&= \frac{\sqrt{18.5184}}{0.9400\sqrt{4}} = 2.2923
\end{aligned}
$$

And

$$
\begin{aligned}
L_{5,\alpha(5,5),2} &= \frac{\sqrt{\chi^2_{\alpha(5,5)/2,n-1}}}{c_4(n)\sqrt{n-1}} = \frac{\sqrt{\chi^2_{0.001954/2,5-1}}}{\sqrt{5-1}} \\
&= \frac{\sqrt{0.0897}}{0.9400\sqrt{4}} = 0.1584,
\end{aligned}
$$

respectively. For other values of n, m and $ICARL_0$, the values of $\alpha(m, n)$, $U_{n,\alpha(m,n),2}$ and $L_{n,\alpha(m,n),2}$ are given in Table 3.

Finally, for $j = 3$ we have $L_{n,\alpha(m,n),3} = \frac{\sqrt{\chi^2_{\alpha(m,n)/2,n-1}}}{\sqrt{n-1}}$ and $U_{n,\alpha,3} = \frac{\sqrt{\chi^2_{1-\alpha(\ ,n)/2,n-1}}}{\sqrt{n-1}}$, and that $\frac{S}{\sigma} \sim \frac{\sqrt{\chi^2_{n-1}}}{\sqrt{n-1}}$. This can be used to calculate $CFAR_3$ as

$$
\begin{aligned}
&CFAR_3\left(X_{03}, m, n, \alpha\left(m, n\right)\right) \\
&= 1 - P\left(L_{n,\alpha(m,n),3}\frac{w_3}{\sigma} < \frac{S_i}{\sigma}\right. \\
&\left. < U_{n,\alpha(m,n),3}\frac{w_3}{\sigma}|\sigma = \sigma_0\right) \\
&= 1 - F_{\chi^2_{n-1}}\left(\chi^2_{1-\alpha(m,n)/2,n-1}\frac{X_{03}}{m\left(n-1\right)}\right) \\
&+ F_{\chi^2_{n-1}}\left(\chi^2_{\alpha(m,n)/2,n-1}\frac{X_{02}}{m\left(n-1\right)}\right)
\end{aligned}
$$

which can be used to determine the required charting constants for this case. For example, for $j = 3$, $m = 5$, $n = 5$, and $ICARL_0 = 370$, the value $\alpha(m, n)$ that satisfies Eq. [5] is 0.001908 Therefore, the corrected charting constants for the Phase II S chart, with $\hat{\sigma}_{03}$ as Phase I estimator, are

$$
\begin{aligned}
U_{5,\alpha(5,5),2} &= \frac{\sqrt{\chi^2_{1-\alpha(5,5)/2,n-1}}}{\sqrt{n-1}} = \frac{\sqrt{\chi^2_{1-0.001908/2,5-1}}}{\sqrt{5-1}} \\
&= \frac{\sqrt{18.5712}}{\sqrt{4}} = 2.1547
\end{aligned}
$$

and

$$
\begin{aligned}
L_{5,\alpha(5,5),2} &= \frac{\sqrt{\chi^2_{\alpha(5,5)/2,n-1}}}{\sqrt{n-1}} = \frac{\sqrt{\chi^2_{0.001908/2,5-1}}}{\sqrt{5-1}} \\
&= \frac{\sqrt{0.8866}}{\sqrt{4}} = 0.1489,
\end{aligned}
$$

respectively. Again, Table 3 gives other values of $\alpha(m, n)$, $U_{n,\alpha(m,n),3}$ and $L_{n,\alpha(m,n),3}$ for different combinations of n, m and $ICARL_0$.

From Table 3, it is interesting to see that when m increases, as one might expect, the $\alpha(m, n)$, $U_{n,\alpha(m,n),j}$, and $L_{n,\alpha(m,n),j}$ values converge to their σ_0 known counterparts α_0, $U_{n,\alpha_0,j}$, and $L_{n,\alpha_0,j}$, respectively.

The analytical approach

While the numerical solutions outlined above are useful, it is interesting to consider an approximation to the charting constants based on the recent work of Goedhart, Schoonhoven, and Does (2016) and Goedhart et al. (2017), which is based on a first-order Taylor approximation of the ICARL. The numerical approach of finding $\alpha(m, n)$ involves numerical integration and solving some nonlinear equations. However, it is also possible to find $\alpha(m, n)$ using a more easily implementable but approximate method. Our approach is to:

(i) specify the values of m, n at hand and the desired nominal $ICARL_0$;
(ii) unify the control charts for dispersion under one chi-square framework, which assumes that the

charting statistic T_{ij} is either exactly or approximately distributed as a scaled chi-square random variable $\frac{\varepsilon_j a_j \sigma \sqrt{\chi^2_{b_j}}}{\sqrt{b_j}}$, where $\chi^2_{b_j}$ is a chi-square random variable with b_j degrees of freedom, ε_j equals the expectation of T_{ij}, and a_j is some constant. Formulae and or values for the constants a_j, b_j, and ε_j are given in Appendix A, and are based on the Patnaik (1950) approximation, similar to w_j;

(iii) use the above chi-square framework to (1) define the control limits and (2) determine the expressions for the $CFAR_j(X_{0j}, m, n, \alpha)$ and the $ICARL_j(m, n, \alpha)$;

(iv) obtain an analytical expression for $\alpha(m, n)$; and

(v) use the resulting value of $\alpha(m, n)$ to adjust the uncorrected charting constants.

Using the approximations in step (ii), we can write $CFAR_j$ more explicitly as

$$\begin{aligned} & CFAR_j\left(X_{0j}, m, n, \alpha\right) \\ & = 1 - P\left(L_{n,\alpha,j} w_j < T_{ij} < U_{n,\alpha,j} w_j | \sigma = \sigma_0\right) \\ & = 1 - P\left(L_{n,\alpha,j} \frac{w_j}{\sigma} < \frac{T_{ij}}{\sigma} < U_{n,\alpha,j} \frac{w_j}{\sigma} | \sigma = \sigma_0\right) \\ & = 1 - P\left(L_{n,\alpha,j} \frac{\varepsilon_{0j} a_{0j} \sqrt{X_{0j}}}{\sqrt{b_{0j}}} < \frac{\varepsilon_j a_j \sqrt{\chi^2_{b_j}}}{\sqrt{b_j}} \right. \\ & \left. < U_{n,\alpha,j} \frac{\varepsilon_{0j} a_{0j} \sqrt{X_{0j}}}{\sqrt{b_{0j}}}\right) \\ & = 1 - P\left(\left(L_{n,\alpha,j}\right)^2 \frac{a_{0j}^2 b_j X_{0j}}{a_j^2 b_{0j}} < \chi^2_{b_j} \right. \\ & \left. < \left(U_{n,\alpha,j}\right)^2 \frac{a_{0j}^2 b_j X_{0j}}{a_j^2 b_{0j}}\right) \\ & = 1 - P\left(\left(\frac{a_j \sqrt{\chi^2_{\alpha/2, b_j}}}{\sqrt{b_j}}\right)^2 \frac{a_{0j}^2 b_j X_{0j}}{a_j^2 b_{0j}} < \chi^2_{b_j} \right. \\ & \left. < \left(\frac{a_j \sqrt{\chi^2_{1-\alpha/2, b_j}}}{\sqrt{b_j}}\right)^2 \frac{a_{0j}^2 b_j X_{0j}}{a_j^2 b_{0j}}\right) \\ & = 1 - P\left(\chi^2_{\alpha/2, b_j} \frac{a_{0j}^2 X_{0j}}{b_{0j}} < \chi^2_{b_j} < \chi^2_{1-\alpha/2, b_j} \frac{a_{0j}^2 X_{0j}}{b_{0j}}\right) \\ & = 1 - F_{\chi^2_{b_j}}\left(\chi^2_{1-\alpha/2, b_j} \frac{a_{0j}^2 X_{0j}}{b_{0j}}\right) \\ & + F_{\chi^2_{b_j}}\left(\chi^2_{\alpha/2, b_j} \frac{a_{0j}^2 X_{0j}}{b_{0j}}\right) \end{aligned} \qquad [6]$$

where $F_{\chi^2_{b_j}}$ represents the CDF of a chi-square variable with b_j degrees of freedom. Consequently, the approximated $ICARL_j$ can be calculated as

$$\begin{aligned} & ICARL_j\left(m, n, \alpha\right) \\ & = \int_0^\infty \left[1 - F_{\chi^2_{b_j}}\left(\chi^2_{1-\alpha/2, b_j} \frac{a_{0j}^2 x}{b_{0j}}\right) \right. \\ & \left. + F_{\chi^2_{b_j}}\left(\chi^2_{\alpha/2, b_j} \frac{a_{0j}^2 x}{b_{0j}}\right)\right]^{-1} f_{\chi^2_{b_{0j}}}(x)\, dx \end{aligned} \qquad [7]$$

where $f_{\chi^2_{b_{0j}}}$ represents the probability density function (PDF) of a chi-square variable with b_{0j} degrees of freedom.

The next step is to determine an analytical expression for $\alpha(m, n)$. In order to do this, we consider a first order Taylor approximation of $ICARL_j(m, n, \alpha(m, n))$, around $\alpha_0 = 1/ICARL_0$, where α_0 is the nominal FAR as before. This gives the approximation

$$\begin{aligned} & ICARL_j\left(m, n, \alpha\left(m, n\right)\right) \\ & = ICARL_j\left(m, n, \alpha_0\right) \\ & + \left(\alpha\left(m, n\right) - \alpha_0\right) \frac{dICARL_j\left(m, n, \alpha = \alpha_0\right)}{d\alpha}. \end{aligned} \qquad [8]$$

Since we want $ICARL_j(m, n, \alpha(m, n)) = ICARL_0$, which equals $\frac{1}{\alpha_0}$, we solve

$$\begin{aligned} \frac{1}{\alpha_0} & = ICARL_j\left(m, n, \alpha = \alpha_0\right) \\ & + \left(\alpha\left(m, n\right) - \alpha_0\right) \frac{dICARL_j\left(m, n, \alpha = \alpha_0\right)}{d\alpha} \end{aligned} \qquad [9]$$

for $\alpha(m, n)$. This yields the approximation

$$\alpha\left(m, n\right) = \frac{1/\alpha_0 - ICARL_j\left(m, n, \alpha = \alpha_0\right)}{\frac{d\left[ICARL_j(m,n,\alpha=\alpha_0)\right]}{d\alpha}} + \alpha_0. \qquad [10]$$

The next step is to determine $\frac{d[ICARL_j(m,n,\alpha)]}{d\alpha}$.

Table 4. The *ICARL* and the *PD* values for the *R* and *S* charts with the charting constants calculated analytically (ANA) and numerical (NUM) for n = 5, $ICARL_0 = 370$ and various values of *m*.

	R chart with σ_0 estimator $\bar{R}/d_2(n)$				S chart with σ_0 estimator $\bar{S}/c_4(n)$				S chart with σ_0 estimator S_p			
m	ICARL NUM	PD	ICARL ANA	PD	ICARL NUM	PD	ICARL ANA	PD	ICARL NUM	PD	ICARL ANA	PD
5	370	0	426	15	370	0	431	17	370	0	444	20
10	370	0	388	5	370	0	390	5	370	0	393	6
20	370	0	375	2	370	0	376	2	370	0	377	2
25	370	0	373	1	370	0	374	1	370	0	375	1
30	370	0	372	1	370	0	373	1	370	0	374	1
50	370	0	371	0	370	0	371	0	370	0	372	0
100	370	0	370	0	370	0	370	0	370	0	370	0
500	370	0	370	0	370	0	370	0	370	0	370	0

From the obtained equation for $ICARL_j(m, n, \alpha)$ it follows that, in order to find its derivative, we need the results

$$\frac{d\chi^2_{1-\frac{\alpha}{2},b}}{d\alpha} = -\left[2 f_{\chi^2_b}\left(\chi^2_{1-\frac{\alpha}{2},b}\right)\right]^{-1} \text{ and}$$

$$\frac{d\chi^2_{\frac{\alpha}{2},b}}{d\alpha} = -\left[2 f_{\chi^2_b}\left(\chi^2_{\frac{\alpha}{2},b}\right)\right]^{-1}.$$

These are obtained using the fact that $[G^{-1}]'(x) = [G'(G^{-1}(x))]^{-1}$, where G and G' denote the CDF of a continuous random variable and its derivative (the PDF), respectively; G^{-1} denotes the inverse of the CDF G and $[G^{-1}]'$ denotes the first derivative of G^{-1} (see for example, Gibbons and Chakraborti, 2010). Thus, we find

$$\frac{d\left[ICARL_j(m, n, \alpha)\right]}{d\alpha} = \int_0^{\infty} -\left[1 - F_{\chi^2_{b_j}}\left(\chi^2_{1-\alpha/2,b_j}\frac{a^2_{0j}x}{b_{0j}}\right) + F_{\chi^2_{b_j}}\left(\chi^2_{\alpha/2,b_j}\frac{a^2_{0j}x}{b_{0j}}\right)\right]^{-2} Q\, f_{\chi^2_{b_{0j}}}(x)\, dx, \quad [11]$$

Where

$$Q = \left[\frac{f_{\chi^2_{b_j}}\left(\chi^2_{1-\alpha/2,b_j}\frac{a^2_{0j}x}{b_{0j}}\right)}{2 f_{\chi^2_{b_j}}\left(\chi^2_{1-\alpha/2,b_j}\right)} + \frac{f_{\chi^2_{b_j}}\left(\chi^2_{\alpha/2,b_j}\frac{a^2_{0j}x}{b_{0j}}\right)}{2 f_{\chi^2_{b_j}}\left(\chi^2_{\alpha/2,b_j}\right)}\right]\frac{a^2_{0j}x}{b_{0j}}.$$

With this result we have all the pieces required to calculate an approximation to $\alpha(m, n)$ from Eq. [10]. Once $\alpha(m, n)$ is found, we can again use it to correct the charting constants for the Montgomery probability limits given earlier. The approximate values of $\alpha(m, n)$, $U_{n,\alpha(m,n),j}$, and $L_{n,\alpha(m,n),j}$, for each chart ($j = 1,2,3$), for different combinations of values of m, n and $ICARL_0 = 370$ and 500 values are tabulated in Table 3.

Note that this approximate result is more general than the provided numerical solutions. In fact, it can be generalized to any combination of Phase I and Phase II estimators. This can be done by determining the required constants a, b, a_0, and b_0 based on the Patnaik (1950) approximation, as described in steps (i) and (ii) of our approach. Moreover, any monotonic increasing function $g(\sigma)$ of σ can be considered, since in that case $P(LCL < T_j < UCL)$ is equivalent to $P(g(LCL) < g(T_j) < g(UCL))$. Hence, our approach can also be applied to S^2 and $\log(S)$ charts.

Comparing the analytical solutions with the numerical solutions it is seen that the approximations from the analytical method are quite accurate and the accuracy increases for higher values of m, as is desirable.

In order to compare the charting constants obtained by the numerical and the analytical methods, we calculated the $ICARL_j(m, n, \alpha(m, n))$ values for each chart, and $ICARL_0 = 370$; $n = 5$ and for various values of m. Table 4 shows the results including the *PD* values relative to 370. As expected, for the numerically calculated probability limits, the *ICARL* values are exactly equal to the nominal value 370. On the other hand, it can be seen that for the analytically calculated probability limits, except for $m = 5$, the *ICARL* values are not more than 6% above the nominal value 370. It can also be seen that as m increases, the *ICARL* values corresponding to the analytical constants converge quickly to 370. This shows that the behavior of the numerically and analytically corrected probability limits is similar.

A numerical illustration

In this section, we illustrate the *R* and *S* charts with the estimated 3-sigma limits, the uncorrected probability limits and the corrected probability limits given in this article. We use a data set from Montgomery (2013)

Table 5. Charting constants and control limits for the R chart when $n=5$; $m=5{,}25$; $ICARL_0=370$ and $\overline{R}=0.3252$.

m = 25	L	U	$LCL=L\overline{R}$	$UCL=U\overline{R}$	Width
3-sigma limits	0	2.1145	0	0.6876	0.6876
Uncorrected probability limits	0.1705	2.3119	0.0555	0.7518	0.6963
Corrected Probability limits (Analytical Approach)	0.1656	2.3294	0.0539	0.7575	0.7036
Corrected Probability limits (Numerical Approach)	0.1660	2.3278	0.0540	0.7570	0.7030
m = 5	L	U	$LCL=L\overline{R}$	$UCL=U\overline{R}$	Width
3-sigma limits	0	2.1145	0	0.6876	0.6876
Uncorrected probability limits	0.1705	2.3119	0.0555	0.7518	0.6963
Corrected Probability limits (Analytical approach)	0.1513	2.3832	0.0492	0.7750	0.7258
Corrected Probability limits (Numerical approach)	0.1569	2.3616	0.0511	0.7676	0.7165

on the measurements of the flow width of a hard bake process. This is a popular data set used in the literature. It contains $m=25$ Phase I subgroups, each of size $n=5$ where $\overline{R}=0.3252$, $\overline{S}=0.1316$, and $S_p=0.1390$. All the Phase II control limits were constructed to achieve the nominal $ICARL_0=370$. Tables 5, 6, and 7 show the calculated limits of the Phase II R and S charts together with their corresponding charting constants L and U. To examine the effect of the number of Phase I subgroups, these tables also include the case m = 5. The charting constants for the corrected limits have been taken from Table 3, while the charting constants for the 3-sigma limits and the uncorrected probability limits have been calculated using their formulas in Appendix A.

For $m=25$, it can be seen that the difference, in width, between the uncorrected and the corrected probability limits, is small. This is as expected, since

Table 6. Charting constants and control limits for the S chart when $n=5$; $m=5{,}25$; $ICARL_0=370$ and $\overline{S}=0.1316$.

$m=25$	L	U	$LCL=L\overline{S}$	$UCL=U\overline{S}$	Width
3-sigma limits	0	2.0890	0	0.2749	0.2749
Uncorrected probability limits	0.1730	2.2442	0.0225	0.2953	0.2728
Corrected Probability limits (Analytical Approach)	0.1680	2.2603	0.0221	0.2975	0.2754
Corrected Probability limits (Numerical Approach)	0.1685	2.2587	0.0222	0.2973	0.2751
$m=5$	L	U	$LCL=L\overline{S}$	$UCL=U\overline{S}$	Width
3-sigma limits	0	2.0890	0	0.2749	0.2749
Uncorrected probability limits	0.1730	2.2442	0.0225	0.2953	0.2728
Corrected Probability limits (Analytical Approach)	0.1513	2.3832	0.0199	0.3136	0.2937
Corrected Probability limits (Numerical Approach)	0.1593	2.2890	0.0210	0.3012	0.2802

Table 7. Charting constants and control limits for the S chart when $n=5$; $m=5{,}25$; $ICARL_0=370$ and $S_p=0.1390$.

$m=25$	L	U	$LCL=LS_p$	$UCL=US_p$	Width
3-sigma limits	0	1.9637	0	0.2732	0.2732
Uncorrected probability limits	0.1626	2.1096	0.0226	0.2935	0.2709
Corrected Probability limits (Analytical Approach)	0.1576	2.1258	0.0219	0.2957	0.2738
Corrected Probability limits (Numerical Approach)	0.1581	2.1240	0.0220	0.2955	0.2735
$m=5$	L	U	$LCL=LS_p$	$UCL=US_p$	Width
3-sigma limits	0	1.9637	0	0.2732	0.2732
Uncorrected probability limits	0.1626	2.1096	0.0226	0.2935	0.2709
Corrected Probability limits (Analytical Approach)	0.1419	2.1788	0.0197	0.3029	0.2832
Corrected Probability limits (Numerical Approach)	0.1489	2.1547	0.0207	0.2997	0.2790

the number of subgroups $m = 25$ is moderately large, and so it improves the performance of the uncorrected probability limits. In addition to this, it can also be seen that the corrected probability limits are a little wider than the uncorrected probability limits. This can be seen even more clearly from the tables constructed assuming $m = 5$. Since the problem with uncorrected probability limits is their high unconditional false alarm rate, widening these control limits helps alleviate the problem.

To summarize, the classical estimated 3-sigma limits should not be used in practice, because the normal approximation to the distribution of the sample range and sample standard deviation is poor. Consequently, as seen from Tables 5–7, the classical estimated 3-sigma limits cannot detect process improvements (only deterioration; since the lower control limit is set to be equal to 0 for subgroup sizes $n \leq 6$). The uncorrected estimated probability limits can still be used if m is large (say $m > 20$). However, it is better to use the corrected charting constants proposed in this article, because they guarantee the expected nominal *ICARL* performance for the value of m and n that one may have. Finally, the analytical method of finding the corrected charting constants is a good approximation to the numerical method.

Out of control performance

The numerical control limits provided here guarantee that the in-control average run-length of the charts is equal to the nominal value of 370 or 500. However, since the corrected limits are wider than the uncorrected probability limits, it is of interest to see whether the correction impacts the out-of-control performance. It may be noted at the outset that such a comparison is not really fair since the in-control performance of the uncorrected limits can be far worse than the nominal.

In order to make this comparison, we compute the ARL for the considered dispersion charts with $n = 5$, for a number of values of the ratio (λ) between the Phase II standard deviation (σ) and the in-control process standard deviation (σ_0), that is, for $\lambda = \sigma/\sigma_0$. In other words, we compute points of the *ARL* profiles of the charts in different cases, where $\lambda = 1$ corresponds to the *ICARL*, and $\lambda \neq 1$ corresponds to the out-of-control *ARL* (*OOCARL*). This is done for several values of m. The results are given in Table 8.

The *ARL*s with the uncorrected and the corrected limits could be easily computed from Eq. [3] by just replacing $CFAR_j$ by the general conditional probability of an alarm, CPA_j. Next, using $\lambda = \sigma/\sigma_0$, and keeping in mind that $\alpha = \alpha\ (m, n)$, we calculate CPA_j for the corrected limits as

$$
\begin{aligned}
CPA_j &= 1 - P\left(L_{n,\alpha,j}\frac{w_j}{\sigma} < \frac{T_{ij}}{\sigma} < U_{n,\alpha,j}\frac{w_j}{\sigma}\right)\\
&= 1 - P\left(L_{n,\alpha,j}\frac{\varepsilon_{0j}a_{0j}\sqrt{X_{0j}}}{\lambda\sqrt{b_{0j}}} < \frac{T_{ij}}{\sigma}\right.\\
&\qquad \left. < U_{n,\alpha,j}\frac{\varepsilon_{0j}a_{0j}\sqrt{X_{0j}}}{\lambda\sqrt{b_{0j}}}\right)\\
&= CFAR_j\left(X_{0j}, m, n, \alpha, \lambda\right).
\end{aligned}
$$

Using the known distributions of T_{ij}/σ this gives

$$
\begin{aligned}
&CPA_1\ (X_{01}, m, n, \alpha, \lambda)\\
&\quad = 1 - F_{W_n}\left(U_{n,\alpha,1}\frac{w_1}{\lambda\sigma_0}\right) + F_{W_n}\left(L_{n,\alpha,1}\frac{w_1}{\lambda\sigma_0}\right)\\
&\quad = 1 - F_{W_n}\left(F_{W_{n,1-\alpha/2}}\frac{a_{01}\sqrt{X_{01}}}{\lambda\sqrt{b_{01}}}\right)\\
&\qquad + F_{W_n}\left(F_{W_{n,\alpha/2}}\frac{a_{01}\sqrt{X_{01}}}{\lambda\sqrt{b_{01}}}\right),\\
&CPA_2\ (X_{02}, m, n, \alpha, \lambda)\\
&\quad = 1 - F_{\chi^2_{n-1}}\left(\left(U_{n,\alpha,2}\frac{w_2\sqrt{n-1}}{\lambda\sigma_0}\right)^2\right)\\
&\qquad + F_{\chi^2_{n-1}}\left(\left(L_{n,\alpha,2}\frac{w_2\sqrt{n-1}}{\lambda\sigma_0}\right)^2\right)\\
&\quad = 1 - F_{\chi^2_{n-1}}\left(\chi^2_{1-\alpha/2,n-1}\frac{a_{02}^2X_{02}}{\lambda^2 b_{02}}\right)\\
&\qquad + F_{\chi^2_{n-1}}\left(\chi^2_{\alpha/2,n-1}\frac{a_{02}^2X_{02}}{\lambda^2 b_{02}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&CPA_3\ (X_{03}, m, n, \alpha, \lambda)\\
&\quad = 1 - F_{\chi^2_{n-1}}\left(\left(U_{n,\alpha,3}\frac{w_3\sqrt{n-1}}{\lambda\sigma_0}\right)^2\right)\\
&\qquad + F_{\chi^2_{n-1}}\left(\left(L_{n,\alpha,3}\frac{w_3\sqrt{n-1}}{\lambda\sigma_0}\right)^2\right)\\
&\quad = 1 - F_{\chi^2_{n-1}}\left(\chi^2_{1-\alpha/2,n-1}\frac{a_{03}^2X_{03}}{\lambda^2 b_{03}}\right)
\end{aligned}
$$

$$+ F_{\chi^2_{n-1}}\left(\chi^2_{\alpha/2,n-1}\frac{a_{02}^2 X_{03}}{\lambda^2 b_{03}}\right).$$

These values can in turn be used as described in the numerical approach, to determine the unconditional ARL as

$$ARL_j\,(m, n, \alpha, \lambda) = \int_0^\infty \left[CPA_j\,(x, m, n, \alpha, \lambda)\right]^{-1} f_{\chi^2_{b_{0j}}}(x)\,dx,$$

where again x is the value of the random variable X_{0j}. Formulae for the CPA_j and ARL_j of the uncorrected limits are the same as above, except that α_0 is used instead of $\alpha(m, n)$.

Table 8 shows the *ICARL* ($\lambda = 1$), the *OOCARL* ($\lambda \neq 1$) and the *PD* values associated with the uncorrected and the corrected estimated probability limits based *R* and *S* charts, for $n = 5, 10$ and various values of λ and m. The *PD* values measure the percentage difference between the unconditional *ARL* values for the estimated σ_0 case and the nominal unconditional *ARL* values. Note that the results for the uncorrected estimated probability limits have been thoroughly discussed by Chen (1998). From Table 8 it can be seen that when the process is *IC* and σ_0 is estimated, using the uncorrected charting constants to construct the uncorrected probability limits gives unconditional *ARL* values that are up to 29% lower than the nominal 370 (corresponding to the σ_0 known case) for $n = 5$ and 32% *lower* than the nominal 370 (corresponding to the σ_0 known case) for $n = 10$, respectively. This means a lot of false alarms. Using the corrected (new) charting constants to construct the probability limits yields the nominal value 370, which is desirable. However, this also leads to larger unconditional ARL values for the corrected charts compared to the uncorrected charts when the process is *OOC*. Interestingly, this difference is smaller for decreases in variability ($\lambda < 1$) than for increases ($\lambda > 1$). In general, both the corrected and uncorrected charts have more difficulty in detecting decreases in variability than increases. It can also be seen that the effect of using either the corrected or uncorrected estimated probability limits is a function of m. In general, increasing m diminishes the effects of parameter estimation on both the *IC* and *OOC* unconditional *ARL* performance for both uncorrected and corrected probability limits, as expected.

To summarize, the corrected estimated probability limits provide a much better *IC* performance than the uncorrected limits, as it yields the nominally specified *ICARL* performance. However, this generally comes with a deterioration of the *OOCARL* performance relative to the uncorrected limits. Note that this tradeoff between IC and OOC performance can be altered by adjusting the value of $ICARL_0$.

Summary and conclusions

Shewhart control charts are often used to monitor process dispersion. However, the standard versions of these charts assume known in-control parameters, which is typically not the case in practice. When the parameters are estimated to set up the control limits, both the *IC* and *OOC* performance of the control charts are affected (Chen 1998). In this article, we have provided corrected control limit constants based on the *ICARL* performance of the probability limits based *R* and *S* charts, to account for the effects of parameter estimation. Two methods are used to find the corrected charting constants. The first method, the numerical approach, involves numerical integration and solving nonlinear equations. The second method, the analytical approach is based on a first-order Taylor approximation to the *ICARL*. Differences in the values obtained with these two methods are small, indicating that the analytical approximations are quite accurate. However, the analytical approach is more general in the sense that it can be applied to any desired estimator. Extensions to other functions of *S*, such as S^2 or log-*S* are straightforward.

The tabulated constants provided here ensure that the unconditional *ICARL* is equal to a pre-specified desired value, taking into account the estimators that are used, the number of Phase I subgroups (m) and the subgroup size (n). However, this *IC* robustness is achieved at the price of a deterioration (increase) in the unconditional *OOCARL*. This deterioration, due to the use of the corrected limits, is negligible for large values of m or large changes of variability.

In conclusion, this article provides the correct charting constants for the popular dispersion charts, for i.i.d. data from a normal distribution, properly accounting for the effects of parameter estimation, in terms of a specified nominal value of the unconditional in-control average run-length. A similar study, for the important case when $n = 1$, is required. Finally, note that in practice, it is possible that the data do not follow a normal distribution. How these corrected limits perform for other distributions and their required modifications

Table 8. The *ARL* and *PD* values associated with the uncorrected and corrected probability limits based *R* and *S* charts for $n = 5, 10$ and various values of λ and *m*.

R Chart with σ_0 estimator $\overline{R}/d_2(n)$

n	m	Type of limit	λ values 0.2		0.5		0.8		1		1.2		1.5		2	
			ARL	PD	ARL	PD	ARL	PD	ARL	PD	ARL	PD	ARL	PD	ARL	PD
5	5	Uncorrected	3	0	68	31	280	−9	269	−27	130	78	25	108	4	33
		Corrected	4	33	93	79	387	26	370	0	175	143	31	158	4	33
	10	Uncorrected	3	0	59	13	304	−1	302	−18	111	52	17	42	4	33
		Corrected	3	0	72	39	374	21	370	0	133	85	19	58	4	33
	20	Uncorrected	3	0	55	6	312	1	327	−11	94	29	14	17	3	0
		Corrected	3	0	62	19	354	15	370	0	104	42	15	25	3	0
	25	Uncorrected	3	0	54	4	313	2	334	−10	90	23	14	17	3	0
		Corrected	3	0	60	15	347	13	370	0	98	34	14	17	3	0
	30	Uncorrected	3	0	54	4	313	2	339	−8	87	19	13	8	3	0
		Corrected	3	0	59	13	342	11	370	0	93	27	14	17	3	0
	50	Uncorrected	3	0	53	2	312	1	349	−5	81	11	13	8	3	0
		Corrected	3	0	56	8	330	7	370	0	85	16	13	8	3	0
	100	Uncorrected	3	0	52	0	310	1	359	−3	76	4	12	0	3	0
		Corrected	3	0	54	4	320	4	370	0	78	7	13	8	3	0
	300	Uncorrected	3	0	52	0	309	0	366	−1	73	0	12	0	3	0
		Corrected	3	0	52	0	312	1	370	0	74	1	12	0	3	0
	500	Uncorrected	3	0	52	0	308	0	368	0	73	0	12	0	3	0
		Corrected	3	0	52	1	310	1	370	0	73	1	12	1	3	0
	1000	Uncorrected	3	0	52	0	308	0	369	0	72	0	12	0	3	0
		Corrected	3	0	52	0	309	0	370	0	72	0	12	0	3	0
	∞	σ *known*	3	0	52	0	308	0	370	0	72	0	12	0	3	0
10	5	Uncorrected	1	0	9	22	166	23	252	−32	94	84	11	57	2	0
		Corrected	1	0	12	71	243	80	370	0	132	159	13	86	2	0
	10	Uncorrected	1	0	8	14	158	17	289	−22	76	49	8	14	2	0
		Corrected	1	0	9	22	201	49	370	0	93	82	9	22	2	0
	20	Uncorrected	1	0	7	0	148	10	319	−14	64	25	8	14	2	0
		Corrected	1	0	8	14	171	27	370	0	72	41	8	14	2	0
	25	Uncorrected	1	0	7	0	146	8	326	−12	62	22	7	0	2	0
		Corrected	1	0	8	14	164	21	370	0	68	33	8	14	2	0
	30	Uncorrected	1	0	7	0	144	7	332	−10	60	18	7	0	2	0
		Corrected	1	0	8	14	160	19	370	0	65	27	8	14	2	0
	50	Uncorrected	1	0	7	0	140	4	345	−7	56	10	7	0	2	0
		Corrected	1	0	7	0	150	11	370	0	59	16	7	0	2	0
	100	Uncorrected	1	0	7	0	138	2	357	−4	54	6	7	0	2	0
		Corrected	1	0	7	0	142	5	370	0	55	8	7	0	2	0
	300	Uncorrected	1	0	7	0	136	1	365	−1	52	2	7	0	2	0
		Corrected	1	0	7	0	137	1	370	0	53	4	7	0	2	0
	500	Uncorrected	1	0	7	0	135	0	367	−1	52	2	7	0	2	0
		Corrected	1	0	7	0	136	1	370	0	52	2	7	0	2	0
	1000	Uncorrected	1	0	7	0	135	0	369	0	52	2	7	0	2	0
		Corrected	1	0	7	0	135	0	370	0	52	2	7	0	2	0
	∞	σ *known*	1		7		135		370		51		7		2	

S* Chart with σ_0 estimator *Sp

n	m	Type of limit	λ values 0.2		0.5		0.8		1		1.2		1.5		2	
			ARL	PD	ARL	PD	ARL	PD	ARL	PD	ARL	PD	ARL	PD	ARL	PD
5	5	Uncorrected	3	0	69	35	290	−6	264	−29	114	75	19	73	3	0
		Corrected	4	33	97	90	410	33	370	0	155	138	24	118	4	33
	10	Uncorrected	3	0	59	16	312	1	298	−19	98	51	14	27	3	0
		Corrected	3	0	73	43	388	26	370	0	117	80	16	45	3	0
	20	Uncorrected	3	0	55	8	317	3	325	−12	83	28	12	9	3	0
		Corrected	3	0	62	22	361	17	370	0	92	42	13	18	3	0
	25	Uncorrected	3	0	54	6	317	3	332	−10	80	23	12	9	3	0
		Corrected	3	0	60	18	353	15	370	0	87	34	12	9	3	0
	30	Uncorrected	3	0	54	6	316	3	337	−9	77	18	12	9	3	0
		Corrected	3	0	59	16	347	13	370	0	83	28	12	9	3	0
	50	Uncorrected	3	0	53	4	314	2	348	−6	72	11	11	0	3	0
		Corrected	3	0	56	10	333	8	370	0	75	15	11	0	3	0

(*Continued on next page*)

Table 8. (Continued)

S Chart with σ_0 estimator $\overline{S}/c_4(n)$

n	m	Type of limit	λ values 0.2 ARL	0.2 PD	0.5 ARL	0.5 PD	0.8 ARL	0.8 PD	1 ARL	1 PD	1.2 ARL	1.2 PD	1.5 ARL	1.5 PD	2 ARL	2 PD
	100	Uncorrected	3	0	52	2	311	1	358	−3	68	5	11	0	3	0
		Corrected	3	0	54	6	321	4	370	0	70	8	11	0	3	0
	300	Uncorrected	3	0	52	2	309	0	366	−1	66	2	11	0	3	0
		Corrected	3	0	52	2	313	2	370	0	66	2	11	0	3	0
	500	Uncorrected	3	0	52	2	309	0	368	−1	65	0	11	0	3	0
		Corrected	3	0	52	2	311	1	370	0	65	0	11	0	3	0
	1000	Uncorrected	3	0	51	0	309	0	369	0	65	0	11	0	3	0
		Corrected	3	0	52	2	309	0	370	0	65	0	11	0	3	0
	∞	*σ known*	3	0	51	0	308	0	370	0	65	0	11	0	3	0
10	5	Uncorrected	1	0	8	33	168	27	252	−32	71	92	6	20	2	0
		Corrected	1	0	11	83	246	86	370	0	99	168	8	60	2	0
	10	Uncorrected	1	0	7	17	156	18	289	−22	56	51	6	20	2	0
		Corrected	1	0	8	33	199	51	370	0	67	81	6	20	2	0
	20	Uncorrected	1	0	7	17	145	10	319	−14	46	24	5	0	2	0
		Corrected	1	0	7	17	167	27	370	0	51	38	5	0	2	0
	25	Uncorrected	1	0	7	17	142	8	326	−12	44	19	5	0	2	0
		Corrected	1	0	7	17	160	21	370	0	48	30	5	0	2	0
	30	Uncorrected	1	0	6	0	140	6	332	−10	43	16	5	0	2	0
		Corrected	1	0	7	17	156	18	370	0	46	24	5	0	2	0
	50	Uncorrected	1	0	6	0	137	4	345	−7	40	8	5	0	2	0
		Corrected	1	0	7	17	146	11	370	0	42	14	5	0	2	0
	100	Uncorrected	1	0	6	0	134	2	357	−4	39	5	5	0	2	0
		Corrected	1	0	6	0	139	5	370	0	40	8	5	0	2	0
	300	Uncorrected	1	0	6	0	132	0	365	−1	37	0	5	0	2	0
		Corrected	1	0	6	0	134	2	370	0	38	3	5	0	2	0
	500	Uncorrected	1	0	6	0	132	0	367	−1	37	0	5	0	2	0
		Corrected	1	0	6	0	133	1	370	0	37	0	5	0	2	0
	1000	Uncorrected	1	0	6	0	132	0	369	0	37	0	5	0	2	0
		Corrected	1	0	6	0	132	0	370	0	37	0	5	0	2	0
	∞	*σ known*	1		6		132		370		37		5		2	

S Chart with σ_0 estimator $\overline{S}/c_4(n)$

n	m	Type of limit	λ values 0.2 ARL	0.2 PD	0.5 ARL	0.5 PD	0.8 ARL	0.8 PD	1 ARL	1 PD	1.2 ARL	1.2 PD	1.5 ARL	1.5 PD	2 ARL	2 PD
5	5	Uncorrected	3	0	67	31	283	−8	270	−27	125	92	22	100	4	33
		Corrected	4	33	91	78	390	27	370	0	167	157	27	145	4	33
	10	Uncorrected	3	0	58	14	306	−1	302	−18	104	60	15	36	3	0
		Corrected	3	0	71	39	376	22	370	0	124	91	17	55	3	0
	20	Uncorrected	3	0	55	8	314	2	328	−11	87	34	12	9	3	0
		Corrected	3	0	61	20	355	15	370	0	96	48	13	18	3	0
	25	Uncorrected	3	0	54	6	314	2	334	−10	83	28	12	9	3	0
		Corrected	3	0	60	18	348	13	370	0	89	37	13	18	3	0
	30	Uncorrected	3	0	54	6	314	2	339	−8	80	23	12	9	3	0
		Corrected	3	0	58	14	343	11	370	0	85	31	12	9	3	0
	50	Uncorrected	3	0	53	4	313	2	350	−5	74	14	11	0	3	0
		Corrected	3	0	56	10	331	7	370	0	77	18	11	0	3	0
	100	Uncorrected	3	0	52	2	311	1	359	−3	69	6	11	0	3	0
		Corrected	3	0	54	6	320	4	370	0	70	8	11	0	3	0
	300	Uncorrected	3	0	52	2	309	0	366	−1	66	2	11	0	3	0
		Corrected	3	0	52	2	312	1	370	0	66	2	11	0	3	0
	500	Uncorrected	3	0	52	2	309	0	368	0	65	0	11	0	3	0
		Corrected	3	0	52	2	310	1	370	0	66	2	11	0	3	0
	1000	Uncorrected	3	0	51	0	308	0	369	0	65	0	11	0	3	0
		Corrected	3	0	52	2	309	0	370	0	65	0	11	0	3	0
	∞	*σ known*	3	0	51	0	308	0	370	0	65	0	11	0	3	0
10	5	Uncorrected	1	0	8	33	163	23	254	−31	76	105	7	40	2	0
		Corrected	1	0	11	83	237	80	370	0	105	184	8	60	2	0

Table 8. (Continued)

			S Chart with σ_0 estimator *Sp*													
			λ values													
n	**m**	**Type of limit**	**0.2**		**0.5**		**0.8**		**1**		**1.2**		**1.5**		**2**	
			ARL	**PD**	**ARL**	**PD**	**ARL**	**PD**	**ARL**	**PD**	**ARL**	**PD**	**ARL**	**PD**	**ARL**	**PD**
	10	**Uncorrected**	1	0	7	17	154	17	291	−21	58	57	6	20	2	0
		Corrected	1	0	8	33	197	49	370	0	70	89	6	20	2	0
	20	**Uncorrected**	1	0	7	17	144	9	320	−14	47	27	5	0	2	0
		Corrected	1	0	7	17	165	25	370	0	52	41	5	0	2	0
	25	**Uncorrected**	1	0	6	0	141	7	327	−12	45	22	5	0	2	0
		Corrected	1	0	7	17	159	20	370	0	49	32	5	0	2	0
	30	**Uncorrected**	1	0	6	0	140	6	333	−10	44	19	5	0	2	0
		Corrected	1	0	7	17	154	17	370	0	47	27	5	0	2	0
	50	**Uncorrected**	1	0	6	0	136	3	345	−7	41	11	5	0	2	0
		Corrected	1	0	7	17	145	10	370	0	43	16	5	0	2	0
	100	**Uncorrected**	1	0	6	0	134	2	357	−4	39	5	5	0	2	0
		Corrected	1	0	6	0	139	5	370	0	40	8	5	0	2	0
	300	**Uncorrected**	1	0	6	0	132	0	366	−1	38	3	5	0	2	0
		Corrected	1	0	6	0	134	2	370	0	38	3	5	0	2	0
	500	**Uncorrected**	1	0	6	0	132	0	367	−1	37	0	5	0	2	0
		Corrected	1	0	6	0	133	1	370	0	37	0	5	0	2	0
	1000	**Uncorrected**	1	0	6	0	132	0	369	0	37	0	5	0	2	0
		Corrected	1	0	6	0	132	0	370	0	37	0	5	0	2	0
	∞	σ *known*	1		6		132		370		37		5		2	

require further investigation, which will be considered elsewhere.

About the authors

Mandla D. Diko obtained his M.Sc. in Statistics from the University of Pretoria. He is currently working as a Ph.D. student in the Department of Operations Management of the University of Amsterdam, the Netherlands. His research interests are in Statistical Process Quality Control.

Rob Goedhart obtained his M.Sc. in econometrics in 2014 at the University of Amsterdam, the Netherlands. He is a Ph.D. student in the Department of Operations Management and consultant at the Institute for Business and Industrial Statistics of the University of Amsterdam, the Netherlands. His current research topic is control charting techniques with estimated parameters.

Subha Chakraborti is a professor of Statistics in the Department of Information Systems, Statistics and Management Science, University of Alabama, USA. He is a Fellow of the ASA and an elected member of the ISI. His research interests are in Nonparametric and Robust Statistical Inference, including applications in Statistical Process Quality Control.

Ronald J. M. M. Does is Professor of Industrial Statistics at the University of Amsterdam, Director of the Institute for Business and Industrial Statistics, Head of the Department of Operations Management, and Director of the Institute of Executive Programmes at the Amsterdam Business School. He is a Fellow of the ASQ and ASA, an elected member of the ISI, and an Academician of the International Academy for Quality. His current research activities include the design of control charts for nonstandard situations, healthcare engineering and operational management methods.

Eugenio K. Epprecht is Professor in the Department of Industrial Engineering at the Pontifical Catholic University of Rio de Janeiro. He is a member of ASQ. His current research activities include the design of control charts with estimated parameters, and industrial engineering.

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Appendix A

Table A1. Formulae for the various constants that are associated with the three spread charts that are considered in this article.

	R chart with σ_0 estimator $\bar{R}/d_2(n)$ ($j=1$)	S chart with σ_0 estimator $\bar{S}/c_4(n)$ ($j=2$)	S chart with σ_0 estimator S_p ($j=3$)
Values of the biased estimator w_j	$w_1 = \bar{R}$	$w_2 = \bar{S}$	$w_3 = S_p$
Values of the unbiasing constants ε_{0j} and ε_j	$\varepsilon_{01} = d_2(n)$ $\varepsilon_1 = d_2(n)$	$\varepsilon_{02} = c_4(n)$ $\varepsilon_2 = c_4(n)$	$\varepsilon_{03} = c_4(m(n-1)+1) \approx 1$ $\varepsilon_3 = c_4(n)$
Values of the unbiased estimator $\hat{\sigma}_{0j}$	$\frac{w_1}{\varepsilon_{01}}$	$\frac{w_2}{\varepsilon_{02}}$	$\frac{w_3}{\varepsilon_{03}}$
Charting statistic T_{ij}	$R \sim W\sigma$	$S \sim \frac{\sqrt{\chi^2_{n-1}}}{\sqrt{n-1}}\sigma$	$S \sim \frac{\sqrt{\chi^2_{n-1}}}{\sqrt{n-1}}\sigma$
Uncorrected probability limits	$L_{n,\alpha,1} = \frac{F_{W_{\alpha/2}}}{\varepsilon_{01}}$ $U_{n,\alpha,1} = \frac{F_{W_{1-\alpha/2}}}{\varepsilon_{01}}$	$L_{n,\alpha,2} = \frac{\sqrt{\chi^2_{\alpha/2,n-1}}}{\varepsilon_{02}\sqrt{n-1}}$ $U_{n,\alpha,2} = \frac{\sqrt{\chi^2_{1-\alpha/2,n-1}}}{\varepsilon_{02}\sqrt{n-1}}$	$L_{n,\alpha,3} = \frac{\sqrt{\chi^2_{\alpha/2,n-1}}}{\varepsilon_{03}\sqrt{n-1}}$ $U_{n,\alpha,3} = \frac{\sqrt{\chi^2_{1-\alpha/2,n-1}}}{\varepsilon_{03}\sqrt{n-1}}$
3-sigma limits (c.f. Montgomery (2013))	$L_{n,\alpha,1} = (1 - 3\frac{d_3(n)}{\varepsilon_1})$ $U_{n,\alpha,1} = (1 + 3\frac{d_3(n)}{\varepsilon_1})$	$L_{n,\alpha,2} = (1 - 3\frac{\sqrt{1-c_4^2(n)}}{\varepsilon_2})$ $U_{n,\alpha,2} = (1 + 3\frac{\sqrt{1-c_4^2(n)}}{\varepsilon_2})$	$L_{n,\alpha,3} = (1 - 3\frac{\sqrt{1-c_4^2(n)}}{\varepsilon_3})$ $U_{n,\alpha,3} = (1 + 3\frac{\sqrt{1-c_4^2(n)}}{\varepsilon_3})$
Chi approximation of T_{ij}	$R \sim \frac{\varepsilon_1 a_1 \sqrt{\chi^2_{b_1}}}{\sqrt{b_1}}$	$S \sim \frac{\varepsilon_2 a_2 \sqrt{\chi^2_{b_2}}}{\sqrt{b_2}}$	$S \sim \frac{\varepsilon_3 a_3 \sqrt{\chi^2_{b_3}}}{\sqrt{b_3}}$
Chartings Constants $L_{n,\alpha,j}, U_{n,\alpha,j}$ Based on the chi approximation of T_{ij}	$L_{n,\alpha,1} = \frac{\varepsilon_1 a_1 \sqrt{\chi^2_{\alpha/2,b_1}}}{\varepsilon_{01}\sqrt{b_1}}$ $U_{n,\alpha,1} = \frac{\varepsilon_1 a_1 \sqrt{\chi^2_{1-\alpha/2,b_1}}}{\varepsilon_{01}\sqrt{b_1}}$	$L_{n,\alpha,2} = \frac{\varepsilon_2 a_2 \sqrt{\chi^2_{\alpha/2,b_2}}}{\varepsilon_{02}\sqrt{b_2}}$ $U_{n,\alpha,2} = \frac{\varepsilon_2 a_2 \sqrt{\chi^2_{1-\alpha/2,b_2}}}{\varepsilon_{02}\sqrt{b_2}}$	$L_{n,\alpha,3} = \frac{\varepsilon_3 a_3 \sqrt{\chi^2_{\alpha/2,b_3}}}{\varepsilon_{03}\sqrt{b_3}}$ $U_{n,\alpha,3} = \frac{\varepsilon_3 a_3 \sqrt{\chi^2_{1-\alpha/2,b_3}}}{\varepsilon_{03}\sqrt{b_3}}$
Variance ($\boldsymbol{V}_{0j}$) of standardized unbiased estimator $\hat{\sigma}_{0j}/\sigma$	$V_{01} = \text{Var}(\frac{\bar{R}}{d_2(n)\sigma}) = \frac{d_3^2(n)}{md_2^2(n)}$	$V_{02} = \text{Var}(\frac{\bar{S}}{c_4(n)\sigma}) = \frac{1-c_4^2(n)}{mc_4^2(n)}$	$V_{03} \approx \text{Var}(\frac{S_p}{\sigma}) = 1 - c_4^2(m(n-1)+1)$
Patnaik and Chen $\boldsymbol{r}_{0j}$ and $\boldsymbol{t}_{0j}$	$r_{01} = (-2 + 2\sqrt{1+2V_{01}})^{-1}$ $t_{01} = V_1 + \frac{1}{16r_{01}^3}$	$r_{02} = (-2 + 2\sqrt{1+2V_{02}})^{-1}$ $t_{01} = V_2 + \frac{1}{16r_{02}^3}$	Not required (distribution is exact)
Value of b_{0j}	$b_{01} = (-2 + 2\sqrt{1+2V_{01}})^{-1}$	$b_{02} = (-2 + 2\sqrt{1+2V_{02}})^{-1}$	$b_{03} = m(n-1)$
Value of a_{0j}	$a_{01} = 1 + \frac{1}{4b_{01}} + \frac{1}{32b_{01}^2} - \frac{5}{128b_{01}^3}$	$a_{02} = 1 + \frac{1}{4b_{02}} + \frac{1}{32b_{02}^2} - \frac{5}{128b_{02}^3}$	$a_{03} = 1$
Variance ($\boldsymbol{V}_j$) of standardized charting statistic $\boldsymbol{T}_j/(\boldsymbol{\varepsilon}_j\boldsymbol{\sigma})$	$V_1 = \text{Var}(\frac{R}{d_2(n)\sigma}) = \frac{d_3^2}{d_2^2}$	$V_2 = \text{Var}(\frac{S}{c_4(n)\sigma}) = \frac{1-c_4^2(n)}{c_4^2(n)}$	$V_3 = \text{Var}(\frac{S}{c_4(n)\sigma}) = \frac{1-c_4^2(n)}{c_4^2(n)}$
Patnaik and Chen $\boldsymbol{r}_j$ and $\boldsymbol{t}_j$	$r_1 = (-2 + 2\sqrt{1+2V_1})^{-1}$ $t_1 = V_1 + \frac{1}{16r_1^3}$	Not required (distribution is exact)	Not required (distribution is exact)
Value of b_j	$b_1 = (-2 + 2\sqrt{1+2V_1})^{-1}$	$b_2 = n-1$	$b_3 = n-1$
Value of a_j	$a_1 = 1 + \frac{1}{4b_1} + \frac{1}{32b_1^2} - \frac{5}{128b_1^3}$	$a_2 = 1/c_4(n)$	$a_3 = 1/c_4(n)$

$$c_4(n) = \left(\frac{2}{n-1}\right)^{1/2} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}$$

Let $W = \frac{R_i}{\sigma}$ and F_{W_n} denote the sample relative range and its distribution function, respectively, then

$$d_2(n) = E(W) = \int_{-\infty}^{\infty} \left(1 - F_{W_n}(w)\right) dw$$

$$E\left(W^2\right) = \int_0^{\infty} \left(1 - F_{W_n}(w)\right) dw^2 = 2\int_0^{\infty} w\left(1 - F_{W_n}(w)\right) dw$$

$$d_3(n) = \sqrt{Var(W)} = \sqrt{E(W^2) - d_2^2(n)}$$

Appendix B

Shewhart R chart with estimator Rbar/d2

```
ICARL0=c(370,500)
n=c(5,10)
m=c(5,10,20,25,30,50,100,300,500,1000)
dd=seq(from=0.001250,to=0.002697,length.out=1250)

library(cubature)
ICARL=function(m,a,n){uclrchart=function(x){qtukey(1-a/2,n,Inf)*aa(m)*sqrt(x)/(sqrt(bb(m)))} # UCL
lclrchart=function(x){qtukey(a/2,n,Inf)*aa(m)*sqrt(x)/(sqrt(bb(m)))} # LCL
PNSrchart=function(x){ptukey(uclrchart(x),n,Inf)-ptukey(lclrchart(x),n,Inf)}AFAR=function(x){1-PNSrchart(x)}
CFAR=function(x){ AFAR(x)^-1*dchisq(x,bb(m))}
b=qchisq(0.99999,bb(m))
adaptIntegrate(CFAR,c(0),c(b),tol=1e-10)[[1]]}

dim1=c("m=5","m=10","m=20","m=25","m=30","m=50","m=100","m=300","m=500","m=1000")
dim2 <- c("alpha", "L", "U")
dim3=c("n=5","n=10")
dim4= c("ICARL0=370","ICARL0=500")
Const=  array(1:120, c(10, 3, 2,2), dimnames=list(dim1, dim2, dim3,dim4))
for(z in 1:length(ICARL0)){
for (l in 1:length(n)){
for (j in 1:length(m)){
for (i in 1:length(dd)){
d2=function(n){
pt=function(w){1-ptukey(w,n,Inf)}
integrate(pt,lower=0,upper=Inf)[[1]]}
d2=d2(n[l])
EW2=function(n){
ptt=function(a){(1-ptukey(sqrt(a),n,Inf))}
integrate(ptt,lower=0,upper=Inf)[[1]]}
d3=function(n){
sqrt(EW2(n)-d2^2)}
d3=d3(n[l])
M=function(m){
d3^2/(m*d2^2)}
r=function(m){
(-2+2*sqrt(1+2*M(m)))^-1}
t=function(m){
M(m)+1/(16*r(m)^3)}
bb=function(m){
(-2+2*sqrt(1+2*t(m)))^-1}
aa=function(m){
1+(1/(4*bb(m)))+(1/(32*bb(m)^2))-(5/(128*bb(m)^3))}
if (ICARL(m[j],dd[i],n[l])<ICARL0[z]) break}
Const[j,,l,z]= c(dd[i], qtukey((dd[i]/2),n[l],Inf)/d2, qtukey(1-(dd[i]/2),n[l],Inf)/d2)
}}}
Const
write.table(Const, "clipboard",sep="\t",col.names=NA)
```

Figure B1. R codes for Table 3 (numerical approach).