ON STRUCTURAL PROPERTIES OF DIFFERENTIAL EQUATIONS

by

JAMES ANDERSON STOVER, JR.

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in the
Department of Mathematics in the
Graduate School of the University of Alabama

UNIVERSITY, ALABAMA

1969

Acknowledgment

I wish to express my sincere gratitude to Professor O. R. Ainsworth, my advisor and dissertation director, for his guidance and inspiration.

Also I would like to thank Professors Robert Plunkett, Henry Miller, and Bill Gray for their helpful comments and suggestions.

Finally, to Beryl Stover, my thanks for her long suffering in typing the manuscript.

Contents

CHAPTER 1. INTRODUCTION AND NOTATION

The description of physical systems by differential equations leads to the problem of obtaining either the solutions of the differential equations or, failing that, obtaining qualitative properties of the solutions. Indeed, in many cases one may be interested only in some qualitative property such as stability or boundedness of the solutions. Much of the recent effort has been to develop techniques for obtaining qualitative information without actually solving the equation. Knowing qualitative properties of the solutions, or even obtaining the solutions themselves does not enable one to predict with certainty the behavior of a physical system, however, since the differential equation itself is only an approximate representation of the physical system. Thus it is necessary to know the behavior of the solutions of all differential equations "near" the given one before one can predict the behavior of the physical system. A property of the solutions of a given differential equation is called a "structural" property if it is also characteristic of the solutions of all differential equations "near" the given one. Thus we can predict that the physical system will be stable, for instance, only if the describing differential equation has structurally stable solutions. This idea is pursued by representing "nearness" as a metric on a set of functions.

Notation. Let A, B be sets, and $f: A \times R \to B$. We shall write f to denote the function, $f(a,t)$ to denote the value of the function at (a,t), and $f(a,\cdot)$ to denote the function $f|\{a\} \times R$. By the notation f_s we shall mean the function defined by $f_s(a,t) = f(a,s+t)$.

CHAPTER II. GENERAL THEOREMS WITHOUT UNIQUENESS

Let W be an open set in R^n and Γ be the class of all continuous functions $f: W\times R \to R^n$. Let ρ be a metric on Γ and denote the metric space by (Γ,ρ). Consider the differential equation

(I) $\frac{dx}{dt} = \dot{x} = f(x,t)$ where $f \varepsilon \Gamma$.

A solution of (I) through the point (y,0) will generally be denoted by $\phi(y,f,\cdot)$, i.e., $\phi(y,f,0) = y$. To denote a solution of (I) through (y,s) we shall write $\phi_{-s}(y,f,\cdot)$ where $\phi_{-s}(y,f,t) = \phi(y,f,t - s)$ for every $t \varepsilon R$. When no misunderstanding can occur, ϕ will be written for $\phi(y,f,\cdot)$.

Definition. We shall say that (I) has unique solutions if for every $(y,s) \varepsilon W\times R$ there exists precisely one solution, $\phi_{-s}(y,f,\cdot)$.

Definition. We shall say that the solutions of (I) are extendable if for each $(y,s) \varepsilon W\times R$, each solution through (y,s) is defined for all $t \varepsilon R$.

We shall often be concerned with the subset Γ_u of Γ defined by $\Gamma_u = \{f \varepsilon \Gamma | \dot{x} = f(x,t)$ has unique solutions$\}$ and the subspace (Γ_u,ρ') where $\rho' = \rho|\Gamma_u$.

Theorem 1. (Peano) Let $f \in \Gamma$ and $(x_o,t_o) \in W\times R$. Then there exists an $\alpha > 0$ such that (1) has a solution through (x_o,t_o) defined on the interval $[t_o,t_o + \alpha]$.

Corollary. Let E be an open subset of $W\times R$ and $|f(x,t)| \leqq M$ on E. If A is any compact subset of E, there exists an $\alpha > 0$, depending on E, A, M, such that if $(x_o,t_o) \in A$ then every solution of (1) through (x_o,t_o) exists on $|t - t_o| \leqq \alpha$.

For a proof of this theorem see [1].

Definition. Let $f \in \Gamma$ and ϕ be a solution of (1) defined on the interval $T \subset R$. The interval T is said to be the maximal interval of definition of the solution if there does not exist a solution ϕ_2 of (1), defined at some end point of T, such that $\phi_1(t) = \phi_2(t)$ for every $t \in T$.

Notation. T_{xf} denotes the maximal interval of definition of the solution $\phi(x,f,\cdot)$.

$T^+_{xf} = \{t \in T_{xf} | t \geqq 0\}$; $T^-_{xf} = \{t \in T_{xf} | t \leqq 0\}$.

The notation ∂T^+_{xf} is used to mean the least upper bound of T^+_{xf} if T^+_{xf} is bounded and $+\infty$ if it is not.

Theorem 2. Let $f \in \Gamma$ and ϕ be a solution of (1) defined on some interval. Then ϕ can be extended to a maximal interval of existence T. Also if T is bounded, then $\phi(t)$ leaves every compact set in W as $t \to \partial T$.

For a proof of this theorem see [1].

Definition. A metric ρ on Γ will be called a C metric if $\{f_n\} \to f$ in (Γ,ρ) implies $|f_n(x,t)-f(x,t)| \to 0$, as $n \to \infty$, uniformly on compact subsets of $W \times R$.

We shall need the following extension of a theorem of Kamke [2].

Theorem 3. Let ρ be a C metric on Γ and $\{f_n\} \to f$. Let $\{x_n\} \to x_o \in W$, $\{t_n\} \to t_o$, ϕ_n be a solution of $\dot{x} = f_n(x,t)$ such that $\phi_n(t_n) = x_n$, and T_n be its maximal interval of existence. Then there exists a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ which converges to a solution ϕ of $\dot{x} = f(x,t)$ such that $\phi(t_o) = x_o$, and the convergence is uniform on compact subsets in the maximal interval of existence, T, of ϕ. Also $T \subset T_{nk}$ for k large enough and if the solution ϕ is unique, then $\{\phi_n(t)\} \to \phi(t)$ uniformly on compact subsets in T.

Proof. Let E_1, E_2, $E_3 \ldots$ be open subsets of $W \times R$ such that $(x_o,t_o) \in E_1$, $\bar{E}_n$ is compact, $\bar{E}_n \subset E_{n+1}$ and $W \times R = \cup E_n$. Since $(x_n,t_n) \to (x_o,t_o)$, there exists an $N > 0$ such that $(x_n,t_n) \in E_1$ for every $n > N$. Since $f_n \to f$, $f_n(x,t)$ converges to $f(x,t)$ uniformly on $\bar{E}_1$ and therefore the $f_n(x,t)$ are uniformly bounded on $\bar{E}_1$. By the corollary to theorem 1, there exists an $\alpha_1 > 0$ such that any ϕ and each ϕ_n exist on the interval $|t-t_n| \leqq \alpha_1$. Let $M > 0$ be such that $|f_n(x,t)| \leqq M$ for every $(x,t) \in E_1$ and for each $n > N$.

Since for $t \in [t_n, t_n + \alpha_1]$,

$$\phi_n(t) = \phi_n(t_n) + \int_{t_n}^{t} f_n(\phi_n(s),s)ds, \text{ and}$$

we have that $|\phi_n(t)| \leqq |x_n| + \alpha_1 M \leqq L + \alpha_1 M$,

where L is a bound on the set E_1.

Further,

$$|\phi_n(t_2) - \phi_n(t_1)| \leqq M(t_2 - t_1) \text{ where } t_n \leqq t_1 < t_2 \leqq t_n + \alpha_1.$$

Thus the $\phi_n(t)$ are uniformly bounded and equi-continuous, and by Ascoli's Theorem there exists a uniformly convergent subsequence $\{\phi_{nk}(t)\}$ of $\{\phi_n(t)\}$. Therefore there exists a ϕ such that

$$\phi(t) = \lim_{k \to \infty} \phi_{nk}(t)$$

$$= \lim_{k \to \infty} [x_{nk} + \int_{t_{nk}}^{t} f_{nk}(\phi_{nk}(s),s)\, ds]$$

$$= x_o + \int_{t_o}^{t} f(\phi(s),s)ds$$

i.e. ϕ is a solution of $\dot{x} = f(x,t)$ on $[t_o, t_o + \alpha_1]$ and $\phi(t_o) = x_o$. Also if the solution ϕ is the only solution of $\dot{x} = f(x,t)$ such that $\phi(t_o) = x_o$, then every convergent subsequence of $\phi_n(t)$ converges to $\phi(t)$.

To show that a subsequence of $\{\phi_n(t)\}$ converges on the interval T, we have that the point $(\phi(t_o + \alpha_1) \in E_1$ and thus a subsequence of $\{n_k\}$ can be chosen (again called n_k) such that $\phi_{nk}(t) \to \phi(t)$ uniformly on $[t_o + \alpha_1,\ t_o + 2\alpha_1]$ where ϕ is a solution of $\dot{x} = f(x,t)$, now defined on $[t_o, t_o + 2\alpha_1]$. This process can be repeated infinitely often to obtain the interval $[0, +\infty)$ or can be repeated only p times where the point $(\phi(t_o + (p-1)\alpha_1), t_o + (p-1)\alpha_1) \varepsilon \bar{E}_1$ but $(\phi(t_o + p\alpha_1), t_o + p\alpha_1)$ does not. In the second case let $s_1 = t_o + p\alpha_1$. Then for some E_j, $(\phi(s_i), s_1) \varepsilon E_j$. The process above is repeated using a suitable α_j until either the interval $[t_o, \infty)$ is obtained or until $\phi(t) \to \partial W$, in which case we have obtained a right maximal interval of existence, T^+. The same argument is used to obtain a left maximal interval of existence T^- and $T = T^+ \cup T^-$. Q.E.D.

The set of all solutions of (1) through (x,0) will be denoted by $\Phi(x,f,\cdot)$. By corollary to theorem 1, there exists an interval T_{xf} containing 0 such that all solutions through (x,0) exist on T_{xf}.

<u>Theorem 4.</u> (Kneser) Let $f \varepsilon \Gamma$ and $t_1 \varepsilon T_{xf}$. Then $\Phi(x,f,t_1)$ is compact and connected.

<u>Theorem 5.</u> (Kamke) Let $f \varepsilon \Gamma$ and $[t_1,t_2] \subset T_{xf}$. Then $\Phi(x,f,[t_1,t_2])$ is compact.

For a proof of these theorems see [3].

Theorem 6. Let $f \varepsilon \Gamma$ and all the solutions through the compact set $A\times\{0\}$, $A \subset W$, of $\dot{x} = f(x,t)$ exist at some $t_1 \varepsilon R$. Then $\Phi(A,f,t_1)$ is compact.

Proof. Suppose $\Phi(A,f,t_1)$ is not bounded. Then there exists a sequence $\{x_n\} \subset A$ and particular solutions ϕ_n such that $|\phi_n(x_n,f,t_1)| \to \infty$ as $n \to \infty$. Since A is compact, there exists a subsequence of $\{x_n\}$, again called $\{x_n\}$, which converges to a point $x \varepsilon A$. By theorem 3, there exists a subsequence $\{\phi_{n_k}(x_{n_k},f,t)\}$ of $\{\phi_n(x_n,f,t_1)\}$ which converges to a solution $\phi(x,f,t_1)$.

But this contradicts the fact that $|\phi_{n_k}(x_{n_k},f,t_1)| \to \infty$, $k \to \infty$, since $\phi(x,f,t_1)$ is defined. Therefore $\Phi(A,f,t_1)$ is bounded.

Suppose $\{y_n\}$ is a sequence in $\Phi(A,f,t_1)$ such that $y_n \to y$. There exists a sequence $\{x_n\}$ in A and particular solutions ϕ_n such that $\phi_n(x_n,f,t_1) = y_n$ for every n. Since A is compact, $\{x_n\}$ has a convergent subsequence; without loss of generality can assume $\{x_n\} \to x$. As above, there exists a subsequence $\{\phi_{n_k}(x_{n_k},f,t_1)\}$ which converges to a solution $\phi(x,f,t_1)$, i.e., the subsequence $\{y_{n_k}\} \to \phi(x,f,t_1) \varepsilon \Phi(A,f,t_1)$. Since $y_n \to y$ also, this gives $y \varepsilon \Phi(A,f,t_1)$, i.e., $\Phi(A,f,t_1)$ is closed.

<u>Theorem 7.</u> Let ρ be a C metric on Γ and F be a compact set in Γ such that all the solutions through (x,0) of $\dot{x} = f(x,t)$ are defined at some $t_1 \in R$, for every $f \in F$. Then the set $\Phi(x,F,t_1)$ is compact.

<u>Proof.</u> Let $\{y_n\}$ be an arbitrary sequence in $\Phi(x,F,t_1)$. If $\{y_n\} \to y$, then there exists a sequence $\{f_n\} \subset F$ and particular solutions ϕ_n such that $\phi_n(x,f_n,t_1) = y_n$. Since $f_n \in F$, there exists a subsequence $\{f_{nk}\}$ of $\{f_n\}$ which converges to an $f \in F$. Therefore,

(1) $\phi_{nk}(x,f_{nk},t_1) = y_{nk} \to y.$

By theorem 3, there exists a subsequence $\{\phi_{nk_i}(x,f_{nk_i},t_1)\}$ which converges to $\phi(x,f,t_1) \in \Phi(x,F,t_1)$. Therefore, $y = \phi(x,f,t_1)$, i.e., $\Phi(x,F,t_1)$ is closed.

If $|y_n| \to \infty$, then in (1), $|y_{nk}| \to \infty$ and $\phi(x,f,\cdot)$ is not defined at t_1 which contradicts the hypothesis. Therefore, $\Phi(x,F,t_1)$ is bounded.

Thus $\Phi(x,F,t_1)$ is compact.

<u>Theorem 8.</u> Let ρ be a C metric on Γ and $A \times F \subset W \times \Gamma$ be connected, with all solutions of $\dot{x} = f(x,t)$ through A at $t = 0$ existing for some $t_1 \in R$. Then $\Phi(A,F,t_1)$ is connected.

<u>Proof.</u> Suppose $\Phi(A,F,t_1)$ is not connected. Then there exists sets U, V such that $\overline{U} \cap V = \emptyset$ and $\overline{V} \cap U = \emptyset$ with $\Phi(A,F,t_1) = U \cup V$.

Let $P = \{(x,f) \in A \times F \mid \text{there exists } \phi(x,f,t_1) \in U\}$

$Q = \{(y,g) \in A \times F \mid \text{there exists } \phi(y,g,t_1) \in V\}$

Since by Kneser's Theorem $\Phi(x,f,t_1)$ is connected, $\phi(x,f,t_1) \in U$ if and only if $\Phi(x,f,t_1) \subset U$, i.e., $P \cap Q = \emptyset$. However, since $A \times F$ is connected, either $\overline{P} \cap U \neq \emptyset$ or $P \cap \overline{Q} \neq \emptyset$. Without loss of generality can assume $\overline{P} \cap Q \neq \emptyset$. Let $(y,g) \in \overline{P} \cap Q$ and $\{(x_n,f_n)\} \subset P$ be a sequence such that $(x_n,f_n) \to (y,g)$. Then by theorem 3 there exists a sequence $\{\phi_n(x_n,f_n,t_1)\}$ which converges to a $\phi(y,g,t_1)$. Now $\phi_n(x_n,f_n,t_1) \in U$; therefore, $\phi(y,g,t_1) \in \overline{U}$. But $\overline{U} \cap V = \emptyset$, i.e., $\phi(y,g,t_1) \notin V$ which contradicts the definition of Q.

Therefore, $\Phi(A,F,t_1)$ is connected.

CHAPTER III. STRUCTURAL PROPERTIES OF UNIQUE SOLUTIONS

In this chapter we consider differential equations

(I) $\dot{x} = f(x,t)$ where $f \in \Gamma_u$, i.e., the solutions of (I) are unique.

Definition. Let (X,d) be a metric space and $\pi: X\times R \to X$ be a function such that

(i) $\pi(x,0) = x$ for every $x \in X$,

(ii) $\pi(\pi(x,t_1),t_2) = \pi(x,t_1 + t_2)$ for every t_1, $t_2 \in R$, $x \in X$,

(iii) π is continuous,

then π is called a dynamical system on X.

Definition. For fixed $x \in X$, the function $\pi_x: R \to X$ defined by $\pi_x(t) = \pi(x,t)$ is called the motion through x. The set of points $\pi(x,R) = \{\pi(x,t) | t \in R\}$ is called the trajectory through x. The positive half trajectory is defined as $\pi(x,R^+)$; the negative half trajectory as $\pi(x,R^-)$.

Remark. For each fixed $t \in R$, the function $\pi_t: X \to X$ defined by $\pi_t(x) = \pi(x,t)$ is a homeomorphism of X onto X. (c.f. [3]).

Notation. $\pi(A,B) = \{\pi(x,t) | x \in A \text{ and } t \in B\}$.

Definition. A subset $M \subset X$ is invariant if $\pi(M,R) \subset M$. Positively invariant and negatively invariant are defined by $\pi(M,R^+) \subset M$ and $\pi(M,R^-) \subset M$ respectively.

Definition. A motion π_x is compact [positively compact] [negatively compact] if $\overline{\pi(x,R)}$ $[\overline{\pi(x,R^+)}]$ $[\overline{\pi(x,R^-)}]$ is compact.

Definition. For any $x \in W$ the omega limit set of x, written Ω_x, is defined by $\Omega_x = \{y \in X |$ there exists a sequence $\{t_n\} \to +\infty$ with $\pi(x,t_n) \to y\}$. Similarly the alpha limit set of x written ∇_x is defined by $\nabla_x = \{y \in X |$ there exists a sequence $\{t_n\} \to -\infty$ with $\pi(x,t_n) \to y\}$.

Remark. $\overline{\pi(x,R^+)} = \pi(x,R^+) \cup \Omega_x$.

Remark. Ω_x is closed and invariant.

Definition. A motion π_x is periodic if there exists $T > 0$ such that $\pi_x(t + T) = \pi_x(t)$ for every $t \in R$.

Definition. A point x is a fixed point or rest point of π if $\pi(x,t) = x$ for every $t \in R$.

Definition. A motion π_x is stable, (positively stable), (negatively stable) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(\pi(x,t), \pi(y,t)) < \varepsilon$ for every $t \in R$, $(t \in R^+)$, $(t \in R^-)$ respectively.

Definition. A motion π_x is (positively) asymptotically stable if it is positively stable and there exists a neighborhood $N(x)$ such that $d(\pi(x,t), \pi(y,t)) \to 0$ as $t \to +\infty$, for every $y \in N$.

Theorem 9. Let $f \in \Gamma$ and define $\pi:\Gamma\times R \to \Gamma$ by $\pi(f,\tau) = f_\tau$ where f_τ is defined by $f_\tau(x,t) = f(x,t + \tau)$. Assume the metric ρ is such that π is continuous. Then π is a dynamical system on Γ.

Proof.

(i) $\pi(f,0) = f_0 = f$

(ii) $\pi(\pi(f,\tau),\sigma) = \pi(f_\tau,\sigma) = (f_\tau)_\sigma = f_{\tau + \sigma} = (f,\tau + \sigma)$

(iii) by assumption.

A version of this theorem is found in Sell [4].

Remark. It is known [5] that there exists a metric ρ on Γ which generates the compact-open topology on Γ and (Γ,ρ) is complete. In this case we have that if $(f_n,t_n) \to (f,t_0)$, then $f_{n t_n}(x,t) = f_n(x,t + t_n) \to f(x,t + t_0)$ uniformly on compact subsets, i.e., $f_{n t_n} \to f_{t_0}$ in the c-o topology on Γ, which shows π is continuous with this topology.

Remark. If $f \in \Gamma_u$ then $f_t \in \Gamma_u$ for every $t \in R$. Therefore $\pi|\Gamma_u$ is a dynamical system on Γ_u.

Remark.

(a) The motion π_f is periodic if and only if f is periodic in t.

(b) f is a fixed point of π if and only if the function f is autonomous.

Definition. The solution $\phi(x,f,\cdot)$ is stable (positively stable), (negatively stable) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies

$|\phi(x,f,t) - \phi(y,f,t)| < \varepsilon$ for every $t \in T_{xf} \cap T_{yf}$ $(T^+_{xf} \cap T^+_{yf})$, $(T^-_{xf} \cap T^-_{yf})$ respectively.

Definition. The solution $\phi(x,f,\cdot)$ is (positively) uniformly stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|\phi(x,f_\tau,t) - \phi(y,f_\tau,t)| < \varepsilon$ for every $\tau \in R$ and for every $t \in T^+_{x,f_\tau} \cap T^+_{y,f_\tau}$.

Definition. A positively extendable solution $\phi(x,f,\cdot)$ is (positively) asymptotically stable if it is positively stable and there exists a neighborhood $N(x)$ such that $|\phi(x,f,t) - \phi(y,f,t)| \to 0$ as $t \to \partial T^+_{yf}$ (or $t \to +\infty$)

Remark. Henceforth we shall drop the (positive) in the above two definitions since this is the only type with which we shall be concerned.

Definition. A solution $\phi(x,f,\cdot)$ is periodic if there exists a $T > 0$ in T_{xf} such that $\phi(x,f,t + T) = \phi(x,f,t)$ for every $t \in T_{xf}$. We say that T is a period of $\phi(x,f,\cdot)$.

Theorem 10. $\phi(x,f,\cdot)$ is asymptotically stable if and only if it is positively stable and there exists a $N(x)$ such that for every $y \in N(x)$, $\phi(y,f,\cdot)$ is positively extendable and $|\phi(x,f,t) - \phi(y,f,t)| \to 0$ as $t \to \infty$.

Proof. Suppose $\phi(x,f,\cdot)$ is asymptotically stable and there does not exist a $N(x)$ such that $\phi(y,f,\cdot)$ is positively extendable for every $y \in N(x)$. Then there exists a sequence $\{y_n\} \to x$ such that for each n, $\phi(y_n,f,\cdot)$ is not positively extendable. Thus there exists a sequence $\{b_n\}$ such that $\phi(y_n,f,t)$ leaves every compact set in $N(x)$ as $t \to b_n$.

Let $N_1(x)$ be a neighborhood of asymptotic stability of $\phi(x,f,\cdot)$.

Then for n large enough, say $n = k$, $y_k \in N_1(x)$.

Let $A = \{\phi(x,f,t_1) | 0 \leqq t_1 \leqq b_k\}$ and let $\delta > 0$ be small enough such that $\overline{S_\delta(A)} \subset W$.

Then $\phi(y_k,f,t)$ leaves $\overline{S_\delta(A)}$ as $t \to b_k$ which contradicts the asymptotic stability of $\phi(x,f,\cdot)$. The remainder of the proof is obvious.

Theorem II. If $\phi(x,f,\cdot)$ is periodic, then it is extendable.

Proof. Suppose $T_{xf} \neq (-\infty, +\infty)$. Then either T_{xf}^{+} is bounded or T_{xf}^{-} is bounded. Suppose b is the least upper bound of T_{xf}^{+}, and $\{t_n\} \to b$, $(t_n < b)$. Then $\phi(x,f,t_n) = \phi(x,f,t_n + T)$, and for n large enough, $t_n + T > b$, a contradiction. On the other hand, if a is the greatest lower bound of T_{xf}^{-} and $\{t_n\} \to a$, $(t_n > a)$ then $\phi(x,f,t_n) \to \partial W$ as $t_n \to a$. But $\phi(x,f,t_n) = \phi(x,f,t_n + T)$, i.e., $\phi(x,f,t_n + T) \to \partial W$ which contradicts the fact that $a + T \in T_{xf}$. Therefore $\phi(x,f,\cdot)$ is extendable.

Definition. A property P (e.g., stability) of the solution $\phi(x,f,\cdot)$ will be called a structural property if there exists a neighborhood of f, V(f), such that P is a property of $\phi(x,g,\cdot)$ for every $g \in V$.

Remark. P is a structural property of $\phi(x,f,\cdot)$ if and only if there exists an $M > 0$ such that $\rho(f,g) < M$ implies P is a property of $\phi(x,g,\cdot)$. The metric ρ thus provides a measure of nearness of two differential equations. If ρ induces the discrete topology on Γ, then every property of a solution is a structural property. On the other hand, if the topology on Γ is the indiscrete topology, the only properties that are structural are those which are properties of $\phi(x,g,\cdot)$ for every $g \in \Gamma$.

Conjecture 1. There exists a (complete) metric on Γ such that asymptotic stability is equivalent to structural stability.

Conjecture 2. There exists a (complete) metric on Γ such that uniform asymptotic stability is equivalent to structural stability.

For a formal definition of structural stability and structural asymptotic stability we have:

Definition. The solution $\phi(x,f,\cdot)$ is structurally stable (structurally asymptotically stable) if there exists a neighborhood $V(f)$ such that for every $g \in V$, $\phi(x,g,\cdot)$ is stable, (asymptotically stable).

Definition. A property P will be called translation invariant if for every $f \in \Gamma_u$, P is a property of $\phi(x,f_t,\cdot)$ whenever P is a property of $\phi(x,f,\cdot)$ and $t \in T_{xf}$.

Theorem 12. P is a structural property of $\phi(x,f,\cdot)$ if and only if there exists a neighborhood $V(f)$ such that P is a structural property of $\phi(x,g,\cdot)$ for every $g \in V(f)$.

Proof. Assume P is a structural property of $\phi(x,f,\cdot)$. Then there exists a neighborhood $V(f)$ such that P is a property of $\phi(x,g,\cdot)$ for every $g \in V(f)$. Therefore, since $V(f)$ is a neighborhood of g we have P is a structural property of $\phi(x,g,\cdot)$.
The converse is immediate since $f \in V$.

Corollary 1. Let P be translation invariant and a structural property of $\phi(x,f,\cdot)$. Let $U_P = \{g \in \Gamma_u | \text{ P is a property of } \phi(x,g,\cdot)\}$. Then interior U_P is not empty and P is a structural property of $\phi(x,g,\cdot)$ for every $g \in$ interior U_p.

Corollary 2. If $f^* \in \Omega_f \cap \Gamma_u$, then either $f^* \in$ interior U_P, i.e., P is a structural property of $\phi(x,f^*,\cdot)$ or $f^* \in \partial U_P$, i.e., every neighborhood $V(f^*)$ contains a point g_1 such that P is a structural property of $\phi(x,g_1,\cdot)$, and also a point g_2 such that P is not a property of $\phi(x,g_2,\cdot)$ (or $g_2 \notin \Gamma_u$).

Theorem 13. If ρ is a C metric on Γ and if there exists an $x \varepsilon W$ such that the set $U_x = \{g \varepsilon \Gamma_u | \phi(x,g,\cdot)$ is stable$\}$ is open in the subspace Γ_u, then Γ_u is not connected.

Proof. Let $\{g_n\} \subset U_x$ and $g_n \to g \varepsilon \Gamma_u$. Suppose $\phi(x,g,\cdot)$ is not stable. Then there exists an $\varepsilon > 0$, and sequences $\{t_k\}$, $\{x_k\} \to x$ such that

$$\varepsilon \leqq |\phi(x,g,t_k) - \phi(x_k,g,t_k)| \leqq |\phi(x,g,t_k) - \phi(x,g_n,t_k)|$$

$$+ |\phi(x,g_n,t_k) - \phi(x_k,g_n,t_k)| + |\phi(x_k,g_n,t_k) - \phi(x_k,g,t_k)|$$

By theorem 3, there exists an $N > 0$ such that $|\phi(x,g,t_k) - \phi(x,g_n,t_k)| < \varepsilon/4$, $n > N$ and $|\phi(x_k,g_n,t_k) - \phi(x_k,g,t_k)| < \varepsilon/4$, $n > N$ with k fixed arbitrarily.

Thus $\varepsilon/2 < |\phi(x,g_n,t_k) - \phi(x_k,g_n,t_k)|$.

Since $x_k \to x$, this contradicts the stability of $\phi(x,g_n,\cdot)$. Therefore the set U_x is closed. Since U_x is clearly a proper subset of Γ_u for x arbitrary, Γ_u is not connected.

Theorem 14. Let ρ be a C metric on Γ. Suppose there exists a neighborhood $N(x)$ such that $\phi(y,f_\tau,\cdot)$ is extendable for every $\tau > 0$, and $y \in N(x)$. If $\phi(x,f,\cdot)$ is positively uniformly stable and if $f^* \in \Gamma_u \cap \Omega_f$, then $\phi(x,f^*,\cdot)$ is positively stable.

Proof. Suppose $\phi(x,f^*,\cdot)$ is not stable. Then there exists a sequence $\{y_n\} \to x$, $\varepsilon > 0$, and sequence $\{t_n\}$ where $t_n \in T^+_{y_n f^*} \cap T^+_{x,f^*}$ such that

(1) $\varepsilon \leqq |\phi(y_n f^*,t_n) - \phi(x,f^*,t_n)|$ for every n.

But;

(2) $|\phi(y_n,f^*,t_n) - \phi(x,f^*,t_n)| \leqq |\phi(y_n,f^*,t_n) - \phi(y_n,f_{\tau_\kappa},t_n)|$

$+ |\phi(y_n,f_{\tau_\kappa},t_n) - \phi(x,f_{\tau_\kappa},t_n)| + |\phi(x,f_{\tau_\kappa},t_n) - \phi(x,f^*,t_n)|$

where n is large enough so that $y_n \in N(x)$ and $\{\tau_\kappa\}$ is a sequence such that $f_{\tau_\kappa} \to f^*$. Consider the first and third terms on the right side of (2). By theorem 3, for each n, there exists τ_{κ_n} such that both these terms are less then $\varepsilon/4$ each, i.e.,

$|\phi(y_n,f^*,t_n) - \phi(x,f^*t_n)| \leqq |\phi(y_n,f^*,t_n) - \phi(y_n,f_{\tau_{\kappa_n}},t_n)|$

$|\phi(y_n,f_{\tau_{\kappa_n}},t_n) - \phi(x,f_{\tau_{\kappa_n}},t_n)| + |\phi(x,f_{\tau_{\kappa_n}},t_n) - \phi(x,f^*t_n)|$

$\varepsilon/4 + \varepsilon/4 + |\phi(y_n,f_{\tau_{\kappa_n}},t) - \phi(x,f_{\tau_{\kappa_n}},t)|$

$\varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4$ whenever n is large enough. This contradicts (1).

Definition. A function $f \varepsilon \Gamma$ will be called asymptotically autonomous if there exists exactly one point f^* in Ω_f.

Theorem 15. If $f \varepsilon \Gamma$ is asymptotically autonomous and the motion $\pi_f(t)$ is asymptotically stable, then f is asymptotically autonomous in a structural sense, i.e., there exists a neighborhood V(f) such that g is asymptotically autonomous for every $g \varepsilon V$.

Proof. If the conclusion were not true there exists a sequence $\{g_n\}$ with $g_n \to f$ and no g_n is asymptotically autonomous. Therefore since $\pi_f(t)$ is asymptotically stable, and $g_n \to f$, there exists integer $N > 0$ such that $\rho(\pi(f,t),\pi(g_N,t)) = \rho(f_t,g_{Nt}) \to 0$ as $t \to \infty$. To simplity notation let $h = g_N$. Then

(a) $\rho(f_t,h_t) \to 0$ as $t \to \infty$.

Let $f^* \varepsilon \Omega_f$ and $h^*, h^{**} \varepsilon \Omega_h$ such that $h^* \neq h^{**}$. Then there exists sequences $\{r_n\}$, $\{s_n\}$ such that

(b) $\{h_{r_n}\} \to h^*$

(c) $\{h_{s_n}\} \to h^{**}$ and

(d) $f_t \to f^*$ as $t \to \infty$.

Let $\alpha = \rho(h^*,h^{**})$. Then using (a), (b), (c), (d) we have

$\rho(f_{r_n},h^*) \leqq \rho(f_{r_n},h_{r_n}) + \rho(h_{r_n},h^*) \to 0$, as $n \to \infty$ and

$\rho(f_{s_n},h^{**}) \leqq \rho(f_{s_n},h_{s_n}) + \rho(h_{s_n},h^{**}) \to 0$, as $n \to \infty$

Thus both $h^*, h^{**} \varepsilon \Omega_f$ which contradicts the hypothesis.

We shall now define the concept of a local dynamical system, which is a generalization of the flow defined by an autonomous differential equation whose solutions do not necessarily have the property of extendability.

This definition is taken from Sell [4].

Let (X,d) be a metric space and suppose for each $x \varepsilon X$ there is associated an open interval $T_x = (\alpha_x, \beta_x)$ containing 0, where possibly $\alpha_x = -\infty$ or $\beta_x = +\infty$. Let $Y = \{(x,t) | x \varepsilon X \text{ and } t \varepsilon T_x\}$. Then a function $\pi: Y \to X$ is said to be a local dynamical system on X if π has the following properties:

(i) $\pi(x,o) = x$ for every $x \varepsilon X$.

(ii) $\pi(\pi(x,s),t) = \pi(x,t+s)$ whenever s, $t+s \varepsilon T_x$ and $t \varepsilon T_{\pi(x,s)}$

(iii) π is continuous

(iv) If $T_x \neq R$, i.e., $\alpha_x \neq -\infty$ or $\beta_x \neq +\infty$, then $\pi(x,t)$ leaves every compact set in X as $t \to \partial T_x$.

Remark. Every dynamical system is a local dynamical system.

The definitions of invariant sets, motions and trajectories of dynamical systems carry over to local dynamical systems except the intervals T_x replace R.

For local dynamical systems we make the following definitions:

Definition. A motion π_x is (positively) stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $d(\pi(x,t),\pi(y,t)) < \varepsilon$ for every $t \in T_x^+ \cap T_y^+$.

Definition. A motion π_x is asymptotically stable if $T_x^+ = [0, +\infty)$, π_x is stable, and there exists a neighborhood $N(x)$ such that $d(\pi(x,t),\pi(y,t)) \to 0$ as $t \to \partial T_y^+$.

Theorem 16. If the metric space X is locally compact, then a motion π_x is asymptotically stable if it is stable and there exists a neighborhood $N(x)$ such that $T_y^+ = [0, +\infty)$ for every $y \in N(x)$ and $d(\pi(x,t),\pi(y,t)) \to 0$ as $t \to +\infty$.

Proof. Suppose π_x is asymptotically stable and there exists a sequence $\{y_n\} \to x$ such that $T_{y_n}^+ = [0,b_n)$, $b_n < +\infty$. Let $N_1(x)$ be the neighborhood of asymptotic stability. Then there exists a K such that $y_k \in N_1(x)$, and there exists a neighborhood $N(\pi(x,b_k))$ such that $\bar{N}$ is compact. By axiom (iv), $\pi(y_k,t)$ leaves $\bar{N}$ as $t \to b_k$ which contradicts the asymptotic stability of π_x.

Define $Z = \{x \in X | T_x = R\}$

Then the local dynamical π restricted to Z becomes a dynamical system on Z. (If Z is not empty.)

Let $Z^+ = \{x \in X | \beta_x = +\infty\}$

and $Z^- = \{x \in X | \alpha_x = -\infty\}$

then $Z = Z^+ \cap Z^-$

If $x \in Z^+$ then the omega limit set Ω_x is defined as for dynamical systems; likewise for $x \in z^-$, the alpha limit set, ∇_x, is defined.

Theorem 17. If π is a local dynamical system and π_x is a positively compact motion, then Ω_x is non empty, compact and invariant. Moreover, for every $y \in \Omega_x$, $T_y = R$.

Proof. Since π_x is positively compact, $\overline{\pi(x,T_x^+)}$ is compact, and by (iv) of the previous definition, $T_x^+ = R^+$. Therefore Ω_x is defined. Let $\{t_n\}$ be any sequence in R^+ such that $t_n \to \infty$. Then the sequence $\{\pi(x,t_n)\}$ is contained in a compact set, (namely $\overline{\pi(x,R^+)}$) and therefore has a convergent subsequence $\{\pi(x,t_{n_k})\}$ which converges to a point $x^* \in \overline{\pi(x,R^+)}$. Therefore $x^* \in \Omega_x$, i.e., Ω_x is not empty. Let $\{x^*_n\}$ be a sequence in Ω_x which converges to a point x_o^*. Let $S_{1/k}(x^*)$ be a sequence of $1/k$ diameter spheres about x_o^* where $k = 1, 2, 3\ldots$. Since for each x_n^* there exists a sequence $\{t_{n_m}\} \to \infty$, with $\pi(x,t_{n_m}) \to x_n^*$ we have that in each sphere $S_{1/k}(x_o^*)$ there exists a point $\pi(x,t_{n_{m_k}})$ and since $1/k \to 0$, we have $t_{n_{m_k}} \to \infty$ and $\pi(x,t_{n_{m_k}}) \to x_o^*$, i.e., $x_o^* \in \Omega_x$. Therefore Ω_x is closed. Since

$\Omega_x \subset \overline{\pi(x,R^+)}$, Ω_x is compact. To show Ω_x is invariant: Let $x^* \in \Omega_x$ and $t \in R$. Let $\{t_n\}$ be the sequence such that $\pi(x,t_n) \to x^*$. We have by continuity of π that $\pi(x,t_n + t) = \pi(\pi(x,t_n),t) \to \pi(x^*,t)$. Since $t_n + t \to \infty$ (t considered fixed), $\pi(x^*,t) \in \Omega_x$, i.e., Ω_x is invariant. The last statement in the theorem now follows from (iv) of the definition.

<u>Corollary.</u> If π is a local dynamical system on X and if there exists a positively compact motion, then Z is not empty and $\pi|Z$ defines a dynamical system on Z.

Let π be a dynamical system on metric space X.

Let M be a non empty, compact, invariant subset of X.

<u>Definition.</u> M is <u>(positively) stable</u> if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\pi(S_\delta(M),R^+) \subset S_\varepsilon(M)$.

<u>Definition.</u> M is a <u>(positive) attractor</u> if there exists an $\alpha > 0$ such that Ω_y is a non empty subset of M whenever $y \in S_\alpha(M)$.

Definition. M is (positively) asymptotically stable if it is a stable attractor.

Negative stability and negative attractors can also be defined. We shall need only the "positive" definition and will use stable to mean positively stable.

By the region of attraction, A(M), of the set M we mean the union of all trajectories with the property that their positive limit sets are non empty and contained in M.

Theorem 18. If M is an attractor, then A(M) is an open invariant set.

Proof. Since M is an attractor there exists an $\delta > 0 \ \varepsilon \ S_\delta(M) \subset A(M)$. Let $y \ \varepsilon \ A(M)$. Then $\pi(y,\tau) \ \varepsilon \ S_\delta(M)$ for some $\tau > 0$. Therefore there exists a neighborhood N of $\pi(y,\tau)$ such that $\pi(N,-\tau)$ is a neighborhood of y and $\Omega_{\pi(N,-\tau)} \subset M$. Therefore $\pi(N,-\tau) \subset A(M)$, i.e., y is an interior point of A(M). Invariance is obvious.

Theorem 19. Let M be an attractor. Then M is stable if and only if A(M) contains no alpha limit points of trajectories in A(M) - M.

For a proof of this theorem see [6].

Theorem 20. Let ρ induce the compact-open topology on Γ. In the space $W\times\Gamma_u$ define $d((x,f),(y,g)) = |x - y| + \rho(f,g)$.
Define $Y = \{(x,f,t)|(x,f) \in W\times\Gamma_u \text{ and } t \in T_{xf}\}$ where T_{xf} is the maximal interval of definition of the solution $\phi(x,f,\cdot)$.
Define $p:Y \to W\times\Gamma_u$ by $p(x,f,t) = (\phi(x,f,t),f_t)$ where $f_t(x,r) = f(x,t + r)$.
Then p is a local dynamical system on $W\times\Gamma_u$.

Proof.

(i) $p(x,f,0) = (\phi(x,f,0),f_0) = (x,f)$.

(ii) $\phi(x,f,t + \tau)$ is the solution of $\frac{dx}{d(t + \tau)} = f(x,t + \tau)$ through the point $\phi(x,f,\tau)$ at $t = 0$.

$\phi(\phi(x,f,\tau),f_\tau,\cdot)$ is the solution of $\frac{dx}{dt} = \frac{dx}{d(t + \tau)} = f_\tau(x,t) = f(x,t + \tau)$

through the point $\phi(x,f,\tau)$ at $t = 0$, i.e., the solutions $\phi_\tau(x,f,\cdot)$ and $\phi(\phi(x,f,\tau),f_\tau,\cdot)$ are solutions of the same differential equation through the same point and thus by uniqueness they are equal.

Therefore,

$p(p(x,f,t),s) = p(\phi(x,f,t),f_t,s) = (\phi(\phi(x,f,t),f_t,s),f_t + s)$
$= (\phi(x,f,t + s),f_t + s) = p(x,f,t + s)$ whenever t, $t + s \in T_{xf}$ and $s \in T_{p(x,f,t)}$.

(iii) Let $\{(x_n,f^n,t_n)\} \to (x,f,t)$ where all points $\in Y$. (The sequence element f^n is written in superscript to distinguish between it and the translation point f_t.)

$p(x_n, f^n, t_n) = (\phi(x_n, f^n, t_n), f^n_{t_n})$ and $p(x,f,t) = (\phi(x,f,t)f_t)$.

$|\phi(x_n, f^n, t_n) - \phi(x,f,t)| \leqq |\phi(x_n, f^n, t_n) - \phi(x,f,t_n)| + |\phi(x,f,t_n) - \phi(x,f,t)|$.

By theorem 3, $|\phi(x_n, f^n, t_n) - \phi(x,f,t_n)| \to 0$, $n \to \infty$ since $\{t_n\}$ is contained in a compact subset of T_{xf} for n large enough. By continuity, $|\phi(x,f,t_n) - \phi(x,f,t)| \to 0$, $n \to \infty$. Therefore $\phi(x_n, f^n, t_n) \to \phi(x,f,t)$. Let $A \times T$ be a compact subset of $W \times R$, and $T' = T \cup \{s + r | s \in T$ and $r = t_n$ or $t\}$. Then $A \times T'$ is compact. Let $\varepsilon > 0$ be arbitrary.

$|f^n_{t_n}(x,s) - f_t(x,s)| = |f^n(x,s + t_n) - f(x,s + t)| \leqq |f^n(x,s + t_n) - f(x,s + t_n)| + |f(x,s + t_n) - f(x,s + t)|$

Since $f^n \to f$ in the compact open topology there exists an N, such that for $n > N$, $|f^n(x,s + t_n) - f(x,s + t_n)| < \varepsilon/2$ for every $(x,s) \in A \times T$, and since $f(x,s)$ is uniformly continuous on $A \times T$, there exists an N_2 such that for every $n > N_2$, $|f(x,s + t_n) - f(x,s + t)| < \varepsilon/2$ on $A \times T$. Therefore $f^n_{t_n}(x,s) \to f_t(x,s)$ uniformly on compact subsets in $W \times R$, i.e., $f^n_{t_n} \to f_t$. Therefore $p(x_n, f^n, t_n) \to p(x,f,t)$, i.e., p is continuous.

(iv) Suppose $T_{xf} \neq R$. Then $p(x,f,t) = (\phi(x,f,t), f_t)$ and by theorem 2, we have that $\phi(x,f,t)$ leaves every compact set in W as $t \to \partial T_{xf}$. Therefore the point $(\phi(x,f,t), f_t)$ leaves every compact set in $W \times \Gamma_u$ as $t \to \partial T_{xf}$.

Theorem 21. Let ρ induce the compact open topology on Γ, $f \in \Gamma_u$, and $f(0,t) = 0$, $t > 0$. Assume that the motion π_f is positively compact and $\Omega_f \subset \Gamma_u$. If $\phi(0,f,\cdot)$ is stable and $\phi(0,f^*,\cdot)$ is asymptotically stable for every $f^* \in \Omega_f$ in a uniform sense, i.e., there exists an $\eta > 0$ such that if $|x| < \eta$ then $|\phi(0,f^*,t) - \phi(x,f^*,t)| \to 0$ as $t \to \infty$ for every $f^* \in \Omega_f$, then $\phi(0,f,\cdot)$ is asymptotically stable.

Proof. Consider $p|W\times\Omega_f$. Let $M = \{0\}\times\Omega_f$. Then M is a stable attractor in the space $W\times\Omega_f$, since $f^*(0,t) \equiv 0$ and therefore $\phi(0,f^*,t) \equiv 0$ for every $f^* \in \Omega_f$. If $A(M)$ is its region of attraction, then there exists a $\beta > 0$ such that the set $\{(x,f^*)|\ |x| < \beta,\ f^* \in \Omega_f\} \subset A(M)$, since $A(M)$ is open. Since $\phi(0,f,\cdot) \equiv 0$ is stable, there exists a $\delta > 0$ such that $|x| < \delta$ implies $|\phi(x,f,t)| < \min(\eta,\beta)$ for every $t \in R^+$, i.e., $\phi(x,f,\cdot)$ is positively compact. Therefore since π_f is positively compact, $\Omega_{(x,f)}$ is non empty and compact and is contained in $A(M)$. Since $\Omega_{(x,f)}$ is compact and invariant, every motion in $\Omega_{(x,f)}$ is compact, thus if $(x^*,f^*) \in \Omega_{(x,f)}$ then the alpha limit set, $\nabla_{(x^*,f^*)}$, of the motion $p_{(x^*,f^*)}$ is non empty and is contained in $\Omega_{(x,f)} \subset A(M)$. By the above theorem we have $\Omega_{(x,f)} \subset M = \{0\}\times\Omega_f$. Therefore $\phi(x,f,t) \to 0$ as $t \to \infty$, i.e., $\phi(0,f,\cdot)$ is asymptotically stable.

Theorem 22. If the motion $p_{(x,f)}$ is asymptotically stable, then $\phi(x,f,\cdot)$ is structurally asymptotically stable.

Proof. If $p_{(x,f)}$ is asymptotically stable then $T^+_{xf} = [0, +\infty)$, i.e., $\Phi(x,f,\cdot)$ is positively extendable, and there exists an $M > 0$ such that $d((xf),(y,g)) < M$ implies $d(p(x,f,t), p(y,g,t)) \to 0$ as $t \to \partial T^+_{yg}$. Suppose $\phi(x,f,\cdot)$ is not structurally stable. Then there exists $\{g_n\} \to f$ such that $\phi(x,g_n,\cdot)$ is not stable, i.e., for each n there exists an $\varepsilon_n > 0$, a sequence $\{x^n_k\} \to x$ as $k \to \infty$, and a sequence $\{t^n_k\}$ such that

(1) $\varepsilon_n \leqq |\phi(x,g_n,t^n_k) - \phi(x^n_k,g_n,t^n_k)|$ for each k.

Since $\phi(x,f,\cdot)$ is positively extendable and $(x,g_n) \to (x,f)$ as $n \to \infty$ we have by theorem 3 that $\phi(x,g_n,\cdot)$ is positively extendable for n large enough. Also since $(x^n_k,g_n) \to (x,g_n)$ as $k \to \infty$, $\phi(x^n_k,g_n,\cdot)$ is positively extendable for k large enough.

Therefore $t^n_k \to \infty$ as $k \to \infty$, since otherwise $\{t^n_k\}$ is contained in a compact set and by theorem 3, (1) cannot hold for k large enough.

We have that

(2) $|\phi(x,g_n,t^n_k) - \phi(x^n_k,g_n,t^n_k)| \leqq |\phi(x,g_n,t^n_k) - \phi(x,f,t^n_k)|$

$+ |\phi(x,f,t^n_k) - \phi(x^n_k,g_n,t^n_k)|$

For n large enough $\rho(f,g_n) < M/2$, and for k large enough $|x - x_k^n| < M/2$. Therefore by hypothesis, each term on the right of the inequality (2) can be made $< \varepsilon_n/3$ for some n, k. This contradicts (1).

Thus $\phi(x,f,\cdot)$ is structurally stable.

If $d((x,f),(y,g)) < M$ then

$|\phi(x,g,t) - \phi(y,g,t)| \leqq |\phi(x,g,t) - \phi(x,f,t)|$

$+ |\phi(x,f,t) - \phi(y,g,t)| \to 0$ as $t \to \infty$ by hypothesis, i.e., $\phi(x,f,\cdot)$ is structurally asymptotically stable.

In conclusion some comments can be made on possible areas of generalization and further study. First, the idea of structural property as defined here has no essential connection with uniqueness. To define structural stability for the non unique solutions, for example, one needs only a practical definition of stability of solution funnels.

As was noted previously, the metric on Γ is intimately connected with the question of whether a property of a solution is structural or not. Conjectures 1 and 2 would seem to be interesting areas of future research.

List of Symbols

1.	R, R^n	The real line and euclidean n-space respectively.
2.	(a,b), $[a,b]$	Open and closed intervals respectively in R.
3.	R^+, R^-	$[0, +\infty)$ and $(-\infty, 0]$ respectively.
4.	$\|x\|$	The euclidean norm of $x \in R^n$.
5.	$N(x)$, $V(f)$	A neighborhood of the point $x \in R^n$ or $f \in \Gamma$ respectively.
6.	∂A	The boundary of a set A.
7.	$\bar{A}$	The closure of a set A.
8.	$S_\varepsilon(A)$	Sphere of radius ε about a set A.

REFERENCES

[1] Hartman, Philip,
Ordinary Differential Equations. Wiley, New York, 1964.

[2] Kamke, E.,
Zur Theorie der Systeme gewohnlicher Differentialgleichungen. II, Acta Math. 58 (1932) 57-85.

[3] Bhatia, N. P. and Szego, G. P.,
Dynamical Systems: Stability Theory and Applications. Springer-Verlag, New York, 1967.

[4] Sell, G. R.,
Nonautonomous Differential Equations and Topological Dynamics I, II. Transactions, American Mathematical Society, Vol. 127, May 1967.

[5] Kelley, J.,
General Topology. Van Nostrand, Princeton, N. J. 1955.

[6] Auslander, J., Bhatia, N. P., and Seibert, P.,
Attractors in Dynamical Systems, Bol. Soc. Mat. Mexicana, 1964.