A CORONA THEOREM FOR CERTAIN SUBALGEBRAS OF $H^{\infty}(\mathbb{D})$

by

JULIE S. RYLE

A DISSERTATION

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ABSTRACT

The corona theorem for the space of bounded analytic functions on the unit disk, $H^{\infty}(\mathbb{D})$, which was proven by Carleson in 1962, states that $\mathbb{D}$ is dense in the maximal ideal space of $H^{\infty}(\mathbb{D})$. This theorem can be reduced to the following result: Let $\{f_i\}_{i=1}^{n} \subset H^{\infty}(\mathbb{D})$ be such that $\sum_{i=1}^{n} |f_i(z)|^2 \geq \varepsilon^2 > 0$ for $z \in \mathbb{D}$. Then there exist $\{g_i\}_{i=1}^{n} \subset H^{\infty}(\mathbb{D})$ such that $\sum_{i=1}^{n} f_i(z) g_i(z) \equiv 1$ in $\mathbb{D}$. Furthermore, if we have the additional condition that $\{f_i\}_{i=1}^{n}$ are such that $\sum_{i=1}^{n} |f_i(z)|^2 \leq 1$, then there exists $B_{\varepsilon} < \infty$ such that $\sum_{i=1}^{n} |g_i(z)|^2 \leq B_{\varepsilon}$ for all $z \in \mathbb{D}$.

In this dissertation, we prove that the corona theorem holds for certain subalgebras of $H^{\infty}(\mathbb{D})$, and we provide estimates for the sizes of the given solutions. Among the algebras we consider are those which contain bounded analytic functions whose k^{th} derivatives vanish at 0 for all $k \in K \subset \mathbb{N}$, which we call $H_K^{\infty}(\mathbb{D})$. We give several properties the set K must have in order for $H_K^{\infty}(\mathbb{D})$ to be an algebra. We then prove the corona theorem in both the vector and matrix cases for these algebras. In fact, in the vector case, we prove the corona theorem using two different techniques. Each technique gives a unique estimate, and one extends our findings to more general algebras.

We also settle a conjecture of Mortini, Sasane, and Wick involving the algebra $\mathbb{C}+BH^{\infty}(\mathbb{D})$, where B is a Blaschke product. We prove the corona theorem in $\mathbb{C}+BH^{\infty}(\mathbb{D})$ holds for an infinite number of functions.

We end with a few suggestions for future research.

LIST OF ABBREVIATIONS AND SYMBOLS

$\mathbb{C}$	the complex numbers
$\mathbb{N}$	the natural numbers
$\mathbb{D}$	the open unit disk, $\{z \in \mathbb{C} : \|z\| < 1\}$
$\overline{\mathbb{D}}$	the closed unit disk, $\{z \in \mathbb{C} : \|z\| \leq 1\}$
$H^{\infty}(\mathbb{D})$	the algebra of bounded, analytic functions on $\mathbb{D}$
$\mathcal{B}(\mathcal{H})$	the set of bounded operators on a Hilbert space $\mathcal{H}$
$A \cup B$	the union of sets A and B
$A \cap B$	the intersection of sets A and B
$A \subset B$	A is a subset of B
$A \oplus B$	the direct sum of A and B
$a \in A$	a is an element of A
$a \notin A$	a is not an element of A
f^T, $\mathcal{A}^T$	the transpose of vector f or matrix $\mathcal{A}$
ran F	the range of F, $\{F(z) : z \in \mathcal{D}\}$
ker F	the kernel of F, $\{z \in \mathcal{D} : F(z) = 0\}$
I_n	the $n \times n$ identity matrix
$\det \mathcal{A}$	the determinant of matrix $\mathcal{A}$

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CONTENTS

Chapter 1
INTRODUCTION AND SUMMARY

Let $H^{\infty}(\mathbb{D})$ denote the collection of bounded analytic functions on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, and let $\boldsymbol{M}$ denote the maximal ideal space of $H^{\infty}(\mathbb{D})$. The corona theorem for $H^{\infty}(\mathbb{D})$, which was proven by Carleson in 1962 $([1])$, states that $\mathbb{D}$ is dense in $\boldsymbol{M}$. This theorem can be reduced to the following result: Let $\{f_i\}_{i=1}^{n} \subset H^{\infty}(\mathbb{D})$ be such that

$$\sum_{i=1}^{n} |f_i(z)|^2 \geq \varepsilon^2 > 0 \text{ for } z \in \mathbb{D}.$$

Then there exist $\{g_i\}_{i=1}^{n} \subset H^{\infty}(\mathbb{D})$ such that

$$\sum_{i=1}^{n} f_i(z)\, g_i(z) \equiv 1 \text{ in } \mathbb{D}.$$

It has also been shown that if we have the additional condition that $\{f_i\}_{i=1}^{n}$ are such that $\sum_{i=1}^{n} |f_i(z)|^2 \leq 1$, then there exists $B_{\varepsilon} < \infty$ such that $\sum_{i=1}^{n} |g_i(z)|^2 \leq B_{\varepsilon}$ for all $z \in \mathbb{D}$ $([13])$.

In this dissertation, we prove that the corona theorem holds for certain subalgebras of $H^{\infty}(\mathbb{D})$, and we provide estimates for the sizes of the given solutions. Among the subalgebras we consider are those which contain functions whose k^{th} derivatives vanish at 0 for all k in some set $K \subset \mathbb{N}$. We prove the corona theorem in both the vector and matrix cases for these algebras.

The content of this dissertation is as follows: in the second chapter, we introduce some background material necessary for the development of our proofs. This material includes the definitions and some important properties of both reproducing kernel Hilbert spaces and multipliers on algebras.

In the third chapter, we present an abbreviated history of the corona theorem. This chapter includes examples of algebras for which the corona theorem holds, as well as an example for which the corona theorem fails.

In the fourth chapter, we introduce some of the algebras for which we will prove the corona theorem. They include the algebra

$$H_K^\infty(\mathbb{D}) \triangleq \left\{ f : \mathbb{D} \to \mathbb{C} : f(z) \in H^\infty(\mathbb{D}) \text{ and, for all } k \in K,\ f^{(k)}(0) = 0 \right\}.$$

Examples of such algebras are given, as well as an example of a set K for which $H_K^\infty(\mathbb{D})$ is not an algebra. Also, many characteristics that the set K must have in order for $H_K^\infty(\mathbb{D})$ to be an algebra are discussed and their necessity proven. Finally, the algebras of both vectors and matrices with entries in $H_K^\infty(\mathbb{D})$ are defined.

In the fifth chapter, another necessary element for our proof of the corona theorem in certain subalgebras of $H^\infty(\mathbb{D})$ is discussed. For a function $F(z)$ in our algebra, we find an operator Q in the same algebra such that ran Q = ker F.

Finally, in the sixth chapter, we prove that the corona theorem holds for certain algebras, including both the vector and matrix versions of $H_K^\infty(\mathbb{D})$, and give estimates on the sizes of the solutions. In fact, in the vector case of $H_K^\infty(\mathbb{D})$, we prove the corona theorem using two different techniques. Each gives a unique estimate, and one extends our findings to more general algebras. We end with a few suggestions for future research.

Chapter 2
REPRODUCING KERNEL HILBERT SPACES AND MULTIPLIERS

2.1 Reproducing Kernel Hilbert Spaces

Let $\mathcal{H}(\mathcal{D})$ be a Hilbert space of complex valued functions on a set $\mathcal{D}$ with an inner product $\langle f, g\rangle_{\mathcal{H}(\mathcal{D})}$ and norm $\|f\|_{\mathcal{H}(\mathcal{D})} = \sqrt{\langle f, f\rangle}$.

Definition 2.1.1 A linear *evaluation functional* over $\mathcal{H}(\mathcal{D})$ is a linear functional $l_z : \mathcal{H}(\mathcal{D}) \to \mathbb{C}$ that evaluates each function $h \in \mathcal{H}(\mathcal{D})$ at the point $z \in \mathcal{D}$, *i.e.*

$$l_z(h) \triangleq h(z).$$

Definition 2.1.2 A functional $l_z : \mathcal{H}(\mathcal{D}) \to \mathbb{C}$ is *bounded* if there exists a $C_z < \infty$ such that, for all $h \in \mathcal{H}(\mathcal{D})$,

$$|l_z(h)| = |h(z)| \leq C_z \|h\|_{\mathcal{H}(\mathcal{D})}.$$

Definition 2.1.3 We say that $\mathcal{H}(\mathcal{D})$ is a *reproducing kernel Hilbert space* on $\mathcal{D}$ if, for every $z \in \mathcal{D}$, the linear evaluation functional $l_z : \mathcal{H}(\mathcal{D}) \to \mathbb{C}$ is bounded.

Example 2.1.1 Hardy space, or the space of holomorphic functions on the unit disc,

$$H^2(\mathbb{D}) = \left\{ f(z) : \mathbb{D} \to \mathbb{C} : \; f(z) = \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} |f_n|^2 < \infty \right\},$$

is a reproducing kernel Hilbert space. For $f, g \in H^2(\mathbb{D})$, we have the inner product $\langle f, g \rangle = \sum_{n=0}^{\infty} f_n \overline{g_n}$ and hence the norm $\|f\|^2 = \sum_{n=0}^{\infty} |f_n|^2$. Let $z \in \mathbb{D}$ and $h \in H^2(\mathbb{D})$. Then we have that, for $|z| < 1$, $h(z) = \sum_{n=0}^{\infty} h_n z^n$, which implies

$$\begin{aligned} |h(z)| &= \left| \sum_{n=0}^{\infty} h_n z^n \right| \\ &\leq \sum_{n=0}^{\infty} |h_n| \cdot |z|^n \\ &\leq \left(\sum_{n=0}^{\infty} |h_n|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{n=0}^{\infty} |z|^{2n} \right)^{\frac{1}{2}} \\ &= C_z \|h\| \end{aligned}$$

where $C_z = \left(\sum_{n=0}^{\infty} |z|^{2n} \right)^{\frac{1}{2}} < \infty$. Thus $h(z)$ is bounded for any fixed $z \in \mathbb{D}$, and so $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space.

Example 2.1.2 The Hilbert space containing square-integrable functions in $[0, 1]$, *i.e.*

$$L^2[0,1] = \left\{ f : [0,1] \to \mathbb{C} : f \text{ is measurable and } \int_0^1 |f(z)|^2 < \infty \right\}$$

is *not* a reproducing kernel Hilbert space. Elements of $L^2[0,1]$ are equivalence classes of functions and cannot be evaluated pointwise.

Suppose $\mathcal{H}(\mathcal{D})$ is a reproducing kernel Hilbert space on $\mathcal{D}$, and let z be fixed in $\mathcal{D}$. Consider the linear evaluation functional $l_z : \mathcal{H}(\mathcal{D}) \to \mathbb{C}$ defined by $l_z(h) = h(z)$. By Definition 2.1.3, we have that l_z is bounded on $\mathcal{H}(\mathcal{D})$. By the Riesz Representation Theorem, every bounded linear functional on $\mathcal{H}(\mathcal{D})$ is given by the inner product with a unique vector in $\mathcal{H}(\mathcal{D})$. Thus we have that for every $w \in \mathcal{D}$, there exists a unique vector $k_w \in \mathcal{H}(\mathcal{D})$ such that, for every $h \in \mathcal{H}(\mathcal{D})$,

$$h(w) \triangleq l_w(h) = \langle h, k_w \rangle_{\mathcal{H}(\mathcal{D})}. \tag{2.1}$$

Definition 2.1.4 We say that this two-variable function $K(z,w) \triangleq k_w(z) : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ is the *reproducing kernel* for $\mathcal{H}(\mathcal{D})$.

Lemma 2.1.1 If $k_w(z)$ is a reproducing kernel for $\mathcal{H}(\mathcal{D})$, then $k_w(z)$ is unique.

Proof Suppose that $k_w(z)$ and $K_w(z)$ are reproducing kernels for $\mathcal{H}(\mathcal{D})$. Then, since (2.1) holds for every $h \in \mathcal{H}(\mathcal{D})$, we can write

$$K_w(z) = \langle K_w, k_z \rangle_{\mathcal{H}(\mathcal{D})}.$$

It follows that

$$\begin{aligned} K_w(z) &= \overline{\langle k_z, K_w \rangle}_{\mathcal{H}(\mathcal{D})} \\ &= \overline{k_z(w)} \\ &= \overline{\langle k_z, k_w \rangle}_{\mathcal{H}(\mathcal{D})} \\ &= \langle k_w, k_z \rangle_{\mathcal{H}(\mathcal{D})} \\ &= k_w(z). \end{aligned}$$

Thus the reproducing kernel for $\mathcal{H}(\mathcal{D})$ is unique. $\square$

Theorem 2.1.1 Let $\mathcal{H}(\mathcal{D})$ be a separable Hilbert space with reproducing kernel $k_w(z)$ and orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Then

$$k_w(z) = \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z).$$

Proof Let $h_n = \sum_{j=0}^{n} \langle k_w, e_j \rangle_{\mathcal{H}(\mathcal{D})} e_j$. Then $h_n \to k_w$ in $\mathcal{H}(\mathcal{D})$, and thus $h_n(z) \to k_w(z)$, *i.e.* we have that

$$\begin{aligned} k_w(z) &= \lim_{n\to\infty} h_n(z) \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} \langle k_w, e_j \rangle_{\mathcal{H}(\mathcal{D})} e_j(z) \\ &= \sum_{n=0}^{\infty} \langle k_w, e_n \rangle_{\mathcal{H}(\mathcal{D})} e_n(z) \\ &= \sum_{n=0}^{\infty} \overline{\langle e_n, k_w \rangle}_{\mathcal{H}(\mathcal{D})} e_n(z) \\ &= \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z). \end{aligned}$$

By the uniqueness of reproducing kernels, we have that the reproducing kernel for $\mathcal{H}(\mathcal{D})$ is

$$k_w(z) = \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z).$$

□

Thus, to state what the reproducing kernel is for any reproducing kernel Hilbert space, we need only know an orthonormal basis for the space. Since we are able to name such a basis for $H^2(D)$, we now use this fact in order to find its reproducing kernel. In addition, we give two more examples of reproducing kernel Hilbert spaces and their reproducing kernels.

Example 2.1.3 The reproducing kernel Hilbert space $H^2(\mathbb{D})$ has orthonormal basis $\{e_n\}_{n=0}^{\infty}$, where $e_n(z) = z^n$ for $n = 0, 1, 2, \ldots$. Hence, for $w, z \in \mathbb{D}$, we have that the reproducing kernel for $H^2(\mathbb{D})$ is

$$\begin{aligned} k_w(z) &= \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z) \\ &= \sum_{n=0}^{\infty} \overline{w^n} z^n \\ &= \sum_{n=0}^{\infty} (\overline{w} z)^n \\ &= \frac{1}{1 - \overline{w} z}. \end{aligned}$$

Example 2.1.4 Bergman space,

$$A^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} f_n z^n, \|f\|^2 = \sum_{n=0}^{\infty} \frac{|f_n|^2}{n+1} < \infty \right\},$$

is a reproducing kernel Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$, where $e_n(z) = \sqrt{n+1}\, z^n$ for $n = 0, 1, 2, \ldots$. Thus, for $w, z \in \mathbb{D}$, we have that the reproducing kernel for $A^2(\mathbb{D})$ is

$$\begin{aligned} k_w(z) &= \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z) \\ k_w(z) &= \sum_{n=0}^{\infty} \overline{\sqrt{n+1}\, w^n} \cdot \sqrt{n+1}\, z^n \\ &= \sum_{n=0}^{\infty} (n+1) (\overline{w} z)^n \\ &= \frac{1}{(1 - \overline{w} z)^2}. \end{aligned}$$

Example 2.1.5 Dirichlet space,

$$\mathcal{D}^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} f_n z^n, \|f\|^2 = \sum_{n=0}^{\infty} (n+1)|f_n|^2 < \infty \right\},$$

is a reproducing kernel Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$, where $e_n(z) = \frac{1}{\sqrt{n+1}} z^n$ for $n = 0, 1, 2, \ldots$. Thus, for $w, z \in \mathbb{D}$, we have that the reproducing kernel for $\mathcal{D}^2(\mathbb{D})$ is

$$\begin{aligned} k_w(z) &= \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z) \\ &= \sum_{n=0}^{\infty} \overline{\frac{1}{\sqrt{n+1}} w^n} \cdot \frac{1}{\sqrt{n+1}} z^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} (\overline{w} z)^n \\ &= \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - \overline{w} z} \right). \end{aligned}$$

Definition 2.1.5 Let $\mathcal{D}$ be a set, and let $K = K(z, w)$ be a two variable function, $K(z, w) : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$. K is called a ***positive definite kernel*** if, for any $N \in \mathbb{N}$, we have

$$\sum_{j=0}^{N} \sum_{k=0}^{N} c_j \overline{c_k} K(z_j, z_k) \geq 0$$

for all $\{z_j\}_{j=0}^{N} \subset \mathcal{D}$ and for all $\{c_j\}_{j=0}^{N} \subset \mathbb{C}$.

Theorem 2.1.2 Suppose $k_w = k(z, w)$ is the reproducing kernel for a Hilbert space $\mathcal{H}(\mathcal{D})$. Then k_w is positive definite.

Proof. Consider $x = \sum_{j=0}^{N} c_j k_{z_j}$. Then

$$\begin{aligned} 0 &\leq \|x\|^2 \\ &= \left\langle \sum_{j=0}^{N} c_j k_{z_j}, \sum_{i=0}^{N} c_i k_{z_i} \right\rangle \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N} \overline{c_i} c_j \left\langle k_{z_j}, k_{z_i} \right\rangle \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N} \overline{c_i} c_j k\left(z_i, z_j\right). \end{aligned}$$

Thus k_w is positive definite. □

2.2 Multipliers

Definition 2.2.6 Let $\mathcal{H}(\mathcal{D})$ be a reproducing kernel Hilbert space of functions on a set $\mathcal{D}$. Let $\varphi \in \mathcal{H}(\mathcal{D})$. For all $f \in \mathcal{H}(\mathcal{D})$, define $M_\varphi : \mathcal{H}(\mathcal{D}) \to \mathcal{D}$ by

$$M_\varphi(f) \triangleq \varphi f.$$

If $M_\varphi \in \mathcal{B}(\mathcal{H}(\mathcal{D}))$, then we say that φ is a *multiplier* for $\mathcal{H}(\mathcal{D})$. We denote the multipliers of $\mathcal{H}(\mathcal{D})$ by $\mathcal{M}(\mathcal{H}(\mathcal{D}))$.

Theorem 2.2.3 If φ is a multiplier for a reproducing kernel Hilbert space $\mathcal{H}(\mathcal{D})$, then φ is bounded on $\mathcal{D}$.

Proof Assume $\mathcal{H}(\mathcal{D})$ has reproducing kernel $k_w(z)$. We first establish the validity of the following claim, which is an interesting and useful correlation between multipliers and reproducing kernels:

Claim 2.2.1 $M_\varphi^*(k_w) = \overline{\varphi(w)}k_w$

Proof of Claim 2.2.1 For all $f \in \mathcal{H}(\mathcal{D})$, we have that

$$\begin{aligned}
\left\langle M_\varphi^*(k_w), f\right\rangle_{\mathcal{H}(\mathcal{D})} &= \langle k_w, M_\varphi(f)\rangle_{\mathcal{H}(\mathcal{D})} \\
&= \langle k_w, \varphi f\rangle_{\mathcal{H}(\mathcal{D})} \\
&= \overline{\langle \varphi f, k_w\rangle_{\mathcal{H}(\mathcal{D})}} \\
&= \overline{\varphi f(w)} \\
&= \overline{\varphi(w) f(w)} \\
&= \overline{\varphi(w)\langle f, k_w\rangle_{\mathcal{H}(\mathcal{D})}} \\
&= \overline{\varphi(w)}\,\langle k_w, f\rangle_{\mathcal{H}(\mathcal{D})} \\
&= \left\langle \overline{\varphi(w)}k_w, f\right\rangle_{\mathcal{H}(\mathcal{D})}.
\end{aligned}$$

□

Thus we have that

$$\begin{aligned}
\left\|\overline{\varphi(w)}k_w\right\|_{\mathcal{H}(\mathcal{D})} &= \left\|M_\varphi^*(k_w)\right\|_{\mathcal{H}(\mathcal{D})} \qquad (2.2) \\
&\leq \left\|M_\varphi^*\right\| \|k_w\|_{\mathcal{H}(\mathcal{D})}.
\end{aligned}$$

But the left hand side of (2.2) is equal to $|\varphi(w)|\,\|k_w\|_{\mathcal{H}(\mathcal{D})}$, and since $k_w \neq 0$ implies that $\|k_w\|_{\mathcal{H}(\mathcal{D})} \neq 0$, dividing by $\|k_w\|_{\mathcal{H}(\mathcal{D})}$ yields

$$|\varphi(w)| \leq \left\|M_\varphi^*\right\| = \|M_\varphi\|$$

for all $w \in \mathcal{D}$. We have by definition that M_φ is bounded. It follows that φ is bounded on $\mathcal{D}$. □

Example 2.2.6 The algebra of multipliers on $H^2(\mathbb{D})$ is isomorphic to $H^\infty(\mathbb{D})$. ([3])

Example 2.2.7 The algebra of multipliers on $A^2(\mathbb{D})$ is also isomorphic to $H^\infty(\mathbb{D})$. ([3])

Example 2.2.8 The algebra of multipliers on $\mathcal{D}^2(\mathbb{D})$ is strictly a subset of $H^\infty(\mathbb{D})$. For example, $\sum_{n=1}^{\infty} \frac{z^{n^3}}{n^2}$ is contained in $H^\infty(\mathbb{D})$ but not in $\mathcal{D}^2(\mathbb{D})$ ([12]).

Chapter 3
THE CORONA THEOREM

In 1941, Japanese mathematician Kakutani proposed a conjecture that later became known as the corona problem $([4])$. Its name is derived from the solar corona, the glowing atmosphere surrounding the sun that is easily visible during a total solar eclipse. Kakutani conjectured that the open unit disc $\mathbb{D}$ is dense in the maximal ideal space of $H^{\infty}(\mathbb{D})$. Here the open unit disk $\mathbb{D}$ is analogous to the sun, and Kakutani claimed that unit circle $\{z : |z| = 1\}$, or the corona, is in essence "empty."

3.1 The Corona Theorem in $H^{\infty}(\mathbb{D})$

Newman showed that the corona problem can be reduced to an interpolation problem $([8])$, which was then proved by Carleson in 1962 $([1])$. The result is the famous Carleson's Corona Theorem, which states that if functions $f_1, f_2, \ldots, f_n \in H^{\infty}(\mathbb{D})$ satisfy

$$|f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_n(z)|^2 \geq \varepsilon^2 > 0$$

for all $z \in \mathbb{D}$, then there exist functions $g_1, g_2, \ldots, g_n \in H^{\infty}(\mathbb{D})$ such that

$$f_1 g_1 + f_2 g_2 + \cdots + f_n g_n \equiv 1$$

in $\mathbb{D}$. He also gave a bound for the size of the solutions. If we have the additional condition that the above $f_1, f_2, \ldots, f_n$ are such that

$$|f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_n(z)|^2 \leq 1,$$

then there exists $B_{\varepsilon,n} < \infty$ such that

$$|g_1(z)|^2 + |g_2(z)|^2 + \cdots + |g_n(z)|^2 \leq B_{\varepsilon,n}$$

for all $z \in \mathbb{D}$.

In 1979, Wolff gave a simplified proof of Carleson's corona theorem, which can be found in Koosis' book ([6]), that made use of Littlewood-Paley expressions. One year late, Rosenblum ([10]) and Tolokonnikov ([14]) independently solved the corona problem for infinitely many functions in $H^\infty(\mathbb{D})$. Fuhrmann extended the $H^\infty(\mathbb{D})$ corona theorem to the finite matrix case ([5]), and Vasyunin (see Nikolskii [9]) to the one-sided infinite matrix case. Uchiyama showed that the corona solution could be estimated by (for ε small)

$$|g_1(z)|^2 + |g_2(z)|^2 + \cdots + |g_n(z)|^2 \leq \frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2}$$

where C_0 is a universal constant (see Trent [13]).

3.2 The Corona Theorem in Other Algebras

Since Carleson's important proof of the corona theorem in $H^\infty(\mathbb{D})$, many have endeavored to solve corona-type problems in other spaces for a finite number of functions, infinitely many functions, or matrices of functions.

Example 3.2.1: Multipliers on Dirichlet Space, $M(\mathcal{D}^2(\mathbb{D}))$

Tolokonnikov proved the corona theorem on the multiplier algebra of Dirichlet space for a finite number of functions (see Nikolskii [9]). Trent then extended this to infinitely many functions ([12]).

Given $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}\left(\mathcal{D}^2\left(\mathbb{D}\right)\right)$, we let $F\left(z\right) = \left(f_1\left(z\right), f_2\left(z\right), \ldots\right)$. We use M_F^C to denote the operator from $\mathcal{D}^2\left(\mathbb{D}\right)$ to $\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}^2\left(\mathbb{D}\right)$ defined by $M_F^C\left(h\right) = \sum_{j=1}^{\infty} f_j h$ for $h \in \mathcal{D}^2\left(\mathbb{D}\right)$. Trent showed that if we have $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}\left(\mathcal{D}^2\left(\mathbb{D}\right)\right)$ with $\left\|M_F^C\right\| \leq 1$ and $0 < \varepsilon^2 < \sum_{j=1}^{\infty} \left|f_j\right|^2$ for all $z \in \mathbb{D}$, then there exists $\{g_j\}_{j=1}^{\infty} \subset \mathcal{M}\left(\mathcal{D}^2\left(\mathbb{D}\right)\right)$ such that:

i. $\sum_{j=1}^{\infty} f_j g_j = 1$ in $\mathbb{D}$ and

ii. $\left\|M_G^C\right\| < \frac{1,500}{\varepsilon^3}$.

Example 3.2.2.: $H_B^{\infty}\left(\mathbb{D}\right)$

Mortini, Sasane, and Wick proved that the finite corona theorem holds in an algebra involving Blaschke products, namely $H_B^{\infty}\left(\mathbb{D}\right)$, defined below ([7]). Also, they conjectured that the corona theorem in $H_B^{\infty}\left(\mathbb{D}\right)$ holds for an infinite number of functions. We settle this conjecture among other results in this dissertation.

Definition 3.2.1 For $a \in \mathbb{D}$, let

$$\varphi_a\left(z\right) = \frac{a - z}{1 - \overline{a}z} : \mathbb{D} \rightarrow \mathbb{D}$$

denote the Möbius transform of the open unit disk $\mathbb{D}$ onto itself. The Blaschke product B with zeros a_k of multiplicity m_k is defined to be

$$B = \prod_{k \geq 1} \left(\frac{\left|a_k\right|}{a_k}\varphi_{a_k}\right)^{m_k} \text{ where } \sum_{k \geq 1} m_k\left(1 - \left|a_k\right|\right) < \infty.$$

Definition 3.2.2 Let a_k, $k \geq 1$, denote the zeros in $\mathbb{D}$ of a Blaschke product B with multiplicity m_k. We denote by $H_B^\infty(\mathbb{D})$ the set of all functions in $H^\infty(\mathbb{D})$ that satisfy the following:

i. for all $j, k : f(a_j) = f(a_k)$

ii. for all k and all $1 \leq m \leq m_k - 1 : f^{(m)}(a_k) = 0$.

An equivalent definition is given by

$$H_B^\infty(\mathbb{D}) \triangleq \mathbb{C} + BH^\infty(\mathbb{D}).$$

In fact, they proved that the corona theorem for a finite number of functions holds more generally for algebras generated by ideals in $H^\infty(\mathbb{D})$. If $\mathbb{I}$ is any proper ideal in $H^\infty(\mathbb{D})$, then the corona theorem holds in the subalgebra of $H^\infty(\mathbb{D})$ defined to be

$$\mathbb{C} + \mathbb{I} \triangleq \{c + \varphi : c \in \mathbb{C} \text{ and } \varphi \in \mathbb{I}\}.$$

Example 3.2.3: Matricial Corona Theorem

Trent and Zhang ([13]) established that, if $\mathcal{A}$ is a multiplier algebra for a reproducing kernel Hilbert space of functions on Ω, and if the corona theorem holds for $\mathcal{A}$, then a one-sided infinite matrix version of the corona theorem also holds. That is, if F is an $m \times \infty$ matrix of elements of $\mathcal{A}$ satisfying:

i. $0 < \varepsilon^2 I_m \leq F(z)F(z)^*$ for all $z \in \Omega$ and

ii. $\max\left\{\|M_F\|, \left\|M_F^T\right\|\right\} = 1,$

then there exists an $m \times \infty$ matrix with entries in $\mathcal{A}$ satisfying:

a. $FG^T = I_m$ and

b. $\max\left\{\|M_G\|, \left\|M_G^T\right\|\right\} < \infty.$

3.3 Subalgebras of $H^\infty(\mathbb{D})$ *for which the Corona Theorem Fails*

Not all subalgebras of $H^\infty(\mathbb{D})$ have corona-type theorems. In 1977, Scheinberg $([11])$ gave subalgebras of $H^\infty(\mathbb{D})$ for which the corona theorem does not hold. Let $A(\mathbb{D})$ be the algebra of functions that are in $H^\infty(\mathbb{D})$ and are continuous on $\overline{\mathbb{D}}$. He first proved the following:

Theorem 3.3.1 If $B(\mathbb{D})$ is a uniformly closed subalgebra of $H^\infty(\mathbb{D})$ such that $B(\mathbb{D}) \supseteq A(\mathbb{D})$ and $1 \in B(\mathbb{D})$, then the following are equivalent:

i. For all $n \in \mathbb{N}$ and for every $f_1(z), f_{2,}(z) \ldots, f_n(z) \in B(\mathbb{D})$ satisfying $\inf_{z\in\mathbb{D}} \{|f_1(z)| + |f_2(z)| + \cdots + |f_n(z)|\} > 0$, there exists $g_1(z), g_{2,}(z) \ldots, g_n(z) \in B(\mathbb{D})$ such that $g_1 f_1 + g_2 f_2 + \cdots + g_n f_n \equiv 1$ in $\mathbb{D}$.

ii. For all $n \in \mathbb{N}$ and for every $f_1(z), f_{2,}(z) \ldots, f_n(z) \in B(\mathbb{D})$ and every complex homomorphism $\varphi \in B(\mathbb{D})$, we have that $\varphi(f_1(z), f_2(z), \ldots, f_n(z))$ is in the closure of $\{f(z) : z \in \mathbb{D}\}$.

iii. For all $n \in \mathbb{N}$ and for every complex homomorphism φ, there is $\lambda \in \overline{\mathbb{D}}$ such that for every $f_1(z), f_{2,}(z) \ldots, f_n(z) \in B(\mathbb{D})$, we have that $\varphi(f_1(z), f_2(z), \ldots, f_n(z)) \in \bigcap_{\varepsilon>0} \{\text{closure of } \{(f_1(z), f_2(z), \ldots, f_n(z)) : z \in \mathbb{D} \text{ and } |z - \lambda| < \varepsilon\}\}.$

iv. The corona theorem is true for $B(\mathbb{D})$.

Scheinberg then proved the existence of subalgebras of $H^{\infty}(\mathbb{D})$ for which the corona theorem is not true.

Theorem 3.3.2 For each $n \in \mathbb{N}$, there exists a uniformly closed subalgebra $B_n(\mathbb{D}) \subseteq H^{\infty}(\mathbb{D})$, $B_n(\mathbb{D}) \supseteq A(\mathbb{D})$ such that $B_n(\mathbb{D})$ satisfies $(iii.)$ above, but does not satisfy the corona theorem.

It does not seem to be known if a corona-type theorem must hold for subalgebras of $H^{\infty}(\mathbb{D})$, which are also multiplier algebras of a reproducing kernel Hilbert space.

Chapter 4
$H_K^{\infty}(\mathbb{D})$, A SUBALGEBRA OF $H^{\infty}(\mathbb{D})$

4.1 Definition of $H_K^{\infty}(\mathbb{D})$

The main purpose of this dissertation is to show that the corona theorem holds for certain subalgebras of $H^{\infty}(\mathbb{D})$ and to provide estimates for the sizes of the given solutions. An example of the type of subalgebras we consider is the algebra of functions in $H^{\infty}(\mathbb{D})$ whose k^{th} derivatives vanish at 0 for all $k \in K \subset \mathbb{N}$, where K is chosen appropriately to give an algebra. We will call this algebra $H_K^{\infty}(\mathbb{D})$. In other words, define

$$H_K^{\infty}(\mathbb{D}) \triangleq \left\{ f : \mathbb{D} \to \mathbb{C} : f(z) \in H^{\infty}(\mathbb{D}) \text{ and, for all } k \in K,\ f^{(k)}(0) = 0 \right\},$$

where K is chosen appropriately to give an algebra. Note that interpolation problems for the algebra $\mathbb{C} + z^2 H^{\infty}(\mathbb{D})$ have been investigated by Davidson, Paulsen, Raghupathi, and Singh. [(2)].

Now, For any K for which $H_K^{\infty}(\mathbb{D})$ is an algebra, we have that $H_K^{\infty}(\mathbb{D})$ is the multiplier algebra for the reproducing kernel Hilbert space on $\mathbb{D}$,

$$H_K^{2}(\mathbb{D}) \triangleq \left\{ f : \mathbb{D} \to \mathbb{C} : f(z) \in H^{2}(\mathbb{D}) \text{ and, for all } k \in K,\ f^{(k)}(0) = 0 \right\}.$$

As such, it seems reasonable to investigate if there is a corona-type problem for $H_K^{\infty}(\mathbb{D})$. Note that the reproducing kernel for $H_K^{2}(\mathbb{D})$ can be found by considering an orthonormal basis for the space, namely $\{e_n\}_{n \in \{0\} \cup \mathbb{N} \setminus K}$, where $e_n = z^n$. This implies that, for $w, z \in \mathbb{D}$, the reproducing kernel for $H^2_K(\mathbb{D})$ is

$$\begin{aligned} k_w(z) &= \sum_{n\in\{0\}\cup\mathbb{N}\setminus K} \overline{e_n(w)}e_n(z) \\ &= \sum_{n\in\{0\}\cup\mathbb{N}\setminus K} \left(\overline{w^n}\right) z^n \\ &= \sum_{n\in\{0\}\cup\mathbb{N}\setminus K} \left(\overline{w}z\right)^n . \end{aligned}$$

First we investigate for what sets K we have that $H^\infty_K(\mathbb{D})$ is an algebra. Given a finite set K, it is easy to determine if $H^\infty_K(\mathbb{D})$ is an algebra, which we illustrate for the reader with a simple example.

Example Functions in $H^\infty(\mathbb{D})$ whose first and third derivatives vanish at 0, *i.e.*

$$H^\infty_{\{1,3\}}(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} : f(z) \in H^\infty(\mathbb{D}) \text{ and } f^{'}(0) = f'''(0) = 0 \right\},$$

make up one such subalgebra. Given $f(z), g(z) \in H^\infty_{\{1,3\}}(\mathbb{D})$, we have:

i. $(f+g)'(0) = (f+g)'''(0) = 0,$

ii. $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

$\Rightarrow (fg)'(0) = 0 \cdot g(0) + f(0) \cdot 0 = 0$, and

iii. $(fg)'''(z) = f'''(z)g(z) + 3f''(z)g'(z) + 3f'(z)g''(z) + f(z)g'''(z)$

$\Rightarrow (fg)'''(0) = 0 \cdot g(0) + 3f''(z) \cdot 0 + 3 \cdot 0 \cdot g''(0) + f(0) \cdot 0 = 0.$

Since $H^\infty_{\{1,3\}}(\mathbb{D})$ is closed under both multiplication and addition, we have that it is indeed an algebra. It follows that it is a multiplier algebra for the reproducing kernel Hilbert space

$H^2_{\{1,3\}}(\mathbb{D})$. For $w, z \in \mathbb{D}$, the reproducing kernel for $H^2_{\{1,3\}}(\mathbb{D})$ is

$$\begin{aligned} k_w(z) &= \sum_{n\in\{0\}\cup\mathbb{N}\setminus\{1,3\}} (\overline{w}z)^n \\ &= \frac{1}{1-\overline{w}z} - \overline{w}z - (\overline{w}z)^3 \\ &= 1 + (\overline{w}z)^2 + \frac{(\overline{w}z)^4}{1-\overline{w}z}. \end{aligned}$$

Example 4.1.1 The functions in $H^\infty(\mathbb{D})$ whose second derivatives vanish at zero, *i.e.*

$$H^\infty_{\{2\}}(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f(z) \in H^\infty(\mathbb{D}) \text{ and } f''(0) = 0\},$$

do *not* form an algebra. We do not know, for arbitrary $f(z), g(z) \in H^\infty_{\{2\}}(\mathbb{D})$, that the second derivative of the product of f and g,

$$(fg)''(z) = f''(z)g(z) + 2f'(z)g'(z) + f(z)g''(z),$$

vanishes at zero. In other words, we do not have that $H^\infty_{\{2\}}(\mathbb{D})$ is closed under multiplication.

4.2 Structure of the Set K

Though we are unable to fully characterize the set K for $H^\infty_K(\mathbb{D})$ to be an algebra, there are certain criteria that K must meet. Some of these criteria are outlined here. For all of the following, assume $K \subseteq \mathbb{N}$ is such that $H^\infty_K(\mathbb{D})$ is an algebra.

Lemma 4.2.1 $k_0 \notin K$ if and only if $\varphi(z) = z^{k_0} \in H^\infty_K(\mathbb{D})$.

Proof

i. For $0 < k < k_0$, $\varphi^{(k)}(z) = \dfrac{k_0!}{(k_0 - k)!} z^{k_0 - k}$, so $\varphi^{(k)}(0) = 0$.

ii. $\varphi^{(k_0)}(z) = k_0!$, so $\varphi^{(k_0)}(0) \neq 0$.

iii. For $k > k_0$, $\varphi^{(k)}(z) = 0$ for all $z \in \mathbb{D}$. $\square$

Lemma 4.2.2 Suppose $j, k \notin K$. Then $j + k \notin K$.

Proof Let $\varphi(z) = z^j$ and $\psi(z) = z^k$. Then $\varphi(z), \psi(z) \in H_K^\infty(\mathbb{D})$. Since $H_K^\infty(\mathbb{D})$ is an algebra, we have that the product $(\varphi\psi)(z) = z^{j+k} \in H_K^\infty(\mathbb{D})$. By Lemma 4.2.1, $j + k \notin K$. $\square$

Corollary 4.2.1 Suppose $j \notin K$. Then $2j, 3j, 4j, ... \notin K$.

Lemma 4.2.3 Suppose n consecutive integers $j, j+1, ..., j+(n-1) \notin K$. Then we can find $n+1$ consecutive integers, namely $nj, nj+1, ..., nj+(n-1), nj+n$, not in K.

Proof

i. By Corollary 4.2.1, $nj \notin K$.

ii. For $k = 1, 2, ..., n-1$, $nj + k = [(n-1)j] + [j+k]$. But by Corollary 4.2.1, we have that $(n-1)j \notin K$, and by supposition, $j + k \notin K$ for any $k = 1, 2, ..., n-1$.

iii. Again, by Corollary 4.2.1,we have that $nj + n = n(j+1) \notin K$. $\square$

Generalization of Lemma 4.2.3 Suppose $j, j+k \notin K$. Then, for any $n = 1, 2, 3, ...$, we have that $nj, nj+k, nj+2k, ..., nj+(n-1)k, nj+nk \notin K$.

Proof By Corollary 4.2.1, $nj \notin K$. Also, for $m = 1, 2, 3, ..., n$, we have that $nj + mk = (n-m)j + m(j+k)$. Now $(n-m)j \notin K$ and $m(j+k) \notin K$ by Corollary 4.2.1. So $(n-m)j + m(j+k) \notin K$ by Lemma 4.2.2. □

Lemma 4.2.4 Suppose $k_0 \in K$. If $1 < j \leq k_0$ satisfies $j \notin K$, then $k_0 - j \in K$.

Proof Suppose $\varphi(z) \in H^\infty_K(\mathbb{D})$. It follows that $\varphi^{(k_0)}(0) = 0$. Let $\psi(z) = z^j$. Then, by Lemma 4.2.1, $\psi(z) \in H^\infty_K(\mathbb{D})$. Since $H^\infty_K(\mathbb{D})$ is an algebra, we have that the product $(\psi\varphi)(z) \in H^\infty_K(\mathbb{D})$. Thus

$$
\begin{aligned}
0 &= (\psi\varphi)^{(k_0)}(0) \\
&= \sum_{p=0}^{k_0} \binom{k_0}{p} \psi^{(p)}(0)\varphi^{(k_0-p)}(0) \\
&= \binom{k_0}{j} \psi^{(j)}(0)\varphi^{(k_0-j)}(0) \\
&= \binom{k_0}{j} j!\varphi^{(k_0-j)}(0) .
\end{aligned}
$$

Thus $\varphi^{(k_0-j)}(0) = 0$, *i.e.* $k_0 - j \in K$. □

Lemma 4.2.5 If K is infinite, then there cannot be two consecutive integers $j, j+1 \notin K$.

Proof Suppose $j, j+1 \notin K$. Then we have that the following integers are not contained in K:

$$j^2 - j = (j-1)j$$

$$j^2 - j + 1 = (j-2)j + (j-1)$$

$$j^2 - j + 2 = (j-3)j + 2(j+1)$$

$$\vdots$$

$$j^2 - 2 = j + (j-2)(j+1)$$

$$j^2 - 1 = (j-1)(j+1)$$

$$j^2 = j \cdot j$$

$$j^2 + 1 = (j-1)j + (j+1)$$

$$j^2 + 2 = (j-2)j + 2(j+1)$$

$$\vdots$$

$$j^2 + j = j(j+1)$$

$$j^2 + j + 1 = j \cdot j + (j+1)$$

$$j^2 + j + 2 = (j-1)j + 2(j+1)$$

$$\vdots$$

Thus K is finite, having fewer than $(j-1)j$ elements. □

Lemma 4.2.6 If $j, k \notin K$ where $\gcd(j,k) = 1$, then K is a finite set.

Proof Since $\gcd(j,k) = 1$, then there exists $M, N \in \mathbb{N}$ such that $Nj - Mk = 1$. We establish the validity of two claims:

Claim 4.2.2 For any $Q \in \mathbb{N}$, $QMk + r \notin K$ for $r = 1, 2, ..., Q$.

Proof of Claim 4.2.2 Let $Q \in \mathbb{N}$ and $r \leq Q$. Then

$$QMk + r = (Q - r)Mk + r(Mk + 1).$$

Now $(Q - r)Mk \notin K$ since it is a multiple of $k \notin K$. Similarly, $r(Mk + 1) \notin K$ since it is a multiple of $Mk + 1 = Nj \notin K$. Thus their sum $QMk + r \notin K$. $\square$

Claim 4.2.3 Any integer greater than M^2k^2 can be written in the form $QMk + r$ for some $Q \in \mathbb{N}$ and $r \leq Q$.

Proof of Claim 4.2.3 Consider $M^2k^2 + p$, $p \geq 1$. If $p < Mk$, then we already have the desired form, for

$$M^2k^2 + p = QMk + r$$

where $Q = Mk$ and $r = p < Q$. If $p \geq Mk$, then write

$$\begin{aligned} M^2k^2 + p &= M^2k^2 + qMk + r \\ &= (Mk + q)Mk + r \end{aligned}$$

where $r < Mk$. Then we have the desired form where $Q = Mk + q$ and $r < Q$. $\square$

Thus, for any $N \geq M^2k^2$, we have that $N \notin K$; hence, K is a finite set. $\square$

The previous lemma tells us that, if $z^j, z^k \in H_K^\infty(\mathbb{D})$ and $\gcd(j, k) = 1$, then $z^n \in H_K^\infty(\mathbb{D})$ for all large n.

Lemma 4.2.7 If $j, k \notin K$ with $\gcd(j, k) = d$, then there exists C_d such that $Nd \notin K$ for any $N \geq C_d$.

Proof Since $\gcd(j, k) = d$, then there exists $M, N \in \mathbb{N}$ such that $Nj - Mk = d$. Let $k = sd$. Again, we first prove two claims:

Claim 4.2.4 For any $Q \in \mathbb{N}$, $QMk + rd \notin K$ for $r = 1, 2, ..., Q$.

Proof of Claim 4.2.4 Let $Q \in \mathbb{N}$ and $r \leq Q$. Then

$$QMk + rd = (Q - r)Mk + r(Mk + d).$$

Now $(Q - r)Mk \notin K$ since it is a multiple of $k \notin K$. Similarly, $r(Mk + d) \notin K$ since it is a multiple of $Mk + d = Nj \notin K$. Thus their sum $QMk + rd \notin K$. □

Claim 4.2.5 For any $n \geq M^2s^2$, we can write nd in the form $QMk + rd$ where $r < Q$.

Proof of Claim 4.2.5 Consider $n = M^2s^2 + p$, $p \geq 1$. If $p < Ms$, then we already have the desired form, for

$$\begin{aligned} nd &= M^2s^2d + pd \\ &= M^2sk + pd \\ &= QMk + rd \end{aligned}$$

where $Q = Ms$ and $r = p < Q$. If $p > Ms$, then write

$$\begin{aligned} nd &= M^2s^2d + pd \\ &= M^2s^2d + Mqsd + rd \\ &= (Ms + q)Mk + rd \end{aligned}$$

where $r < Ms$. Then we have the desired form where $Q = Ms + q$ and $r < Q$. □

Thus, for any $N \geq M^2s^2$, we have that $Nd \notin K$. □

The previous lemma tells us that, if $z^j, z^k \in H_K^\infty(\mathbb{D})$ with $\gcd(j, k) = d$, then $z^{nd} \in H_K^\infty(\mathbb{D})$ for all large n.

Theorem 4.2.1 Suppose $K \subset \mathbb{N}$ is infinite. Then for some integers N and d, we have that $z^{Nd}, z^{(N+1)d}, z^{(N+2)d}, \ldots \in H_K^\infty(\mathbb{D})$, and if $z^k \in H_K^\infty(\mathbb{D})$ where $k \geq Nd$, then $k = Md$ for some integer M.

Proof Let $K \subset \mathbb{N}$ be an infinite set for which $H_K^\infty(\mathbb{D})$ is an algebra. Let $p, q \notin K$ and $\gcd(p,q) = d$. Then, by Lemma 4.2.7, there exists $N \in \mathbb{N}$ such that $z^{nd} \in H_K^\infty(\mathbb{D})$ for all $n \geq N$. Suppose $z^k \in H_K^\infty(\mathbb{D})$, where $Md < k < (M+1)d$ for some $M \geq N$. Let $\gcd(Md, k) = d_1$, with $Md = jd_1$ and $k = ld_1$. Then we have that

$$\begin{aligned} k &< (M+1)d \\ k &< Md + d \\ k - Md &< d \\ ld_1 - jd_1 &< d \\ (l-j)d_1 &< d. \end{aligned}$$

But

$$\begin{aligned} Md &< k \\ jd_1 &< ld_1 \\ j &< l \\ l - j &> 0 \\ l - j &\geq 1. \end{aligned}$$

So we have that $d_1 < d$. Then, by Lemma 4.2.7, there exists $N_1 \in \mathbb{N}$ such that $z^{nd_1} \in H_K^\infty(\mathbb{D})$ for all $n \geq N_1$. $\square$

Definition 4.2.1 For $N, d \in \mathbb{N}$, we say that the infinite set $K \subset \mathbb{N}$ *stabilizes* at $k = Nd$ if

i. $z^{(N-1)d} \notin H^\infty_K(\mathbb{D})$,

ii. z^{Nd}, $z^{(N+1)d}$, $z^{(N+2)d}$, ... $\in H^\infty_K(\mathbb{D})$, and

iii. if $z^k \in H^\infty_K(\mathbb{D})$ where $k \geq Nd$, then $k = Md$ for some integer M.

Theorem 4.2.2 Let $K \subset \mathbb{N}$ be infinite. Suppose that we have, for some $N \in \mathbb{N}$ and some $d \in \mathbb{N}$, $z^{nd} \in H^\infty_K(\mathbb{D})$ for all $n \geq N$ (condition $(ii.)$ above). Additionally, suppose that, if $z^p \in H^\infty_K(\mathbb{D})$ for $p \geq Nd$, then $p = Md$ for some $M \in \mathbb{N}$ (condition $(iii.)$ above). Then $z^k \in H^\infty_K(\mathbb{D})$ implies that $k = jd$ for some $j \in \mathbb{N}$.

Proof Suppose $z^k \in H^\infty_K(\mathbb{D})$, and k is not a multiple of d. Then $z^{mk} \in H^\infty_K(\mathbb{D})$ for all $m \in \mathbb{N}$. Now there exists $Q \in \mathbb{N}$ with $\gcd(Q, d) = 1$ such that $Qk \geq Nd$. Now d does not divide Qk, and $z^{Qk} \in H^\infty_K(\mathbb{D})$. Thus we have a contradiction, and so k must be a multiple of d. $\square$

Theorems 4.2.1 and 4.2.2 taken together tell us that, if the set $K \subset \mathbb{N}$ is infinite, then all elements of $\mathbb{N}\backslash K$ must be a multiple of some integer d, *i.e.* if $\mathbb{N}\backslash K = \{k_1, k_{2,} k_{3,...}\}$, then $k_i = n_i d$ where $n_i \in \mathbb{N}$ for $i = 1, 2, 3, \ldots$. Furthermore, we have that K must stabilize at some point, say at $n_j d$. So $n_{k+1} = n_k + 1$ for all $k \geq j$. Thus

$$\mathbb{N}\backslash K = \{n_1 d, n_2 d, \ldots n_{j-1} d, n_j d, (n_j + 1)\, d, (n_j + 2)\, d, \ldots\},$$

and elements of $H^\infty_K(\mathbb{D})$ have the form

$$F(z) = f_0 + f_1 z^{n_1 d} + f_2 z^{n_2 d} + \cdots + f_j z^{n_j d} + f_{j+1} z^{(n_j+1)d} + f_{j+2} z^{(n_j+2)d} + \cdots \tag{4.1}$$

where $f_i \in \mathbb{C}$. Letting $w = z^d$ yields that (4.1) becomes

$$\begin{aligned} F_1(w) &= f_0 + f_1 w^{n_1} + f_2 w^{n_2} + \cdots + f_j w^{n_j} + f_{j+1} w^{n_j+1} + f_{j+2} w^{n_j+2} + \cdots \\ &= f_0 + f_1 w^{n_1} + f_2 w^{n_2} + \cdots + \sum_{k=0}^{\infty} f_{j+k} w^{n_j+k}. \end{aligned}$$

Thus $F_1(w)$ is contained in the algebra $H^\infty_{K_1}(\mathbb{D})$ where

$$K_1 = \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n_2 - 1, n_2 + 1, \ldots, n_j - 1\}$$

is a finite set. Thus any algebra $H^\infty_K(\mathbb{D})$ where K is infinite can be reduced to a finite case.

Thus the problem of finding a corona solution in $H^\infty_K(\mathbb{D})$ where K is infinite can be reduced to two simpler steps. First, solve the corresponding problem in $H^\infty_{K_1}(\mathbb{D})$ where K_1 is finite as above. Then, take those solutions in $H^\infty_{K_1}(\mathbb{D})$ and compose them with z^d in order to get the solution in $H^\infty_K(\mathbb{D})$.

Example 4.2.2 Let $\mathbb{N}\backslash K = \{6, 8, 2n : n \geq 6\}$, so $K = \{1, \ldots, 5, 7, 9, 10, 11, 2n+1 : n \geq 6\}$ is an infinite set. Consider $H^\infty_K(\mathbb{D})$, which is indeed an algebra. Note that $H^\infty_K(\mathbb{D})$ stabilizes at $k = 2 \cdot 6 = 12$. We have that $F(z) \in H^\infty_K(\mathbb{D})$ if it has the form

$$F(z) = f_0 + f_6 z^6 + f_8 z^8 + \sum_{n=6}^{\infty} f_{2n} z^{2n} \tag{4.2}$$

where $f_i \in \mathbb{C}$. Let $w = z^2$. Then (4.2) becomes

$$F_1(w) = f_0 + f_6 w^3 + f_8 w^4 + \sum_{n=6}^{\infty} f_{2n} w^n.$$

Now $F_1(w)$ is contained in the algebra $H^\infty_{K_1}(\mathbb{D})$ where $K_1 = \{1, 2, 5\}$ is a finite set. Thus the task of finding a corona solution for $F(z)$ in $H^\infty_K(\mathbb{D})$ reduces to finding a corona solution for $F_1(w)$ in $H^\infty_{K_1}(\mathbb{D})$, say $G_1(z)$, and composing this solution with z^2. Then we have that $G(z) \triangleq G_1(z^2)$ is the desired corona solution for $F(z)$ in $H^\infty_K(\mathbb{D})$.

4.3 *Vector and Matrix Versions of* $H_K^\infty(\mathbb{D})$

We also consider the algebra comprised of vectors whose entries lie in $H_K^\infty(\mathbb{D})$. This algebra is of the form

$$\overset{n}{\underset{j=1}{\oplus}} H_K^\infty(\mathbb{D}) = \left\{ \{f_j\}_{j=1}^n : f_j(z) \in H_K^\infty(\mathbb{D}) \text{ for } j = 1, 2, ..., n \text{ and } \sup_{z\in\mathbb{D}} \sum_{j=1}^n |f_j(z)|^2 < \infty \right\}.$$

We will call this algebra $\mathcal{H}_{K,n}^\infty(\mathbb{D})$. Multiplication here is entrywise. Elements of $\mathcal{H}_{K,n}^\infty(\mathbb{D})$ are $n-$vectors, which we will write as "row" vectors, with entries in $H_K^\infty(\mathbb{D})$. Thus $F(z) \in \mathcal{H}_{K,n}^\infty(\mathbb{D})$ implies that $F(z) \in \mathcal{M}(\mathbb{C}^n, \mathbb{C})$ and that it has the form

$$F(z) = (f_1(z), f_2(z), ..., f_n(z))$$

for $n < \infty$, where $f_j(z) \in H_K^\infty(\mathbb{D})$ for $j = 1, 2, ..., n$. Note that we could have $n = \infty$, in which case $F(z) \in \mathcal{M}(l^2, \mathbb{C})$ is an infinite vector of the form

$$F(z) = (f_1(z), f_2(z), ...)$$

where $f_j(z) \in H_K^\infty(\mathbb{D})$ for $j \in \mathbb{N}$.

Similarly, we consider the matrix version of this algebra,

$$\left[\overset{m}{\underset{k=1}{\oplus}} \mathcal{H}_{K,n}^\infty(\mathbb{D})\right]^T = \left\{ [\{F_k\}_{k=1}^m]^T : F_k(z) \in \mathcal{H}_{K,n}^\infty(\mathbb{D}) \text{ for } k = 1, 2, ..., m \right\},$$

elements of which are $m \times n$ matrices with $n \geq m$ of the form

$$\mathcal{F}(z) = \begin{bmatrix} F_1(z) \\ F_2(z) \\ \vdots \\ F_m(z) \end{bmatrix} = \begin{bmatrix} f_{1,1}(z) & f_{1,2}(z) & \cdots & f_{1,n}(z) \\ f_{2,1}(z) & f_{2,2}(z) & \cdots & f_{2,n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(z) & f_{m,2}(z) & \cdots & f_{m,n}(z) \end{bmatrix}$$

where $F_k(z) \in \mathcal{H}_{K,n}^\infty(\mathbb{D})$ for $k = 1, 2, ..., m$, ***i.e.*** $f_{k,j} \in H_K^\infty(\mathbb{D})$ for $k = 1, 2, ..., m$ and $j = 1, 2, ..., n$. We will call this algebra $\left[\mathcal{H}_{K,n}^\infty(\mathbb{D})\right]_m$. Again, we could have $n = \infty$, in which case we have $\left[\mathcal{H}_{K,\infty}^\infty(\mathbb{D})\right]_m$, an $m \times \infty$ matrix with rows in $\mathcal{H}_{K,\infty}^\infty(\mathbb{D})$.

Chapter 5
REPRESENTATION OF KERNELS OF MATRICES WITH ELEMENTS IN THE ALGEBRA AS RANGES OF MATRICES WITH ENTRIES IN THE ALGEBRA

In general terms, the problem we have is that, for a certain function F in an algebra $\mathcal{A}$, we want to find a solution $A \in \mathcal{A}$ such that $FA = 1$. Now our algebra $\mathcal{A}$ is a subspace of a larger algebra $\mathcal{X}$ for which the problem has been solved. In other words, we know we can find a solution G_0 in $\mathcal{X}$ such that $FG_0 = 1$. If we can we find a suitable operator Q in $\mathcal{A}$ so that ker $F =$ ran Q, then we will let $A = G_0 - QX$ for some $X \in \mathcal{X}$ chosen appropriately so that $A \in \mathcal{A}$. This will be our desired solution, since

$$\begin{aligned} FA &= F(G_0 - QX) \\ &= FG_0 - FQX \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

In this chapter, we find such an operator Q. We find the operator X in the following chapter.

Let $\{a_j\}_{j=1}^{\infty}, \{b_j\}_{j=1}^{\infty} \in l^2$ and $\underline{a} = (a_1, a_2, ...), \underline{b} = (b_1, b_2, ...) \in \mathcal{B}(l^2, \mathbb{C})$. For $\underline{c} = (c_1, c_2, ...) \in \mathcal{B}(l^2, \mathbb{C})$, define $Q_{\underline{c}}^{(0)} = (c_1, c_2, ...)$, and let $S : l^2 \to l^2$ be the backward shift operator with $S(\underline{c}) = (c_2, c_3, ...)$. Let

$$A_k^{(1)}(\underline{c}) = \begin{matrix} \overset{k-1}{\underset{1}{\oplus}} \mathbb{C} \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \\ \mathbb{C} \left\{ \right. \\ \overset{\infty}{\underset{k+1}{\oplus}} \mathbb{C} \left\{ \right. \end{matrix} \left[\begin{matrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots \\ & -Q_{S^{(k)}(\underline{c})}^{(0)} & & \\ & c_k I & & \end{matrix} \right].$$

Define $Q_{\underline{c}}^{(1)} = \left(A_1^{(1)}(\underline{c}), A_2^{(1)}(\underline{c}), ...\right) : \overset{\infty}{\underset{1}{\oplus}} l^2 \to l^2$. We know (see Trent [12]) that, for all $\underline{a}, \underline{b} \in l^2$, we have

Theorem 5.0.1 $\quad < \underline{a}, \underline{b} > I = Q_{\underline{a}}^{(0)} Q_{\underline{b}}^{(0)*} I = Q_{\underline{b}}^{(0)*} Q_{\underline{a}}^{(0)} + Q_{\underline{a}}^{(1)} Q_{\underline{b}}^{(1)*}.$

We give an alternative treatment for the extension of Theorem 5.0.1 to the case of general n. Define

$$\begin{aligned} H_0 &= \mathbb{C}, \\ H_1 &= \overset{\infty}{\underset{1}{\oplus}} H_0 = l^2, \\ H_2 &= \overset{\infty}{\underset{1}{\oplus}} H_1, \\ &\vdots \\ H_{n+1} &= \overset{\infty}{\underset{1}{\oplus}} H_n. \end{aligned}$$

Assume that for $j = n-1, n$, we have

$$A_k^{(j)}(\underline{c}) = \begin{matrix} \overset{k-1}{\underset{1}{\oplus}} H_j \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \\ H_j \left\{ \right. \\ \overset{\infty}{\underset{k+1}{\oplus}} H_j \left\{ \right. \end{matrix} \left[\begin{matrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots \\ & -Q_{S^{(k)}(\underline{c})}^{(j-1)} & & \\ & c_k I & & \end{matrix} \right]$$

and $Q_{\underline{c}}^{(j)} = \left(A_1^{(j)}\left(\underline{c}\right), A_2^{(j)}\left(\underline{c}\right), ...\right) : H_{j+1} \to H_j$ such that for all $\underline{a}, \underline{b} \in l^2$,

$$< \underline{a}, \underline{b} > I = Q_{\underline{b}}^{(n-1)*} Q_{\underline{a}}^{(n-1)} + Q_{\underline{a}}^{(n)} Q_{\underline{b}}^{(n)*}.$$

(Note that for $j = 0, 1$, the above is as in Theorem 5.0.1.) We want to find $Q_{\underline{a}}^{(n+1)}, Q_{\underline{b}}^{(n+1)}$ such that

$$< \underline{a}, \underline{b} > I = Q_{\underline{b}}^{(n)*} Q_{\underline{a}}^{(n)} + Q_{\underline{a}}^{(n+1)} Q_{\underline{b}}^{(n+1)*}.$$

Let

$$A_k^{(n+1)}(\underline{c}) = \begin{array}{r} \overset{k-1}{\underset{1}{\oplus}} H_{n+1} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ H_{n+1} \left\{ \begin{array}{c} \\ \end{array} \right. \\ \overset{\infty}{\underset{k+1}{\oplus}} H_{n+1} \left\{ \begin{array}{c} \\ \end{array} \right. \end{array} \left[\begin{array}{cccc} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots \\ & -Q_{S^{(k)}(\underline{c})}^{(n)} & & \\ & c_k I & & \end{array} \right].$$

Define $Q_{\underline{c}}^{(n+1)} = \left(A_1^{(n+1)}\left(\underline{c}\right), A_2^{(n+1)}\left(\underline{c}\right), ...\right)$. So $Q_{\underline{c}}^{(n+1)} : H_{n+2} \to H_{n+1}$. Consider

$$Q_{\underline{b}}^{(n)*} Q_{\underline{a}}^{(n)} + Q_{\underline{a}}^{(n+1)} Q_{\underline{b}}^{(n+1)*}. \tag{5.1}$$

We observe that formally

$$Q_{\underline{a}}^{(n+1)} = \left[\begin{array}{cc} -Q_{S(\underline{a})}^{(n)} & 0 \\ a_1 I & Q_{S(\underline{a})}^{(n+1)} \end{array} \right] \text{ and } Q_{\underline{b}}^{(n+1)*} = \left[\begin{array}{cc} -Q_{S(\underline{b})}^{(n)*} & \overline{b_1} I \\ 0 & Q_{S(\underline{b})}^{(n+1)*} \end{array} \right].$$

Similarly,

$$Q_{\underline{a}}^{(n)} = \left[\begin{array}{cc} -Q_{S(\underline{a})}^{(n-1)} & 0 \\ a_1 I & Q_{S(\underline{a})}^{(n)} \end{array} \right] \text{ and thus } Q_{\underline{b}}^{(n)*} = \left[\begin{array}{cc} -Q_{S(\underline{b})}^{(n-1)*} & \overline{b_1} I \\ 0 & Q_{S(\underline{b})}^{(n)*} \end{array} \right].$$

So we have that $(5.1) =$

$$\left[\begin{array}{cc} -Q_{S(\underline{b})}^{(n-1)*} & \overline{b_1} I \\ 0 & Q_{S(\underline{b})}^{(n)*} \end{array} \right] \left[\begin{array}{cc} -Q_{S(\underline{a})}^{(n-1)} & 0 \\ a_1 I & Q_{S(\underline{a})}^{(n)} \end{array} \right] + \left[\begin{array}{cc} -Q_{S(a)}^{(n)} & 0 \\ a_1 I & Q_{S(\underline{a})}^{(n+1)} \end{array} \right] \left[\begin{array}{cc} -Q_{S(\underline{b})}^{(n)*} & \overline{b_1} I \\ 0 & Q_{S(\underline{b})}^{(n+1)*} \end{array} \right]$$

$$= \left[\begin{array}{cc} a_1 \overline{b_1} I + Q_{S(\underline{b})}^{(n-1)*} Q_{S(\underline{a})}^{(n-1)} + Q_{S(\underline{a})}^{(n)} Q_{S(\underline{b})}^{(n)*} & 0 \\ 0 & a_1 \overline{b_1} I + Q_{S(\underline{b})}^{(n)*} Q_{S(\underline{a})}^{(n)} + Q_{S(\underline{a})}^{(n+1)} Q_{S(\underline{b})}^{(n+1)*} \end{array} \right].$$

But by our inductive hypothesis,

$$a_1\overline{b_1}I + Q_{S(\underline{b})}^{(n-1)*}Q_{S(\underline{a})}^{(n-1)} + Q_{S(\underline{a})}^{(n)}Q_{S(\underline{b})}^{(n)*} = a_1\overline{b_1}I + S(\underline{a})\,S(\underline{b})^*\,I = <\underline{a},\underline{b}> I.$$

Repeating this argument p times yields that $(5.1) =$

$$\begin{bmatrix} <\underline{a},\underline{b}> I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & <\underline{a},\underline{b}> I & 0 \\ 0 & \cdots & 0 & \sum_{i=1}^{p} a_i\overline{b_i}I + Q_{S^p(\underline{b})}^{(n)*}Q_{S^p(\underline{a})}^{(n)} + Q_{S^p(\underline{a})}^{(n+1)}Q_{S^p(\underline{b})}^{(n+1)*} \end{bmatrix}.$$

Since this holds for all $p = 1, 2, ...$, we get that

Theorem 5.0.2 $<\underline{a},\underline{b}> I = Q_{\underline{b}}^{(n)*}Q_{\underline{a}}^{(n)} + Q_{\underline{a}}^{(n+1)}Q_{\underline{b}}^{(n+1)*}.$

It follows that $Q_{\underline{c}}^{(n+1)} \in \mathcal{B}(H_{n+2}, H_{n+1})$. Another way to write Theorem 5.0.2 is, for $\underline{u}, \underline{v} \in l^2$,

$$\left(\underline{u}\,\underline{v}^T\right) I = Q_{\underline{v}}^{(n)T}Q_{\underline{u}}^{(n)} + Q_{\underline{u}}^{(n+1)}Q_{\underline{v}}^{(n+1)T}. \tag{5.2}$$

This will be useful for our proof of the following theorem, which will be utilized in our proof of the corona theorem for the matrix case, $\left[\mathcal{H}_{K,\infty}^{\infty}(\mathbb{D})\right]_m$.

Theorem 5.0.3 Suppose $F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$ and $G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{pmatrix}$ where $f_i, g_i \in l^2$ for $i = 1, 2, \ldots, m$. Define $Q_F = Q_{f_1}^{(1)}Q_{f_2}^{(2)}\cdots Q_{f_m}^{(m)}$, and likewise $Q_G = Q_{g_1}^{(1)}Q_{g_2}^{(2)}\cdots Q_{g_m}^{(m)}$. If $FG^T = I_m$, then

$$G^TF + Q_FQ_G^T = I.$$

Proof Let

$$F = \begin{pmatrix} F_n \\ f_{n+1} \end{pmatrix} \text{ where } F_n = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \text{ and } G = \begin{pmatrix} G_n \\ g_{n+1} \end{pmatrix} \text{ where } G_n = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix},$$

with $f_i, g_i \in l^2$ for $i = 1, 2, \ldots, n+1$. Suppose $FG^T = I_{n+1}$ in $\mathbb{D}$. Define $Q_F \triangleq Q_{f_1}^{(1)} Q_{f_2}^{(2)} \cdots Q_{f_n}^{(n)} Q_{f_{n+1}}^{(n+1)}$ and $Q_G \triangleq Q_{g_1}^{(1)} Q_{g_2}^{(2)} \cdots Q_{g_{n+1}}^{(n+1)}$. Consider

$$\begin{aligned} &G^T F + Q_F Q_G^T \\ &\quad = \begin{pmatrix} G_n^T & g_{n+1}^T \end{pmatrix} \begin{pmatrix} F_n \\ f_{n+1} \end{pmatrix} + Q_{f_1}^{(1)} Q_{f_2}^{(2)} \ldots Q_{f_n}^{(n)} \underbrace{Q_{f_{n+1}}^{(n+1)} Q_{g_{n+1}}^{(n+1)T}}_{a_1} Q_{g_n}^{(n)T} \ldots Q_{g_2}^{(2)T} Q_{g_1}^{(1)T}. \end{aligned} \quad (5.3)$$

From (5.2), we have that

$$\left(f_{n+1} g_{n+1}^T\right) I = Q_{g_{n+1}}^{(n)T} Q_{f_{n+1}}^{(n)} + Q_{f_{n+1}}^{(n+1)} Q_{g_{n+1}}^{(n+1)T}.$$

But $FG^T = I_{n+1}$ implies that $f_{n+1} g_{n+1}^T = 1$ in $\mathbb{D}$. Thus

$$I - Q_{g_{n+1}}^{(n)T} Q_{f_{n+1}}^{(n)} = Q_{f_{n+1}}^{(n+1)} Q_{g_{n+1}}^{(n+1)T} = a_1.$$

Substitution into (5.3) yields

$$\begin{aligned} &G^T F + Q_F Q_G^T \\ &= G_n^T F_n + g_{n+1}^T f_{n+1} + Q_{F_n} Q_{G_n}^T - Q_{f_1}^{(1)} Q_{f_2}^{(2)} \ldots \underbrace{Q_{f_n}^{(n)} Q_{g_{n+1}}^{(n)T}}_{a_2} \underbrace{Q_{f_{n+1}}^{(n)} Q_{g_n}^{(n)T}}_{a_3} \ldots Q_{g_2}^{(2)T} Q_{g_1}^{(1)T}. \end{aligned} \quad (5.4)$$

We utilize the identity (5.2) again to substitute for the terms a_2 and a_3:

$$\left(f_n g_{n+1}^T\right) I = Q_{g_{n+1}}^{(n-1)T} Q_{f_n}^{(n-1)} + Q_{f_n}^{(n)} Q_{g_{n+1}}^{(n)T}.$$

But $f_n g_{n+1}^T = 0$, so we have

$$-Q_{g_{n+1}}^{(n-1)T} Q_{f_n}^{(n-1)} = Q_{f_n}^{(n)} Q_{g_{n+1}}^{(n)T} = a_2.$$

Similarly,

$$-Q_{g_n}^{(n-1)T} Q_{f_{n+1}}^{(n-1)} = Q_{f_{n+1}}^{(n)} Q_{g_n}^{(n)T} = a_3.$$

Substitution into (5.4) yields

$$G^T F + Q_F Q_G^T$$
$$= G_n^T F_n + g_{n+1}^T f_{n+1} + Q_{F_n} Q_{G_n}^T - Q_{f_1}^{(1)} Q_{f_2}^{(2)} \dots Q_{g_{n+1}}^{(n-1)T} Q_{f_n}^{(n-1)} Q_{g_n}^{(n-1)T} Q_{f_{n+1}}^{(n-1)} \dots Q_{g_2}^{(2)T} Q_{g_1}^{(1)T}.$$

Continuing in this manner will give us the equation

$$G^T F + Q_F Q_G^T$$
$$= G_n^T F_n + g_{n+1}^T f_{n+1} + Q_{F_n} Q_{G_n}^T - g_{n+1}^T f_1 Q_{f_2}^{(1)} \dots Q_{f_n}^{(n-1)} Q_{g_n}^{(n-1)T} \dots Q_{g_2}^{(1)T} g_1^T f_{n+1}.$$

But $f_1 Q_{f_2}^{(1)} \dots Q_{f_n}^{(n-1)} Q_{g_n}^{(n-1)T} \dots Q_{g_2}^{(1)T} g_1^T = \det FG^T$ $([13])$, so the above equation becomes

$$\begin{aligned} G^T F + Q_F Q_G^T &= G_n^T F_n + g_{n+1}^T f_{n+1} + Q_{F_n} Q_{G_n}^T - g_{n+1}^T \cdot \det FG^T \cdot f_{n+1} \\ &= G_n^T F_n + g_{n+1}^T f_{n+1} + Q_{F_n} Q_{G_n}^T - g_{n+1}^T \cdot 1 \cdot f_{n+1} \\ &= G_n^T F_n + Q_{F_n} Q_{G_n}^T. \end{aligned}$$

Iteration of the above argument yields

$$\begin{aligned} G^T F + Q_F Q_G^T &= G_{n-1}^T F_{n-1} + Q_{F_{n-1}} Q_{G_{n-1}}^T \\ &= G_{n-2}^T F_{n-2} + Q_{F_{n-2}} Q_{G_{n-2}}^T \\ &\vdots \\ &= G_2^T F_2 + Q_{F_2} Q_{G_2}^T \\ &= g_1^T f_1 + Q_{f_1} Q_{g_1}^T. \end{aligned}$$

But, by definition, $Q_{f_1} Q_{g_1}^T = Q_{f_1}^{(1)} Q_{g_1}^{(1)T}$. From (5.2), we have that $\left(f_1 g_1^T\right) I = g_1^T f_1 + Q_{f_1}^{(1)} Q_{g_1}^{(1)T}$. Since $f_1 g_1^T = 1$, we have

$$\begin{aligned} G^T F + Q_F Q_G^T &= g_1^T f_1 + Q_{f_1} Q_{g_1}^T \\ &= g_1^T f_1 + Q_{f_1}^{(1)} Q_{g_1}^{(1)T} \\ &= \left(f_1 g_1^T\right) I \\ &= I. \end{aligned}$$

□

Chapter 6
THE CORONA THEOREM IN $H_K^\infty(\mathbb{D})$

Let $F \in \mathcal{A} \subset \mathcal{X}$ have a corona solution in $\mathcal{X}$, say G_0. In the previous chapter, we found for F an operator $Q \in \mathcal{A}$ such that ker $F =$ ran Q. We now find an operator $X \in \mathcal{X}$ such that $G_0 - QX \in \mathcal{A}$.

6.1 The Corona Theorem in $\mathcal{H}^\infty_{\{1\},\infty}(\mathbb{D})$

In order to make clear for the reader the first method which we use, we begin with the special case of $\mathcal{H}^\infty_{\{1\},\infty}(\mathbb{D})$. Note that though we consider an algebra of infinite vectors, the following method can also be applied to the algebra of finite vectors, $\mathcal{H}^\infty_{\{1\},n}(\mathbb{D})$.

Let $F(z) \in \mathcal{H}^\infty_{\{1\},\infty}(\mathbb{D}) \subset \overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$ with

$$0 < \varepsilon^2 \leq F(z)F(z)^* \leq 1$$

for all $z \in \mathbb{D}$. Then we know that there is a corona solution for $F(z)$, say $G(z)$, which lies in $\overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$, *i.e.* for all $z \in \mathbb{D}$, we have that

$$F(z)\,G(z)^T \equiv 1.$$

We want to use $G(z)$ to find a corona solution in $\mathcal{H}^\infty_{\{1\},\infty}(\mathbb{D})$. Note that, since $G(z) \in \overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$, it can be represented by the power series

$$G(z) = \underline{G_0} + \underline{G_1}z + \underline{G_2}z^2 + \ldots,$$

where $\underline{G_i}$ is a constant vector for $i = 0, 1, 2, \ldots$. Eliminating the z term from $G(z)$ would yield a function that lies in $\mathcal{H}^{\infty}_{\{1\},\infty}(\mathbb{D})$ as desired. Let

$$X(z)^T = \frac{Q^*_{F(0)} G'(0)^T}{F(0)F(0)^*} z \in \bigoplus_1^{\infty} H^{\infty}(\mathbb{D}),$$

where we have suppressed notation by writing $Q^{(1)}_{F(z)}$ as $Q_{F(z)}$. Consider

$$V(z)^T \triangleq G(z)^T - Q_{F(z)} X(z)^T. \tag{6.1}$$

Then we have that

i. $F(z)V(z)^T \equiv 1$ in $\mathbb{D}$:

$$\begin{aligned} F(z)V(z)^T &= F(z)\left[G(z)^T - Q_{F(z)} X(z)^T\right] \\ &= F(z)G(z)^T - F(z)Q_{F(z)} X(z)^T \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

for all $z \in \mathbb{D}$.

ii. $V(z) \in \mathcal{H}^{\infty}_{\{1\},\infty}(\mathbb{D})$:

In other words, we want to show that $V'(0) = \underline{0}$. We prove that $V'(0)^T = \underline{0}^T$.

Differentiating (6.1) yields

$$\begin{aligned} V'(z)^T &= G'(z)^T - Q'(z) X(z)^T - Q_{F(z)} X'(z)^T \Rightarrow \\ V'(0)^T &= G'(0)^T - Q'(0) X(0)^T - Q_{F(0)} X'(0)^T. \end{aligned}$$

But $X'(0)^T = \dfrac{Q^*_{F(0)} G'(0)^T}{F(0)F(0)^*}$. Also, $Q_{F(z)} \in \mathcal{H}^{\infty}_{\{1\},\infty}(\mathbb{D})$, so $Q'(0) = 0$. This implies that

$$V'(0)^T = G'(0)^T - \frac{Q_{F(0)} Q^*_{F(0)}}{F(0)F(0)^*} G'(0)^T. \tag{6.2}$$

Now, by Theorem 5.0.1, we have that

$$\begin{aligned} Q^{(0)}_{F(z)}Q^{(0)*}_{F(z)}I &= Q^{(0)*}_{F(z)}Q^{(0)}_{F(z)} + Q^{(1)}_{F(z)}Q^{(1)*}_{F(z)} \\ F(z)\,F(z)^*\,I &= F(z)^*\,F(z) + Q^{(1)}_{F(z)}Q^{(1)*}_{F(z)}. \end{aligned}$$

By supposition, $F(z)\,F(z)^* \geq \varepsilon^2 > 0$. Thus we have the following identity:

$$I = \frac{F(z)^*F(z)}{F(z)F(z)^*} + \frac{Q^{(1)}_{F(z)}Q^{(1)*}_{F(z)}}{F(z)F(z)^*}$$

Suppressing the superscript notation again, evaluating at $z = 0$, and multiplying by $G'(0)^T$, we get that

$$G'(0)^T = \frac{F(0)^*F(0)}{F(0)F(0)^*}G'(0)^T + \frac{Q_{F(0)}Q^*_{F(0)}}{F(0)F(0)^*}G'(0)^T. \tag{6.3}$$

But by supposition, we know that $F(z)G(z)^T \equiv 1$. Differentiating yields

$$\begin{aligned} F'(z)G(z)^T + F(z)G'(z)^T &\equiv 0 \text{ in } \mathbb{D} \Rightarrow \\ F'(0)G(0)^T + F(0)G'(0)^T &= 0 \\ F(0)G'(0)^T &= 0 \end{aligned}$$

So (6.3) becomes

$$\begin{aligned} G'(0)^T &= \frac{F(0)^*}{F(0)F(0)^*}\cdot 0 + \frac{Q_{F(0)}Q^*_{F(0)}}{F(0)F(0)^*}G'(0)^T \\ G'(0)^T &= \frac{Q_{F(0)}Q^*_{F(0)}}{F(0)F(0)^*}G'(0)^T \\ \underline{0}^T &= G'(0)^T - \frac{Q_{F(0)}Q^*_{F(0)}}{F(0)\,F(0)^*}G'(0)^T \end{aligned}$$

Applying this to (6.2) gives us that $V'(0)^T = \underline{0}^T$, so $V(z) \in \mathcal{H}^\infty_{\{1\},\infty}(\mathbb{D})$.

Thus we have found a corona solution, namely $V(z)$, where

$$V(z)^T = G(z)^T - \frac{Q_{F(z)}Q^*_{F(0)}}{F(0)F(0)^*}G'(0)^T z,$$

with $V(z) \in \mathcal{H}^{\infty}_{\{1\},\infty}(\mathbb{D})$ as desired. We can use the original corona estimate $||G||_{\infty}$ to estimate the size of $V(z)$. Due to Uchiyama , we have that $||G||_{\infty} \leq \frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2}$ for ε small (see Trent [13]) . This gives us that

$$||V||_{\infty} \leq 2\,||G||_{\infty} \leq 2\left(\frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2}\right).$$

6.2 The Corona Theorem in $\mathcal{H}^{\infty}_{K,\infty}(\mathbb{D})$

Now we wish to find a corona solution in a similar algebra, but with a general K. We use an inductive method like the above. Again, though the algebra we consider is one of infinite vectors, the method can be used for one of finite vectors.

Lemma 6.2.1 Suppose $K_n = \{k_1, k_2, \ldots, k_n\} \subset \mathbb{N}$ where $k_i > k_j$ for $i > j$ is such that $H^{\infty}_{K_n}(\mathbb{D})$ is an algebra. Also, define $K = K_n \cup \{k\}$, $k > k_n$, where k is chosen appropriately so that $H^{\infty}_{K}(\mathbb{D})$ is an algebra. Let $F(z) \in \mathcal{H}^{\infty}_{K,\infty}(\mathbb{D}) \subset \bigoplus_{1}^{\infty} H^{\infty}(\mathbb{D})$. Assume

$$0 < \varepsilon^2 \leq F(z)F(z)^* \leq 1$$

for all $z \in \mathbb{D}$. Suppose a corona solution has already been found for $\mathcal{H}^{\infty}_{K_n,\infty}(\mathbb{D})$, *i.e.* there exists $G(z) \in \mathcal{H}^{\infty}_{K_n,\infty}(\mathbb{D})$ such that $F(z)G(z)^T \equiv 1$ in $\mathbb{D}$, with $\|G\|_{\infty} \leq C_{\varepsilon} < \infty$. Then there exists a corona solution $V(z) \in \mathcal{H}^{\infty}_{K,\infty}(\mathbb{D})$ such that for all $z \in \mathbb{D}$,

$$F(z)\,V(z)^T \equiv 1,$$

and $\|V\|_{\infty} \leq 2\,\|G\|_{\infty}$.

Proof We wish to make use of $G(z)$ in order to find a solution to the corona problem in $\mathcal{H}^{\infty}_{K,\infty}(\mathbb{D})$. Note that, since $G(z) \in \mathcal{H}^{\infty}_{K_n,\infty}(\mathbb{D})$ where $K_n = K \setminus \{k\}$, it can be written in a power series expansion that includes the term $\underline{G_k} z^k$ where $\underline{G_k}$ is a constant vector. Eliminating this term from $G(z)$ would yield a function that lies in $\mathcal{H}^{\infty}_{K,\infty}(\mathbb{D})$ as desired. Let

$$X(z)^T = \frac{1}{k!}\frac{Q^*_{F(0)}G^{(k)}(0)^T}{F(0)F(0)^*}z^k \in H^{\infty}_{K_1}(\mathbb{D}),$$

and consider

$$V(z)^T \triangleq G(z)^T - Q_{F(z)}X(z)^T. \tag{6.4}$$

Then we have that:

i. $F(z)V(z)^T \equiv 1$, as in the previous argument.

ii. $V(z) \in \mathcal{H}^{\infty}_{K_n,\infty}(\mathbb{D})$:

In other words, we want to show that $V^{(k_i)}(0) = \underline{0}$ for each $k_i \in K_n$. We prove that $V^{(k_i)}(0)^T = \underline{0}^T$. Differentiating (6.4) k_i times yields

$$\begin{aligned} V^{(k_i)}(z)^T &= G^{(k_i)}(z)^T - \sum_{j=0}^{k_i}\binom{k_i}{j}Q^{(k_i-j)}(z)X^{(j)}(z)^T \Rightarrow \\ V^{(k_i)}(0)^T &= G^{(k_i)}(0)^T - \sum_{j=0}^{k_i}\binom{k_i}{j}Q^{(k_i-j)}(0)X^{(j)}(0)^T. \end{aligned}$$

Now $G(z) \in \mathcal{H}^{\infty}_{K_n,\infty}(\mathbb{D})$, so $G^{(k_i)}(0) = \underline{0}^T$ for all $k_i \in K_n$. Also,

$$X^{(j)}(0)^T = \frac{1}{(k-j)!}\frac{Q^*_{F(0)}G^{(k)}(0)^T}{F(0)F(0)^*}\cdot 0^{k-j} = \underline{0}^T$$

for all $j \leq k_i < k$. So $V^{(k_i)}(0)^T = \underline{0}^T$ for all $k_i \in K_n$, and hence $V(z) \in \mathcal{H}^{\infty}_{K_n,\infty}(\mathbb{D})$.

iii. $V(z) \in \mathcal{H}^{\infty}_{K,\infty}(\mathbb{D})$:

We need to prove that $V^{(k)}(0) = \underline{0}$. Now

$$\begin{aligned} V^{(k)}(z)^T &= G^{(k)}(z)^T - \sum_{j=0}^{k}\binom{k}{j}Q^{(k-j)}(z)X^{(j)}(z)^T \Rightarrow \\ V^{(k)}(0)^T &= G^{(k)}(0)^T - \sum_{j=0}^{k}\binom{k}{j}Q^{(k-j)}(0)X^{(j)}(0)^T. \end{aligned}$$

Again, for all $j < k$, we have that $X^{(j)}(0)^T = \underline{0}^T$. Thus

$$V^{(k)}(0)^T = G^{(k)}(0)^T - Q_{F(0)}X^{(k)}(0)^T. \tag{6.5}$$

But we have that

$$X^{(k)}(0)^T = \frac{Q^*_{F(0)}G^{(k)}(0)^T}{F(0)F(0)^*},$$

which gives us that (6.5) is

$$\begin{aligned} V^{(k)}(0)^T &= G^{(k)}(0)^T - Q_{F(0)} \cdot \frac{Q^*_{F(0)}G^{(k)}(0)^T}{F(0)F(0)^*} \\ V^{(k)}(0)^T &= G^{(k)}(0)^T - \frac{Q_{F(0)}Q^*_{F(0)}}{F(0)F(0)^*}G^{(k)}(0)^T. \end{aligned} \tag{6.6}$$

Now multiplying the identity

$$I = \frac{F(z)^*F(z)}{F(z)F(z)^*} + \frac{Q_{F(z)}Q^*_{F(z)}}{F(z)F(z)^*}$$

by $G^{(k)}(0)^T$, we get that

$$G^{(k)}(0)^T = \frac{F(0)^*F(0)}{F(0)F(0)^*}G^{(k)}(0)^T + \frac{Q_{F(0)}Q^*_{F(0)}}{F(0)F(0)^*}G^{(k)}(0)^T \tag{6.7}$$

But $F(z)\,G(z)^T \equiv 1$ in $\mathbb{D}$. Differentiating k times gives us that, for all $z \in \mathbb{D}$,

$$\begin{aligned} \sum_{j=0}^{k} \binom{k}{j} F^{(k-j)}(z)G^{(j)}(z)^T &\equiv 0 \Rightarrow \\ \sum_{j=0}^{k} \binom{k}{j} F^{(k-j)}(0)G^{(j)}(0)^T &= 0. \end{aligned} \tag{6.8}$$

Since $G(z) \in \mathcal{H}^\infty_{K_n,\infty}(\mathbb{D})$, we know that $G^{(j)}(0) = \underline{0}$ for all $j \in K_n$. Suppose $j \notin K$, $j < k$. Now since $k \in K$, we have by Lemma 4.2.4 that $k - j \in K$, and hence $F^{(k-j)}(0) = \underline{0}$ for all such j. Since $\{0, 1, 2, \ldots, k\} = K_n \cup \{j : j \notin K,\ j < k\} \cup \{k\}$, it follows that (6.8) is equivalent to

$$\begin{aligned}\sum_{j\in K_n}\binom{k}{j}F^{(k-j)}(0)G^{(j)}(0)^T+\sum_{j\notin K,\,j<k}\binom{k}{j}F^{(k-j)}(0)G^{(j)}(0)^T+F(0)G^{(k)}(0)^T &= 0\\ \sum_{j\in K_n}\binom{k}{j}F^{(k-j)}(0)\cdot\underline{0}^T+\sum_{j\notin K,\,j<k}\binom{k}{j}\cdot\underline{0}\cdot G^{(j)}(0)^T+F(0)G^{(k)}(0)^T &= 0\\ F(0)G^{(k)}(0)^T &= 0.\end{aligned}$$

This implies that (6.7) is

$$\begin{aligned}G^{(k)}(0)^T &= \frac{F(0)^*}{F(0)F(0)^*}\cdot 0+\frac{Q_{F(0)}Q_{F(0)}^*}{F(0)F(0)^*}G^{(k)}(0)^T\\ G^{(k)}(0)^T &= \frac{Q_{F(0)}Q_{F(0)}^*}{F(0)F(0)^*}G^{(k)}(0)^T\\ \underline{0}^T &= G^{(k)}(0)^T-\frac{Q_{F(0)}Q_{F(0)}^*}{F(0)F(0)^*}G^{(k)}(0)^T\end{aligned}$$

Applying this to (6.6) gives us that

$$V^{(k)}(0)^T=\underline{0}^T.$$

So $V(z)\in\mathcal{H}^\infty_{K,\infty}(\mathbb{D})$.

Thus we have found a corona solution, namely $V(z)$ where

$$V(z)^T=G(z)^T-\frac{1}{k!}\frac{Q_{F(z)}Q_{F(0)}^*}{F(0)F(0)^*}G^{(k)}(0)^Tz^k,$$

with $V(z)\in\mathcal{H}^\infty_{K,\infty}(\mathbb{D})$ as desired. In order to estimate the size of $V(z)$, we make use of the estimate $\|G\|_\infty$, and have that

$$\|V\|_\infty\le 2\,\|G\|_\infty\,.$$

$\square$

Theorem 6.2.1 Suppose $K = \{k_1, k_2, \ldots, k_n, k_{n+1}\} \subset \mathbb{N}$ where $k_i > k_j$ for $i > j$ is such that $H_K^\infty(\mathbb{D})$ is an algebra. Let $F(z) \in \mathcal{H}_{K,\infty}^\infty(\mathbb{D}) \subset \overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$. Assume that, for all $z \in \mathbb{D}$,

$$0 < \varepsilon^2 \leq F(z)F(z)^* \leq 1.$$

Then there exists a corona solution $V(z) \in \mathcal{H}_{K,\infty}^\infty(\mathbb{D})$ such that for all $z \in \mathbb{D}$,

$$F(z)\, V(z)^T \equiv 1,$$

and $\|V\|_\infty \leq 2^{n+1} \frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2}$ for ε small.

Proof Since $\mathcal{H}_{K,\infty}^\infty(\mathbb{D}) \subset \overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$, we have by Carleson's corona theorem that there exists a corona solution for $F(z)$ that lies in $\overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$. Call this solution $G_0(z)$. From Uchiyama, we have that $||G_0||_\infty \leq \frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2}$ for ε small. Using the technique found in the proof of Lemma 6.2.1, we can build a solution that lies in the algebra $\mathcal{H}_{K_1,\infty}^\infty(\mathbb{D})$ where $K_1 = \{k_1\}$. This solution is $G_1(z)$ where

$$G_1(z)^T \triangleq G_0(z)^T - \frac{1}{k_1!} \frac{Q_{F(z)} Q_{F(0)}^*}{F(0)F(0)^*} G_0^{(k_1)}(0)^T z^{k_1}.$$

We have that $||G_1||_\infty \leq 2\, ||G_0||_\infty$.

Similarly, we can use $G_1(z)$ to build a solution that lies in $\mathcal{H}_{K_2,\infty}^\infty(\mathbb{D})$ where $K_2 = \{k_1, k_2\}$. We call this solution $G_2(z)$ where

$$G_2(z)^T \triangleq G_1(z)^T - \frac{1}{k_2!} \frac{Q_{F(z)} Q_{F(0)}^*}{F(0)F(0)^*} G_1^{(k_2)}(0)^T z^{k_2}$$

and note that $||G_2||_\infty \leq 2\, ||G_1||_\infty \leq 2^2\, ||G_0||_\infty$.

Continuing in this manner, we observe that we can build a solution for the corona problem for $F(z)$ in $\mathcal{H}_{K,\infty}^\infty(\mathbb{D})$, namely $G_{n+1}(z)$, with

$$||G_{n+1}||_\infty \leq 2^{n+1}\, ||G_0||_\infty \leq 2^{(n+1)} \left(\frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2} \right).$$

$\square$

Note that this estimate is based on the number of elements in the set K, which in this case is $n+1$.

Now we consider a similar algebra but employ another technique. This technique will provide us with different, though not necessarily better, estimates than those found in Lemma 6.2.1. However, this approach is useful in that it can handle not only the algebra $H^{\infty}_{K,\infty}(\mathbb{D})$, but also algebras formed as follows: fix a finite Blaschke product, B, and consider

$$C_0 + C_1 B^{n_1} + C_2 B^{n_2} + \cdots + C_{N-1} B^{n_{k-1}} + B^{n_k} f(z)$$

where $f(z) \in H^{\infty}(\mathbb{D})$ and $C_i \in \mathbb{C}$. Let $N_k = \{n_i\}_{i=1}^{k}$ be chosen so that the above functions form an algebra, and call it $H^{\infty,B}_{N_k}(\mathbb{D})$. Then the following techniques give an "infinite" corona theorem for $H^{\infty,B}_{N_k}(\mathbb{D})$.

Let $F(z) \in \mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})$ where

$$H^{\infty}_{K_j}(\mathbb{D}) = \left\{ C_0 + C_1 z^{p_1} + \cdots + C_{j-1} z^{p_{j-1}} + z^{p_j} h(z) : \begin{array}{c} C_n \in \mathbb{C} \text{ for } n \leq j-1, \\ h(z) \in H^{\infty}(\mathbb{D}), \\ p_i > p_k \text{ for } i > k. \end{array} \right\}$$

Assume that, for all $z \in \mathbb{D}$,

$$0 < \varepsilon^2 \leq F(z) F(z)^* \leq 1.$$

Suppose we have a corona solution for $F(z)$, say $G(z)$, which lies in $\mathcal{H}^{\infty}_{K_{j-1},\infty}(\mathbb{D})$ where

$$H^{\infty}_{K_{j-1}}(\mathbb{D}) = \left\{ C_0 + C_1 z^{p_1} + \cdots + C_{j-1} z^{p_{j-2}} + z^{p_{j-1}} h(z) : \begin{array}{c} C_n \in \mathbb{C} \text{ for } n \leq j-2, \\ h(z) \in H^{\infty}(\mathbb{D}), \\ p_i > p_k \text{ for } i > k. \end{array} \right\}.$$

In addition, assume that we have that $\|G\|_{\infty} \leq C_{\varepsilon} < \infty$. Now we know that

$$F(z)\, G(z)^T \equiv 1$$

in $\mathbb{D}$. We want to use $G(z)$ in order to find a corona solution in $\mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})$. It suffices to find $X(z) \in \bigoplus_{1}^{\infty} H^{\infty}(\mathbb{D})$ such that

$$\left[G(z)^T - Q_{F(z)} X(z)^T \right]^T \in \mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D}).$$

We know from Theorem 5.0.1 that

$$\begin{aligned} F(z)\,G(z)^T I &= G(z)^T F(z) + Q_{F(z)}Q_{G(z)}^T \Rightarrow \\ I &= G(z)^T F(z) + Q_{F(z)}Q_{G(z)}^T \end{aligned}$$

in $\mathbb{D}$. Also, $F(z)$ has the form

$$F(z) = \underline{F_0} + \underline{F_1}z^{p_1} + \cdots + \underline{F_{j-1}}z^{p_{j-1}} + z^{p_j}F_R(z),$$

where $\underline{F_i}$ is a constant vector for $1 \leq i \leq j-1$ and $F_R(z) \in \overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$. Thus

$$\begin{aligned} I &= G(z)^T\left[\underline{F_0} + \underline{F_1}z^{p_1} + \cdots + \underline{F_{j-1}}z^{p_{j-1}} + z^{p_j}F_R(z)\right] + Q_{F(z)}Q_{G(z)}^T \\ G(z)^T\underline{F_0} + Q_{F(z)}Q_{G(z)}^T &= I - G(z)^T\left[\underline{F_1}z^{p_1} + \underline{F_2}z^{p_2} + \cdots + \underline{F_{j-1}}z^{p_{j-1}} + z^{p_j}F_R(z)\right]. \end{aligned} \tag{6.9}$$

Now, $\underline{F_0} = (f_1, f_2, ...)$ where the $f_i \in \mathbb{C}$ are not all zero. Suppose $f_k \neq 0$, and define $\underline{F_0}e_k^T \triangleq (f_k)_0$ where $e_k = \left(0, \ldots, 0, \underbrace{1}_{k^{th}\text{ entry}}, 0, \ldots\right)$. Then multiplying (6.9) by $\dfrac{e_k^T}{(f_k)_0}$ yields

$$G(z)^T + Q_{F(z)}\frac{Q_{G(z)}^T e_k^T}{(f_k)_0} = \frac{e_k^T}{(f_k)_0} - \frac{G(z)^T}{(f_k)_0}\left(\underline{F_1}z^{p_1} + \underline{F_2}z^{p_2} + \cdots + \underline{F_{j-1}}z^{p_{j-1}} + z^{p_j}F_R(z)\right)e_k^T. \tag{6.10}$$

Now $G(z)^T$ has the form

$$G(z)^T = \underline{G_0^T} + \underline{G_1^T}z^{p_1} + \cdots + \underline{G_{j-2}^T}z^{p_{j-2}} + z^{p_{j-1}}G_R(z)^T$$

where $\underline{G_i}$ is a constant vector for $1 \leq i \leq j-2$ and $G_R(z) \in \overset{\infty}{\underset{1}{\oplus}} H^\infty(\mathbb{D})$. Thus

$$\begin{aligned} & G(z)^T\left(\underline{F_1}z^{p_1} + \underline{F_2}z^{p_2} + \cdots + \underline{F_{j-1}}z^{p_{j-1}} + z^{p_j}F_R(z)\right) \\ = \; & \left(\underline{G_0^T} + \underline{G_1^T}z^{p_1} + \cdots + \underline{G_{j-2}^T}z^{p_{j-2}} + z^{p_{j-1}}G_R(z)^T\right) \\ & \times\left(\underline{F_1}z^{p_1} + \underline{F_2}z^{p_2} + \cdots + \underline{F_{j-1}}z^{p_{j-1}} + z^{p_j}F_R(z)\right) \\ = \; & P(z) + \underbrace{z^{p_{j-1}}G_R(z)^T\left[\underline{F_1}z^{p_1} + \underline{F_2}z^{p_2} + \cdots + \underline{F_{j-1}}z^{p_{j-1}} + z^{p_j}F_R(z)\right]}_{\triangleq R(z)} \end{aligned}$$

where $P(z) \in \mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})$. But

$$z^{p_{j-1}} z^{p_i} = z^{p_{j-1}+p_i} \in H^{\infty}_{K_j}(\mathbb{D})$$

for $1 \leq i \leq j$, so $p_{j-1} + p_i \geq p_j$ for $1 \leq i \leq j$. Therefore $R(z) \in \mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})$, hence

$$G\left(z\right)^T \left(\underline{F_1} z^{p_1} + \underline{F_2} z^{p_2} + \cdots + \underline{F_{j-1}} z^{p_{j-1}} + z^{p_j} F_R(z)\right) \in \mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D}).$$

It follows that the transpose of the right hand side of (6.10) is in $\mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})$ as desired. Thus we have found a suitable $X(z) \in \overset{\infty}{\underset{1}{\oplus}} H^{\infty}(\mathbb{D})$, where $X\left(z\right)^T = -\dfrac{Q^T_{G(z)} e^T_k}{(f_k)_0}$, that gives us a solution to the corona problem in $\mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})$, namely $V(z)$ where

$$V(z)^T \triangleq G\left(z\right)^T + Q_{F(z)} \frac{Q^T_{G(z)} e^T_k}{(f_k)_0}.$$

We can use $||G||_{\infty}$ to estimate the size of $V(z)$:

$$||V||_{\infty} \leq \left(1 + \frac{1}{\max\limits_{i\in\mathbb{N}} |(f_i)_0|}\right) ||G||_{\infty} . \qquad \text{(Estimate A)}$$

Note that this estimate is based on the size of the entries of the constant vector, $\underline{F_0} = (f_1, f_2, ...)$.

If we apply the technique used in the proof of Lemma 6.2.1 to this problem, we observe that it would take m iterations where $m = p_j - p_{j-1} - 1$ in order to have a corona solution that is contained in $\mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})$. Therefore, the estimate for the corona solution found using that technique, call it $U\left(z\right)$, would be

$$||U||_{\infty} \leq 2^{p_j - p_{j-1} - 1} ||G||_{\infty} . \qquad \text{(Estimate B)}$$

In some cases, Estimate A may be better than Estimate B. For example, if the constant vector $\underline{F_0}$ has any entry greater than 1, then this is indeed the case. However, if all of the entries of $\underline{F_0}$ are sufficiently small, then Estimate B may be better than Estimate A. Note that Estimate B depends on the size of the gap between p_{j-1} and p_j. The closer the integer p_j is to p_{j-1}, the better the estimate.

6.3 *The Corona Theorem in* $\left[\mathcal{H}^{\infty}_{K,\infty}(\mathbb{D})\right]_m$

The previous argument can be altered and applied to the matrix versions of both our algebra $H^{\infty}_{K}(\mathbb{D})$ and the previously defined algebras involving Blaschke products, $H^{\infty,B}_{N_k}(\mathbb{D})$. Let $\mathcal{F}(z)$ be an $m\times\infty$ matrix in $\left[\mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})\right]_m$ where

$$H^{\infty}_{K_j}(\mathbb{D}) = \left\{ C_0 + C_1 z^{p_1} + \cdots + C_{j-1} z^{p_{j-1}} + z^{p_j} h(z) : \begin{array}{c} C_n \in \mathbb{C} \text{ for } n \le j-1, \\ h(z) \in H^{\infty}(\mathbb{D}), \\ p_i > p_k \text{ for } i > k. \end{array} \right\}$$

Assume

$$0 < \varepsilon^2 I_m \le \mathcal{F}(z)\mathcal{F}(z)^* \le I_m$$

for all $z \in \mathbb{D}$. Suppose we have a corona solution for $\mathcal{F}(z)$, say $\mathcal{G}(z)$, which lies in $\left[\mathcal{H}^{\infty}_{K_{j-1},\infty}(\mathbb{D})\right]_m$ where

$$H^{\infty}_{K_{j-1}}(\mathbb{D}) = \left\{ C_0 + C_1 z^{p_1} + \cdots + C_{j-1} z^{p_{j-2}} + z^{p_{j-1}} h(z) : \begin{array}{c} C_n \in \mathbb{C} \text{ for } n \le j-2, \\ h(z) \in H^{\infty}(\mathbb{D}), \\ p_i > p_k \text{ for } i > k. \end{array} \right\}.$$

Assume an estimate for the size of $\mathcal{G}(z)$ is known, i.e. $\|\mathcal{G}\|_{\infty} \le C_{\varepsilon} < \infty$. Now we have that

$$\mathcal{F}(z)\mathcal{G}(z)^T \equiv I_m$$

in $\mathbb{D}$. We want to use $\mathcal{G}(z)$ in order to find a corona solution in $\left[\mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})\right]_m$. It suffices to find $\mathcal{X}(z)$, an $m\times\infty$ matrix with entries in $H^{\infty}(\mathbb{D})$, such that

$$\left[\mathcal{G}(z)^T - Q_{\mathcal{F}(z)}\mathcal{X}(z)^T\right]^T \in \left[\mathcal{H}^{\infty}_{K_j,\infty}(\mathbb{D})\right]_m.$$

We know from Theorem 5.0.3 that we have the identity

$$\mathcal{G}(z)^T \mathcal{F}(z) + Q_{\mathcal{F}(z)} Q^T_{\mathcal{G}(z)} = I$$

in $\mathbb{D}$. Also, $\mathcal{F}(z)$ has the form

$$\mathcal{F}(z) = \mathcal{F}_0 + \mathcal{F}_1 z^{p_1} + \mathcal{F}_2 z^{p_2} + \cdots + \mathcal{F}_{j-1} z^{p_{j-1}} + z^{p_j} \mathcal{F}_R(z),$$

where $\mathcal{F}_i$ is a constant $m \times \infty$ matrix for $1 \leq i \leq j-1$ and $\mathcal{F}_R(z)$ is an $m \times \infty$ matrix with entries in $H^\infty(\mathbb{D})$. It follows that

$$\begin{aligned} I &= \mathcal{G}(z)^T \left[\mathcal{F}_0 + \mathcal{F}_1 z^{p_1} + \cdots + \mathcal{F}_{j-1} z^{p_{j-1}} + z^{p_j} \mathcal{F}_R(z)\right] + Q_{\mathcal{F}(z)} Q^T_{\mathcal{G}(z)} \\ \mathcal{G}(z)^T \mathcal{F}_0 + Q_{\mathcal{F}(z)} Q^T_{\mathcal{G}(z)} &= I - \mathcal{G}(z)^T \left[\mathcal{F}_1 z^{p_1} + \cdots + \mathcal{F}_{j-1} z^{p_{j-1}} + z^{p_j} \mathcal{F}_R(z)\right] \qquad (6.11) \end{aligned}$$

Now, we have that for all $z \in \mathbb{D}$,

$$\mathcal{F}(z)\mathcal{F}(z)^* \geq \varepsilon^2 I_m.$$

Thus

$$\det \mathcal{F}(z)\mathcal{F}(z)^* \geq \varepsilon^{2m}$$

for all $z \in \mathbb{D}$, and so

$$\det \mathcal{F}_0 \mathcal{F}_0^* \geq \varepsilon^{2m}$$

since $\mathcal{F}(0) = \mathcal{F}_0$. Let $\mathcal{I}_m$ denote the set of increasing $m-$tuples of positive integers. Then we have that

$$\det \mathcal{F}_0 \mathcal{F}_0^* = \sum_{\pi_k \in \mathcal{I}_m} \det \mathcal{F}_0 E_{\pi_k} \mathcal{F}_0^*,$$

so

$$\det \mathcal{F}_0 E_{\pi_k} \mathcal{F}_0^* \geq \delta > 0$$

for some $\pi_k \in \mathcal{I}_m$. (In the case where $\mathcal{F}(z)$ is an $m \times n$ matrix with $m \leq n < \infty$, we have the estimate $\det \mathcal{F}_0 E_{\pi_k} \mathcal{F}_0^* \geq \frac{\varepsilon^{2m}}{\binom{n}{m}}$.) Define

$$\mathcal{A} \triangleq \mathcal{F}_0 E_{\pi_k} \mathcal{F}_0^*.$$

Note that $\mathcal{A}$ is invertible.

Multiplying (6.11) by $E_{\pi_k}\mathcal{F}_0^*\mathcal{A}^{-1}$ yields:

$$\begin{aligned}&\mathcal{G}(z)^T + Q_{\mathcal{F}(z)}Q_{\mathcal{G}(z)}^T E_{\pi_k}\mathcal{F}_0^*\mathcal{A}^{-1} \\ &\quad = E_{\pi_k}\mathcal{F}_0^*\mathcal{A}^{-1} - \mathcal{G}(z)^T\left[\mathcal{F}_1 z^{p_1} + \mathcal{F}_2 z^{p_2} + ... + \mathcal{F}_{j-1}z^{p_{j-1}} + z^{p_j}\mathcal{F}_R(z)\right]E_{\pi_k}\mathcal{F}_0^*\mathcal{A}^{-1} \quad (3.4)\end{aligned}$$

We wish to show that the transpose of the right hand side of (3.4) is in the algebra, $\left[\mathcal{H}_{K_j,\infty}^{\infty}(\mathbb{D})\right]_m$. Now $E_{\pi_k}\mathcal{F}_0^*\mathcal{A}^{-1}$ is a constant matrix. Since $\mathcal{G}(z) \in \left[H_{K_{j-1},\infty}^{\infty}(\mathbb{D})\right]_m$, it has the form

$$\mathcal{G}(z) = \mathcal{G}_0 + \mathcal{G}_1 z^{p_1} + \cdots + \mathcal{G}_{j-2}z^{p_{j-2}} + z^{p_{j-1}}\mathcal{G}_R(z).$$

It follows that

$$\begin{aligned}&\mathcal{G}(z)^T\left[\mathcal{F}_1 z^{p_1} + \mathcal{F}_2 z^{p_2} + \cdots + \mathcal{F}_{j-1}z^{p_{j-1}} + z^{p_j}\mathcal{F}_R(z)\right] \\ =\ &\left[\mathcal{G}_0^T + \mathcal{G}_1^T z^{p_1} + \cdots + \mathcal{G}_{m-2}^T z^{p_{j-2}} + z^{p_{j-1}}\mathcal{G}_R(z)^T\right] \\ &\times\left[\mathcal{F}_1 z^{p_1} + \mathcal{F}_2 z^{p_2} + \cdots + \mathcal{F}_{j-1}z^{p_{j-1}} + z^{p_j}\mathcal{F}_R(z)\right] \\ =\ &\mathcal{P}(z) + \underbrace{z^{p_{j-1}}\mathcal{G}_R(z)^T\left[\mathcal{F}_1 z^{p_1} + \mathcal{F}_2 z^{p_2} + \cdots + \mathcal{F}_{j-1}z^{p_{j-1}} + z^{p_j}\mathcal{F}_R(z)\right]}_{\triangleq \mathcal{R}(z)}\end{aligned}$$

where $\mathcal{P}(z) \in \left[\mathcal{H}_{K_j,\infty}^{\infty}(\mathbb{D})\right]_m$. But

$$z^{p_{j-1}}z^{p_i} = z^{p_{j-1}+p_i} \in H_{K_j}^{\infty}(\mathbb{D})$$

for $1 \leq i \leq j$, so $p_{j-1} + p_i \geq p_j$ for $1 \leq i \leq j$. Therefore $\mathcal{R}(z) \in \left[\mathcal{H}_{K_j,\infty}^{\infty}(\mathbb{D})\right]_m$, hence

$$\mathcal{G}(z)^T\left[\mathcal{F}_1 z^{p_1} + \cdots + \mathcal{F}_{j-1}z^{p_{j-1}} + z^{p_j}\mathcal{F}_R(z)\right] \in \left[\mathcal{H}_{K_j,\infty}^{\infty}(\mathbb{D})\right]_m.$$

It follows that the transpose of the right hand side of (3.4) is in $\left[\mathcal{H}_{K_j,\infty}^{\infty}(\mathbb{D})\right]_m$ as desired. Thus we have found a corona solution for $\mathcal{F}(z)$, namely $\mathcal{V}(z) \in \left[\mathcal{H}_{K_j,\infty}^{\infty}(\mathbb{D})\right]_m$, where

$$\begin{aligned}\mathcal{V}(z)^T &\triangleq \mathcal{G}(z)^T + Q_{\mathcal{F}(z)}Q_{\mathcal{G}(z)}^T E_{\pi_k}\mathcal{F}_0^*\mathcal{A}^{-1} \\ &= \mathcal{G}(z)^T + Q_{\mathcal{F}(z)}Q_{\mathcal{G}(z)}^T E_{\pi_k}\mathcal{F}_0^*\left(\mathcal{F}_0 E_{\pi_k}\mathcal{F}_0^*\right)^{-1}.\end{aligned}$$

We can use $\|\mathcal{G}\|_\infty$ in order to estimate the size of $\mathcal{V}(z)$:

$$\|\mathcal{V}\|_\infty \leq \left(1 + \inf_{\pi_k \in \mathcal{I}_m} \left\|\left(\mathcal{F}_0 E_{\pi_k} \mathcal{F}_0^*\right)^{-1}\right\|_\infty\right) \|\mathcal{G}\|_\infty .$$

6.4 *Future Research*

As is the nature of research, this dissertation leads us to yet more questions. We describe a few ideas for future research here.

While we give many properties in Section 4.2 that a set $K \subset \mathbb{N}$ must possess in order for $H_K^\infty(D)$ to be an algebra, it is not clear how to completely characterize all such sets K. Is there a systematic way to build them so that all appropriate sets are captured?

One result that can be improved upon is to reduce the size of the estimate found in our first technique of solving the corona problem for $F(z) \in \mathcal{H}_{K,\infty}^\infty(\mathbb{D})$ in Section 6.2. Recall that when the number of elements in K is n, our estimate for the corona solution $G(z)$ is

$$\|G\|_\infty \leq 2^{(n+1)} \left(\frac{C_0}{\varepsilon^2} \ln \frac{1}{\varepsilon^2}\right)$$

for ε small. This estimate is less than ideal as it involves geometric growth.

Lastly, we observe that the second technique in Section 6.2 allowed us to extend the corona theorem to other algebras, such as $H_{N_k}^{\infty,B}(\mathbb{D})$ defined above. We conjecture that a similar technique can be used to give a corona theorem for an infinite version of the algebras $\mathbb{C} + \mathbb{I}$, where $\mathbb{I}$ is a closed ideal in $H^\infty(\mathbb{D})$.

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