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Canonical Quantization of Chiral Bosons

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We perform the canonical quantization for a system of non-Abelian chiral bosons. We show that unlike in the Wess-Zumino-Witten model, the left- and right-handed current densities do not simultaneously span Kac-Moody algebras in the quantum theory. At the critical values of the coupling constants, only one Kac-Moody algebra results.

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Chiral bosons in two dimensions are an important ingredient for heterotic string theories. The action principle[1,2] and quantization[3] of Abelian chiral bosons have recently been given in the literature. The situation is not as complete for non-Abelian chiral bosons. In the Hamilton formalism, the theory should consist of a single Kac-Moody algebra. A Lagrangian for the system was recently proposed by Sonnenschein[4]; it is the non-Abelian generalization of the system of Floreanini and Jackiw.[2] In this Letter, we perform a canonical quantization of this Lagrangian and show that it leads to the desired Hamiltonian description.

The relevant action of Ref. 4 is

$$S[g]=-\frac{\sqrt{2}\lambda}{4\pi}\int_{M^2}d^2x\,\mathrm{Tr}(\partial_- g\,\partial_1 g^{-1})+S_{\mathrm{WZ}}[g]\,, \quad (1)$$

$$S_{\mathrm{WZ}}[g]=\frac{n}{12\pi}\int_{D^3}\mathrm{Tr}(g^{-1}dg)^3\,, \quad (2)$$

where $g=g(x)$ takes values in a semisimple Lie group G, $\partial_-=(1/\sqrt{2})(\partial_0-\partial_1)$, and the boundary ∂D^3 of the three-dimensional disk D^3 is two-dimensional space-time M^2 whose coordinates are given by $x=(x^0,x^1)$. Semiclassical arguments require n to be an integer.[5] Initially, we shall assume that the coefficients λ and n are unrelated. It is the intent of this Letter to show that a Kac-Moody algebra results only when the coupling constants take the critical values $\lambda=\pm n$. It is plausible that a consistent quantum theory is only possible for these couplings. In a recent work, Rai and Rodgers[6] utilized a geometrical construction due to Balachandran[7] to show that actions like that of Eq. (1) with $\lambda=\pm n$ can be written as the integral of the symplectic two-form on the space of coadjoint orbits of the Kac-Moody group. It is then perhaps natural to expect that the Kac-Moody algebra appears in a canonical quantization.

Because the Wess-Zumino action (2) is invariant under diffeomorphisms, $S[g]$ possesses the same space-time symmetries as the corresponding Abelian action obtained in Ref. 2 [cf. Eq. (20)]. Both are invariant under $x^a\to x'^a=L^a{}_b x^b$ where $[L^a{}_b]$ has two solutions,

$$\begin{pmatrix}\mu & \mu^{-1}-\mu\\ 0 & \mu^{-1}\end{pmatrix}\ \text{and}\ \begin{pmatrix}\nu & \nu^{-1}-\nu\\ \nu & -\nu\end{pmatrix},$$

μ and ν being real parameters.

The system described by Eq. (1) as well as the Wess-Zumino-Witten (WZW) model[5] are obtained from the reparametrization-invariant action

$$-\frac{\lambda}{4\pi}\int_{M^2}d^2x\,\sqrt{\gamma}\gamma^{ab}\mathrm{Tr}\partial_a g\,\partial_b g^{-1}+S_{\mathrm{WZ}}[g]\,,$$

where γ is the space-time metric. The latter results after substitution of the conformal gauge, while Eq. (1) is obtained from

$$[\gamma_{ab}]\sim\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}.$$

The two gauges are not equivalent since, as we shall demonstrate, they describe physically distinct systems.

To obtain the equations of motion for this system, we perform the variation $\delta g=i\epsilon g$ in the action (1), where ϵ is an infinitesimal Lie algebra element. As a result of this variation,

$$0=(\lambda+n)\partial_1(\partial_0 g g^{-1})+(\lambda-n)\partial_0(\partial_1 g g^{-1})-2\lambda\partial_1(\partial_1 g g^{-1})\,. \quad (3)$$

This equation of motion simplifies at the critical values

$$\partial_1(\partial_- g g^{-1})=\partial_-(g^{-1}\partial_1 g)=0\,,\ \text{for}\ \lambda=n\,, \quad (4)$$

$$\partial_1(g^{-1}\partial_- g)=\partial_-(\partial_1 g g^{-1})=0\,,\ \text{for}\ \lambda=-n\,. \quad (5)$$

The former (latter) equation implies that the current J_R^μ (J_L^μ) is conserved, where

$$J_R^0=-J_R^1=\frac{in}{2\pi}g^{-1}\partial_1 g\,,\quad J_L^0=-J_L^1=\frac{in}{2\pi}\partial_1 g g^{-1}\,.$$

In addition to a simplification of the equations of motion, the imposition of the critical values implies that

$S[g]$ is invariant under the following transformations:

$$g \to h_1(x^0) g h_2(x^+)^{-1} \quad \text{for } \lambda = n\,, \tag{6}$$

$$g \to h_3(x^+) g h_4(x^0)^{-1} \quad \text{for } \lambda = -n\,, \tag{7}$$

where h_1, h_2, h_3, and h_4 take values in G.

We note that loop-group transformations (6) and (7) are functions of x^+ and x^0, whereas in the WZW model (at the critical values) the analogous loop-group symmetry transformations are functions of the two light-cone variables x^+ and x^-. In the WZW model, a canonical quantization of the system (with a Hamiltonian H generating evolution in x^0) shows the presence of two independent Kac-Moody algebras. These algebras are spanned by the left- and right-handed current densities J_L^0 and J_R^0. It is generally assumed that these current densities completely span the phase space.

On the other hand, for the system defined by the action (1), a canonical quantization at the critical values shows the existence of just one Kac-Moody algebra. It is associated with a single non-Abelian current density, either J_L^0 or J_R^0 (the choice depending on the coupling constants). The action (1) differs from the WZW-model action in that it is linear in time (x^0) derivatives. As a result, constraints exist in the Hamiltonian formulation of the theory which effectively cut the number of phase-space variables in half. It is plausible that the reduced phase space is completely spanned by either J_L^0 or J_R^0.

Before proceeding with the canonical formalism, we first need an expression for the Lagrangian density associated with the action (2). It is known that there exists no global Lagrangian density associated with $S_{\rm WZ}$. At best, we can define a Lagrangian density L_u (associated with $S_{\rm WZ}$) on local coordinate patches u of the group manifold G. On any such coordinate patch, $S_{\rm WZ}$ is the integral of an exact three-form. Following Witten,[5] we can thus take L_u integrated over coordinate patch u to be the integral of a two-form A which is a local function of $g \in G$. Let the group elements g be parametrized by angles ζ^i. Then

$$\int L_u\, d^2x = \int_u A = \tfrac{1}{2} \int_u A_{ij}(\zeta)\, d\zeta^i \wedge d\zeta^j\,. \tag{8}$$

$A_{ij}(\zeta) = -A_{ji}(\zeta)$ can be regarded as antisymmetric potentials on G. The associateds Kalb-Ramond fields are

$$F_{ijk} \equiv \frac{\partial}{\partial \zeta^i} A_{jk} + \frac{\partial}{\partial \zeta^j} A_{ki} + \frac{\partial}{\partial \zeta^k} A_{ij}\,. \tag{9}$$

These fields, as well as the action (8), are invariant under

$$A_{ij} \to A_{ij} + \frac{\partial}{\partial \zeta^i} \Lambda_i\,, \quad \Lambda_i = \Lambda_i(\zeta)\,. \tag{10}$$

The values of F_{ijk} can be determined by comparing variations of (2) with (8). Upon varying the Wess-Zumino action (2), we find

$$\begin{aligned} \delta S_{\rm WZ}[g] &= \frac{n}{4\pi} \int_{D^3} {\rm Tr}(dg\,g^{-1})^2 d(\delta g\,g^{-1}) = \frac{n}{4\pi} \int_{M^2} {\rm Tr}\, dg\,g^{-1} d(\delta g\,g^{-1}) \\ &= \frac{n}{4\pi} \int_{M^2} {\rm Tr} \left(\frac{\partial g}{\partial \zeta^i} g^{-1} \frac{\partial g}{\partial \zeta^j} g^{-1} \frac{\partial g}{\partial \zeta^k} g^{-1} \right) d\zeta^i \wedge d\zeta^j \delta\zeta^k\,, \end{aligned} \tag{11}$$

where variations of (8) give

$$\delta \int L_u\, d^2x = \tfrac{1}{2} \int_u F_{ijk}\, d\zeta^i \wedge d\zeta^j \delta\zeta^k\,. \tag{12}$$

Thus, we may make the identification

$$F_{jik} = \frac{n}{4\pi} {\rm Tr} \left(\frac{\partial g}{\partial \zeta^i} g^{-1} \left[\frac{\partial g}{\partial \zeta^j} g^{-1}, \frac{\partial g}{\partial \zeta^k} g^{-1} \right] \right). \tag{13}$$

This expression is valid globally, i.e., over the entire group manifold G, in contrast to A_{ij} which can only be expressed over local coordinate patches u of G.

The total Lagrangian density for the system (1) is

$$L = \frac{\sqrt{2}\lambda}{4\pi} {\rm Tr}(g^{-1}\partial_- g\, g^{-1}\partial_1 g) + A_{ij}(\zeta)\partial_0\zeta^i \partial_1\zeta^j\,. \tag{14}$$

We define canonical momenta π_i in the usual way

$$\pi_i = \frac{\delta L}{\delta \partial_0 \zeta_i}\,. \tag{15}$$

The canonical momenta are not invariant under gauge transformations (10). To rectify this situation, we define the gauge-invariant momenta $v_i = \pi_i - A_{ij}\partial_1\zeta^j$. Poisson brackets involving v_i are

$$\{\zeta^i(x^1), v_j(x'^1)\} = \delta^i_j\, \delta(x^1 - x'^1)\,, \tag{16}$$

$$\{v_i(x^1), v_j(x'^1)\} = F_{ijk}\, \partial_1\zeta^k \delta(x^1 - x'^1)\,. \tag{17}$$

The momenta are subject to the primary constraints

$$\phi_i \equiv v_i - \frac{\lambda}{4\pi} {\rm Tr} g^{-1} \frac{\partial g}{\partial \zeta^i} g^{-1} \partial_1 g \approx 0\,. \tag{18}$$

Since the Lagrangian is linear in time derivatives of ζ^i, the Hamiltonian is a linear combination of constraints ϕ_i, $H = \int dx^1 \rho^i(x^1)\phi_i(x^1)$. The condition that $\{\phi_i(x^1), H\} \approx 0$ does not lead to secondary constraints, but rather restricts the multiplier ρ^i. Thus, Eq. (18) yields the complete set of constraints on ζ^i and v_i.

We now perform a change in variables. From the coordinates and momenta, we can construct two sets of variables $\{R_i\}$ and $\{L_i\}$ whose Poisson brackets yield two

Kac-Moody algebras. The construction is as follows:

$$R_i \equiv M_{ij}^{-1} v_j + \frac{in}{4\pi}(g^{-1}\partial_1 g)_i \,, \tag{19}$$

$$L_i \equiv -N_{ij}^{-1} v_j + \frac{in}{4\pi}(\partial_1 g g^{-1})_i \,, \tag{20}$$

where $(g^{-1}\partial_1 g)_i \equiv \mathrm{Tr}(T_i g^{-1}\partial_1 g)$ and $(\partial_1 g g^{-1})_i \equiv \mathrm{Tr}(T_i \,\partial_1 g g^{-1})$, T_i being generators of G normalized such that $[T_i,T_j]=ic_{ij}^k T_k$ and $\mathrm{Tr}T_iT_j=\delta_{ij}$. The matrices M_{ij} and N_{ij} are defined via the relations

$$g^{-1}\frac{\partial g}{\partial \zeta^i} = iM_{ij}T_j \,, \quad N_{ik}=M_{ij}\mathrm{Tr}(T_k g T_j g^{-1}). \tag{21}$$

M_{ij} and N_{ij} are shown to be nonsingular in Ref. 8. From the Poisson bracket (16), it is easy to check that R_i (L_i) generate right (left) translations on $g(\zeta)$,

$$\{g(x^1),R_i(x'^1)\} = ig(x^1)T_i\,\delta(x^1-x'^1)\,, \tag{22}$$

$$\{g(x^1),L_i(x'^1)\} = -iT_i g(x^1)\delta(x^1-x'^1)\,. \tag{23}$$

From (16) and (17), we further find

$$\{R_i(x^1),R_j(x'^1)\} = c_{ij}^k R_k(x^1)\delta(x^1-x'^1) - \frac{n}{4\pi}\delta_{ij}\left(\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x'^1}\right)\delta(x^1-x'^1)\,, \tag{24}$$

$$\{L_i(x^1),L_j(x'^1)\} = c_{ij}^k L_k(x^1)\delta(x^1-x'^1) + \frac{n}{4\pi}\delta_{ij}\left(\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x'^1}\right)\delta(x^1-x'^1)\,, \tag{25}$$

$$\{R_i(x^1),L_j(x'^1)\} = 0\,, \tag{26}$$

corresponding to two Kac-Moody algebras.

In the WZW model $\{R_i\}$ and $\{L_i\}$ span two independent Kac-Moody algebras. Here, however, $\{R_i\}$ and $\{L_i\}$ are not independent due to constraints (18). Furthermore, we note that $\{\phi_i\}$ contain second-class constraints; so before concluding that the quantum theory contains a Kac-Moody algebra, we must first replace the Poisson brackets in (24)–(26) by Dirac brackets.

Since we wish to obtain a Kac-Moody algebra in the quantum theory, we ask the following. When is the Dirac-bracket algebra for variables R_i (L_i) identical to the Poisson-bracket algebra for R_i (L_i)? A sufficient condition for the above is that R_i (L_i) have vanishing Poisson brackets with the constraints. We will see that R_i and L_i cannot simultaneously commute with the constraints. Consequently, at most, one set of variables spans a Kac-Moody algebra in the quantum theory.

To check when R_i has vanishing Poisson brackets with ϕ_j, it is convenient to rewrite the constraints according to

$$\phi_i^R \equiv M_{ij}^{-1}\phi_j = R_i - \frac{i}{4\pi}(n+\lambda)(g^{-1}\partial_1 g)_i \approx 0\,. \tag{27}$$

Now use (22) and (24) to show that

$$\{R_i(x^1),\phi_j^R(x'^1)\} = c_{ij}^k\phi_k^R(x^1)\delta(x^1-x'^1) + \frac{1}{4\pi}(\lambda-n)\delta_{ij}\frac{\partial}{\partial x^1}\delta(x^1-x'^1)\,. \tag{28}$$

We see that R_i has (weakly) vanishing Poisson brackets with the constraints at the critical values $\lambda=n$. At these values for the coupling constants, a Kac-Moody algebra is realized by Dirac brackets of the variables $R_i \approx (in/2\pi)(g^{-1}\partial_1 g)_i$ with themselves. From Eq. (4), these are the same critical values which lead to the conservation of the current J_R^μ.

We repeat the above procedure for the variables L_i. Now rewrite the constraints (18) according to

$$\phi_i^L \equiv -N_{ij}^{-1}\phi_j = L_i - \frac{i}{4\pi}(n-\lambda)(\partial_1 g g^{-1})_i \approx 0\,. \tag{29}$$

From Poisson brackets (23) and (25), we find that

$$\{L_i(x^1),\phi_j^L(x'^1)\} = c_{ij}^k\phi_k^L(x^1)\delta(x^1-x'^1) + \frac{1}{4\pi}(n+\lambda)\delta_{ij}\frac{\partial}{\partial x^1}\delta(x^1-x'^1)\,. \tag{30}$$

L_i has (weakly) vanishing Poisson brackets with the constraints when $\lambda=-n$. At these critical values, a Kac-Moody algebra is realized by Dirac brackets of the variables $L_i \approx (in/2\pi)(\partial_1 g g^{-1})_i$ with themselves. From Eq. (5), these are the same critical values that lead to the conservation of J_L^μ.

From Eq. (27) [Eq. (29)], we note that the constraints reduce to $R_i(x^1)\approx 0$ [$L_i(x^1)\approx 0$] when $\lambda=-n$ [$\lambda=n$]. Thus, at the critical values of the coupling constants, the Poisson-bracket algebra of the constraints with themselves also corresponds to a Kac-Moody algebra. The result that the degrees of freedom L_i [R_i] commute with the constraints at $\lambda=-n$ [$\lambda=n$] follows simply from Eq. (26). Among the conditions $R_i(x^1)\approx 0$ [$L_i(x^1)\approx 0$] are the first-class constraints

$$R_i^0 \equiv \int_0^{2\pi} dx^1 R_i(x^1) \approx 0, \quad L_i^0 \equiv \int_0^{2\pi} dx^1 L_i(x^1) \approx 0\,,$$

where we now assume periodic conditions in x^1, $0 \leq x^1 < 2\pi$. From Eq. (22) [Eq. (23)], R_i^0 [L_i^0] generates time-dependent group transformations on the right [left] of g. These are identical to the gauge transformations by h_4 in Eq. (7) [h_1 in Eq. (6)].

Following standard canonical procedure, we can set the second-class constraints strongly to zero and quantize only the gauge-invariant variables or observables. Then for $\lambda = -n$, we eliminate

$$R_i^m \equiv \int_0^{2\pi} dx^1 R_i(x^1) e^{imx^1}, \quad m \neq 0,$$

and quantize the variables

$$L_i^{m'} \equiv \int_0^{2\pi} dx^1 L_i(x^1) e^{im'x^1}, \quad m' = \text{integer},$$

since the latter commute with R_i^0. The quantum representation for $L_i^{m'}$ are well known. [The set of observables $\{L_i^{m'}\}$ is actually overcomplete. Using $L_i = (in/2\pi) \times (\partial_1 g g^{-1})_i$ and periodic boundary conditions in g, we obtain the nonlocal condition

$$P \exp\left[-\frac{2\pi i}{n} \int_0^{2\pi} L_i(x^1) T_i \, dx^1 \right] = P \exp\left[\int_0^{2\pi} \partial_1 g g^{-1} dx^1 \right] = 1,$$

where P denotes path ordering. The meaning of this condition and its implications for the quantum representations of the theory is obscure to us. Alternatively, we may drop the condition if we do not insist on periodic boundary conditions for $g(x^1)$.]

Thus far our treatment has not required a critical number of dimensions for the target space. Critical dimensions for the WZW model followed from demanding conformal invariance. Although the system given in Eq. (1) is not invariant under the usual conformal transformations, it is invariant under local reparametrizations in x^1 when the coupling constants take their critical values. For instance, when $\lambda \equiv -n$, $\partial_1 g g^{-1}$ is a function of only x^+ [from Eq. (5)] and we can choose $h_3 = 1 + i\epsilon \partial_1 g g^{-1}$ and $h_4 = 1$ in Eq. (7), where $\epsilon = \epsilon(x^+)$ is small. This yields $\delta g = -i\epsilon \partial_1 g$ corresponding to reparametrization in x^1. The quantum generators of this symmetry are obtained via the Sugawara construction.[9] However, as usual, the symmetry is anomalous for all dimensions but the critical one. Analysis of the ghost Lagrangian for this system shows that the critical dimensions are identical to those found recently by Mezincescu and Nepomechie,[10] even though their system looks quite different from ours.

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[1] W. Siegel, Nucl. Phys. **B238**, 307 (1984).

[2] R. Floreanini and R. Jackiw, Phys. Rev. Lett. **59**, 1873 (1987).

[3] C. Imbimbo and A. Schwimmer, Phys. Lett. B **193**, 445 (1987); J. M. F. Labastida and M. Pernici, Nucl. Phys. B **297**, 557 (1988); **306**, 516 (1988); M. Bernstein and J. Sonnenschein, Phys. Rev. Lett. **60**, 1772 (1988); M. E. V. Costa and H. O. Girotti, Phys. Rev. Lett. **60**, 1773 (1988); H. O. Girotti, M. Gomes, V. Kurak, V. O. Rivelles, and A. J. da Silva, Phys. Rev. Lett. **60**, 1913 (1988); M. N. Sanielevici, G. W. Semenoff, and Y.-S. Wu, MIT Report No. CTP 1584 (to be published).

[4] J. Sonnenschein, Nucl. Phys. **B309**, 752 (1988). Also see, Y. Frishman and J. Sonnenschein, Nucl. Phys. **B301**, 346 (1988).

[5] E. Witten, Commun. Math. Phys. **92**, 455 (1984).

[6] Balram Rai and V. G. J. Rodgers, State University of New York at Stony Brook Report No. ITP-SB-88-84 (to be published); State University of New York at Syracuse Report No. SU4228-398 (to be published).

[7] A. P. Balachandran (private communication).

[8] A. P. Balachandran, G. Marmo, B. Skagerstam, and A. Stern, *Gauge Symmetries and Fibre Bundles,* Lecture Notes in Physics Vol. 188 (Springer-Verlag, Berlin 1983).

[9] P. Goddard and D. Olive, Int. J. Mod. Phys. A **1**, 303 (1988).

[10] L. Mezincescu and R. I. Nepomechie, Phys. Rev. D **37**, 3067 (1988). Also see, S. Belluci, M. F. L. Golterman, and D. N. Petcher, Florida State University Report No. FSCU-SCRI-88-53 (to be published).