

COVERINGS OF PROFINITE GRAPHS

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ABSTRACT

We define a covering of a profinite graph to be a projective limit of a system of covering maps of finite graphs. With this notion of covering, we develop a covering theory for profinite graphs which is in many ways analogous to the classical theory of coverings of abstract graphs. For example, it makes sense to talk about the universal cover of a profinite graph and we show that it always exists and is unique. We define the profinite fundamental group of a profinite graph and show that a connected cover of a connected profinite graph is the universal cover if and only if its profinite fundamental group is trivial.

DEDICATION

I dedicate this dissertation to my mother Sucheta Acharyya, my father Sudip Kumar Acharyya, my aunt Jhuma Acharyya and my uncle Pradip Kumar Acharyya. From my very childhood my parents inspired and motivated me to pursue higher education. My mother's incredible support, sacrifice and encouragement for success in all stages of my educational career played an extremely important role towards proceeding in accordance with my goals. My father being a Mathematician and a professor in a University, always tried to spread his enthusiasm and passion for teaching and learning Mathematics within me from the beginning of my career. From the very first day in my life my aunt and uncle also raised me as their own child. Their immense love, support, blessings and sacrifice pushed me forward throughout my career.

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CHAPTER 1

Introduction

In this work we develop the idea of coverings for abstract graphs and the topological covering space theory in the category of Profinite Graphs which is in many senses similar to the idea of a Galois Cover, first studied and coined by Pavel Zalesskii [8]. We define a more general covering of profinite graphs to be a projective limit of a system of covering maps of finite graphs. With this notion of covering, we developed a covering theory for profinite graphs which is in many ways analogous to the classical theory of coverings of abstract graphs and topological covering spaces. We first note that some elementary properties that are known to be true for abstract graphs also make sense in our construction. For example, a covering map $f: \Gamma \rightarrow \Delta$ of profinite graphs is locally bijective, and if Δ is connected in the profinite sense then f is surjective. Moreover, f is a quotient map in that case. We also found that there is a uniformly continuous homeomorphism between each fiber in a path component of Δ . We give a necessary and sufficient condition for a continuous map of profinite graphs to lift to a profinite covering graph. Our condition is a projective analogue of the well-known condition in the topological theory of covering spaces. As in the classical theory, we first observe that the usual uniqueness of lifts result holds for coverings of profinite graphs. We define the covering transformations in a similar way as in the classical theory and found that the group of covering transformations of a connected covering graph is a profinite group that acts freely and uniformly equicontinuously on the covering graph.

In light of our definition of a covering of profinite graphs we define a regular covering of profinite graphs to be a projective limit of regular coverings of finite path

connected graphs. At this point we show that our regular cover turns to be what Zalesskii has called a Galois cover. We observe that every covering transformation h in the group of covering transformations of a regular covering graph has a resolution $h = \varprojlim h_i, i \in I$, the directed set as in the definition of the regular covering. We prove that the action of the group of covering transformation of a regular covering graph is continuous, residually free and is simply transitive on each vertex fiber. Conversely, we show that if a profinite group acts continuously and residually freely on a profinite connected graph Γ , without edge inversions, then the corresponding orbit mapping is a regular covering map of profinite graphs; moreover the profinite group turns out to be the group of covering transformations of the corresponding covering graph. We also give a criterion for a map of profinite graphs to be a regular covering in terms of Good Pairs of compatible cofinite entourages. We observe that the base graph of a regular covering graph is not only a quotient but also a uniform quotient of the regular covering graph. We also show that, any closed subgroup H of G , the group of covering transformations of a regular covering graph, gives rise to a profinite covering graph of the corresponding base graph, and this is again a regular covering if H is a normal subgroup of G . Moreover, the original regular covering graph is a regular covering arising in that way with $H = G$.

Since profinite graphs are not necessarily path connected, fundamental groups may not be as useful a tool as in the classical covering theory of path connected graphs and spaces. We see however, that there are some aspects of fundamental groups that carry over to the profinite analogue of the fundamental group. Just as we have defined most notions here, we define this first in the special case of a finite discrete graph. Then we extended this idea to an arbitrary profinite graph by taking the projective limit of the profinite fundamental groups of its finite discrete quotient graphs. We notice that for any map of profinite graphs there exists a natural induced homomorphism of the profinite analogue of the fundamental groups. This induced

homomorphism is injective if the original map of profinite graphs is a covering of profinite graphs; and the induced homomorphism is an isomorphism of groups if the original map of profinite graphs is an isomorphism of profinite graphs.

We define a universal covering of profinite connected graphs by way of a universal mapping property as in the case of universal coverings of abstract connected graphs and path connected topological spaces. We first notice that if the universal cover of a profinite connected graph exists then it is unique up to isomorphism of profinite covering graphs. We prove that if the profinite analogue of the fundamental group of a connected profinite covering graph based at some vertex is trivial, then it is the universal profinite covering graph. We give a construction of the universal profinite covering graph for any connected profinite graph. Then, by examining the graphs arising from this construction, we get a better understanding of the properties of universal profinite covering graphs. In particular, we show that a universal covering of profinite graphs is a regular covering. From our construction we also notice that a connected covering graph is the universal covering of the base graph if and only if the profinite analogue of the fundamental group of that covering graph based at any of its vertices is trivial. Applying results of regular coverings and good pairs to the universal covering of a connected profinite graph Δ leads to a characterization of all profinite covering graphs of Δ in terms of closed subgroups of the group of covering transformations of the universal profinite covering graph of Δ .

CHAPTER 2

Preliminaries

2.1. Uniform Topological Spaces

In this subsection we discuss uniform topological spaces. For details, see [2] and [4].

DEFINITION 2.1. *A uniform space (X, Φ) is a set X along with a non-empty family Φ of subsets of $X \times X$, that satisfies the following:*

- (1) *If U is in Φ then U contains the diagonal $D(X) = \{(x, x) \in X \times X \mid x \in X\}$ of X .*
- (2) *If U is in Φ and V is a subset of $X \times X$ such that $U \subseteq V$, then V is in Φ .*
- (3) *If $U, V \in \Phi$, then $U \cap V \in \Phi$.*
- (4) *If U is in Φ , then there exists $V \in \Phi$ such that for all $(x, y), (y, z) \in V, (x, z) \in U$.*
- (5) *If U is in Φ then $U^{-1} = \{(y, x) \in X \times X \mid (x, y) \in U\}$ is in Φ .*

Φ is called the uniform structure or uniformity of X and its elements are called entourages. We define $U[x] = \{y \in X \mid (x, y) \in U\}$, for all $U \in \Phi$ and $x \in X$. If $(x, y) \in U$ we say that x and y are U **close**. If $A \subseteq X$ is such that for all $(x, y) \in A \times A, x, y$ are U -close we say that A is U -**small**. An entourage will be called symmetric iff $U = U^{-1}$.

A **fundamental system** of entourages for a uniformity Φ over X is a subfamily $\Psi \subseteq \Phi$ such that for all $U \in \Phi$, there exists $V \in \Psi$ with $V \subseteq U$. So given any fundamental system of entourages, Ψ , we have $\Phi = \{U \subseteq X \times X \mid V \subseteq U, \text{ for some } V \in \Psi\}$ is the uniform system corresponding to or generated by Ψ .

Every uniform space has a fundamental system of symmetric entourages.

2.2. *Let X be a compact Hausdorff topological space. Then $\Phi = \{U \subseteq X \times X \mid D(X) \subseteq O \subseteq U \text{ for some open set } O \subseteq X \times X\}$ is the unique uniformity over X which admits the original topology on X .*

2.3. *Let $f: X \rightarrow Y$ be a continuous map where both X, Y are compact Hausdorff spaces. Then f is uniformly continuous with respect to the unique induced uniformities over X and Y , which admit the corresponding topologies.*

2.2. Profinite and Residually Finite Groups

In this section we are summarizing some well known results from [3]. In the study of residually finite group it is often useful to give the group a topology as follows:

Let G be a residually finite group and let Σ be a separating filter base of normal subgroups of finite index in G ,

By this we mean that Σ is a set of finite index normal subgroups of G satisfying,

(1). For each pair $N_1, N_2 \in \Sigma$, there exists a member N_3 of Σ such that N_3 is a subset of $N_1 \cap N_2$.

(2). The intersection of all members of Σ is 1.

Then G can be given, in exactly one way, the structure of a Hausdorff topological group in which Σ is a basis of neighborhoods of the identity. The resulting topological group is referred to as a cofinite group. Thus a cofinite group is simply a topological group in which the identity has a basis of neighborhoods consisting of normal subgroups of finite index, which will necessarily be both open and closed. A topology on a group G determined by a set Σ of finite index normal subgroups of G satisfying (1) and (2) above is referred to as a cofinite topology. And when the resulting topological group is profinite we will call it a profinite topological group. Any

compact topological group H containing a cofinite group G as a dense subgroup of itself is called a completion of G .

2.4. *Let G be a cofinite group contained as a dense subgroup of a topological group \widehat{G} , Let H be any compact topological group, and let $\phi: G \rightarrow H$ be a continuous homomorphism.*

Then:

- (1) ϕ can be extended uniquely to a continuous map $\widehat{\phi}: \widehat{G} \rightarrow H$.
- (2) $\widehat{\phi}$ is a group homomorphism.
- (3) $\widehat{\phi}(\widehat{G}) = \overline{\phi(G)}$
- (4) $\widehat{\phi}$ is injective iff ϕ is an algebraic and topological embedding. In that case, $\widehat{\phi}$ determines an algebraic and topological isomorphism between \widehat{G} and $\overline{\phi(\widehat{G})}$.

2.5. *Let G be an abstract residually finite group and let τ_1, τ_2 be cofinite topologies on G . Suppose that $\tau_1 \geq \tau_2$. Let G_i be a completion of G in the topology τ_i , where i runs over the set $\{1, 2\}$. Then the identity map id_G on G extends uniquely to a continuous epimorphism $\bar{i}: G_1 \rightarrow G_2$. Furthermore \bar{i} is an isomorphism iff $\tau_1 = \tau_2$.*

2.6. *The completion of a cofinite group G is unique up to an isomorphism extending the identity map on G .*

Following Hartley [3] we will refer to the completion of the residually finite group G with respect to its finest cofinite topology as the UNIVERSAL COMPLETION.

2.7. *Let G be an abstract residually finite group, let \widehat{G} be the universal completion of G , and let \overline{G} be a completion of G in an arbitrary cofinite topology. Then there is a continuous epimorphism $\widehat{G} \rightarrow \overline{G}$ extending the identity map on G .*

2.8. *Existence: Let G be a cofinite group and let I be the separating filter base of open normal subgroups of finite index in G . Then the profinite group $\widehat{G} = \varprojlim_{K \in I} G/K$*

is a compact cofinite group and the natural map $G \rightarrow \widehat{G}$ that sends $x \in G$ to the element whose component in G/K is xK , embeds G as a dense subgroup of \widehat{G} .

2.9. Let G be a dense cofinite subgroup of a topological group H and let I be the set of all open normal subgroups of G . Then

- i. H is cofinite and $\{\overline{K} \mid K \in I\}$ is the set of all open normal subgroups of H .
- ii. If $K \in I$, then $\overline{K} \cap G = K$.

2.10. Let G be a cofinite group.

- i. If H is a subgroup of G , then H is cofinite (with the induced topology).
- ii. If K is a topological group and $\phi: G \rightarrow K$ is a continuous epimorphism, then K is cofinite and every open normal subgroup of K is the image of an open normal subgroup of G .

2.3. Equivalence Relations and Cofinite spaces

In this section we are summarizing some results from the preprint of our works in [1].

2.11. Let X, Y be two sets and $f: X \rightarrow Y$ be a function of sets. Then f as a binary relation over $X \times Y$ is given by $f = \{(x, y) \in X \times Y \mid f(x) = y\}$. Let $f^{-1} = \{(y, x) \in Y \times X \mid f(x) = y\}$. Then the following properties are true,

- $(f \times f)[S] = fSf^{-1}$, for all relations S on X and $(f \times f)^{-1}[R] = f^{-1}Rf$, for all relations R over Y .
- If $K_f = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$ then $K_f = (f^{-1})f = (f \times f)^{-1}[D(Y)]$ is an equivalence relation.
- $f(f^{-1}) \subseteq D(Y)$

THEOREM 2.12 (Correspondence Theorem). *If $f: X \rightarrow Y$ is a map, then $(f \times f)^{-1}[R]$ is an equivalence relation on X , for all equivalence relation R over Y and if f*

is a Surjection, $(f \times f)[S]$ is an equivalence relation on Y , for all equivalence relation S over X , that contains K_f . Moreover if R and S have finitely many equivalence classes in Y and X respectively then $(f \times f)^{-1}[R]$ and $(f \times f)[S]$ have finitely many equivalence classes too, in X and Y , respectively.

DEFINITION 2.13 (cofinite equivalence relation). Let X be a topological space. A *cofinite equivalence relation* on X is an equivalence relation R such that the quotient space X/R is a finite discrete space.

DEFINITION 2.14 (Cofinite entourage). Let X be a uniform space. By a *cofinite entourage* on X we will mean an entourage R which is also a cofinite equivalence relation on X .

2.15. Let X and Y be uniform spaces. Then,

- (1) The intersection $R_1 \cap R_2$ of two cofinite entourages R_1, R_2 of X is also a cofinite entourage.
- (2) Let S be an equivalence relation on X . If S contains a cofinite entourage, then S itself is a cofinite entourage.
- (3) If R_1, R_2 are commuting equivalence relations on a space X , and if one of R_1, R_2 is a cofinite entourage, then the product $R_1 R_2$ is also a cofinite entourage.
- (4) If $f: X \rightarrow Y$ is a uniformly continuous map and R is a cofinite entourage of Y , then $(f \times f)^{-1}[R]$ is a cofinite entourage of X .
- (5) If X is compact and Hausdorff, then every equivalence relation R on X , where R is an open subset of $X \times X$, is a cofinite entourage.

DEFINITION 2.16 (Cofinite space). A *cofinite (uniform) space* is a uniform space X whose cofinite entourages form a fundamental system of entourages (i.e., every entourage of X contains a cofinite entourage).

Let X be a cofinite space and K an equivalence relation over X . Let $q: X \rightarrow X/K$ be the corresponding quotient map. Then, $K_q = \{(x, y) \in X \times X \mid q(x) = q(y)\} = K$. Let I be a fundamental system of cofinite entourages over X . Let $I' = \{R \in I \mid K \subseteq R\}$. By the correspondence theorem, the collection $J = \{(q \times q)[R] \mid R \in I'\}$ is a fundamental system of entourages for a uniformity on the set of equivalence classes X/K and each such $(q \times q)[R]$ has finitely many equivalence classes. We call this uniformity the *quotient uniformity of X modulo K* .

In general, the topology induced by the quotient uniformity of X modulo K is not as fine as the quotient topology on X/K . For this reason, we write $X//K$ for the set X/K endowed with the quotient uniformity of X modulo K and the topology it induces, reserving the notation X/K for the quotient space (with the quotient topology).

DEFINITION 2.17 (Uniform quotient space). If K is an equivalence relation on a cofinite space X , then $X//K$ is called the *uniform quotient space of X modulo K* .

2.4. Inverse limits of compact Hausdorff spaces

In this section we are summarizing some well known results from Profinite Groups; see by John S. Wilson [7] and our work in [1] for proofs and more details.

2.18. Let (X_i, ϕ_{ij}) be an inverse system of topological spaces indexed by a directed set I . Let X denote the inverse limit of (X_i, ϕ_{ij}) and let $\phi_i: X \rightarrow X_i$ be the canonical map for each $i \in I$.

- (1) The family of sets $\phi_i^{-1}(U_i)$, where $i \in I$ and U_i is open in X_i , is a basis for the topology of X .
- (2) Let A be a subset of X and write $A_i = \phi_i(A)$ for each $i \in I$. Then

$$\overline{A} = \bigcap_{i \in I} \phi_i^{-1}(\overline{A_i}) = \varprojlim \overline{A_i}.$$

- (3) If A is a subset of X satisfying $\phi_i(A) = X_i$ for all $i \in I$, then A is dense in X .
- (4) If $f: Y \rightarrow X$ is a function from a space Y , then f is continuous if and only if each composition $\phi_i f$ is continuous.

2.19. Let (X_i, ϕ_{ij}) be an inverse system of non-empty compact Hausdorff spaces indexed by a directed set I . Then the inverse limit $X = \varprojlim X_i$ has the following properties:

- (1) X is a non-empty compact Hausdorff space.
- (2) $\phi_i(X) = \bigcap_{j \geq i} \phi_{ij}(X_j)$ for each $i \in I$.
- (3) If A, B are disjoint closed subsets of X , then there exists $i \in I$ such that $\phi_i(A), \phi_i(B)$ are disjoint closed subsets of X_i .
- (4) If Y is a discrete space and $f: X \rightarrow Y$ is a continuous map, then f factors through some X_k ; i.e., for some $k \in I$ there is a continuous map $h: X_k \rightarrow Y$ such that $f = h\phi_k$.

2.20. The following conditions are equivalent for any compact Hausdorff space X :

- (1) X is totally disconnected;
- (2) the clopen subsets of X form a basis for its topology;
- (3) $\bigcap \{R \mid R \text{ is a co-discrete equivalence relation on } X\}$ is equal to the diagonal of $X \times X$;
- (4) X is Hausdorff cofinite space, when endowed with the unique uniform structure;
- (5) X is the inverse limit of an inverse system (X_i, ϕ_{ij}) of finite discrete spaces.

DEFINITION 2.21 (Profinite space). A compact Hausdorff space X that satisfies the equivalent conditions of the previous result is called a *profinite space*.

We will always assume that a profinite space X is endowed with the unique uniform structure that induces its topology, and hence, X is a Hausdorff cofinite space. Thus profinite spaces are precisely the compact, Hausdorff cofinite spaces.

2.5. Finite Graphs

Let us first recall some basic theory of abstract graphs. A path of length $n \geq 1$ in a graph Γ is finite string of edges $p = e_1 \cdots e_n \in E(\Gamma)^*$ such that $t(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$. The source and target of this path p are the vertices $s(p) = s(e_1)$ and $t(p) = t(e_n)$. In addition, for each vertex x of Γ , there is a trivial path ϵ_x of length zero with $s(\epsilon_x) = t(\epsilon_x) = x$. We write \bar{p} to mean $\bar{e}_n \cdots \bar{e}_1$, where \bar{e} is the inverse of e or tracing e in the reverse direction, from $t(e)$ to $s(e)$ and define $s(\bar{e}) = t(e)$, $t(\bar{e}) = s(e)$. We say that Γ is path connected if there is a path in Γ joining any two vertices.

If p, q are paths of length m, n in Γ with $t(p) = s(q)$, then their concatenation pq is a path of length $m + n$ whose source is $s(p)$ and whose target is $t(q)$.

Two paths p and q in Γ are homotopic, written $p \simeq q$, if there is a finite sequence of paths

$$p = p_0, p_1, \dots, p_k = q$$

passing from p to q such that each p_{i+1} is obtained from p_i by deleting or inserting a part of the form $e\bar{e}$ for some $e \in E(\Gamma)$. Homotopic paths have the same source and target vertices and concatenation of paths respects homotopy classes. For each vertex x of Γ , the set of homotopy classes $[p]$ of paths p whose sources and targets coincide with x forms a group $\pi_1(\Gamma, x)$ called the fundamental group of Γ based at x . The multiplication in this group is given by $[p][q] = [pq]$. The identity element is $1 = [\epsilon_x]$. The inverse of the homotopy class $[p]$ of a path $p = e_1 \cdots e_n$ is given by $[p]^{-1} = [\bar{p}]$, where $\bar{p} = \bar{e}_n \cdots \bar{e}_1$.

A map of graphs $f: \Gamma \rightarrow \Delta$ extends to a mapping of paths that preserves length and respects products: if $p = e_1 \cdots e_n$ is a path in Γ , then $f(p) = f(e_1) \cdots f(e_n)$ is

a path in Δ and $f(\epsilon_x) = \epsilon_{f(x)}$. Furthermore, if p, q are homotopic paths in Γ , then $f(p), f(q)$ are homotopic paths in Δ . Thus f determines a homomorphism of groups

$$f^*: \pi_1(\Gamma, x) \rightarrow \pi_1(\Delta, f(x))$$

given by $f^*([p]) = [f(p)]$. Now we cite some well known results in the theory of abstract graphs: See for example, Stallings [6].

2.22. (*Uniqueness of lifting*): If $f: \Gamma \rightarrow \Delta$ is a local injective map of graphs, Σ a path connected graph and $g_1, g_2: \Sigma \rightarrow \Delta$ are maps of graphs such that $fg_1 = fg_2$, and if there is a vertex v of Σ such that $g_1(v) = g_2(v)$, then $g_1 = g_2$.

2.23. (*General lifting*): Let $f: \Gamma \rightarrow \Delta$ be a covering, $g: \Sigma \rightarrow \Delta$ a map of graphs, and Σ a path connected graph. If u, v are vertices of Γ, Σ respectively such that $f(u) = g(v) = b$, then there exists a unique map of graphs $h: \Sigma \rightarrow \Gamma$ such that $h(v) = u$ and $f \circ h = g$, if and only if $g^*(\pi_1(\Sigma, v)) \subseteq f^*(\pi_1(\Gamma, u))$.

$$\begin{array}{ccc}
 & & (\Gamma, u) \\
 & \nearrow h & \downarrow f \\
 (\Sigma, v) & \xrightarrow{g} & (\Delta, b)
 \end{array}$$

2.24. (*Existence of coverings*): If $f: \Gamma \rightarrow \Delta$ is a covering of abstract graphs and u a vertex of Γ , then $f^*: \pi_1(\Gamma, u) \rightarrow \pi_1(\Delta, f(u))$ is injective. If Δ is path connected, v a vertex and $H \leq \pi_1(\Delta, v)$ a subgroup, then there exists a covering $p: (\Sigma, a) \rightarrow (\Delta, v)$ where Σ is path connected, with vertex a , such that $p(a) = v$ and $p^*(\pi_1(\Sigma, a)) = H$. Moreover, any two such coverings are isomorphic (in the standard sense). The index of H in $\pi_1(\Delta, v)$ is equal to the cardinality of $p^{-1}(v)$.

2.25. Let $p: \Gamma \rightarrow \Delta$ be a covering map where Γ and Δ are path connected abstract graphs and let $b \in V(\Delta)$. Then the following are equivalent:

- (1) The action of $\text{Aut}(\Gamma, p)$, the group of covering transformations of p , is transitive on the fiber $p^{-1}(b)$;
- (2) for any $a \in p^{-1}(b)$, $p^*(\pi_1(\Gamma, a))$ is a normal subgroup of $\pi_1(\Delta, b)$;
- (3) for all $a, a' \in p^{-1}(b)$, $p^*(\pi_1(\Gamma, a)) = p^*(\pi_1(\Gamma, a'))$.

DEFINITION 2.26. A covering with the properties described in the previous note is called a regular or normal or Galois covering.

2.6. Topological Graphs

Throughout the rest of this section we cite some concepts and results from our previous work in [1]. A topological graph is a topological space Γ that is partitioned into two closed subsets $V(\Gamma)$ and $E(\Gamma)$ together with two continuous functions $s, t: E(\Gamma) \rightarrow V(\Gamma)$ and a continuous function $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ satisfying the following properties: for every $e \in E(\Gamma)$,

- (1) $\bar{e} \neq e$ and $\bar{\bar{e}} = e$;
- (2) $t(\bar{e}) = s(e)$ and $s(\bar{e}) = t(e)$.

The elements of $V(\Gamma)$ are called vertices. An element $e \in E(\Gamma)$ is called a (directed) edge with source $s(e)$ and target $t(e)$; the edge \bar{e} is called the reverse or inverse of e .

A map of graphs $f: \Gamma \rightarrow \Delta$ is a function that maps vertices to vertices, edges to edges, and preserves sources, targets, and inverses of edges. Analogously, we will call a map of graphs a graph isomorphism if and only if it is a bijection.

An orientation of a topological graph Γ is a closed subset $E^+(\Gamma)$ consisting of exactly one edge in each pair $\{e, \bar{e}\}$ of edges of Γ . In this situation, setting $E^-(\Gamma) = \{e \in E(\Gamma) \mid \bar{e} \in E^+(\Gamma)\}$ we see that $E(\Gamma)$ is a disjoint union of the two closed (hence also open) subsets $E^+(\Gamma)$, $E^-(\Gamma)$.

DEFINITION 2.27 (Compatible equivalence relation). An equivalence relation R on a graph Γ is *compatible* if the following properties hold:

- (1) $R = R_V \cup R_E$ where R_V, R_E are equivalence relations on $V(\Gamma), E(\Gamma)$, that are precisely the restriction of R ;
- (2) if $(e_1, e_2) \in R$, then $(s(e_1), s(e_2)) \in R, (t(e_1), t(e_2)) \in R$, and $(\bar{e}_1, \bar{e}_2) \in R$;
- (3) for all $e \in E(\Gamma)$, $(e, \bar{e}) \notin R$.

2.28. *If K is a compatible equivalence relation on Γ , $K[x] = \{y \in \Gamma \mid (x, y) \in K\}$ for any $x \in \Gamma$, then there is a unique way to make Γ/K into a graph such that the canonical map $\Gamma \rightarrow \Gamma/K$ is a map of graphs. It is defined by setting $s(K[e]) = K[s(e)]$, $t(K[e]) = K[t(e)]$, and $\overline{K[e]} = K[\bar{e}]$. Conversely, if Δ is a graph and $f: \Gamma \rightarrow \Delta$ is a surjective map of graphs, then $K = f^{-1}f = \{(a, b) \in \Gamma \times \Gamma \mid f(a) = f(b)\}$ is a compatible equivalence relation on Γ and f induces an isomorphism of graphs $\Gamma/K \cong \Delta$.*

2.29. *If R_1 and R_2 are compatible equivalence relations on Γ , then so is $R_1 \cap R_2$.*

2.30. *Let R be any cofinite equivalence relation on a topological graph Γ . Then there exists a compatible cofinite equivalence relation S on Γ such that $S \subseteq R$.*

2.31. *Let Γ be a topological graph with a specified closed orientation $E^+(\Gamma)$. Then for any cofinite equivalence relation R on Γ , there exists a compatible orientation preserving cofinite equivalence relation S on Γ such that $S \subseteq R$.*

2.32. *If Γ is a compact Hausdorff totally disconnected topological graph, then its compatible cofinite equivalence relations form a fundamental system of entourages for the unique uniform structure that induces the topology of Γ .*

DEFINITION 2.33 (Profinite graph). A compact Hausdorff totally disconnected topological graph Γ is called a *profinite graph*.

As for any compact Hausdorff space, we will view a profinite graph as a uniform space endowed with the unique uniformity that induces its topology. Thus, Corollary 2.32 states that the collection of all compatible cofinite equivalence relations on a profinite graph Γ form a fundamental system of entourages.

2.7. Cofinite graphs

By a uniform topological graph we mean a topological graph Γ endowed with a uniform structure that induces its topology such that Γ is the uniform sum of its uniform subspaces $V(\Gamma)$, $E(\Gamma)$ and the maps $s, t: E(\Gamma) \rightarrow V(\Gamma)$ and $\bar{\cdot}: E(\Gamma) \rightarrow E(\Gamma)$ are uniformly continuous.

2.34. If $f: \Gamma \rightarrow \Delta$ is a uniformly continuous map of uniform topological graphs then for any compatible cofinite equivalence relation R over Δ , $(f \times f)^{-1}(R)$ is a compatible cofinite equivalence relation over Γ .

DEFINITION 2.35 (Cofinite graph). A *cofinite graph* is an abstract graph Γ endowed with a Hausdorff uniformity such that the compatible cofinite entourages of Γ form a fundamental system of entourages (i.e. every entourage of Γ contains a compatible cofinite entourage).

2.36. *Cofinite graphs are uniform topological graphs.*

2.37. *Every profinite graph is a cofinite graph.*

2.38. *Let Γ be a cofinite graph and let Z be a cofinite space. Then a map $f: \Gamma \rightarrow Z$ is uniformly continuous if and only if both the restrictions $f|_{V(\Gamma)}$ and $f|_{E(\Gamma)}$ are uniformly continuous.*

2.8. Completions of cofinite graphs

2.39. Let Γ be a cofinite graph contained as a dense subgraph in a compact Hausdorff topological graph $\bar{\Gamma}$. Then given any compact Hausdorff topological graph Δ and any uniformly continuous map of graphs $\varphi: \Gamma \rightarrow \Delta$,

- (1) $\overline{V(\Gamma)} = V(\bar{\Gamma})$ and $\overline{E(\Gamma)} = E(\bar{\Gamma})$; and
- (2) there exists a unique continuous map of graphs $\bar{\varphi}: \bar{\Gamma} \rightarrow \Delta$ extending φ .

REMARK 1. As in the Note 2.4, $\bar{\varphi}(\bar{\Gamma}) = \overline{\varphi(\Gamma)}$.

In light of Note 2.39 we make the following definition.

DEFINITION 2.40 (Completion). Let Γ be a cofinite graph. Then any compact Hausdorff topological graph $\bar{\Gamma}$ that contains Γ as a dense subgraph is called a *completion* of Γ .

Analogous to some theorems in Hartley's paper at [3] we have the following results.

2.41 (Uniqueness of completions). The completion of a cofinite graph Γ is unique up to an isomorphism extending the identity map on Γ .

2.42 (Existence of completions). Let Γ be a cofinite graph and let I be a fundamental system of compatible cofinite entourages of Γ , directed by the opposite of inclusion. Then the inverse limit $\hat{\Gamma} = \varprojlim \Gamma/R$ ($R \in I$) is a compact Hausdorff topological graph and the natural map $\Gamma \rightarrow \hat{\Gamma}$ embeds Γ as a dense subgraph of $\hat{\Gamma}$.

2.43. Let $\bar{\Gamma}$ be the completion of a cofinite graph Γ and let R be a compatible cofinite entourage of Γ . Then \bar{R} is a compatible cofinite entourage of $\bar{\Gamma}$ and $\bar{R} \cap (\Gamma \times \Gamma) = R$.

2.44. Let Γ be a cofinite graph and let I be the filter base of all compatible cofinite entourages of Γ . Then the completion $\bar{\Gamma}$ is also a cofinite graph and $\{\bar{R} \mid R \in I\}$ is the filter base of all compatible cofinite entourages of $\bar{\Gamma}$.

2.9. Profinite Connectedness

DEFINITION 2.45. A cofinite graph Γ is *cofinitely connected* if for each compatible cofinite entourage R on Γ , the quotient graph Γ/R is path connected.

2.46. *The following statements are equivalent for any cofinite graph Γ :*

- (1) Γ is *cofinitely connected*;
- (2) Γ is *not the uniform sum of two disjoint nonempty subgraphs*.

2.10. Groups Acting on Cofinite Graphs

DEFINITION 2.47. Let G be a group and Γ be a cofinite graph. We say that the group G acts over Γ if and only if

- (1) For all x in Γ , for all $g \in G$, $g.x \in \Gamma$;
- (2) For all x in Γ , for all g_1, g_2 in G , $g_1.(g_2.x) = (g_1g_2).x$;
- (3) For all x in Γ , $1.x = x$;
- (4) For all v in $V(\Gamma)$, for all g in G , $g.v$ is in $V(\Gamma)$ and for all e in $E(\Gamma)$, for all g in G , $g.e$ is in $E(\Gamma)$;
- (5) For all e in $E(\Gamma)$, for all g in G , $g.s(e) = s(g.e)$, $g.t(e) = t(g.e)$, $g.(\bar{e}) = \overline{g.e}$;
- (6) There exists a G -invariant orientation $E^+(\Gamma)$ of Γ .

Note that the aforesaid group action restricted to a singleton group element $g \in G$ can be treated as a well defined map of graphs, $\Gamma \rightarrow \Gamma$ taking $x \mapsto g.x$.

DEFINITION 2.48. A group G is said to act uniformly equicontinuously over a cofinite graph Γ , if and only if for each entourage W over Γ there exists an entourage V over Γ such that for all g in G , $(g \times g)[V] \subseteq W$.

2.49. *If G acts uniformly equicontinuously over a cofinite graph Γ , then there exists a fundamental system of entourages consisting of G -invariant compatible cofinite*

entourages over Γ , i.e., for all entourage U over Γ there exists a compatible cofinite entourage R over Γ such that for all $g \in G$, $(g \times g)[R] \subseteq R \subseteq U$.

DEFINITION 2.50. We say a group G acts on a cofinite space Γ faithfully, if for all g in $G \setminus \{1\}$ there exists x in Γ such that $gx \neq x$ in Γ .

2.51. *Let G acts on a cofinite graph Γ uniformly equicontinuously. Then G acts on Γ/R and G/N_R acts on Γ/R as well, where R is a G -invariant compatible cofinite entourage over Γ and N_R is the kernel of the action of G over Γ/R . If $\{R \mid R \in I\}$ is a fundamental system of G -invariant compatible cofinite entourages over Γ , then $\{N_R \mid R \in I\}$ forms a fundamental system of cofinite congruences for some uniformity over G .*

DEFINITION 2.52. We say that a group G acts on a cofinite graph Γ residually freely, if there exists a fundamental system of G -invariant compatible cofinite entourages R over Γ such that the induced group action of G/N_R over Γ/R is a free action.

2.53. $N_R[1] = \{g \in G \mid (g, 1) \in N_R \text{ where } 1 \text{ is the identity element of } G\}$ is a normal subgroup of finite index in G and $G/N_R[1]$ is isomorphic with G/N_R . More generally, if N is a congruence on G , then $N[1]$ is a normal subgroup of G and $G/N[1] \cong G/N$.

2.54. *The induced uniform topology over G , as in Note 2.51, is Hausdorff if and only if G acts faithfully over Γ .*

2.55. *Suppose that G is a group acting uniformly equicontinuously on a cofinite graph Γ and give G the induced uniformity, as in the Lemma 2.51. Then the action $G \times \Gamma \rightarrow \Gamma$ is uniformly continuous.*

2.56. *If G acts faithfully on Γ and as in Lemma 2.51 , then \widehat{G} acts on $\widehat{\Gamma}$ uniformly equicontinuously.*

2.57. *The uniformities on \widehat{G} obtained by Φ_1 and Φ_2 ,are equivalent where $\Phi_1 = \{N_{\overline{R}} \mid R \in I\}$, $\Phi_2 = \{\overline{N_R} \mid R \in I\}$ and I is a fundamental system of G -invariant compatible cofinite entourage over a cofinite graph Γ induced by a uniform equicontinuous action of the group G over Γ .*

CHAPTER 3

Definition and elementary properties

By a covering of profinite graphs, we will mean a projective limit of an inverse system of locally bijective maps of finite discrete graphs. To make this precise, we first discuss inverse systems of maps of cofinite graphs in general.

Let I be a directed set. An *inverse system of maps of cofinite graphs* $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$ indexed by I consists of:

- (i) for each $i \in I$, a continuous map of cofinite graphs $f_i: \Gamma_i \rightarrow \Delta_i$;
- (ii) for each $j \geq i$, continuous maps of graphs $\phi_{ij}: \Gamma_j \rightarrow \Gamma_i$ and $\psi_{ij}: \Delta_j \rightarrow \Delta_i$ such that the diagram

$$\begin{array}{ccc} \Gamma_j & \xrightarrow{\phi_{ij}} & \Gamma_i \\ f_j \downarrow & & \downarrow f_i \\ \Delta_j & \xrightarrow{\psi_{ij}} & \Delta_i \end{array}$$

commutes;

- (iii) for all $k \geq j \geq i$, we require that $\phi_{ik} = \phi_{ij}\phi_{jk}$ and $\psi_{ik} = \psi_{ij}\psi_{jk}$, furthermore for all $i \in I$, $\phi_{ii} = id_{\Gamma_i}$, $\psi_{ii} = id_{\Delta_i}$.

In other words, an inverse system of maps of cofinite graphs is an inverse system in the category whose objects are continuous maps of cofinite graphs and whose morphisms are pairs of continuous maps of graphs that form commutative diagrams.

An *inverse limit* of the system $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$ consists of:

- (i) a continuous map of cofinite graphs $f: \Gamma \rightarrow \Delta$ and two families of continuous maps of cofinite graphs $(\phi_i: \Gamma \rightarrow \Gamma_i)_{i \in I}$ and $(\psi_i: \Delta \rightarrow \Delta_i)_{i \in I}$ such that: for

each $i \in I$, the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\phi_i} & \Gamma_i \\ f \downarrow & & \downarrow f_i \\ \Delta & \xrightarrow{\psi_i} & \Delta_i \end{array}$$

commutes, and for each $j \geq i$, $\phi_i = \phi_{ij}\phi_j$ and $\psi_i = \psi_{ij}\psi_j$.

- (ii) If for another continuous map of cofinite graphs $f': \Gamma' \rightarrow \Delta'$ and two families of continuous maps of cofinite graphs $(\phi'_i: \Gamma' \rightarrow \Gamma_i)_{i \in I}$ and $(\psi'_i: \Delta' \rightarrow \Delta_i)_{i \in I}$ such that, for each $i \in I$, the diagram

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\phi'_i} & \Gamma_i \\ f' \downarrow & & \downarrow f_i \\ \Delta' & \xrightarrow{\psi'_i} & \Delta_i \end{array}$$

commutes, and for each $j \geq i$, $\phi'_i = \phi_{ij}\phi'_j$ and $\psi'_i = \psi_{ij}\psi'_j$, then there exist unique continuous maps of graphs

$$h: \Gamma' \rightarrow \Gamma, g: \Delta' \rightarrow \Delta$$

such that the following diagram commutes.

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\phi_i} & \Gamma_i & & \\ \downarrow f & \swarrow h & \Gamma' & \searrow \phi'_i & \downarrow f_i \\ & & \downarrow f' & & \\ \Delta & \xrightarrow{\psi_i} & \Delta_i & & \\ & \swarrow g & \Delta' & \searrow \psi'_i & \\ & & \downarrow & & \end{array}$$

If the inverse limit exists, then it is unique up to a natural isomorphism of graphs.

For if there exist two such inverse limits

- (i) $f: \Gamma \rightarrow \Delta$ and two families of continuous maps of cofinite graphs $(\phi_i: \Gamma \rightarrow \Gamma_i)_{i \in I}$ and $(\psi_i: \Delta \rightarrow \Delta_i)_{i \in I}$
- (ii) $f': \Gamma' \rightarrow \Delta'$ and two families of continuous maps of cofinite graphs $(\phi'_i: \Gamma' \rightarrow \Gamma_i)_{i \in I}$ and $(\psi'_i: \Delta' \rightarrow \Delta_i)_{i \in I}$

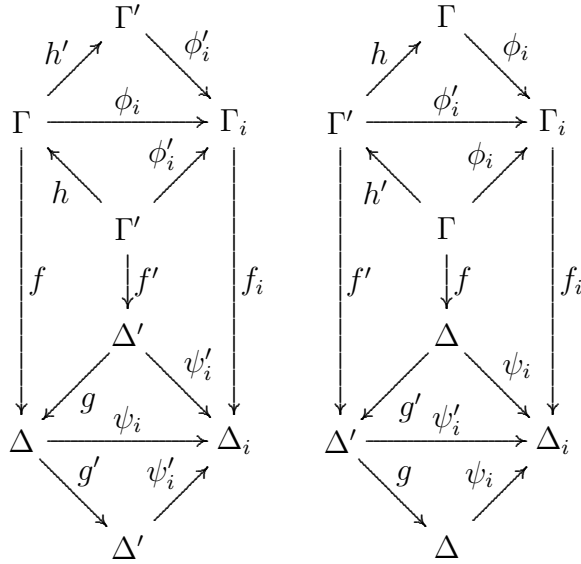
then there exist unique continuous maps of cofinite graphs

$$h: \Gamma' \rightarrow \Gamma, g: \Delta' \rightarrow \Delta$$

and

$$h': \Gamma \rightarrow \Gamma', g': \Delta \rightarrow \Delta'$$

so that the following diagrams commute.



Hence, by uniqueness

$$h \circ h' = id_{\Gamma}, h' \circ h = id_{\Gamma'}$$

$$g \circ g' = id_{\Delta}, g' \circ g = id_{\Delta'}$$

Thus it follows that the inverse limit is unique up to isomorphism of graphs.

To see that such inverse limits exists, we can construct the inverse limit of an inverse system $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$ of maps of cofinite graphs as follows:

Let $\Gamma = \varprojlim(\Gamma_i, \phi_{ij})$ and $\Delta = \varprojlim(\Delta_i, \psi_{ij})$; denote the canonical projections by $\phi_i: \Gamma \rightarrow \Gamma_i$ and $\psi_i: \Delta \rightarrow \Delta_i$. Then the compositions $(f_i \circ \phi_i: \Gamma \rightarrow \Delta_i)_{i \in I}$ form a compatible family of continuous maps of graphs and thus induce a continuous map of graphs $f: \Gamma \rightarrow \Delta$. We now show that these maps satisfy the requirements of the inverse limit.

For, suppose there exist another map of cofinite graphs $f': \Gamma' \rightarrow \Delta'$ and two families of continuous maps of cofinite graphs $(\phi'_i: \Gamma' \rightarrow \Gamma_i)_{i \in I}$ and $(\psi'_i: \Delta' \rightarrow \Delta_i)_{i \in I}$ such that for each $i \in I$ and $j \geq i$, $\phi'_i = \phi_{ij}\phi'_j$, $\psi'_i = \psi_{ij}\psi'_j$. Then since $\Gamma = \varprojlim \Gamma_i$, $\Delta = \varprojlim \Delta_i$, there exist unique continuous maps of graphs $g: \Delta' \rightarrow \Delta$, $h: \Gamma' \rightarrow \Gamma$ such that the following diagrams commute.

$$\begin{array}{ccc}
 & \Delta & \\
 g \nearrow & & \downarrow \psi_i \\
 \Delta' & \xrightarrow{\psi'_i} & \Delta_i
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \Gamma & \\
 h \nearrow & & \downarrow \phi_i \\
 \Gamma' & \xrightarrow{\phi'_i} & \Gamma_i
 \end{array}$$

So we have for all $i \in I$,

$$\psi_i \circ g \circ f' = \psi'_i \circ f' = f_i \circ \phi'_i = f_i \circ \phi_i \circ h = \psi_i \circ f \circ h$$

Thus $g \circ f' = f \circ h$. This completes all the requirements of the inverse limit.

Since the inverse limit is unique up to isomorphism of cofinite graphs, we write

$$f = \varprojlim(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$$

or $f = \varprojlim f_i$ when the system of maps is understood. Our definition of a covering can now be stated precisely as follows.

DEFINITION 3.1 (Covering). A map of profinite graphs $f: \Gamma \rightarrow \Delta$ is a *covering* if it is the inverse limit of an inverse system of maps of graphs $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$, where each $f_i: \Gamma_i \rightarrow \Delta_i$ is a locally bijective map of finite discrete graphs. In this situation, we call the pair (Γ, f) a *profinite covering graph* of Δ .

For the rest of this section, let I be a directed set and let $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$ be an inverse system of locally bijective maps of finite discrete graphs, indexed by I . Then the inverse limit $f: \Gamma \rightarrow \Delta$ of this system is a covering map of profinite graphs. Denote the canonical projections by $\phi_i: \Gamma \rightarrow \Gamma_i$ and $\psi_i: \Delta \rightarrow \Delta_i$.

- REMARKS 3.2. (1) $\varprojlim \Gamma_i = \Gamma = \varprojlim \phi_i(\Gamma)$, $\varprojlim \Delta_i = \Delta = \varprojlim \psi_i(\Delta)$
(2) $\{R_i \mid i \in I\}$, $\{S_i \mid i \in I\}$ form fundamental systems of compatible cofinite entourages for Γ , Δ respectively where $R_i = \phi_i^{-1}\phi_i$, $S_i = \psi_i^{-1}\psi_i$, for each $i \in I$.
(3) We may assume that the canonical maps $\psi_i: \Delta \rightarrow \Delta_i$ are surjective, and thus $\Delta_i = \psi_i(\Delta) = \Delta/S_i$ and $\Gamma_i = f_i^{-1}[(\psi_i(\Delta))] = f_i^{-1}(\Delta_i)$.
(4) If both Δ and Γ are connected, then we may assume that each Δ_i and each Γ_i is path connected.

PROOF. We will now provide brief proofs of the above remarks. We frequently use a few facts about inverse limits of profinite graphs listed as Mathematical Preliminaries.

- (1) Since Γ is closed, in itself $\varprojlim \phi_i(\Gamma) = \Gamma$ and similarly, $\Delta = \varprojlim \psi_i(\Delta)$.
(2) $R_i = \phi_i^{-1}\phi_i$, $S_i = \psi_i^{-1}\psi_i$, are compatible cofinite entourages over Γ and Δ respectively, for each $i \in I$. For all $i \in I$ and all $\gamma, \delta \in \Gamma$, $\phi_i(\gamma) = \phi_i(\delta)$ if and only if $(\gamma, \delta) \in R_i$, that is if and only if $R_i[\gamma] = R_i[\delta]$. Thus without loss of generality one can take $\Gamma/R_i = \phi_i(\Gamma)$ by identifying y with $R_i[x]$, for all $y \in \phi_i(\Gamma)$ whenever $\phi_i(x) = y_i$ and similarly $\Delta/S_i = \psi_i(\Delta)$. Thus we can successfully claim that $\varprojlim \Gamma/R_i = \Gamma$ and $\varprojlim \Delta/S_i = \Delta$.

Now let R be any compatible cofinite entourage over Γ and let $\eta_R: \Gamma \rightarrow \Gamma/R$ be the natural quotient map. Since Γ/R is discrete there exists some R_i such that the following diagram commutes for some continuous map of graphs $\phi_{RR_i}: \Gamma/R_i \rightarrow \Gamma/R$.

$$\begin{array}{ccc} \Gamma & & \\ \downarrow \phi_i & \searrow \eta_R & \\ \Gamma/R_i & \xrightarrow{\phi_{RR_i}} & \Gamma/R \end{array}$$

Clearly, for $a, b \in \Gamma$, $(a, b) \in R_i$ implies $\phi_i(a) = \phi_i(b)$. This implies $\phi_{RR_i}\phi_i(a) = \phi_{RR_i}\phi_i(b)$; hence $\eta_R(a) = \eta_R(b)$, so $(a, b) \in R$. Hence $R_i \subseteq R$. Thus $\{R_i \mid i \in I\}$ forms a fundamental system of compatible cofinite entourages over Γ . Similarly $\{S_i \mid i \in I\}$ forms a fundamental systems of compatible cofinite entourages over Δ .

- (3) Let us denote $f_i|_{f_i^{-1}[\psi_i(\Delta)]}$ by g_i , $\psi_i(\Delta)$ by Δ'_i and $f_i^{-1}[(\psi_i(\Delta))]$ by Γ'_i . Thus $f_i|_{f_i^{-1}[(\psi_i(\Delta))]}: f_i^{-1}[(\psi_i(\Delta))] \rightarrow \psi_i(\Delta)$ will now be viewed as $g_i: \Gamma'_i \rightarrow \Delta'_i$. It follows that for all $x \in \Gamma'_i$, $g_i(x) \in \psi_i(\Delta) = \Delta'_i$. So, g_i is well defined. Now g_i is locally injective as f_i is so. For the local surjectiveness of g_i , let $x_i \in V(\Gamma'_i)$ and e_i in $L_{g_i(x_i)}$ in $\Delta'_i \subseteq \Delta$. So, by local bijectivity of f_i , there exists a unique e'_i in L_{x_i} in Γ_i such that $f_i(e'_i) = e_i \in \Delta'_i = \psi_i(\Delta)$. So $(e'_i) \in f_i^{-1}[(\psi_i(\Delta))] = \Gamma'_i$.

Now as $f_i[\phi_i(\Gamma)] = \psi_i[f(\Gamma)] \subseteq \psi_i(\Delta)$ one can say that $\phi_i(\Gamma) \subseteq f_i^{-1}[(\psi_i(\Delta))] = \Gamma'_i$, for all i in I . Also for $i, j \in I$ with $j \geq i$, if $x_j \in f_j^{-1}[(\psi_j(\Delta))]$ then $f_j(x_j) \in \psi_j(\Delta)$. So, $f_j(x_j) = \psi_j(y)$ for some $y \in \Delta$. That is, $\psi_{ij}f_j(x_j) = \psi_{ij}\psi_j(y) = \psi_i(y) \in \psi_i(\Delta)$. This means $f_i\phi_{ij}(x_j) = \psi_i(y) \in \psi_i(\Delta)$. So, $\phi_{ij}(x_j) \in f_i^{-1}[(\psi_i(\Delta))]$. Thus

$$\phi_{ij}|_{f_j^{-1}[\psi_j(\Delta)]}: f_j^{-1}[\psi_j(\Delta)] \rightarrow f_i^{-1}[\psi_i(\Delta)]$$

is well defined and hence by our earlier notation

$$(\Gamma'_i, \phi_{ij})_{i,j \in I, j \geq i}$$

(where, by a little abuse of notation, we are assuming $\phi_{ij} = \phi_{ij}|_{\Gamma'_j}$) forms an inverse system of finite discrete graphs where $\phi_i(\Gamma) \subseteq \Gamma'_i \subseteq \Gamma_i$, for all i in I . Thus $\Gamma = \varprojlim \phi_i(\Gamma) = \varprojlim \Gamma'_i = \varprojlim \Gamma_i$. So without loss of generality, we are able to replace Δ_i by $\Delta'_i = \psi_i(\Delta) = \Delta/S_i$ and Γ_i by $\Gamma'_i = f_i^{-1}(\psi_i(\Delta))$. Hence, if Δ is connected, then each Δ/S_i is path connected and we may assume that each Δ_i is path connected.

- (4) Let us choose a vertex $v = (v_i)_{i \in I}$ in $\Gamma = \varprojlim \Gamma_i$ where, without loss of generality, we may assume $(f_i^{-1}(\Delta_i)) = \Gamma_i$ and $\psi_i(\Delta) = \Delta_i$. Since Δ is cofinitely connected, each Δ_i is path connected. Let us choose Γ'_i as the path component of Γ_i containing v_i . Let us now denote $f_i|_{\Gamma'_i}$ by g_i . Then $g_i: \Gamma'_i \rightarrow \Delta_i$ is well-defined. Now g_i is locally injective as f_i is so. Let $x_i \in V(\Gamma'_i)$. Consider $g_i(x_i) = y_i \in V(\Delta_i)$ and $e_i \in L_{y_i}$. Then there exists a unique e'_i in L_{x_i} in Γ_i such that $f_i(e'_i) = e_i$. Since $x_i \in \Gamma'_i$ and Γ'_i is a path component, $e'_i \in \Gamma'_i$. Thus g_i is locally surjective and hence is locally bijective. Now $\phi_{ij}(v_j) = v_i \in \Gamma'_i$ for any $i, j \in I$ with $j \geq i$. Again, ϕ_{ij} being a map of graphs and Γ'_j being path connected, $\phi_{ij}(\Gamma'_j)$ is a path connected graph containing v_i . So $\phi_{ij}(\Gamma'_j) \subseteq \Gamma'_i$. Thus $(\Gamma'_i, \phi_{ij})_{i,j \in I, j \geq i}$, (where by a little abuse of notation we are assuming $\phi_{ij} = \phi_{ij}|_{\Gamma'_j}$) forms an inverse system of maps of finite discrete graphs. Also, since Γ is profinitely connected, each $\phi_i(\Gamma) = \Gamma/R_i$ is path connected, and $v_i \in \Gamma'_i \cap \phi_i(\Gamma)$. It follows that $\phi_i(\Gamma) \subseteq \Gamma'_i \subseteq \Gamma_i$. So, $\Gamma = \varprojlim \phi_i(\Gamma) = \varprojlim (\Gamma'_i) = \varprojlim \Gamma_i = \Gamma$.

Thus without loss of generality we may assume that each $\Gamma_i = \Gamma'_i$ is path connected and also each Δ_i is path connected.

□

In the following lemma, we list some elementary properties of coverings.

LEMMA 3.3. *Let $f: \Gamma \rightarrow \Delta$ be a covering of profinite graphs. Then the following properties hold:*

- (a) *f is a locally bijective map of graphs;*
- (b) *if Δ is connected, then f is surjective and thus f is a quotient map;*
- (c) *For each edge $e \in \Delta$, we define a map $a \mapsto a.e$ from $f^{-1}(s(e))$ to $f^{-1}(t(e))$ by requiring that $s(\tilde{e}).e = t(\tilde{e})$ for all \tilde{e} in $f^{-1}(e)$. Then $a \mapsto a.e$ is an isomorphism of uniform spaces.*

PROOF. We have $f = \varprojlim (f_i : \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$, where each f_i is locally bijective.

- (a) We claim f is locally injective: Let $x = (x_i)_{i \in I} \in V(\Gamma)$. Let $e_1 = (e_{1,i})_{i \in I}, e_2 = (e_{2,i})_{i \in I}$ with $e_1, e_2 \in L_x$ in Γ , and $f(e_1) = f(e_2)$. So, $\psi_i f(e_1) = \psi_i f(e_2)$ for all $i \in I$. Thus we have $(f_i \phi_i)(e_1) = (f_i \phi_i)(e_2), \forall i \in I$, so $f_i(e_{1,i}) = f_i(e_{2,i}), \forall i \in I$. Now $s(e_{1,i}) = (x_i) = s(e_{2,i}), \forall i \in I$. So $f_i : \Gamma_i \rightarrow \Delta_i$ being locally injective, we have $e_{1,i} = e_{2,i}, \forall i \in I$. Thus $e_1 = e_2$.

Also f is locally surjective: Let $x = (x_i)_{i \in I} \in V(\Gamma)$. Then f , being a map of graphs, $f(x) \in V(\Delta)$. Let $f(x) = y = (y_i)_{i \in I} \in V(\Delta)$. Let $e = (e_i)_{i \in I} \in L_{f(x)} = L_y$. So, $e_i \in L_{y_i}, \forall i \in I$. Now $y_i = \psi_i f(x) = (f_i \phi_i)(x) = f_i(x_i), \forall i \in I$. Now, as each f_i is locally bijective, there exists a unique edge $e'_i \in L_{x_i}$ in Γ_i such that $f_i(e'_i) = e_i$. It remains to check that $e' = (e'_i)_{i \in I} \in \Gamma$ as $s(e') = s((e'_i)_{i \in I}) = (s(e'_i))_{i \in I} = (x_i)_{i \in I} = x$. For all $i, j \in I, f_i \phi_{ij}(e'_j) = \psi_{ij} f_j(e'_j) = \psi_{ij}(e_j) = e_i$, where $j \geq i$. Also $s(\phi_{ij}(e'_j)) = \phi_{ij}(s(e'_j)) = \phi_{ij}(x_j) = x_i$ and each f_i is locally bijective. Thus $\phi_{ij}(e'_j) = e'_i, \forall i, j \in I$ where $j \geq i$.

Now $\psi_i f(e') = f_i \phi_i(e') = f_i(e'_i) = e_i, \forall i \in I$. This means $f(e') = e$.

Hence the result follows.

- (b) As in remark (4) above, we may assume that each Δ_i is path connected. Since (Γ_i, f_i) is a (nonempty) covering graph of the path connected graph Δ_i , it follows that each f_i is surjective. Let $y \in \Delta$ be given and write $y_i = \psi_i(y)$ for $i \in I$. Then $X_i = f_i^{-1}(y_i)$ is nonempty and compact for each $i \in I$. Note that for all $i \leq j$, $\phi_{ij}(X_j) \subseteq X_i$. For, if $x_j \in X_j$, $f_j(x_j) = y_j$. So, $y_i = \psi_{ij}(y_j) = \psi_{ij} f_j(x_j) = f_i \phi_{ij}(x_j)$. So we have an inverse system of nonempty compact Hausdorff spaces (X_i, ϕ'_{ij}) , where each $\phi'_{ij}: X_j \rightarrow X_i$ is the restriction of ϕ_{ij} . Hence $X = \varprojlim X_i$ is nonempty and $X \subseteq \Gamma$. Choose any $x \in X$ and write $\phi_i(x) = x_i$ for all $i \in I$, then $f_i(\phi_i(x)) = y_i$. This means $\psi_i f(x) = y_i$ for all $i \in I$. Hence $f(x) = y$.

Moreover since f is a continuous map from a compact space Γ to a Hausdorff space Δ it is a closed continuous surjection and hence a quotient map.

- (c) Let $\tau: f^{-1}(s(e)) \rightarrow f^{-1}(t(e))$ denote the mapping taking $a \mapsto a.e$ where $a.e = t(\tilde{e})$ whenever $s(\tilde{e}) = a$, for all \tilde{e} in $f^{-1}(e)$. Clearly τ is well defined as $f(a.e) = f(t(\tilde{e})) = t(f(\tilde{e})) = t(e)$ and there exists a unique $\tilde{e} \in L_a$ for any $a \in f^{-1}(s(e))$. Also τ is a bijection with inverse $\eta: f^{-1}(t(e)) \rightarrow f^{-1}(s(e))$, the mapping taking $b \mapsto b.\tilde{e}$.

By Remark 2 above, $\{R_i \mid i \in I\}$ forms a fundamental system of entourages over Γ . Consider $R_i = \phi_i^{-1} \phi_i$ for some $i \in I$. Let, $a_1, a_2 \in f^{-1}(s(e))$ and $(a_1, a_2) \in R_i$. Then, $\phi_i(a_1) = \phi_i(a_2)$. Also $\phi_i(a_1).\psi_i(e) = \phi_i(a_2).\psi_i(e)$ where we consider the map $\tau_i: f_i^{-1}(s(\psi_i(e))) \rightarrow f_i^{-1}(t(\psi_i(e)))$ in the corresponding finite graph level behaving exactly as in the situation of τ . Now if $\tilde{e}_1, \tilde{e}_2 \in f^{-1}(e)$, $a_1, a_2 \in f^{-1}(s(e))$, then $\phi_i(a_1).\psi_i(e) = t\phi_i(\tilde{e}_1) = \phi_i(t(\tilde{e}_1)) = \phi_i(a_1.e)$ and $\phi_i(a_2).\psi_i(e) = t\phi_i(\tilde{e}_2) = \phi_i(t(\tilde{e}_2)) = \phi_i(a_2.e)$. So, since we

have $\phi_i(a_1).\psi_i(e) = \phi_i(a_2).\psi_i(e)$, it follows that $(a_1.e, a_2.e) \in R_i$. Thus τ is uniformly continuous. Hence the result follows.

□

CHAPTER 4

General lifting criterion

In this section we give a necessary and sufficient condition for a continuous map of profinite graphs to lift to a profinite covering graph. Our condition is a projective analogue of the well-known condition in the topological theory of covering spaces. As in that classical theory, we first observe that the usual uniqueness of lifts result holds for coverings of profinite groups.

Let $f: \Gamma \rightarrow \Delta$ be a covering of profinite graphs and fix an inverse system $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$ of locally bijective maps of finite discrete graphs, indexed by I , such that $f = \varprojlim f_i$. Denote the canonical projections by $\phi_i: \Gamma \rightarrow \Gamma_i$ and $\psi_i: \Delta \rightarrow \Delta_i$ and assume, as we may, that each ψ_i is surjective and identify Δ_i with Δ/S_i where $S_i = \psi_i^{-1}\psi_i$.

LEMMA 4.1. *Let $f: \Gamma \rightarrow \Delta$ be a covering of profinite graphs and let Σ be a connected profinite graph. If $h_1, h_2: \Sigma \rightarrow \Gamma$ are continuous maps of graphs such that $fh_1 = fh_2$, and if $h_1(c) = h_2(c)$ for some $c \in \Sigma$, then $h_1 = h_2$.*

PROOF. Before we actually start proving this lemma, let us notice a general fact about maps of profinite graphs and compatible cofinite entourages.

Let $f: \Gamma \rightarrow \Delta$ be a map of profinite graphs and R, S be two compatible cofinite entourages over Γ, Δ respectively such that $(f \times f)[R] \subseteq S$. Then $f_{SR}: \Gamma/R \rightarrow \Delta/S$ defined via $f_{SR}(R[x]) = S[f(x)]$, for all $x \in \Gamma$ is a well defined map of graphs. We will only prove the fact that f_{SR} is well defined as the rest of the claim is obvious. For if $R[x] = R[y]$, then $(x, y) \in R$, yielding $(f(x), f(y)) \in (f \times f)[R] \subseteq S$. Thus $S[f(x)] = S[f(y)]$, that is $f_{SR}(R[x]) = f_{SR}(R[y])$.

Now let us start proving the original lemma. Without loss of generality we may assume that $c \in V(\Sigma)$. Let $h_1(c) = h_2(c)$. Let us take $x \in \Sigma$. Consider $h_1(x)$ and $h_2(x)$. Let $i \in I$. Set

$$T_i = (h_1 \times h_1)^{-1}(R_i) \bigcap (h_2 \times h_2)^{-1}(R_i)$$

Then T_i is a compatible cofinite entourage over Σ . Since Σ is profinitely connected, Σ/T_i is path connected. Let $g = fh_1 = fh_2$. Then, $(g \times g)[T_i] = (f \times f)((h_1 \times h_1)[T_i]) \subseteq (f \times f)[R_i] \subseteq S_i$. For if $(x, y) \in R_i$ then $\phi_i(x) = \phi_i(y)$, so $f_i\phi_i(x) = f_i\phi_i(y)$. Hence, $\psi_i f(x) = \psi_i f(y)$. This implies $(f(x), f(y)) \in S_i$. So, now consider the natural maps of graphs $g_{S_i T_i}: \Sigma/T_i \rightarrow \Delta/S_i$ and $f_{S_i R_i}: \Gamma/R_i \rightarrow \Delta/S_i$. Also, we observe that $(h_1 \times h_1)[T_i] \subseteq R_i, (h_2 \times h_2)[T_i] \subseteq R_i$. So, again consider the natural maps of graphs $(h_1)_{R_i T_i}: \Sigma/T_i \rightarrow \Gamma/R_i$, and $(h_2)_{R_i T_i}: \Sigma/T_i \rightarrow \Gamma/R_i$. We note that $f_{S_i R_i}(h_1)_{R_i T_i} = f_{S_i R_i}(h_2)_{R_i T_i}$ since $fh_1 = fh_2$, and as $h_1(c) = h_2(c), (h_1)_{R_i T_i}(T_i[c]) = (h_2)_{R_i T_i}(T_i[c])$. Now we have Γ/R_i is embedded inside Γ_i as a subgraph, and so is Δ/S_i inside Δ_i , where $f_i: \Gamma_i \rightarrow \Delta_i$ is a local bijection of finite graphs and the restriction of f_i over Γ/R_i is $f_{S_i R_i}$. Hence by the results of abstract graph theory it follows that $(h_1)_{R_i T_i} = (h_2)_{R_i T_i}$ since Σ/T_i is path connected. Thus we have for any $x \in \Sigma, R_i[h_1(x)] = (h_1)_{R_i T_i}(T_i(x)) = (h_2)_{R_i T_i}(T_i(x)) = R_i[h_2(x)]$ for any $i \in I$. Hence, $(h_1(x), h_2(x)) \in R_i$, for all $i \in I$. Thus, $h_1(x) = h_2(x)$, for all $x \in \Sigma$. \square

THEOREM 4.2. *Suppose Σ is a connected profinite graph, $g: \Sigma \rightarrow \Delta$ is a continuous map of graphs, and let a, b, c be vertices of Γ, Δ, Σ such that $f(a) = b = g(c)$. Then there exists a unique continuous map of graphs $h: \Sigma \rightarrow \Gamma$ such that $h(c) = a$ and $fh = g$ if and only if for each $i \in I$, there exists a compatible cofinite entourage T_i of Σ such that $(g \times g)[T_i] \subseteq S_i$ and $g_{S_i T_i} \pi_1(\Sigma/T_i, T_i[c]) \subseteq f_i \pi_1(\Gamma_i, a_i)$, where $g_{S_i T_i}: (\Sigma/T_i, T_i[c]) \rightarrow (\Delta_i, b_i)$ is the natural map, with $\Delta_i = \Delta/S_i, b_i = \psi_i(b)$.*

$$\begin{array}{ccc}
& & \Gamma, a \\
& \nearrow h & \downarrow f \\
\Sigma, c & \xrightarrow{g} & \Delta, b
\end{array}$$

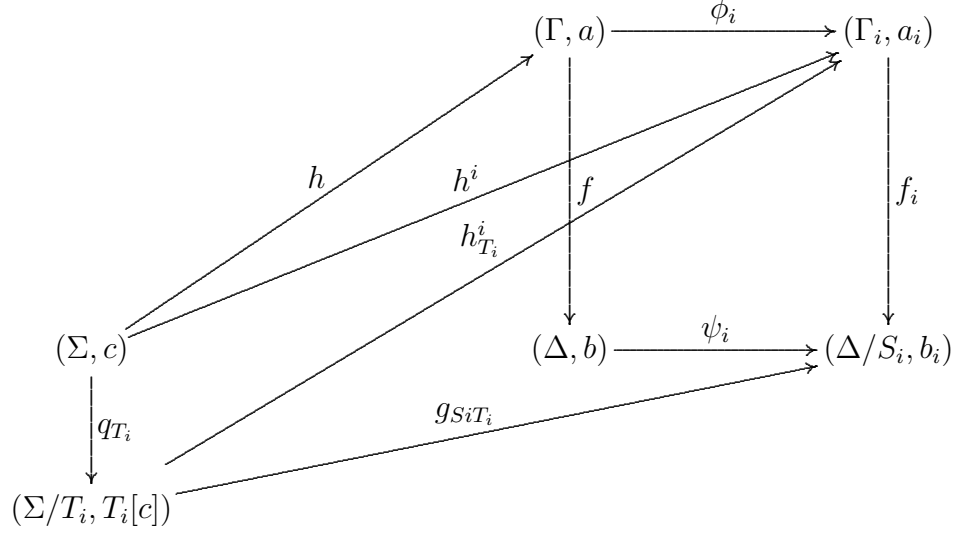
Note that for each $i \in I$, the containment of the images of the fundamental groups in the condition of the theorem takes place in the group $\pi_1(\Delta/S_i, S_i[b])$.

PROOF. Let us first assume that for each $i \in I$, there exists a compatible cofinite entourage T_i of Σ such that $(g \times g)[T_i] \subseteq S_i$ and $g_{S_i T_i} \pi_1(\Sigma/T_i, T_i[c]) \subseteq f_i \pi_1(\Gamma_i, a_i)$, where, by abuse of notation,

$$g_{S_i T_i}: \pi_1(\Sigma/T_i, T_i[c]) \rightarrow \pi_1(\Delta_i, b_i)$$

is the induced map of fundamental groups obtained from the map of abstract graphs $g_{S_i T_i}: (\Sigma/T_i, T_i[c]) \rightarrow (\Delta_i, b_i)$ with $g_{S_i T_i}(T_i[x]) = S_i[g(x)] = \psi_i(g(x))$. This is well defined as $(g \times g)[T_i] \subseteq S_i = \psi_i^{-1}\psi_i$. Let $a = (a_i)_{i \in I}, b = (b_i)_{i \in I}$. Then $f_i(a_i) = b_i$, for all $i \in I$ as in the last result. Since Σ is profinitely connected Σ/T_i is path connected. As $f_i: (\Gamma_i, a_i) \rightarrow (\Delta_i, b_i)$ is a local bijection and $g_{S_i T_i} \pi_1(\Sigma/T_i, T_i[c]) \subseteq f_i \pi_1(\Gamma_i, a_i)$, by the general lifting criterion of finite graphs there exists a unique lift $h_{T_i}^i: (\Sigma/T_i, T_i[c]) \rightarrow (\Gamma_i, a_i)$ such that $f_i h_{T_i}^i = g_{S_i T_i}$. Define $h^i: \Sigma \rightarrow \Gamma_i$ by $h^i = h_{T_i}^i q_{T_i}$ where $q_{T_i}: \Sigma \rightarrow \Sigma/T_i$ is the natural quotient map. Then h^i is a continuous map of graphs as $h_{T_i}^i$ is a continuous map of finite discrete graphs and q_{T_i} is a quotient map

of graphs and hence both of them are continuous.



We should now check that h^i is independent of the choice of T_i . Let there exists another compatible cofinite entourage W_i over Σ such that the above is true. Then there exists a continuous map of graphs

$$h_{W_i}^i : (\Sigma/W_i, W_i[c]) \rightarrow (\Gamma_i, a_i)$$

such that $f_i h_{W_i}^i = g_{S_i W_i}$. Then $h_{W_i}^i q_{W_i}(c) = a_i = h_{T_i}^i q_{T_i}(c)$ where $q_{W_i} : \Sigma \rightarrow \Sigma/W_i$ is the natural quotient map of graphs. As Σ is a profinitely connected graph and $f_i : (\Gamma_i, a_i) \rightarrow (\Delta_i, b_i)$ is a covering map of finite graphs and it is a covering map of profinite graphs by Lemma 4.1, $h_{T_i}^i q_{T_i} = h_{W_i}^i q_{W_i} = h^i$. So, now we define $h : \Sigma \rightarrow \Gamma$ as $h(x) = (h^i(x))_{i \in I}$. Let us check that $h(x) \in \Gamma$. So, we have to show that for any $x \in \Sigma, i \leq j \in I, \phi_{ij} h^j(x) = h^i(x)$. By Lemma 4.1 it is enough to show that $\phi_{ij} h^j(c) = h^i(c)$ and $f_i \phi_{ij} h^j = f_i h^i$. Now $\phi_{ij} h^j(c) = \phi_{ij}(a_j) = a_i = h^i(c)$. Also $f_i h^i(x) = f_i h_{T_i}^i q_{T_i}(x) = g_{S_i T_i} q_{T_i}(x) = \psi_i g(x) = \psi_{ij} \psi_j g(x) = \psi_{ij} g_{S_j T_j}(T_j[x]) = \psi_{ij} f_j h_{T_j}^j(T_j[x]) = \psi_{ij} f_j h^j(x) = f_i \phi_{ij} h^j(x)$, where T_j is a compatible cofinite entourage over Σ satisfying $(g \times g)[T_j] \subseteq S_j$ and $g_{S_j T_j} \pi_1(\Sigma/T_j, T_j[c]) \subseteq f_j \pi_1(\Gamma_j, a_j)$. Also $h^j, h_{T_j}^j$ are defined accordingly. Thus, $f_i h^i = f_i \phi_{ij} h^j$. So, $h(x) \in \Gamma$. Here h , being a

continuous map of profinite graphs (as each h^i is so), is also a uniformly continuous map of profinite graphs. So it remains to see that $fh(x) = g(x)$, for all $x \in \Sigma$. Now for any $i \in I$, $\psi_i fh(x) = f_i \phi_i h(x) = f_i h^i(x) = f_i h_{T_i}^i(T_i[x]) = g_{S_i T_i} q_{T_i}(x) = \psi_i g(x)$, for all $x \in \Sigma$. Thus $fh = g$. Also, the choice of h is unique by Lemma 4.1.

Conversely, let such h exist. Let $i \in I$ and let R_i, S_i be as in remark 2 to Definition 3.1. Then $T_i = (g \times g)^{-1}(S_i) \cap (h \times h)^{-1}(R_i)$ is a compatible cofinite entourage over Σ with $(g \times g)[T_i] \subseteq S_i$. Let us define $h_i: \Sigma/T_i \rightarrow \Gamma_i$ via $h_i(T_i[x]) = \phi_i h(x)$. Then h_i is well defined as $T_i \subseteq (h \times h)^{-1}(R_i)$. Furthermore $h_i(T[c]) = \phi_i h(c) = a_i$. Now $f_i h_i(T_i[x]) = f_i \phi_i h(x) = \psi_i fh(x) = \psi_i g(x) = g_{S_i T_i}(T_i[x])$. So, $f_i h_i = g_{S_i T_i}$. Thus by the lifting criterion of finite graph theory, since f_i is a local bijection of finite graphs and Σ/T_i is path connected, $g_{S_i T_i} \pi_1(\Sigma/T_i, T_i[c]) \subseteq f_i \pi_1(\Gamma_i, a_i)$.

□

CHAPTER 5

Covering transformations

We begin by defining homomorphisms and isomorphisms of covering graphs of a given profinite graph in the usual way. Here, and elsewhere, let Δ be a fixed profinite graph.

DEFINITION 5.1. Let (Γ_1, f_1) and (Γ_2, f_2) be profinite covering graphs of Δ . A *homomorphism* from (Γ_1, f_1) to (Γ_2, f_2) is a continuous map of graphs $h: \Gamma_1 \rightarrow \Gamma_2$ such that $f_2 h = f_1$. A homomorphism h from (Γ_1, f_1) to (Γ_2, f_2) is called an *isomorphism* if there exists a homomorphism g from (Γ_2, f_2) to (Γ_1, f_1) such that both gh and hg are identity maps.

An isomorphism h of a profinite covering graph (Γ, f) of Δ to itself is called an automorphism or *covering transformation*. The set of all covering transformations of (Γ, f) is a group under composition of maps; we denote it by $\text{Aut}(\Gamma, f)$.

As an immediate consequence of Lemma 4.1 we see that: if (Γ_1, f_1) and (Γ_2, f_2) are connected profinite covering graphs of Δ , and if h_1 and h_2 are homomorphisms from (Γ_1, f_1) to (Γ_2, f_2) such that $h_1(a) = h_2(a)$ for some $a \in \Gamma_1$, then $h_1 = h_2$. Applying this to covering transformations, we obtain the following.

LEMMA 5.2. *If $f: \Gamma \rightarrow \Delta$ is a covering of profinite graphs and Γ is connected, then the group of covering transformations $\text{Aut}(\Gamma, f)$ acts freely on Γ .*

Recall that a group action of an abstract group G on a uniform space X is said to be *uniformly equicontinuous* if the set of translations $\{x \mapsto g \cdot x \mid g \in G\}$ is a uniformly

equicontinuous family of functions from X to itself, that is, if for each entourage W of X , there is an entourage V of X such that $(g \times g) \cdot V \subseteq W$ for all $g \in G$.

LEMMA 5.3. *If $f: \Gamma \rightarrow \Delta$ is a covering of profinite graphs and Γ is connected, then the group $\text{Aut}(\Gamma, f)$ acts uniformly equicontinuously on Γ .*

PROOF. Let $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij})$ be an inverse system of locally bijective maps of finite discrete graphs such that $f = \varprojlim f_i$. Suppose that I is the directed index set corresponding to this inverse system. Then for $i \in I$ and each $\alpha \in \text{Aut}(\Gamma, f)$, the map $\phi_i \alpha$ is a lift of $\psi_i f$ to the finite covering graph (Γ_i, f_i) of Δ_i .

$$\begin{array}{ccccc}
 & & \Gamma & \xrightarrow{\phi_i} & \Gamma_i \\
 & \nearrow \alpha & \downarrow f & & \downarrow f_i \\
 \Gamma & \xrightarrow{f} & \Delta & \xrightarrow{\psi_i} & \Delta_i
 \end{array}$$

Since, by Lemma 4.1, two lifts of $\psi_i f$ are equal if they agree at one point, it follows that the set of maps $A_i = \{\phi_i \alpha: \Gamma \rightarrow \Gamma_i \mid \alpha \in \text{Aut}(\Gamma, f)\}$ is finite, since Γ_i is finite. For each $g = \phi_i \alpha: \Gamma \rightarrow \Gamma_i$ in A_i , let $T_g = g^{-1}g = \alpha^{-1} \phi_i^{-1} \phi_i \alpha = (\alpha \times \alpha)^{-1}[R_i]$, where $R_i = \phi_i^{-1} \phi_i$. Since R_i is a compatible cofinite entourage of Γ and α is uniformly continuous, it follows that the (finite) intersection $T = \bigcap_{g \in A_i} T_g$ is a compatible cofinite entourage of Γ . Now, for each $\alpha \in \text{Aut}(\Gamma, f)$, $\phi_i \alpha = g$ for some $g \in A_i$ and hence $(\alpha \times \alpha)[T] \subseteq (\alpha \times \alpha)[T_g] \subseteq R_i$. However, the set of all R_i , $i \in I$, is a fundamental system of entourages of Γ . Thus it follows that the action of $\text{Aut}(\Gamma, f)$ on Γ is uniformly equicontinuous. \square

Given a connected profinite covering graph (Γ, f) of Δ , the previous lemma shows that $\text{Aut}(\Gamma, f)$ acts uniformly equicontinuously on Γ . Thus, by Note 2.51, $\text{Aut}(\Gamma, f)$, with the uniformity induced by its action on Γ , is a cofinite group. From here on, we will always endow the group of covering transformations of a connected profinite

covering graph with its cofinite group structure arising in this way. Moreover, we next prove that even more is true.

LEMMA 5.4. *If $f: \Gamma \rightarrow \Delta$ is a covering of profinite graphs and Γ is connected, then $\text{Aut}(\Gamma, f)$ is a profinite group.*

PROOF. Observe that $G = \text{Aut}(\Gamma, f)$ acts uniformly equicontinuously on Γ . Hence, \widehat{G} , the profinite completion of G with respect to the induced separating filter base of cofinite congruences obtained from the action of G over Γ , also acts over Γ uniformly equicontinuously. In order to show that G is compact we wish to show that $\widehat{G} \subseteq G$. Let $\alpha \in \widehat{G}$. Then since the action of \widehat{G} over Γ preserves the graph structure of Γ , $\alpha: \Gamma \rightarrow \Gamma$ is a map of graphs. Also α is continuous as the group action of \widehat{G} over Γ is uniformly continuous, by Note 2.56. Also $\alpha^{-1} \in \widehat{G}$, as \widehat{G} is a group. So, $\alpha: \Gamma \rightarrow \Gamma$ is a continuous map of graphs with continuous inverse.

Now in order to show $\alpha \in G$ we have to prove $f\alpha = f$. Note that $\alpha = (N_R[\alpha_R])_{R \in J}$ where J is the fundamental system of G -invariant compatible cofinite entourages over Γ , N_R is the cofinite congruence over G corresponding to the compatible cofinite entourage R , due to the group action $G \times \Gamma \rightarrow \Gamma$, and $\alpha_R \in G$, for all $R \in J$. So, for any $i \in I$, there exists a G -invariant compatible cofinite entourage $R \subseteq R_i$. So, we can obtain a well-defined map of graphs $f_{S_i R}: \Gamma/R \rightarrow \Delta/S_i$, as $R \subseteq R_i$ implies that $(f \times f)[R] \subseteq (f \times f)[R_i] \subseteq S_i$, since $f_i R_i[x] = f_i \phi_i(x) = \psi_i f(x) = S_i(f(x))$ and if $(x, y) \in R_i$ then $f_i R_i[x] = f_i R_i[y]$, which means $S_i[f(x)] = S_i[f(y)]$. Now for all $i \in I$, and for all $x \in \Gamma$, since $f \circ \alpha_R = f$

$$S_i[f\alpha(x)] = f_{S_i R}(R[\alpha(x)]) = f_{S_i R}(R[\alpha_R(x)]) = S_i[f\alpha_R(x)] = S_i[f(x)]$$

That means that for all $i \in I$, $\psi_i f\alpha(x) = \psi_i f(x)$. Thus $f \circ \alpha = f$. □

Combining the above lemmas, we have the following theorem.

THEOREM 5.5. *Let $f: \Gamma \rightarrow \Delta$ be a covering of profinite graphs, where Γ is connected. Then the group of covering transformations $\text{Aut}(\Gamma, f)$ is a profinite group that acts freely and uniformly equicontinuously on Γ .*

CHAPTER 6

Regular coverings

Recall the classical definition of a regular covering of abstract path connected graphs: A covering $f: \Gamma \rightarrow \Delta$ of path connected graphs is called a regular covering if for any vertex a of Γ , the image of the induced monomorphism of fundamental groups $f(\pi_1(\Gamma, a))$ is a normal subgroup of $\pi_1(\Delta, f(a))$. Equivalently, it is well known that a covering $f: \Gamma \rightarrow \Delta$ of path connected graphs is regular if and only if its group of covering transformations $\text{Aut}(\Gamma, f)$ acts transitively on the fiber $f^{-1}(b)$ for any vertex b of Δ .

In light of our definition of a covering of profinite graphs (Definition 3.1), it seems only natural to define a regular covering of profinite graphs to be a projective limit of regular coverings of finite path connected graphs. This is what we do in this section. However, in the next section we will see that there is an equivalent (more intrinsic) way to describe regular coverings.

DEFINITION 6.1 (Regular covering). A map $f: \Gamma \rightarrow \Delta$ of profinite graphs is a *regular covering* if it is the inverse limit of an inverse system of maps of graphs $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij}), i, j \in I, j \geq i$ where each $f_i: \Gamma_i \rightarrow \Delta_i$ is a regular covering of path connected finite discrete graphs. In this situation, we call the pair (Γ, f) a *regular profinite covering graph* of Δ .

Throughout this chapter let $f: \Gamma \rightarrow \Delta$ be a regular covering of profinite graph which is the inverse limit of an inverse system of maps of graphs $(f_i: \Gamma_i \rightarrow \Delta_i, \phi_{ij}, \psi_{ij}), i, j \in I, j \geq i$ where each $f_i: \Gamma_i \rightarrow \Delta_i$ is a regular covering of path connected finite discrete graphs. We know that in this situation, all Γ_i, Δ_i are path

connected as in remark 4 in Chapter 3. Consider $R_i = \phi_i^{-1}\phi_i, S_i = \psi_i^{-1}\psi_i$ as in remark 2 in Chapter 3. So, Γ/R_i is isomorphic to $\phi_i(\Gamma) \subseteq \Gamma_i$, and $\Delta_i = \Delta/S_i$ as in remark 3 in Chapter 3. Also we will assume everywhere in this chapter that $G = \text{Aut}(\Gamma, f)$, the group of covering transformations of the regular profinite covering graph Γ corresponding to f .

We first show that, just as for regular coverings of path connected graphs and spaces in the classical sense, the group of covering transformations of a regular profinite covering graph acts transitively on each fiber.

LEMMA 6.2. *If a_1, a_2 are vertices of Γ such that $f(a_1) = f(a_2)$, then there exists a unique covering transformation α of (Γ, f) such that $\alpha(a_1) = a_2$.*

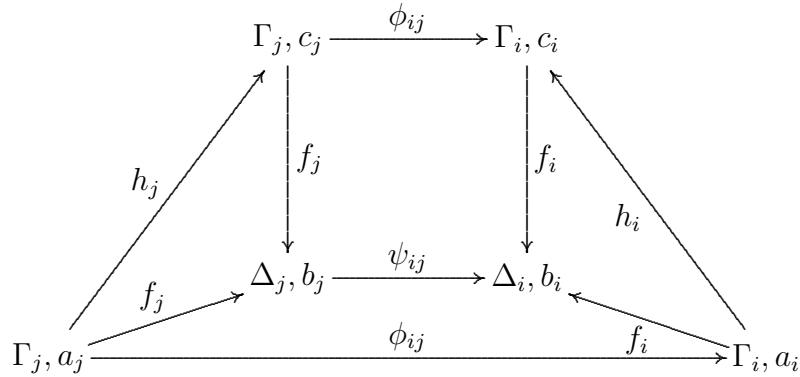
PROOF. Let $f(a_1) = f(a_2) = b = (b_i)_{i \in I}$ and $a_1 = (a_{1,i})_{i \in I}, a_2 = (a_{2,i})_{i \in I}$. Then for all $i \in I, f_i(a_{1,i}) = b_i = f_i(a_{2,i})$. Now Γ/R_i is isomorphic to $\phi_i(\Gamma) \subseteq \Gamma_i$. Also $f_i: \Gamma_i \rightarrow \Delta_i$ is a map of graphs and $\Delta_i = \Delta/S_i$. Moreover f_i restricted to Γ/R_i can be viewed as the natural map of graphs $f_{S_i R_i}: \Gamma/R_i \rightarrow \Delta/S_i, (f \times f)(R_i) \subseteq S_i$. Now, $f_i: (\Gamma_i, a_{1,i}) \rightarrow (\Delta_i, b_i)$ is a regular covering of finite graphs. So $f_i \pi_1(\Gamma_i, a_{1,i}) = f_i \pi_1(\Gamma_i, a_{2,i})$. Thus, $f_{S_i R_i}(\pi_1(\Gamma/R_i, a_{1,i})) \subseteq f_i \pi_1(\Gamma_i, a_{1,i}) \subseteq f_i \pi_1(\Gamma_i, a_{2,i})$. Hence, by Theorem 4.2, there exists a unique continuous map of profinite connected graphs $\alpha: \Gamma \rightarrow \Gamma$ such that $\alpha(a_1) = a_2$ and $f\alpha = f$. Similarly, there exists a unique continuous map of profinite connected graphs $\beta: \Gamma \rightarrow \Gamma$ such that $\beta(a_2) = a_1$ and $f\beta = f$. So we have the following commutative diagram.

$$\begin{array}{ccc}
 & & (\Gamma, a_2) \\
 & \nearrow \alpha & \downarrow f \\
 (\Gamma, a_1) & \xrightarrow{f} & (\Delta, b)
 \end{array}$$

Since $\alpha\beta(a_2) = a_2, \beta\alpha(a_1) = a_1$ Lemma 4.1 implies that $\alpha\beta = \beta\alpha = \text{identity}$ of $\text{Aut}(\Gamma, f)$. Hence $\alpha \in \text{Aut}(\Gamma, f)$. \square

LEMMA 6.3. *Given any $h \in G, h = \varprojlim h_i$ where h_i is a covering transformation of the finite covering graph (Γ_i, f_i) and $\{R_i \mid i \in I\}$ is a fundamental system of G -invariant compatible cofinite entourages over Γ .*

PROOF. Consider $a \in V(\Gamma)$ and let $h(a) = c$. Let $f(c) = fh(a) = f(a) = b$, where $a = (a_i)_{i \in I}, b = (b_i)_{i \in I}, c = (c_i)_{i \in I}$. Then it follows that $f_i(a_i) = b_i = f_i(c_i)$ for all $i \in I$. Since $f_i: (\Gamma_i, c_i) \rightarrow (\Delta_i, b_i)$ is a regular covering of finite path connected graphs $f_i[\pi_1(\Gamma_i, c_i)] = f_i[\pi_1(\Gamma_i, a_i)]$ for each $i \in I$. So by the lifting criteria of Path Connected Coverings, for each $i \in I$, there exists a unique bijective map of graphs $h_i: (\Gamma_i, a_i) \rightarrow (\Gamma_i, c_i)$ such that $f_i h_i = f_i$. Define $\alpha: \Gamma \rightarrow \Gamma$ as $\alpha(x) = (h_i(x_i))_{i \in I}$ where $x = (x_i)_{i \in I}$. We prove that α is well defined. Now, $a \in \Gamma$ implies $\phi_{ij}(a_j) = a_i$, for all $i, j \in I, j \geq i$. Also $h_i \phi_{ij}(a_j) = c_i = \phi_{ij} h_j(a_j)$ and $f_i h_i \phi_{ij} = f_i \phi_{ij} = \psi_{ij} f_j = \psi_{ij} f_j h_j = f_i \phi_{ij} h_j$.



So, by Lemma 4.1, $h_i \phi_{ij} = \phi_{ij} h_j$. Thus we have $\phi_{ij} h_j(x_j) = h_i \phi_{ij}(x_j) = h_i(x_i)$. Hence α is a well defined continuous map of graphs as each h_i is so. Also $f\alpha(x) = f(h_i(x_i))_{i \in I}$. Now, for any $i \in I, \psi_i f\alpha(x) = \psi_i f(h_i(x_i))_{i \in I} = f_i \phi_i(h_i(x_i))_{i \in I} = f_i h_i(x_i) = f_i(x_i) = f_i \phi_i(x) = \psi_i f(x)$. Thus, $f\alpha = f$. Also, $\alpha(a) = c = h(a)$. Thus by the uniqueness part in Lemma 4.1 we have $\alpha = h$. Again, let $x = (x_i)_{i \in I}, y = (y_i)_{i \in I}$

and $(x, y) \in R_i$. This implies that $x_i = y_i$. So, $h_i(x_i) = h_i(y_i)$. This implies $\phi_i h(x) = \phi_i h(y)$. So, $(h(x), h(y)) \in R_i$, for all $h \in G$.

□

LEMMA 6.4. *Let $f: \Gamma \rightarrow \Delta$ be a regular covering of profinite graphs. Then Γ has a fundamental system of entourages consisting of G -invariant compatible cofinite equivalence relations R such that the induced faithful action of G/N_R on Γ/R is free, where N_R is the kernel of the induced action of G on Γ/R .*

PROOF. f is a covering of a profinite connected graph by Lemma 5.3, so G acts uniformly equicontinuously over Γ . Hence Γ has a fundamental system of G -invariant compatible cofinite entourages.

Consider the group action of G/N_{R_i} over Γ/R_i which is induced by the action of G over Γ . Let $N_{R_i}[h]R_i[x] = R_i[x]$ for some $h \in G$ and some $R_i[x] \in \Gamma/R_i$. So, $R_i[hx] = R_i[x]$. This means $x_i = R_i[x] = R_i[hx] = h_i(x_i)$, where $x = (x_i)_{i \in I}$, $h = \varprojlim h_i$ as in the previous lemma. Since h_i fixes one element of Γ_i , $h_i(y_i) = y_i$, for all $y_i \in \Gamma_i$, as $f_i \circ h_i = h_i$. This means $(hy, y) \in R_i$ for all $y \in \Gamma$. This implies $(h, 1_G) \in N_{R_i}$. Thus $N_{R_i}[h] = N_{R_i}[1_G]$, the identity element of G/N_{R_i} . Hence the action of G/N_{R_i} over Γ/R_i is free. □

For convenience, we give a name to the property that we just saw holds for the action of the group of covering transformations of a regular profinite covering graph.

DEFINITION 6.5. We say that an action of a group G on a profinite graph Γ is *residually free* if there exists a fundamental system I of G -invariant compatible cofinite entourages of Γ such that for each $R \in I$, the induced faithful action of G/N_R on Γ/R is free, where N_R is the kernel of the action of G on Γ/R .

The following theorem uses this terminology and summarizes what we have shown so far for regular coverings.

THEOREM 6.6. *Let $f: \Gamma \rightarrow \Delta$ be a regular covering of profinite graphs. Then G is a profinite group whose action on Γ is continuous, residually free, and simply transitive on the fiber $f^{-1}(b)$, for any vertex b of Δ .*

In the next section we prove that, conversely, every continuous residually free group action of a profinite group on a connected profinite graph gives rise to a regular covering.

CHAPTER 7

Good pairs and residually free group actions

DEFINITION 7.1. Let $f: \Gamma \rightarrow \Delta$ be a map of profinite graphs and let R, S be compatible cofinite entourages of Γ, Δ . We call (R, S) a *good pair* for f if $(f \times f)[R] \subseteq S$ and the induced map $f_{SR}: \Gamma/R \rightarrow \Delta/S$ is locally bijective. If in addition, f_{SR} is a regular covering of finite path connected graphs, then we say that the good pair (R, S) is *regular*.

A family I of good pairs for a map $f: \Gamma \rightarrow \Delta$ of profinite graphs is called a *fundamental system* of good pairs if for all entourages V, W of Γ, Δ , there exists $(R, S) \in I$ such that $(R, S) \subseteq (V, W)$. A fundamental system of regular good pairs for f is defined similarly.

LEMMA 7.2. *If there exists a fundamental system I of good pairs (respectively, regular good pairs) for a map $f: \Gamma \rightarrow \Delta$ of profinite graphs, then $f: \Gamma \rightarrow \Delta$ is a covering (respectively, regular covering).*

PROOF. The fundamental system I is a directed set under the partial ordering given by the opposite of inclusion of pairs. Now $\{R \mid (R, S) \in I\}$ forms a fundamental system of compatible cofinite entourages over Γ and $\{S \mid (R, S) \in I\}$ is a fundamental system of compatible cofinite entourages over Δ , as for any entourage T over Γ , and for $\Delta \times \Delta$ playing the role of an entourage over Δ , there is $(R, S) \in I$ such that $R \subseteq T$. Similarly the claim for Δ also follows. So, $\Gamma = \varprojlim \Gamma/R$ where R runs through $\{R \mid (R, S) \in I\}$ and $\Delta = \varprojlim \Delta/S$ where S runs through $\{S \mid (R, S) \in I\}$. And for each $(R, S), (R_1, S_1), (R_2, S_2) \in I, (R_2, S_2) \subseteq (R_1, S_1)$, we have the two commutative

diagrams as follows:

$$\begin{array}{ccc}
\Gamma/R_2 & \xrightarrow{\phi_{R_1 R_2}} & \Gamma/R_1 \\
\downarrow f_{S_2 R_2} & & \downarrow f_{S_1 R_1} \\
\Delta/S_2 & \xrightarrow{\psi_{S_1 S_2}} & \Delta/S_1
\end{array}
\qquad
\begin{array}{ccc}
\Gamma & \xrightarrow{\phi_R} & \Gamma/R \\
\downarrow f & & \downarrow f_{SR} \\
\Delta & \xrightarrow{\psi_S} & \Delta/S
\end{array}$$

where each $f_{SR} : \Gamma/R \rightarrow \Delta/S$ is locally bijective (respectively, a regular covering), and thus f is a covering (resp. regular covering) of profinite graphs. \square

THEOREM 7.3. *Let G be a profinite group acting continuously and residually freely on a connected profinite graph Γ , without edge inversions. Then Γ/G is a profinite graph and the orbit mapping $f : \Gamma \rightarrow \Gamma/G$ is a regular covering of profinite graphs. Moreover, $G = \text{Aut}(\Gamma, f)$ and they are isomorphic as profinite groups.*

PROOF. Since G is acting residually freely over the profinite graph Γ , by Definition 6.5, there exists a fundamental system I of G -invariant compatible cofinite entourages R over Γ . Consider the natural orbit map of graphs $f : \Gamma \rightarrow \Gamma/G$ and the following diagram

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\phi_R} & \Gamma/R \\
\downarrow f & & \downarrow f_R \\
\Gamma/G & \xrightarrow{\zeta_R} & (\Gamma/R)/G
\end{array}$$

where $f_R : \Gamma/R \rightarrow (\Gamma/R)/G$ is the quotient map of graphs corresponding to the group action of G over Γ/R defined by $g.R[x] = R[gx]$. This action is well defined since R is G -invariant and this action also preserves the graph structure of Γ/R , without inverting edges. Also ϕ_R is the natural quotient map from Γ to Γ/R , and $\zeta_R : \Gamma/G \rightarrow (\Gamma/R)/G$ is defined by $\zeta_R([x]) = f_R \phi_R(x)$, for all $[x] \in \Gamma/G$, where $[x]$ is the orbit of $x \in \Gamma$ in the quotient graph Γ/G . The map ζ_R is well-defined, for if $x, y \in \Gamma$ are such that $[x] = [y]$, in Γ/G , then $y = gx$ for some g in G . Hence,

$R[y] = \phi_R(y) = \phi_R(gx) = R[gx] = g.\phi_R[x]$. Hence $f_R\phi_R(y) = f_Rg.\phi_R(x) = f_R\phi_R(x)$. This means $\zeta_R[x] = \zeta_R[y]$. Also then the above diagram is a commutative diagram of continuous maps of graphs, where ζ_R is continuous since f is a quotient map and $\zeta_R f = f_R\phi_R$ is a continuous map of graphs. Write $T_R = \zeta_R^{-1}\zeta_R$. and $K = f^{-1}f$. Let's first prove some important claims.

Claim (1) $RK = KR$, for all $R \in I$.

Proof: $K = \{(x, gx) \mid x \in \Gamma, g \in G\}$. So $(x, z) \in RK$ implies there exists $y \in \Gamma$ such that $(x, y) \in K, (y, z) \in R$. So $y = gx$ for some g in G , then $(gx, z) \in R$. So, $(x, g^{-1}z) \in R$, since R is G -invariant. Again $(g^{-1}z, z) \in K$. So, $(x, z) \in KR$. Thus $RK \subseteq KR$. Similarly, $KR \subseteq RK$. Hence, $RK = KR$.

Claim (2) $(f \times f)[R] = T_R$ for all $R \in I$.

Proof: Let $(x, y) \in R$. Then $\phi_R(x) = \phi_R(y)$. So, $f_R\phi_R(x) = f_R\phi_R(y)$. This means $\zeta_R f(x) = \zeta_R f(y)$. Hence $(f(x), f(y)) \in \zeta_R^{-1}\zeta_R = T_R$. So, $(f \times f)[R] \subseteq T_R$. Let $(u, v) \in T_R$. Since $u, v \in \Gamma/G = f(\Gamma)$, there exists $x, y \in \Gamma$ such that $f(x) = u, f(y) = v$ and so $f_R R[x] = f_R\phi_R(x) = \zeta_R(f(x)) = \zeta_R(u) = \zeta_R(v) = \zeta_R(f(y)) = f_R\phi_R(y) = f_R R[y]$. So, there exists $g \in G$ such that $g.R[x] = R[y]$, that is $R[gx] = R[y]$. So, $(gx, y) \in R$. Thus $(f(gx), f(y)) \in (f \times f)R$. This means $(u, v) \in (f \times f)R$ since $f(gx) = f(x) = u, f(y) = v$. Thus, $(f \times f)R \subseteq T_R$.

Claim (3) K is closed in $\Gamma \times \Gamma$

Proof: Consider the map $h: G \times \Gamma \rightarrow \Gamma \times \Gamma$ defined by $h(g, x) = (x, gx)$. This is a continuous map from a profinite space to profinite space. So, $K = \text{Image of } h$ is closed in $\Gamma \times \Gamma$.

Claim(4) $\Gamma/G = \varprojlim (\Gamma/R)/G$ where R runs through I , the fundamental system of G invariant compatible cofinite entourages over Γ .

Proof: We know $\Gamma = \varprojlim \Gamma/R$ where $R \in I$. Let's choose $R, S \in I$ with $S \subseteq R$. Then consider the natural map of graphs $\chi_{RS} : (\Gamma/S)/G \rightarrow (\Gamma/R)/G$. The map χ_{RS} is continuous since each of $(\Gamma/R)/G, (\Gamma/S)/G$ has the discrete topology and each f_R is a quotient map. So, $((\Gamma/R)/G, \chi_{RS}, S \subseteq R, R, S \in I)$ is an inverse system of continuous map of finite discrete graphs. For $R \in I$ consider $\zeta_R : \Gamma/G \rightarrow (\Gamma/R)/G$ as defined earlier. Hence we have the following commutative diagram of continuous map of graphs.

$$\begin{array}{ccc}
 & \Gamma/G & \\
 \zeta_S \swarrow & & \searrow \zeta_R \\
 (\Gamma/S)/G & \xrightarrow{\chi_{RS}} & (\Gamma/R)/G
 \end{array}$$

Then there exists a unique continuous map of graphs $\chi : \Gamma/G \rightarrow \varprojlim (\Gamma/R)/G$, where $R \in I$, such that the following diagram commutes for all $R \in I$

$$\begin{array}{ccc}
 & \varprojlim (\Gamma/R)/G & \\
 \chi \nearrow & & \searrow \chi_R \\
 \Gamma/G & \xrightarrow{\zeta_R} & (\Gamma/R)/G
 \end{array}$$

where $\chi_R : \varprojlim (\Gamma/R)/G \rightarrow (\Gamma/R)/G$ is the canonical projection. Then for all $R \in I, \chi_R \chi(\Gamma/G) = \zeta_R(\Gamma/G) = (\Gamma/R)/G$, since each ζ_R is onto. So, $\chi(\Gamma/G)$ is dense in $\varprojlim (\Gamma/R)/G$. Again, χ is a continuous map from a compact space to a Hausdorff space. So $\chi(\Gamma/G)$ is closed in $\varprojlim (\Gamma/R)/G$. Hence, χ is onto. Let $\chi([x]) = \chi([y])$ for $[x], [y] \in \Gamma/G$. This implies $\zeta_R f(x) = \zeta_R f(y)$ for all $R \in I$. This means $(f(x), f(y)) \in T_R$ for all $R \in I$. This implies $(x, y) \in (f \times f)^{-1}(T_R) = f^{-1}T_R f = f^{-1}fRf^{-1}f = KRK = RKK = RK$, by Claim (1) and Claim (2). Thus $(x, y) \in \bigcap_{R \in I} RK = \overline{K}^{\Gamma \times \Gamma} = K$ as K is

closed in $\Gamma \times \Gamma$. Hence $[x] = [y]$ in Γ/G . Thus χ is injective. Since χ is a continuous bijective map of graphs from a compact topological graph to a Hausdorff topological graph, χ is a graph isomorphism.

Since, for all $R \in I$, G/N_R acts freely on Γ/R with quotient graph $(\Gamma/R)/G$, the orbit map $f_R: \Gamma/R \rightarrow (\Gamma/R)/G$ is a regular covering of finite graphs.

So, combining Claim (1) through Claim (5), we deduce that Γ/G is a profinite graph and $f' = (\varprojlim_{R \in I} f_R): \Gamma \rightarrow \Gamma/G$ is a regular covering.

Now we have a continuous map of profinite graphs $f: \Gamma \rightarrow \Gamma/G$ and two families of continuous maps of cofinite graphs $(\phi_R: \Gamma \rightarrow \Gamma/R)_{R \in I}$ and $(\zeta_R: \Gamma/G \rightarrow (\Gamma/R)/G)_{R \in I}$ such that for each $R \in I$, the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\phi_R} & \Gamma/R \\ f \downarrow & & \downarrow f_R \\ \Gamma/G & \xrightarrow{\zeta_R} & (\Gamma/R)/G \end{array}$$

commutes; and for each $R, S \in I, S \subseteq R$, $\phi_R = \phi_{RS}\phi_S$ and $\zeta_R = \chi_{RS}\zeta_S, T_S \subseteq T_R$. Thus there exist unique continuous maps of profinite graphs

$$\Xi: \Gamma \rightarrow \Gamma, \xi: \Gamma/G \rightarrow \Gamma/G$$

such that the diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\phi_R} & \Gamma/R & & \\ \downarrow & \swarrow \Xi & \nearrow \phi_R & & \downarrow \\ \Gamma & & \Gamma & & \Gamma \\ \downarrow f' & & \downarrow f & & \downarrow f_R \\ \Gamma/G & & \Gamma/G & & \Gamma/R/G \\ \downarrow & \swarrow \xi & \searrow \zeta_R & & \downarrow \\ \Gamma/G & \xrightarrow{\xi \zeta_R} & (\Gamma/R)/G & & \end{array}$$

commutes. However, since the above diagram also commutes with $\Xi = \text{id}_\Gamma$ and $\xi = \text{id}_{\Gamma/G}$ we obtain $f' = f' \circ \text{id}_\Gamma = \text{id}_{\Gamma/G} \circ f = f$ and the orbit mapping $f: \Gamma \rightarrow \Gamma/G$ is a regular covering of profinite graphs.

Now, let $g \in G$. The action of G over Γ is continuous, so the restriction $\{g\} \times \Gamma \rightarrow \Gamma$ is also continuous. Clearly, g defines an automorphism of the profinite covering graph Γ . Also, $fg(x) = f(x)$ since f is the orbit map induced by the action of G over Γ . Thus, $g \in \text{Aut}(\Gamma, f)$. So, $G \subseteq \text{Aut}(\Gamma, f)$. Now, let $h \in \text{Aut}(\Gamma, f)$, and let $h(a) = b$ for some vertex a in Γ . Then, $f(a) = fh(a) = f(b)$. Since f is the orbit map induced by the action of G over Γ there exists $g \in G$ such that $b = g.a$. Again, $g \in \text{Aut}(\Gamma, f)$ by the above. This means $g.a = g(a) = b = h(a)$. Thus, by uniqueness of lifts, $h = g \in G$. Thus, $\text{Aut}(\Gamma, f) \subseteq G$. So, it remains to show that the identity map from G to $\text{Aut}(\Gamma, f)$ is continuous. For this, note that the group action of G over Γ is uniformly continuous since it is continuous and $G \times \Gamma$ is compact. So, for any compatible cofinite entourage R over Γ there is a cofinite congruence S on G with respect to the original profinite topology over G we started with and a compatible cofinite entourage T over Γ such that for all $(g, h) \in S$ and for all $(x, y) \in T, (gx, hy) \in R$. Hence for all $(g, h) \in S, (gx, hx) \in R$, since by the reflexivity of $T, (x, x) \in T$, for all $x \in \Gamma$. So, $S \subseteq N_R$. Since $\{N_R \mid R \in I\}$ form a fundamental system of entourages of $\text{Aut}(\Gamma, f)$, it follows that the identity map from G to $\text{Aut}(\Gamma, f)$ is uniformly continuous and hence a homeomorphism as G and $\text{Aut}(\Gamma, f)$ are compact Hausdorff spaces. Hence, the result follows.

□

We next show that, amazingly, the converse of the lemma 7.2 is true for regular coverings of profinite graphs; we do not know whether or not the converse holds for non-regular coverings. Hence, regular coverings can be characterized as follows.

THEOREM 7.4. *A map of profinite graphs $f: \Gamma \rightarrow \Delta$ is a regular covering if and only if there exists a fundamental system of regular good pairs for f .*

PROOF. Since f is a covering map of profinite graphs with Γ profinitely connected, by Lemma 5.3, $G = \text{Aut}(\Gamma, f)$ acts over Γ uniformly equicontinuously and hence continuously. By Lemma 6.4, G acts residually freely over Γ . So, by Theorem 7.3, the orbit map $q: \Gamma \rightarrow \Gamma/G$ is a regular covering of profinite graphs and each orbit map $f_R: \Gamma/R \rightarrow (\Gamma/R)/G$, induced by the action of G over Γ/R , is a regular covering map of finite connected graphs. We show that there is an isomorphism of topological graphs between Γ/G and Δ . Define $\theta: \Gamma/G \rightarrow \Delta$ via $\theta([x]) = f(x)$ where $[x]$ is the orbit of x under the orbit map $q: \Gamma \rightarrow \Gamma/G$. Then θ is well defined as $[x] = [y]$ implies that there is a g in G such that $y = gx$. Then $f(y) = f(gx) = f(x)$. So, $\theta[x] = \theta[y]$. Let $\theta[x] = \theta[y]$. So, $f(x) = f(y)$. Hence by Theorem 6.6, there exists $g \in G$ such that $y = gx$. Thus $[x] = [y]$ and θ is injective. Clearly, θ is surjective since Δ profinitely connected implies that f is surjective. The map θ is a map of graphs as f is so and the action of G over Γ preserves the graph structure of Γ . Also θ is continuous since $\theta q = f$ and f is continuous while q is a quotient map. Thus θ is a continuous, bijective map of graphs from a compact topological graph Γ/G to a Hausdorff topological graph Δ , and so θ is an isomorphism of topological graphs. Now let $R_i = \phi_i^{-1}\phi_i, S_j = \psi_j^{-1}\psi_j$ be any two compatible cofinite entourages over Γ and Δ respectively, where R_i is G -invariant. Consider $S_k = \psi_k^{-1}\psi_k$ and suppose $S_k \subseteq S_i, S_j$. Suppose $\phi_k^{-1}\phi_k = R_k$ is the G -invariant compatible cofinite entourage over Γ with $R_k \subseteq R_i, R_j$. Also $(f \times f)(R_k) = T_{R_k}$, as in the Claim (2) in the proof of Theorem 7.3 and $(f \times f)(R_k) \subseteq S_k$. This implies that $T_{R_k} \subseteq S_k$. So we have $(R_k, T_{R_k}) \subseteq (R_i, S_j)$,

where $f_{R_k}: \Gamma/R_k \rightarrow (\Gamma/R_k)/G$ is a regular covering map finite graphs. As in the proof of Theorem 7.3 (R, T_R) is a regular good pair for each G -invariant compatible cofinite entourage R over Γ . Hence it follows that if $f: \Gamma \rightarrow \Delta$ is a regular covering of profinite graphs then there exists a fundamental system of regular good pairs for f . By considering Lemma 7.2 we obtain the converse part and thus the theorem. \square

COROLLARY 7.5. *If $f: \Gamma \rightarrow \Delta$ is a regular covering of profinite graphs then it is a uniform quotient map.*

PROOF. By the above Theorem, Δ is isomorphic to Γ/G , where $G = \text{Aut}(\Gamma, f)$. Then, since we know there is already a continuous bijection of profinite graphs, $i: \Gamma/G \rightarrow \Gamma//K$, we only need to show that $\Gamma//K$ is Hausdorff, since Γ/G is compact so i will be a graph isomorphism of profinite graphs. Now as in the claim (4) of the Theorem 7.3, we have K is closed and hence $K = \bigcap_{R \in I} RKR = \bigcap_{R \in I} RRR$, (R being G -invariant) $= \bigcap_{R \in I} RK$. Hence $\Gamma//K$ is Hausdorff and our claim follows. \square

LEMMA 7.6. *Let $f: \Gamma \rightarrow \Delta$ be a regular covering map of profinite connected graphs. Then for any closed subgroup H of $G = \text{Aut}(\Gamma, f)$, we have the commutative diagram*

$$\begin{array}{ccc}
 & & \Gamma \\
 & \swarrow h & \downarrow f \\
 \Gamma/H & \xrightarrow{f_H} & \Gamma/G \cong \Gamma//G
 \end{array}$$

where f_H, h are a covering map and a regular covering map of profinite connected graphs respectively. Also f_H is a regular covering if H is a normal subgroup of G .

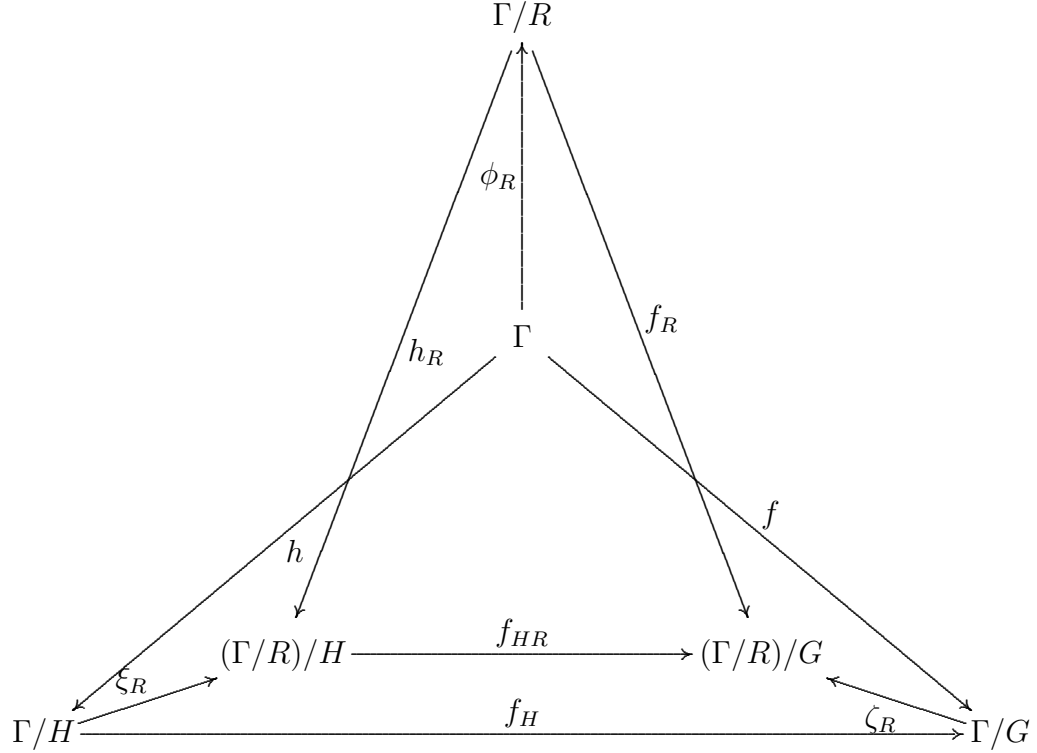
PROOF. By Theorem 6.6, since the group action $G \times \Gamma \rightarrow \Gamma$ is a continuous, residually free action without edge inversion, the restriction of the group action $H \times$

$\Gamma \rightarrow \Gamma$ is also a continuous, residually free action without edge inversion. Each G -invariant compatible cofinite entourage R over Γ is also an H -invariant compatible cofinite entourage. Thus, as in Theorem 7.3, each $h_R: \Gamma/R \rightarrow (\Gamma/R)/H$, the natural quotient map is a regular covering of finite path connected graphs. Also H is closed in the compact space G , so H is compact with respect to the subspace topology of H inherited from G . Thus it follows that $h = \varprojlim_{R \in I} h_R$ is a regular covering of profinite connected graphs.

As a consequence, we have a new commutative diagram of maps of finite path connected graphs, namely

$$\begin{array}{ccc}
 & \Gamma/R & \\
 & \swarrow h_R & \searrow f_R \\
 (\Gamma/R)/H & \xrightarrow{f_{HR}} & (\Gamma/R)/G
 \end{array}$$

Then f_{HR} is a well-defined map of graphs as $H \leq G$ and, since f_R, h_R are locally bijective and each of $\Gamma/R, (\Gamma/R)/H, (\Gamma/R)/G$ are path connected, f_{HR} is also locally bijective which proves that we have a covering of profinite connected graphs $f_H: \Gamma/H \rightarrow \Gamma/G$, since $\Gamma/H = \varprojlim (\Gamma/R)/H$ and $\Gamma/G = \varprojlim (\Gamma/R)/G$, by Claim (5) of Theorem 7.3. We have



Thus, for all $\delta \in \Gamma$, if $\delta = (\delta_R)_{R \in I}$, then $f_H(h(\delta)) = f_H(h((\delta_R)_{R \in I})) = f_H((h_R(\delta_R))_{R \in I}) = (f_{HR}((h_R(\delta_R)))_{R \in I}) = (f_R(\delta_R))_{R \in I} = f((\delta_R)_{R \in I}) = f(\delta)$.

Now, let H be a normal subgroup of G . In order to show f_H is a regular covering we wish to show that each f_{HR} is a regular covering of finite path connected graphs. Let us denote any element of $(\Gamma/R)/H$ by $[R[x]]_H$ for $R[x] \in \Gamma/R$ and any element of $(\Gamma/R)/G$ by $[R[x]]_G$ for $R[x] \in \Gamma/R$. Now we claim that G acts on $(\Gamma/R)/H$ via covering transformation, where we take the natural action of G over $(\Gamma/R)/H$ that takes $(g, [R[x]]_H)$ to $[R[gx]]_H$, for any $g \in G, [R[x]]_H \in (\Gamma/R)/H$. The action is well defined: For if $[R[x]]_H = [R[y]]_H$, there exists $h \in H$ with $R[y] = h.R[x] = R[hx]$. So, for any $g \in G, g.[R[y]]_H = g.R[hx]_H = gh.[R[x]]_H = ghg^{-1}[R[gx]]_H = [R[gx]]_H$, since H is a normal subgroup of G . Also, for any $g \in G, g.[R[x]]_H = g.[R[y]]_H$ implies that $[R[gx]]_H = [R[gy]]_H$, so, there exists $h \in H$ with $[R[gx]]_H = h.[R[gy]]_H$. So $[R[x]]_H = g^{-1}[R[gx]]_H = g^{-1}[R[hgy]]_H = [R[g^{-1}hgx]]_H = [R[y]]_H$. Thus the

group action defines an injective map of graphs. Also, it is surjective, since for any $[R[x]]_H \in (\Gamma/R)/H$, $g.[R[g^{-1}x]]_H = [R[x]]_H$. Again, $f_{HR}(g.[R[x]]_H) = [R[gx]]_G = [R[x]]_G = f_{HR}([R[x]]_H)$. Hence the claim follows. Thus by theory of path connected regular coverings f_{HR} is a regular covering of path connected graphs.

□

CHAPTER 8

Profinite fundamental groups and the Universal coverings

The goal of the next section is to construct a universal covering of any connected profinite graph Δ . Since such a graph is not necessarily path connected, fundamental groups are not as useful tool as in the classical covering theory of path connected graphs and spaces. However, the profinite analogue of the fundamental group that we introduce in this section does retain one valuable aspect: it provides a convenient way of determining whether a covering is a universal covering.

We define a profinite analogue of the fundamental group of an abstract graph. Just as we have defined most notions here, we define this first in the special case of a finite discrete graph and then extend to an arbitrary profinite graph by taking the projective limit of the profinite fundamental groups of its finite discrete quotient graphs.

DEFINITION 8.1. Let Δ be a profinite graph and let b , be a vertex of Δ . The *profinite fundamental group* $\widehat{\pi}_1(\Delta, b)$ of Δ based at b is defined as follows:

- (a) If Δ is finite, then $\widehat{\pi}_1(\Delta, b)$ is the profinite completion of the ordinary fundamental group $\pi_1(\Delta, b)$ with respect to the collection of all cofinite congruences over $\pi_1(\Delta, b)$.
- (b) In the general case, let J be a fundamental system of compatible cofinite entourages of Δ and define $\widehat{\pi}_1(\Delta, b) = \varprojlim \widehat{\pi}_1(\Delta/S, S[b])$, where S runs through J .

REMARKS 8.2. *The structure, defined above is well defined. We first see that with suitable interpretations of the notations, $(\widehat{\pi}_1(\Delta/S, S[b]), \phi_{SR} \mid S, R \in J, R \geq S$*

if and only if $R \subseteq S$) is an inverse system of profinite groups and continuous group homomorphisms. For any pair $R, S \in J$, with $R \subseteq S$, let ϕ_{SR} from $\pi_1(\Delta/R, R[a])$ to $\pi_1(\Delta/S, S[b])$ be the group homomorphism induced by the natural map of graphs ϕ_{SR} from $(\Delta/R, R[b])$ to $(\Delta/S, S[b])$. Now each ϕ_{SR} from $\pi_1(\Delta/R, R[b])$ to $\pi_1(\Delta/S, S[b])$ is uniformly continuous, as each $\pi_1(\Delta/R, R[b])$, $\pi_1(\Delta/S, S[b])$ are cofinite groups with respect to the uniformity induced by the fundamental system of all cofinite congruences over themselves. Thus, there exists a unique extension of ϕ_{SR} to a continuous group homomorphism of profinite groups (and hence uniform continuous) $\widehat{\phi}_{SR}$ from $\widehat{\pi}_1(\Delta/R, R[b])$ to $\widehat{\pi}_1(\Delta/S, S[b])$. For any $R \subseteq S \subseteq T$ in J , $\phi_{TS}\phi_{SR} = \phi_{TR}$. Thus by the uniqueness of extensions $\widehat{\phi}_{TS}\widehat{\phi}_{SR} = \widehat{\phi}_{TR}$ and $\widehat{\phi}_{RR} = \text{id}$ on $\widehat{\pi}_1(\Delta/R, R[b])$. Hence, without loss of generality, with a little abuse of notations denoting $\widehat{\phi}_{SR}$ simply by ϕ_{SR} , $(\widehat{\pi}_1(\Delta/S, S[b]), \phi_{SR} \mid S, R \in J, R \geq S \text{ if and only if } R \subseteq S)$ is an inverse system of profinite groups and uniformly continuous group homomorphisms. Thus $\widehat{\pi}_1(\Delta, b) = \varprojlim \widehat{\pi}_1(\Delta/S, S[b])$, where S runs through J , is well defined.

Next we define homomorphisms on profinite fundamental groups induced by continuous map of profinite graphs.

DEFINITION 8.3. Let $f: \Gamma \rightarrow \Delta$ be a continuous map of profinite graphs. First consider the case when Γ and Δ are finite. Let f denote the induced homomorphism on ordinary fundamental groups $f: \pi_1(\Gamma, a) \rightarrow \pi_1(\Delta, f(a))$ for any vertex a in Γ . Then, f is continuous in the cofinite topologies in which every normal subgroup of finite index is open. Hence f determines a continuous homomorphism of the profinite completions $f^*: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, f(a))$. Now we consider the general case where Γ and Δ are any profinite graphs. For any pair of compatible cofinite entourages R and S over Γ and Δ respectively, let us call (R, S) a *half good pair* relative to f if and only if $(f \times f)[R] \subseteq S$, that is if and only if there is well-defined map of finite graphs $f_{SR}: \Gamma/R \rightarrow \Delta/S$ determined by f . Then, $J = \{(R, S) \mid (R, S) \text{ is a half good pair}\}$

pair relative to f }, endowed with reverse inclusion, i.e. $(R, S) \geq (T, Q)$ if and only if $R \subseteq T$ and $S \subseteq Q$, forms a directed set. Then $f = \varprojlim f_{SR}, ((R, S) \in J)$. So, for $(R, S), (T, Q) \in J$ with $T \subseteq R, Q \subseteq S$ we have the commutative diagram

$$\begin{array}{ccc}
 \Gamma/T & \xrightarrow{\phi_{RT}} & \Gamma/R \\
 \downarrow f_{QT} & & \downarrow f_{SR} \\
 \Delta/Q & \xrightarrow{\psi_{SQ}} & \Delta/S
 \end{array}$$

where ϕ_{RT}, ψ_{SQ} are map of the quotient graphs defined in the natural way. So, the above commutative diagram leads to another commutative diagram of continuous homomorphism of profinite groups.

$$\begin{array}{ccc}
 \widehat{\pi}_1(\Gamma/T, T[a]) & \xrightarrow{\phi_{RT}^*} & \widehat{\pi}_1(\Gamma/R, R[a]) \\
 \downarrow f_{QT}^* & & \downarrow f_{SR}^* \\
 \widehat{\pi}_1(\Delta/Q, Q[f(a)]) & \xrightarrow{\psi_{SQ}^*} & \widehat{\pi}_1(\Delta/S, S[f(a)])
 \end{array}$$

So, now we can define $f^* = \varprojlim f_{SR}^*: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, f(a))$

LEMMA 8.4. *For any two continuous maps of profinite graphs f from (Γ, a) to (Δ, b) and g from (Δ, b) to (Σ, c) for any vertex $a \in V(\Gamma), f(a) = b \in V(\Delta), g(b) = c \in V(\Sigma)$, consider the corresponding induced maps $f^*: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, b)$, $g^*: \widehat{\pi}_1(\Delta, b) \rightarrow \widehat{\pi}_1(\Sigma, c)$ and the induced map $(gf)^*: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Sigma, c)$. Then $(gf)^* = g^*f^*$.*

PROOF. Let $I = \{(R, S) \mid (R, S) \text{ half good pair rel } f\}$, $J = \{(S, T) \mid (S, T) \text{ half good pair rel } g\}$, $K = \{(R, T) \mid (R, T) \text{ half good pair rel } gf\}$. First, let us prove that $L = \{S \mid (R, S), (S, T) \text{ are half good pair rel } f \text{ and } g \text{ respectively for some } R \text{ so that } (R, S) \in I \text{ and for some } T \text{ so that } (S, T) \in J\}$ is a fundamental system of compatible cofinite entourages over Δ . For let us choose any compatible cofinite entourage S' over Δ . Then for any compatible cofinite entourage T over Σ , $(g \times g)^{-1}(T) \cap S' = S$ is such that (S, T) is a half good pair rel g , where $S \subseteq S'$. And for $R = (f \times f)^{-1}(S)$, (R, S) is a half good pair rel f . Thus $S \in L$. So, L is a fundamental system of compatible cofinite entourages over Δ . Hence, Δ can be viewed as $\varprojlim \Delta/S$ where S runs through L . Now for given any $S \in L$, corresponding $(R, S) \in I, (S, T) \in J, (R, T) \in K$. Again for $(R, T) \in K$, let us define $S = (g \times g)^{-1}(T)$ and since $R \subseteq (gf \times gf)^{-1}(T) = (f \times f)^{-1}((g \times g)^{-1}[T]) = (f \times f)^{-1}[S]$, we get $(R, S) \in I, (S, T) \in J$. In any case consider the corresponding natural map of graphs $f_{SR}: \Gamma/R \rightarrow \Delta/S, g_{TS}: \Delta/S \rightarrow \Sigma/T$ and the corresponding induced map of fundamental groups of the finite graphs $f_{SR}: \pi_1(\Gamma/R, R[a]) \rightarrow \pi_1(\Delta/S, S[b]), g_{TS}: \pi_1(\Delta/S, S[b]) \rightarrow \pi_1(\Sigma/T, T[c])$. Then it follows that $g_{TS}f_{SR} = (gf)_{RT}$ in the the fundamental group level. Now, as $f_{SR}, g_{TS}, (gf)_{RT}$ are uniformly continuous, their unique extension are also uniformly continuous and we have $g_{TS}f_{SR} = (gf)_{RT}$. Hence we have $(gf)^* = g^*f^*$.

□

LEMMA 8.5. *If $f: \Gamma \rightarrow \Delta$ is an isomorphism of profinite graphs, then for any vertex a in Γ the induced homomorphism $f^*: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, f(a))$ is an isomorphism of profinite groups.*

PROOF. Since f is an isomorphism of profinite graphs, it has a continuous inverse $f^{-1}: \Delta \rightarrow \Gamma$ and hence $ff^{-1} = id_\Gamma, f^{-1}f = id_\Delta$. So, by the above lemma, f^* has a continuous inverse. Thus f^* is an isomorphism of profinite groups. □

From now we will denote the induced homomorphism $f^*: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, f(a))$ simply by $f: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, f(a))$ whenever $f: \Gamma \rightarrow \Delta$ is a continuous map of profinite graphs and a is a vertex in Γ .

LEMMA 8.6. *If $f: \Gamma \rightarrow \Delta$ is a covering of profinite graphs, then for any vertex a in Γ the induced homomorphism $f: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, f(a))$ is a continuous monomorphism.*

PROOF. Now if f is a covering map of profinite graphs then each f_i , as in the definition of coverings of profinite graphs, is a local bijection and R_i, S_i are as in remark 2 right after the definition. Let's denote the restriction of f_i over Γ/R_i simply by $f_{SR}: \Gamma/R \rightarrow \Delta/S$ where we set $R_i = R, S_i = S$. Then f_{SR} is well-defined map of graphs which is also locally injective. So, the induced map of fundamental groups f_{SR} are topological and algebraic embedding with respect to the cofinite topologies, where each normal subgroup of finite index is open. Hence f_{SR} extends to a unique uniform continuous mono morphism $\widehat{f}_{SR}: \widehat{\pi}_1(\Gamma/R, R[a]) \rightarrow \widehat{\pi}_1(\Delta/S, S[f(a)])$ for all $(R, S) \in J$, where by abuse of notation we can denote \widehat{f}_{SR} by f_{SR} . Thus $f: \widehat{\pi}_1(\Gamma, a) \rightarrow \widehat{\pi}_1(\Delta, f(a))$ is a continuous monomorphism. \square

LEMMA 8.7. *Let Γ be a connected profinite graph. Let $a \in V(\Gamma)$. Then, $\widehat{\pi}_1(\Gamma, a) = 1$ if and only if for each compatible cofinite entourage R over Γ and subgroup H of finite index in $\pi_1(\Gamma/R, R[a])$ there exists a compatible cofinite entourage $R' \subseteq R$ such that $\phi_{RR'}(\pi_1(\Gamma/R', R'[a])) \subseteq H$.*

PROOF. Let us first assume that $\widehat{\pi}_1(\Gamma, a) = 1$. Then, let H be any subgroup of finite index in $\pi_1(\Gamma/R, R[a])$ for some compatible cofinite entourage R over Γ . Without loss of generality we can assume that H is a normal subgroup of finite index since every subgroup of finite index contains such a normal subgroup. Let the closure of H in $\widehat{\pi}_1(\Gamma/R, R[a])$ be denoted by \overline{H} . So, \overline{H} is also open normal subgroup of finite

index in $\widehat{\pi}_1(\Gamma/R, R[a])$. Also, for any compatible cofinite entourage R over Γ ,

$$\phi_R(\widehat{\pi}_1(\Gamma, a)) = 1 = \bigcap_{R'' \geq R} \phi_{RR''} \widehat{\pi}_1(\Gamma/R'', R''[a])$$

Hence,

$$\bigcap_{R'' \geq R} [\phi_{RR''}(\widehat{\pi}_1(\Gamma/R'', R''[a]))] \setminus \overline{H} = \emptyset.$$

Now each $\phi_{RR''}$ is a continuous map from a compact space to a Hausdorff space. So each $\phi_{RR''}(\widehat{\pi}_1(\Gamma/R'', R''[a]))$ is a closed subset of $\widehat{\pi}_1(\Gamma/R, R[a])$. So, $\phi_{RR''}(\widehat{\pi}_1(\Gamma/R'', R''[a])) \setminus \overline{H}$ is closed subset of the compact space $\widehat{\pi}_1(\Gamma/R, R[a])$. Thus by the finite intersection property there exists finite number of compatible cofinite entourages R_1, R_2, \dots, R_n over Γ , with $R_i \subseteq R$, for all $i = 1, 2, \dots, n$, such that $\bigcap_{i=1}^n [\phi_{RR_i}(\widehat{\pi}_1(\Gamma/R_i, R_i[a]))] \setminus \overline{H} = \emptyset$. So, $\bigcap_{i=1}^n [\phi_{RR_i}(\widehat{\pi}_1(\Gamma/R_i, R_i[a]))] \subseteq \overline{H}$. Thus for the compatible cofinite entourage $R' = \bigcap_{i=1}^n R_i \subseteq R$ over Γ ,

$$\phi_{RR'}(\widehat{\pi}_1(\Gamma/R', R'[a])) \subseteq \bigcap_{i=1}^n [\phi_{RR_i}(\widehat{\pi}_1(\Gamma/R_i, R_i[a]))] \subseteq \overline{H}$$

So, $\phi_{RR'}(\widehat{\pi}_1(\Gamma/R', R'[a])) \cap \pi_1(\Gamma/R, R[a]) \subseteq \overline{H} \cap \pi_1(\Gamma/R, R[a])$. Hence $\phi_{RR'}(\pi_1(\Gamma/R', R'[a])) \subseteq \phi_{RR'}(\widehat{\pi}_1(\Gamma/R', R'[a])) \cap \pi_1(\Gamma/R, R[a]) \subseteq H$.

Conversely, for each compatible cofinite entourage R over Γ and H , a subgroup of finite index in $\pi_1(\Gamma/R, R[a])$ there exists a compatible cofinite entourage $R' \subseteq R$ such that $\phi_{RR'}(\pi_1(\Gamma/R', R'[a])) \subseteq H$. Choose any $\gamma \in \widehat{\pi}_1(\Gamma, a)$, then, for any compatible cofinite entourage R over Γ , consider

$$\phi_R(\gamma) \in \phi_R(\widehat{\pi}_1(\Gamma, a)) = \bigcap_{R' \geq R} [\phi_{RR'}(\widehat{\pi}_1(\Gamma/R', R'[a]))]$$

Now let \overline{N} be a normal subgroup of finite indexed in $\widehat{\pi}_1(\Gamma/R, R[a])$. Then, $\overline{N} \cap \pi_1(\Gamma/R, R[a]) = N$, say, is a normal subgroup of finite index in $\pi_1(\Gamma/R, R[a])$. So, by the given condition, there exists a compatible cofinite entourage R' over Γ , with

$R' \subseteq R$ such that $\phi_{RR'}(\pi_1(\Gamma/R', R'[a])) \subseteq N$. Thus $\phi_{RR'}(\widehat{\pi}_1(\Gamma/R', R'[a])) \subseteq \overline{N}$. Hence by our earlier work we have,

$$\phi_R(\gamma) \in \bigcap_{R' \geq R} [\phi_{RR'}(\widehat{\pi}_1(\Gamma/R', R'[a]))] \subseteq \phi_{RR'}(\widehat{\pi}_1(\Gamma/R', R'[a])) \subseteq \overline{N}$$

Thus, $\phi_R(\gamma) \in \bigcap \overline{N}$, where \overline{N} runs through the family of all normal subgroups of finite index in $\widehat{\pi}_1(\Gamma/R, R[a])$. This is residually finite as is $\pi_1(\Gamma/R, R[a])$. So, $\phi_R(\gamma) \in \bigcap \overline{N} = 1$. This is true for any compatible cofinite entourage R over Γ . Thus, we have $\gamma = 1$. So, $\widehat{\pi}_1(\Gamma/R, R[a]) = 1$. \square

THEOREM 8.8. *Let $g: \tilde{\Delta} \rightarrow \Delta$ be a covering of connected profinite graphs and let c be a vertex of $\tilde{\Delta}$. Let $\widehat{\pi}_1(\tilde{\Delta}, c) = 1, g(c) = b$. If $f: \Gamma \rightarrow \Delta$ is a covering map of profinite graphs with $f(a) = b$ for some vertex $a \in V(\Gamma)$, then there exists a unique lift $h: \tilde{\Delta} \rightarrow \Gamma$ such that $fh = g, h(c) = a$.*

PROOF. As in the definition of covering of profinite graphs, choose $i \in I$. Let, $T = (g \times g)^{-1}(S_i)$. Then $(g \times g)(T) \subseteq S$ where we are actually setting $S_i = S$ and we can think about the natural map of graphs g_{ST} from $(\tilde{\Delta}/T, T[c])$ to $(\Delta/S, S[b])$ and also the corresponding group homomorphism in the fundamental group level as $g_{ST}: \pi_1(\tilde{\Delta}/T, T[c]) \rightarrow \pi_1(\Delta/S, S[b])$. Now, $f_i(\pi_1(\Gamma_i, a_i))$ is a subgroup of finite index in $\pi_1(\Delta/S, S[b])$, where $a = (a_i)_{i \in I} \in V(\Gamma)$ and f_i is as in the definition of coverings of profinite graphs. So, $H = g_{ST}^{-1}(f_i(\pi_1(\Gamma_i, a_i)))$ is a subgroup of finite index in $\pi_1(\tilde{\Delta}/T, T[c])$. So, by the previous lemma, there exists a compatible cofinite entourage T' over $\tilde{\Delta}$ such that $T' \subseteq T$ and $\chi_{TT'}(\pi_1(\tilde{\Delta}/T', T'[c])) \subseteq H$, where the natural group homomorphism $\chi_{TT'}$ from $\pi_1(\tilde{\Delta}/T', T'[c])$ to $\pi_1(\tilde{\Delta}/T, T[c])$ is induced by the corresponding natural map of graphs $\chi_{TT'}$ from $(\tilde{\Delta}/T', T'[c])$ to $(\tilde{\Delta}/T, T[c])$. So $g_{ST}(\chi_{TT'}(\pi_1(\tilde{\Delta}/T', T'[c]))) \subseteq f_i \pi_1(\Gamma_i, a_i)$. Now since $T' \subseteq T$ implies $(g \times g)T' \subseteq (g \times g)T \subseteq S$ and $g_{ST} \circ \chi_{TT'} = g_{ST'}: \Delta/T' \rightarrow \Delta/S$ we deduce $g_{ST'}(\pi_1(\tilde{\Delta}/T', T'[c]))$ is

a subset of $f_i(\pi_1(\Gamma_i, a_i))$, where $g_{ST'}$ is the group homomorphism obtained from the corresponding natural map of graph. Thus our claim follows by Theorem 4.2. \square

LEMMA 8.9. *If Γ is a connected profinite graph and $\widehat{\pi}_1(\Gamma, a_0) = 1$ for some vertex a_0 , then $\widehat{\pi}_1(\Gamma, a) = 1$ for every vertex a .*

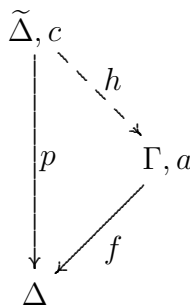
PROOF. We show that the property of the previous lemma holds for any vertex a given that it holds for the vertex a_0 . Let R be a compatible cofinite entourage over Γ and let H be a normal subgroup of finite index in $\pi_1(\Gamma/R, R[a])$. Choose a path γ in Γ/R from $R[a_0]$ to $R[a]$. Let $\gamma_*: \pi_1(\Gamma/R, R[a_0]) \rightarrow \pi_1(\Gamma/R, R[a])$ denote the isomorphism given by $\gamma_*(\alpha) = \gamma^{-1}\alpha\gamma$. Then $\gamma_*^{-1}(H) = \gamma H\gamma^{-1}$ is a normal subgroup of finite index in $\pi_1(\Gamma/R, R[a_0])$. So, by the previous lemma, there exists a compatible cofinite entourage $R' \subseteq R$ such that the image of $\phi_{RR'}: \pi_1(\Gamma/R', R'[a_0]) \rightarrow \pi_1(\Gamma/R, R[a])$ lies in $\gamma H\gamma^{-1}$. Choose a path β in Γ/R' from $R'[a_0]$ to $R'[a]$. We then have the following commutative diagram of continuous group homomorphisms.

$$\begin{array}{ccc}
\pi_1(\Gamma/R', R'[a_0]) & \xrightarrow{\phi_{RR'}} & \pi_1(\Gamma/R, R[a_0]) \\
\downarrow \beta_* & & \downarrow (\phi_{RR'}\beta)_* \\
\pi_1(\Gamma/R', R'[a]) & \xrightarrow{\phi_{RR'}} & \pi_1(\Gamma/R, R[a])
\end{array}$$

So, since β_* is an isomorphism we have that the image of the bottom homomorphism lies in the subgroup $(\phi_{RR'}\beta)_*(\gamma H\gamma^{-1}) = (\phi_{RR'}\beta)^{-1}\gamma H\gamma^{-1}(\phi_{RR'}\beta) = H$ as $\gamma^{-1}(\phi_{RR'}\beta) \in \pi_1(\Gamma/R, R[a])$ and H is a normal subgroup of $\pi_1(\Gamma/R, R[a])$. Therefore, applying the previous lemma, we have $\widehat{\pi}_1(\Gamma, a) = 1$.

\square

DEFINITION 8.10. Let Δ be a connected profinite graph. A connected profinite covering graph $(\tilde{\Delta}, p)$ of Δ is called a *universal profinite covering graph* of Δ provided that the following universal property holds: if (Γ, f) is a connected profinite covering graph of Δ and c, a are vertices of $\tilde{\Delta}, \Gamma$ with $p(c) = f(a)$, then there exists a homomorphism of coverings h from $(\tilde{\Delta}, p)$ to (Γ, f) such that $h(c) = a$.



REMARKS 8.11. *We have the following observations:*

- (1) *The homomorphism h from $(\tilde{\Delta}, p)$ to (Γ, f) with $h(c) = a$ for any given covering (Γ, f) of Δ and vertices c, a of $\tilde{\Delta}, \Gamma$ respectively, with $p(c) = f(a)$ is unique by Lemma 4.1.*
- (2) *If $(\tilde{\Delta}, p)$ and $(\tilde{\Delta}', p')$ are two universal covering graphs of a connected profinite graph Δ , then $(\tilde{\Delta}, p)$ and $(\tilde{\Delta}', p')$ are isomorphic covering graphs. Hence, we can refer to the universal covering graph of Δ , once we know that it exists.*
- (3) *Applying Theorem 8.8 and Lemma 8.9 we can observe that a covering of connected profinite graphs $p: \tilde{\Delta} \rightarrow \Delta$ is a universal covering if $\hat{\pi}_1(\tilde{\Delta}, a) = 1$ for any vertex a in $\tilde{\Delta}$.*

PROOF. We just prove Remark 2. Consider any vertex a of $\tilde{\Delta}$. Let, $p(a) = b$. Then p' is surjective by lemma 3.3 (b), since Δ is profinitely connected. So there is a vertex a' of $\tilde{\Delta}'$ such that $p'(a') = b$. Since $(\tilde{\Delta}, p)$ is a universal covering of Δ there exists a homomorphism of coverings $h: (\tilde{\Delta}, a) \rightarrow (\tilde{\Delta}', a')$, so that $p'h = p$. Also since $(\tilde{\Delta}', p')$ is a universal covering there is another homomorphism of coverings

$h': (\tilde{\Delta}', a') \rightarrow (\tilde{\Delta}, a)$ so that $ph' = p'$. Thus, $ph'h = p'h = p$ and $p'hh' = ph' = p'$. Furthermore, $h \circ h'(a') = a', h' \circ h(a) = a$. So, by Lemma 4.1, $h \circ h' = \text{id}_{\tilde{\Delta}'}$, and $h' \circ h = \text{id}_{\tilde{\Delta}}$. \square

CHAPTER 9

Existence of profinite coverings

The goal of this section is to give a construction of the universal profinite covering graph for any connected profinite graph Δ . Then by examining the graphs arising from this construction, we get a better understanding of the properties of universal profinite covering graphs. In particular, we show that a universal covering of profinite graphs is a regular covering. Hence, the results in the chapters on Regular Coverings and of Good pairs and residually free group actions can be applied and this leads to a characterization of all connected profinite covering graphs of a connected profinite graph Δ in terms of closed subgroups of the group $\text{Aut}(\tilde{\Delta}, p)$ of covering transformations of the universal profinite covering graph of Δ .

THEOREM 9.1. *Let Δ be a connected profinite graph. Then its universal profinite covering graph $(\tilde{\Delta}, p)$ exists and it is regular. Furthermore, if (Σ, g) is any profinite covering graph of Δ and h is a homomorphism of coverings from $(\tilde{\Delta}, p)$ to (Σ, g) , then the pair $(\tilde{\Delta}, h)$ is a regular profinite covering graph of Σ .*

PROOF. Without loss of generality we can assume that $\Delta = \varprojlim \Delta_i$ where i runs through I a directed set corresponding to a fundamental system of compatible cofinite entourages over Δ and each $\Delta_i = \psi_i(\Delta) = \Delta/S$ with $\psi_i^{-1}\psi_i = S_i = S$ and ψ_i being the canonical projection from Δ to Δ_i . So, consider a typical $\Delta/S = \Delta_i$, and $S[b] \in V(\Delta/S)$ where $b \in V(\Delta)$. Let $J = \{(S, N) \mid N \text{ a normal subgroup of finite index in } \pi_1(\Delta/S, S[b]), S = S_i \text{ for some } i \in I\}$. Then by path connected coverings there exists a finite connected graph Γ_{SN} , a vertex $a_{SN} \in V(\Gamma_{SN})$, and a regular covering of finite graphs f_{SN} from (Γ_{SN}, a_{SN}) to $(\Delta/S, S[b])$, so that $f_{SN}(\pi_1(\Gamma_{SN}, a_{SN})) = N$.

Let us now define an order over J by $(S_j, N_j) \geq (S_i, N_i)$ if and only if $j \geq i$ in I and $\psi_{ij}(N_j) \subseteq N_i$, where $\psi_{ij}: \pi_1(\Delta_j, b_j) \rightarrow \pi_1(\Delta_i, b_i)$ is the group homomorphism induced by the map of graphs $\psi_{ij}: (\Delta_j, b_j) \rightarrow (\Delta_i, b_i)$, where $(\Delta_i, \psi_{ij})_{i,j \in I, j \geq i}$ is the corresponding inverse system and the vertex b can also be viewed as $(b_i)_{i \in I}$.

We should first show that J is a directed set. Clearly, the relation ' \geq ' is reflexive as $\psi_{ii} = \text{id}_{\pi_1(\Delta_i, b_i)}$ and it is transitive, for if for $i, j, k \in I, k \geq j \geq i, (S_k, N) \geq (S_j, M) \geq (S_i, L)$ then $S_k \subseteq S_j \subseteq S_i$. Also, $\psi_{ij}(M) \subseteq L, \psi_{jk}(N) \subseteq M$. So, $\psi_{ik}(N) = \psi_{ij}\psi_{jk}(N) \subseteq \psi_{ij}(M) \subseteq L$. Thus $(S_k, N) \geq (S_i, L)$. Consider again, $(S_i, L), (S_j, M) \in J$. Then, consider $S_k = S_i \cap S_j$, so $S_k \subseteq S_i$ and $S_k \subseteq S_j$. Consider, $N = \psi_{ik}^{-1}(L) \cap \psi_{jk}^{-1}(M)$. Now, L, M are normal subgroups of finite index in $\pi_1(\Delta_i, b_i)$ and $\pi_1(\Delta_j, b_j)$ respectively, since ψ_{ik}, ψ_{jk} are group homomorphisms and the intersection of two normal subgroups of finite index is again a normal subgroup of finite index. So, we see that N is a normal subgroup of finite index in $\pi_1(\Delta_k, b_k)$. Now $\psi_{ik}(N) = \psi_{ik}(\psi_{ik}^{-1}(L) \cap \psi_{jk}^{-1}(M)) \subseteq \psi_{ik}\psi_{ik}^{-1}(L) \cap \psi_{ik}\psi_{jk}^{-1}(M) \subseteq L$. In a similar way $\psi_{jk}(N) \subseteq M$. Thus, we have $(S_k, N) \geq (S_i, L)$ and (S_j, M) . This proves that J is a directed set.

Let's now consider any two $(S, N), (P, M) \in J$ with $(P, M) \geq (S, N)$. Then $\psi_{SP}f_{PM}: (\Gamma_{PM}, a_{PM}) \rightarrow (\Delta/S, S[b])$ is a map of finite path connected graphs and $f_{SN}: (\Gamma_{SN}, a_{SN}) \rightarrow (\Delta/S, S[b])$ is a local bijection of finite path connected graphs such that

$$\psi_{SP}f_{PM}(\pi_1(\Gamma_{PM}, a_{PM})) = \psi_{SP}(M) \subseteq N = f_{SN}(\pi_1(\Gamma_{SN}, a_{SN})).$$

By virtue of the lifting criterion of finite graphs, there exists a unique map of graphs $\phi_{SNPM}: (\Gamma_{PM}, a_{PM}) \rightarrow (\Gamma_{SN}, a_{SN})$ such that

$$f_{SN}\phi_{SNPM} = \psi_{SP}f_{PM}$$

that is, the following diagram commutes.

$$\begin{array}{ccc}
(\Gamma_{PM}, a_{PM}) & \xrightarrow{\phi_{SNPM}} & (\Gamma_{SN}, a_{SN}) \\
\downarrow f_{PM} & & \downarrow f_{SN} \\
(\Delta/P, P[b]) & \xrightarrow{\psi_{SP}} & (\Delta/S, S[b])
\end{array}$$

So, for any $(S, N) \in J$, $\phi_{SNSN}: (\Gamma_{SN}, a_{SN}) \rightarrow (\Gamma_{SN}, a_{SN})$ is an identity map of graph by uniqueness of lifting in finite graph theory. Again, by uniqueness of lifting it follows that for any $(S, N), (P, M), (Q, L)$ in J if $(P, M) \geq (S, N)$ and $(Q, L) \geq (P, M)$ then

$$\phi_{SNPM}\phi_{PMQL} = \phi_{SNQL}$$

since $\psi_{SP}\psi_{PQ} = \psi_{SQ}$.

Thus we have an inverse system of finite discrete graphs $\{(\Gamma_{SN}, a_{SN})\}$ along with a family of continuous maps of graphs

$$\{\phi_{SNPM} \mid (S, N), (P, M) \in J, (P, M) \geq (S, N)\}$$

So let us define $\tilde{\Delta} = \varprojlim_{(S, N) \in J} (\Gamma_{SN}, a_{SN})$ and $p = \varprojlim_{(S, N) \in J} f_{SN}$. From the construction, $p: (\tilde{\Delta}, a) \rightarrow (\Delta, b)$ is a regular cover of profinite graphs.

Now let $v \in V(\tilde{\Delta})$ where $v = (v_{SN})_{(S, N) \in J}$, we claim that $\hat{\pi}_1(\tilde{\Delta}, v) = 1$.

Proof of the claim: Let $\phi_{SN}: (\tilde{\Delta}, v) \rightarrow (\Gamma_{SN}, v_{SN})$ be the canonical projection map. Let $R_{SN} = \phi_{SN}^{-1}\phi_{SN}$. Then we can identify $(\tilde{\Delta}/R_{SN}, R_{SN}[v])$ with $\phi_{SN}((\tilde{\Delta}, v))$, as a subgraph of (Γ_{SN}, v_{SN}) and $R_{SN}[v] = v_{SN}$. Hence we can view $K = \{R_{SN} \mid (S, N) \in J\}$ as a fundamental system of compatible cofinite entourages over $\tilde{\Delta}$.

Now for any R_{SN} in K , $\tilde{\Delta}/R_{SN}$ is a finite discrete graph and $\phi_{SN}: (\tilde{\Delta}, v) \rightarrow (\tilde{\Delta}/R_{SN}, R_{SN}[v])$ is a continuous map of graphs. Thus for any $(P, M) \in J$ with $(P, M) \geq (S, N)$ and for the continuous map of graphs

$\phi_{SNPM}: (\Gamma_{PM}, v_{PM})$ to $(\tilde{\Delta}/R_{SN}, R_{SN}[v])$ the following diagram commutes.

$$\begin{array}{ccc}
 (\tilde{\Delta}, v) & \xrightarrow{\phi_{SN}} & (\tilde{\Delta}/R_{SN}, R_{SN}[v]) \\
 & \searrow \phi_{PM} & \nearrow \phi_{SNPM} \\
 & & (\Gamma_{PM}, v_{PM})
 \end{array}$$

Let $R_{PM} = \phi_{PM}^{-1}\phi_{PM}$. Hence, $R_{PM} \subseteq R_{SN}$ by the above commutative diagram. Now let N_1 be a normal subgroup of finite index in $\pi_1((\tilde{\Delta}/R_{SN}, R_{SN}[v]))$. Then $\phi_{SNPM}^{-1}(N_1)$ is a normal subgroup of finite index in $\pi_1(\Gamma_{PM}, v_{PM})$. Hence $f_{PM}(\phi_{SNPM}^{-1}(N_1))$ is a subgroup of finite index in $\pi_1(\Delta/P, P[p(v)])$ as $f_{PM}(\pi_1(\Gamma_{PM}, v_{PM}))$ is a subgroup of finite index in $\pi_1(\Delta/P, P[p(v)]) \cong \pi_1(\Delta/P, P[b])$ as Δ/P is path connected. Let H a normal subgroup of finite index in $\pi_1(\Delta/P, P[p(v)])$ be such that $H \subseteq f_{PM}(\phi_{SNPM}^{-1}(N_1)) \cap M$. Hence there exists (P, H) in J such that $f_{PH}: (\Gamma_{PH}, v_{PH}) \rightarrow (\Delta/P, P[p(v)])$ is a regular covering of finite graphs

and $f_{PH}(\pi_1(\Gamma_{PH}, v_{PH})) = H$. Then we have the following commutative diagram.

$$\begin{array}{ccc}
 (\tilde{\Delta}, v) & \xrightarrow{\phi_{SN}} & (\tilde{\Delta}/R_{SN}, R_{SN}[v]) \\
 \searrow \phi_{PH} & & \nearrow \phi_{SNPM} \\
 & \phi_{PM} & \\
 (\Gamma_{PH}, v_{PH}) & \xrightarrow{\phi_{PMPH}} & (\Gamma_{PM}, v_{PM}) \\
 \searrow f_{PH} & & \downarrow f_{PM} \\
 & & (\Delta/P, P[p(v)])
 \end{array}$$

Thus

$$f_{PM}(\phi_{PMPH}(\pi_1((\Gamma_{PH}, v_{PH})))) \subseteq H$$

Hence

$$\phi_{PMPH}(\pi_1((\Gamma_{PH}, v_{PH}))) \subseteq f_{PM}^{-1}(H) \subseteq \phi_{SNPM}^{-1}(N_1)$$

Calling $\phi_{PH}^{-1}\phi_{PH} = R_{PH}$, as earlier, we have now $R_{PH} \subseteq R_{PM} \subseteq R_{SN}$ and

$$\phi_{SNPH}(\pi_1(\tilde{\Delta}/R_{PH}, R_{PH}[v])) = \phi_{SNPM}(\phi_{PMPH}(\pi_1(\tilde{\Delta}/R_{PH}, R_{PH}[v])))$$

$$\subseteq \phi_{SNPM}(\phi_{PMPH}(\pi_1((\Gamma_{PH}, v_{PH})))) \subseteq \phi_{SNPM}((\phi_{SNPM}^{-1}(N_1)) \subseteq N_1$$

Hence, by the Lemma 8.7, $\hat{\pi}_1(\tilde{\Delta}, v) = 1$. So, by a remark at the end of Chapter 6 we have $\tilde{\Delta}$ is the universal covering of Δ .

Let $g: \Gamma \rightarrow \Delta$ be any covering of profinite connected graphs. Then, by Chapter 5, $\Gamma = \tilde{\Delta}/H$ for some closed subgroup H of $G = \text{Aut}(\tilde{\Delta}, p)$. So, as in Chapter 5, since

H acts residually freely over $\tilde{\Delta}$ we can view $g: \Gamma \rightarrow \Delta$ as the orbit map $h: \tilde{\Delta} \rightarrow \tilde{\Delta}/H$ which is a regular covering. Hence the result follows. \square

Thus in the light of Theorem 8.8, Lemma 6.8, Theorem 9.1, we have the following theorem

THEOREM 9.2. *Let Δ be a connected profinite graph. Then a covering map $p: \tilde{\Delta} \rightarrow \Delta$ is the universal covering of Δ if and only if $\hat{\pi}_1(\tilde{\Delta}, v) = 1$ for any vertex $v \in V(\tilde{\Delta})$*

THEOREM 9.3. *For any covering $f: \Gamma \rightarrow \Delta$ of connected profinite graphs, Γ can be viewed as $\tilde{\Delta}/H$ where $p: \tilde{\Delta} \rightarrow \Delta$ is the universal covering map and H is a closed subgroup of $G = \text{Aut}(\tilde{\Delta}, p)$.*

PROOF. Let $f: \Gamma \rightarrow \Delta$ be a covering map of profinite connected graphs. Then, as in the proof of Theorem 9.1, there exists a unique lift $h: \tilde{\Delta} \rightarrow \Gamma$, where h is a regular covering map of profinite connected graphs and $fh = p$. Then, as in the proof of Theorem 7.4, Γ is isomorphic to $\tilde{\Delta}/H$, where $H = \text{Aut}(\tilde{\Delta}, h)$.

Now, H is a subgroup of G . For if $x \in H$ then $hx = h$. So, now $px = fhx = fh = p$. Thus $x \in G$. Let us now show that H is a closed subgroup of G . Since H is a closed subgroup of G all G -invariant compatible cofinite entourages R over Γ are also H -invariant. So, the fundamental system I of G -invariant compatible cofinite entourages over Γ is also a fundamental system of H -invariant compatible cofinite entourages. Thus as R runs through I , the cofinite entourages $\{(h_1, h_2) \in H \times H \mid (h_1x, h_2x) \in R, \forall x \in \Gamma\} = N_R \cap (H \times H)$ form a fundamental system of entourages of $\text{Aut}(\tilde{\Delta}, h) = H$. It follows that H is a uniform subspace of G . So, by Theorem 7.3, H is compact and G is Hausdorff. Hence H is closed in G . \square

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