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Hidden quantum group symmetry in the chiral model

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Abstract

We apply the $SL(2, \mathbb{C})$ lattice Kac–Moody algebra of Alekseev, Faddeev and Semenov-Tian-Shansky to obtain a new lattice description of the $SU(2)$ chiral model in two dimensions. The system has a global quantum group symmetry and it can be regarded as a deformation of two different theories. One is the non-abelian Toda lattice, which is obtained in the limit of infinite central charge, while the other is a non-standard Hamiltonian description of the chiral model, which is obtained in the continuum limit. © 1997 Elsevier Science B.V.

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1. Introduction

The two-dimensional chiral model is an example of a field theory which has an infinite number of conserved charges, yet its quantization is problematic. The reason is due to the theory being ‘non-ultralocal’, which means that Schwinger-like terms appear in the algebra of the Lax matrices. This in turn leads to difficulties in defining the Poisson brackets of the conserved charges constructed out of the monodromy matrices.

A number of proposals for removing the above ambiguities in dealing with non-ultralocal models have been made [1–6]. One which we wish to pursue here is to put the theory on the lattice. (Our lattice will be one dimensional, with time remaining continuous.) Here we shall examine two different schemes for discretizing the $SU(2)$ chiral model. The first we shall study is already known and it is based on the non-abelian Toda lattice [2]. It leads to the standard Hamiltonian formalism in the continuum limit, and the classical theory has a global canonical symmetry. The second is new and is based

on the lattice Kac–Moody algebra of Alekseev, Faddeev and Semenov-Tian-Shansky [7] (also see Refs. [8–10]). The continuum limit of this theory gives a non-standard Hamiltonian formulation of the $SU(2)$ chiral model found by Rajeev which is based on the $SL(2, \mathbb{C})$ current algebra [11,12,6,13]. The description based on the lattice Kac–Moody algebra is canonically inequivalent to the previous one based on the non-abelian Toda lattice. Rather, it is a deformation of the previous description. The classical lattice Kac–Moody algebra has a global Lie–Poisson symmetry (the classical counterpart of a quantum group symmetry [14]) and it is characterized by a central charge κ . Upon taking the limit of infinite central charge one recovers the previous description.

A quantum group symmetry was shown previously to exist in the WZNW model [15]. Here we conclude that a quantum group symmetry can be present in a theory that contains no Wess–Zumino term. We shall not investigate the question of conserved charges and integrability for the lattice theories in this article, but intend it to be the subject for a future article.

We begin in Section 2 by reviewing the continuum theory of the chiral model. The discretization of the standard Hamiltonian formalism is given in Section 3. In Section 4 we write down the classical version of the lattice algebra of Alekseev, Faddeev and Semenov-Tian-Shansky. We then examine the $*$ operation for this system and find a consistent algebra only for the case of a real central charge. After confirming that its continuum limit is the $SL(2, \mathbb{C})$ Kac–Moody algebra, we show that in the limit of infinite central charge the lattice algebra coincides with that of the non-abelian Toda lattice. A Lie–Poisson gauge transformation is revealed for the lattice algebra in Section 5. In Section 6 we specify a Hamiltonian for this system which agrees with Ref. [11] in the continuum limit and the Toda lattice Hamiltonian in the limit of infinite central charge. The Hamiltonian breaks the local Lie–Poisson symmetry of the Poisson brackets to a global one. In Section 7 we consider restoring the local symmetry in order to write a Lie–Poisson lattice gauge theory, and in so doing we recover previously found systems [16,17]. Here we shall also show how the non-local algebra of Ref. [7] can be expressed entirely in terms of a local algebra and that the latter is related to the algebra of the classical double. Some concluding remarks are made in Section 8.

2. The continuum theory

In two dimensions, the $SU(2)$ chiral model dynamics can be specified in terms of the currents I_α and J_α , $\alpha = 1, 2, 3$, the equations of motion being

$$\begin{aligned} \dot{I}_\alpha &= J'_\alpha, \\ \dot{J}_\alpha &= I'_\alpha - \epsilon_{\alpha\beta\gamma} I_\beta J_\gamma, \end{aligned} \quad (1)$$

where the dot denotes the time derivative, the prime denotes the space derivative, and $\epsilon_{\alpha\beta\gamma}$ are the structure constants for $SU(2)$. [Here and throughout this paper we specialize to the case of $SU(2)$.] The equations can be generalized to include effects of a Wess–

Zumino term, but we shall not consider such a modification here. (For a discussion see Section 7.)

We shall be concerned with two canonically inequivalent Hamiltonian descriptions of the chiral model, which we refer to as the ‘standard’ and ‘alternative’ formulations. In the standard Hamiltonian formulation the equations of motion (1) are recovered from the Hamiltonian

$$H_0 = -\frac{1}{4\chi^2} \int dx (I_\alpha I_\alpha + J_\alpha J_\alpha), \tag{2}$$

and Poisson brackets

$$\begin{aligned} \frac{1}{2\chi^2} \{I_\alpha(x), I_\beta(y)\}_0 &= \varepsilon_{\alpha\beta\gamma} I_\gamma(x) \delta(x-y), \\ \frac{1}{2\chi^2} \{I_\alpha(x), J_\beta(y)\}_0 &= \varepsilon_{\alpha\beta\gamma} J_\gamma(x) \delta(x-y) - \delta_{\alpha\beta} \delta'(x-y), \\ \frac{1}{2\chi^2} \{J_\alpha(x), J_\beta(y)\}_0 &= 0, \end{aligned} \tag{3}$$

χ being an arbitrary constant.

The alternative canonical formalism for the chiral model was introduced in [11,12,6]. It replaces (2) and (3) by¹

$$H_{1/\xi} = -\frac{1}{4\chi^2(1+\xi^{-2})^2} \int dx (I_\alpha I_\alpha + J_\alpha J_\alpha), \tag{4}$$

$$\begin{aligned} \frac{1}{2\chi^2} \{I_\alpha(x), I_\beta(y)\}_{1/\xi} &= (1+\xi^{-2}) \varepsilon_{\alpha\beta\gamma} I_\gamma(x) \delta(x-y), \\ \frac{1}{2\chi^2} \{I_\alpha(x), J_\beta(y)\}_{1/\xi} &= (1+\xi^{-2}) \varepsilon_{\alpha\beta\gamma} J_\gamma(x) \delta(x-y) - (1+\xi^{-2})^2 \delta_{\alpha\beta} \delta'(x-y), \\ \frac{1}{2\chi^2} \{J_\alpha(x), J_\beta(y)\}_{1/\xi} &= -\xi^{-2} (1+\xi^{-2}) \varepsilon_{\alpha\beta\gamma} I_\gamma(x) \delta(x-y), \end{aligned} \tag{5}$$

for a real constant ξ . For finite ξ the Poisson structure (5) is canonically inequivalent to (3). On the other hand, the Hamiltonian (4) and Poisson structure (5) reduces to (2) and (3), respectively, when $\xi \rightarrow \infty$. Therefore (4) and (5) give a one-parameter deformation of the standard canonical formalism.

The Poisson bracket algebra (5) is equivalent to the $SL(2, \mathbb{C})$ Kac–Moody algebra. To see this we can write

$$I_\alpha = 2\chi^2(1+\xi^{-2}) \tilde{I}_\alpha, \quad J_\alpha = -\frac{2\chi^2}{\xi} (1+\xi^{-2}) \tilde{J}_\alpha, \tag{6}$$

and then (5) in terms of the currents $\tilde{I}_\alpha(x)$ and $\tilde{J}_\alpha(x)$ becomes

$$\begin{aligned} \{\tilde{I}_\alpha(x), \tilde{I}_\beta(y)\} &= -\{\tilde{J}_\alpha(x), \tilde{J}_\beta(y)\} \\ &= \varepsilon_{\alpha\beta\gamma} \tilde{I}_\gamma(x) \delta(x-y), \end{aligned}$$

¹ In comparing with Ref. [12], the parameters ρ and τ of that reference are given by $\rho = 0$ and $\tau^2 = -\xi^{-2}$.

$$\{\tilde{I}_\alpha(x), \tilde{J}_\beta(y)\} = \epsilon_{\alpha\beta\gamma} \tilde{J}_\gamma(x) \delta(x-y) + \frac{\xi}{2\chi^2} \delta_{\alpha\beta} \partial_x \delta(x-y). \tag{7}$$

Note that this is not the most general $SL(2, \mathbb{C})$ Kac–Moody algebra as a second central term is allowed in the algebra (see the discussion in Section 7).

In the section which follows we examine the discretization of the standard Hamiltonian formalism, while the discretization of the alternative Hamiltonian formalism is given in Sections 4–6.

3. The non-abelian Toda lattice

Here we show that the lattice version of the standard Hamiltonian description defined by (2) and (3) can be formulated in terms of the non-abelian Toda lattice [2]. Once again we specialize to the case of $SU(2)$. Then this system can be written in terms of 2×2 matrices G_n and B_n , $n = 1, 2, \dots, N$, N being the total number of lattice sites, where G_n is traceless and antihermitian, while B_n is an $SU(2)$ matrix. For their Poisson brackets we take

$$\begin{aligned} \frac{2}{\chi^2} \{ G_n, G_m \} &= [C, G_m] \delta_{n,m}, \\ \frac{2}{\chi^2} \{ G_n, B_m \} &= C B_m \delta_{n,m-1} - B_m C \delta_{n,m}, \\ \{ B_n, B_m \} &= 0, \end{aligned} \tag{8}$$

where we utilize tensor product notation. Here

$$G_n = G_n \otimes \mathbf{1}, \quad G_m = \mathbf{1} \otimes G_m, \quad B_n = B_n \otimes \mathbf{1}, \quad B_m = \mathbf{1} \otimes B_m$$

and $C = \sigma_\alpha \otimes \sigma_\alpha$ is adjoint invariant. This algebra is non-local due to the interactions between neighboring sites in the second Poisson bracket. Nevertheless, it can be reexpressed in terms of a local algebra, more specifically, in terms of variables which span the product space of cotangent bundles of $SU(2)$. We show how to do this in Section 7.

The continuum limit of the non-abelian Toda lattice algebra (8) is the Poisson algebra defined by the brackets (3). To see this we write

$$G_n = -ia\sigma_\alpha \mathcal{I}_\alpha(x_n)/2, \quad B_n = \exp\{-ia\sigma_\alpha \mathcal{J}_\alpha(x_n)/2\}, \tag{9}$$

σ_α being the Pauli matrices, and make the identification of $\mathcal{I}_\alpha(x)$ and $\mathcal{J}_\alpha(x)$ with $I_\alpha(x)$ and $J_\alpha(x)$.

Canonical transformations are present for the Poisson brackets (8). The latter are preserved under

$$G_n \rightarrow G'_n = v_n^{-1} G_n v_n, \quad B_n \rightarrow B'_n = v_{n-1}^{-1} B_n v_n, \quad v_n \in SU(2). \tag{10}$$

These transformations correspond to $SU(2)$ gauge (or local) symmetries as we can associate an $SU(2)$ group element v_n with each link on the lattice. Furthermore, from the Poisson brackets (8) they are generated by the set of all G_n .

For the lattice dynamics we need to specify the Hamiltonian, which we denote by H_0^{lat} . We take

$$H_0^{\text{lat}} = \frac{1}{2\alpha\chi^2} \sum_n \text{Tr} (G_n^2 + B_n + B_n^\dagger), \tag{11}$$

from which we recover (up to an infinite constant) the chiral model Hamiltonian H_0 in the $a \rightarrow 0$ limit. Regarding the symmetries, we note that $\text{Tr} B_n$ and hence H_0^{lat} are not invariant under the most general canonical transformations (10) of the Poisson brackets (8) (although $\text{Tr} G_n^2$ is invariant). Rather, they are preserved only under the action of the global subgroup, i.e. where (10) is restricted by $v_1 = v_2 = \dots = v_N = v$. The generator of this subgroup is $G = \sum_{n=1}^N G_n$. It is conserved provided we take suitable boundary conditions. We can either demand that the underlying manifold on which the lattice is constructed is a circle, whereby the boundary conditions are periodic $B_{n+N} = B_n$, or we can require that $B_n \rightarrow 0$ as $n \rightarrow 1, N$ at a suitable rate. This is seen from the equations of motion which follow from (11):

$$\begin{aligned} a\dot{G}_n &= \frac{1}{2} [B_{n+1} - B_n - B_{n+1}^\dagger + B_n^\dagger]_{t\ell}, \\ a\dot{B}_n &= B_n G_n - G_{n-1} B_n, \end{aligned} \tag{12}$$

where $[A]_{t\ell}$ denotes the traceless part of 2×2 matrix A , i.e., $[A]_{t\ell} = A - \frac{1}{2} \text{Tr} (A) \times \mathbf{1}$, $\mathbf{1}$ denoting the 2×2 unit matrix. Then

$$a\dot{G} = \frac{1}{2} [B_{N+1} - B_1 - B_{N+1}^\dagger + B_1^\dagger]_{t\ell}, \tag{13}$$

which vanishes after applying the boundary conditions.

The Poisson brackets (8) and Hamiltonian (11) give a lattice formulation of the standard Hamiltonian description of the chiral model. In the next three sections we develop a lattice formulation of the alternative Hamiltonian description of the chiral model. Since from (7) it is based on the $SL(2, \mathbb{C})$ Kac–Moody algebra we must address the discretization of this algebra, which is done in the following section.

4. Discretization of the $SL(2, \mathbb{C})$ Kac–Moody algebra

Here we first write down the classical version of the lattice algebra of Alekseev, Faddeev and Semenov-Tian-Shansky [7]. It is characterized by a central charge κ which we will relate to the parameters ξ and χ appearing in the alternative Hamiltonian formulation of the chiral model. We examine the $*$ operation for this system and find a consistent algebra only for the case of real κ . We then confirm that continuum limit agrees with the $SL(2, \mathbb{C})$ Kac–Moody algebra of the alternative Hamiltonian formalism given by (7). We further show that in the limit $\kappa \rightarrow \infty$ the lattice algebra coincides with that of the non-abelian Toda lattice (8).

4.1. The classical lattice algebra

The classical version of the discretized $SL(2, \mathbb{C})$ Kac–Moody algebra [7] (also see Refs. [8,9]) is given in terms of $SL(2, \mathbb{C})$ group matrices $d_n^{(-)}$, where n again labels the lattice points. $d_n^{(-)}$ satisfy the Poisson brackets:

$$\begin{aligned} \{ d_{n_1}^{(-)}, d_{n_2}^{(-)} \} = & - \left(d_{n_1}^{(-)} d_{n_2}^{(-)} r + \tilde{r} d_{n_1}^{(-)} d_{n_2}^{(-)} \right) \delta_{n,m} \\ & + d_{n_1}^{(-)} r d_{n_2}^{(-)} \delta_{n,m-1} + d_{n_2}^{(-)} \tilde{r} d_{n_1}^{(-)} \delta_{n,m+1}. \end{aligned} \quad (14)$$

Here $d_{n_1}^{(-)}$, $d_{n_2}^{(-)}$, r and \tilde{r} denote 4×4 matrices with $d_{n_1}^{(-)} = d_n^{(-)} \otimes \mathbf{1}$, $d_{n_2}^{(-)} = \mathbf{1} \otimes d_m^{(-)}$ and r, \tilde{r} given by

$$r = \frac{i}{2\kappa} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & 4 & -1 & \\ & & & 1 \end{pmatrix}, \quad \tilde{r} = -r^T, \quad (15)$$

T denoting transpose. κ is a constant which serves the role of the central charge which we will later relate to ξ and χ . It can be checked that the Poisson brackets are skew symmetric and satisfy the Jacobi identity. In addition, $\det d_n^{(-)}$ is in the center of the algebra and hence can be set to unity. The difference of r and \tilde{r} is adjoint invariant,

$$C = -i\kappa(r - \tilde{r}). \quad (16)$$

Ignoring the interactions between neighboring sites, i.e. the last two terms in (14), the Poisson bracket relations for $d_n^{(-)}$ define the classical double algebra at every site n on the one-dimensional lattice [8,18–22]. However, due to the interactions the full space is not simply a product of classical doubles, and it is non-local. In Section 7 we shall show how to rewrite this algebra in terms of a local one and we show that the latter is related to the classical double algebra.

4.2. * Operation

Because $d_n^{(-)}$ are complex matrices, the relations (14) are insufficient for determining the entire algebra. That is, we must enlarge the algebra to include the brackets of the $d_n^{(-)}$'s with their hermitian conjugates $d_n^{(-)\dagger}$, or equivalently, with

$$d_n^{(+)} = d_n^{(-)\dagger^{-1}}. \quad (17)$$

Properties like the Jacobi identity and skew symmetry should remain intact when we make this enlargement. In addition, we require that the Poisson structure is preserved under complex conjugation. This means that if α and β are any two matrix elements of $d_n^{(-)}$ or $d_n^{(+)}$, then

$$\{\alpha, \beta\}^* = \{\alpha^*, \beta^*\}. \tag{18}$$

We have found solutions to the above requirements only in the case of real κ , which we now assume. For the brackets of the $d_n^{(-)}$'s with $d_n^{(+)}$'s we take the following:

$$\begin{aligned} \{d_{1n}^{(-)}, d_{2m}^{(+)}\} = & - \left(d_{1n}^{(-)} d_{2m}^{(+)} r + r d_{1n}^{(-)} d_{2m}^{(+)} \right) \delta_{n,m} \\ & + d_{1n}^{(-)} r d_{2m}^{(+)} \delta_{n,m-1} + d_{2m}^{(+)} r d_{1n}^{(-)} \delta_{n,m+1}. \end{aligned} \tag{19}$$

An alternative way to write these Poisson brackets is

$$\begin{aligned} \{d_{1n}^{(+)}, d_{2m}^{(-)}\} = & - \left(d_{1n}^{(+)} d_{2m}^{(-)} \tilde{r} + \tilde{r} d_{1n}^{(+)} d_{2m}^{(-)} \right) \delta_{n,m} \\ & + d_{1n}^{(+)} \tilde{r} d_{2m}^{(-)} \delta_{n,m-1} + d_{2m}^{(-)} \tilde{r} d_{1n}^{(+)} \delta_{n,m+1}. \end{aligned} \tag{20}$$

This equation is obtained from (19) using the property of skew symmetry, switching the indices m and n , as well as the order of the vector spaces in the tensor product, $1 \rightleftharpoons 2$. Upon switching the order of the vector spaces, $r \rightarrow r^T$ and $\tilde{r} \rightarrow \tilde{r}^T$. Let us verify property (18) by taking the Poisson bracket of $d_n^{(-)\dagger}$ with $d_n^{(+)\dagger}$. Using (17),

$$\begin{aligned} \{d_{1n}^{(-)\dagger}, d_{2m}^{(+)\dagger}\} = & \{d_{1n}^{(+)-1}, d_{2m}^{(-)-1}\} \\ = & d_{1n}^{(+)-1} d_{2m}^{(-)-1} \{d_{1n}^{(+)}, d_{2m}^{(-)}\} d_{1n}^{(+)-1} d_{2m}^{(-)-1} \\ = & - \left(d_{1n}^{(-)\dagger} d_{2m}^{(+)\dagger} \tilde{r} + \tilde{r} d_{1n}^{(-)\dagger} d_{2m}^{(+)\dagger} \right) \delta_{n,m} \\ & + d_{2m}^{(+)\dagger} \tilde{r} d_{1n}^{(-)\dagger} \delta_{n,m-1} + d_{1n}^{(-)\dagger} \tilde{r} d_{2m}^{(+)\dagger} \delta_{n,m+1}. \end{aligned} \tag{21}$$

By comparing (21) with (19), we see that the property (18) is satisfied and hence the Poisson structure is preserved under complex conjugation. Note that here we need κ to be real.

In addition to the relations (14) and (19), we can obtain the Poisson brackets for $d_n^{(+)}$ with $d_m^{(+)}$ by taking the complex conjugate of (14), again assuming the property (18). We find

$$\begin{aligned} \{d_{1n}^{(+)}, d_{2m}^{(+)}\} = & - \left(d_{1n}^{(+)} d_{2m}^{(+)} r + \tilde{r} d_{1n}^{(+)} d_{2m}^{(+)} \right) \delta_{n,m} \\ & + d_{1n}^{(+)} \tilde{r} d_{2m}^{(+)} \delta_{n,m-1} + d_{2m}^{(+)} r d_{1n}^{(+)} \delta_{n,m+1}. \end{aligned} \tag{22}$$

The brackets (14), (19) and (22) completely specify the Poisson structure. We note that r and \bar{r} in the first two terms can be interchanged in Eqs. (14) and (22), due to C being an adjoint invariant. It can be checked that the complete set of Poisson brackets (14), (19) and (22) are skew symmetric and satisfy the Jacobi identity, and that $\det d_n^{(\pm)}$ is in the center of the algebra.

4.3. The continuum limit

To recover the $SL(2, \mathbb{C})$ Kac–Moody algebra in the continuum limit we set

$$d_n^{(-)} = \exp \left\{ \frac{a}{\kappa} j(x_n) \right\}, \quad d_n^{(+)} = \exp \left\{ -\frac{a}{\kappa} j(x_n)^\dagger \right\}, \quad (23)$$

where a is the lattice spacing and $j(x_n)$ is the current evaluated at the lattice site x_n . Next we do an expansion in a/κ . From the Poisson brackets (14), we get

$$\begin{aligned} \left\{ j(x_n)_1, j(x_m)_2 \right\} = & -\frac{\kappa^2}{a^2} \left((r + \bar{r}) \delta_{n,m} - r \delta_{n+1,m} - \bar{r} \delta_{n-1,m} \right) \\ & - \frac{\kappa}{2a} \left([r, j(x_n)_1 - j(x_m)_2] \delta_{n+1,m} \right. \\ & \left. - [\bar{r}, j(x_n)_1 - j(x_m)_2] \delta_{n-1,m} \right) + O(1). \end{aligned} \quad (24)$$

Now taking the limit $a \rightarrow 0$,

$$\left\{ j(x)_1, j(y)_2 \right\} = -i[C, j(x)_1] \delta(x-y) + i\kappa C \partial_x \delta(x-y). \quad (25)$$

From the Poisson brackets (19),

$$\begin{aligned} \left\{ j(x_n)_1, j(x_m)_2^\dagger \right\} = & \frac{\kappa^2}{a^2} r (2\delta_{n,m} - \delta_{n+1,m} - \delta_{n-1,m}) \\ & + \frac{\kappa}{2a} [r, j(x_n)_1 + j(x_m)_2^\dagger] (\delta_{n+1,m} - \delta_{n-1,m}) + O(1). \end{aligned} \quad (26)$$

Now taking the limit $a \rightarrow 0$,

$$\left\{ j(x)_1, j(y)_2^\dagger \right\} = 0. \quad (27)$$

Finally, after substituting

$$j(x) = \tilde{I}_\alpha(x) \sigma_\alpha + i \tilde{J}_\alpha(x) \sigma_\alpha, \quad \xi = \chi^2 \kappa, \quad (28)$$

where $\tilde{I}_\alpha(x)$ and $\tilde{J}_\alpha(x)$ are real-valued currents, in (25) and (27), we get the $SL(2, \mathbb{C})$ Kac–Moody algebra (7). Here we see that we obtain only a single central term in the current algebra due to the restriction of κ being real.

Canonical symmetries are present for the $SL(2, \mathbb{C})$ Kac–Moody algebra. The Poisson brackets (7) are preserved under the global $SL(2, \mathbb{C})$ transformations

$$j(x) \rightarrow j(x)' = v j(x) v^{-1}, \quad v \in SL(2, \mathbb{C}). \tag{29}$$

4.4. The Toda lattice limit

The non-abelian Toda lattice algebra (8) results from the Poisson brackets (14), (19) and (22) in the limit of infinite central charge κ . For this we need to make the $SL(2, \mathbb{C})$ group matrices $d_n^{(\pm)}$ depend on κ in a manner which we show below. We first parametrize (locally) the $SL(2, \mathbb{C})$ group matrices $d_n^{(\pm)}$ in terms of matrices in the subgroups $SU(2)$ and $SB(2, \mathbb{C})$ (the Borel group). We denote them by B_n and $\ell_n^{(\pm)}$, respectively. We then write

$$d_n^{(\pm)} = B_n \ell_n^{(\pm)}. \tag{30}$$

From (17) it follows that

$$\ell_n^{(+)} = \ell_n^{(-)\dagger^{-1}}. \tag{31}$$

The Poisson brackets (14), (19) and (22) for $d_n^{(\pm)}$ are recovered if we make the following choice of brackets for B_n and $\ell_n^{(\pm)}$:

$$\begin{aligned} \{ B_n, B_m \} &= -[r, B_n B_m] \delta_{n,m}, \\ \{ \ell_n^{(\pm)}, \ell_m^{(\pm)} \} &= [r, \ell_n^{(\pm)} \ell_m^{(\pm)}] \delta_{n,m}, \\ \{ \ell_n^{(+)}, \ell_m^{(-)} \} &= [\tilde{r}, \ell_n^{(+)} \ell_m^{(-)}] \delta_{n,m}, \\ \{ \ell_n^{(-)}, B_m \} &= -B_m r \ell_n^{(-)} \delta_{n,m} + \ell_n^{(-)} r B_m \delta_{n,m-1}, \\ \{ \ell_n^{(+)}, B_m \} &= -B_m \tilde{r} \ell_n^{(+)} \delta_{n,m} + \ell_n^{(+)} \tilde{r} B_m \delta_{n,m-1}. \end{aligned} \tag{32}$$

Using (31), the fifth equation is the hermitian conjugate of the fourth equation. Interactions between neighboring sites occur only in these brackets, once again making this a non-local algebra. In Section 7 we show how this algebra can be reexpressed in terms of a local one.

Next we write

$$\ell_n^{(-)} = \exp \left\{ -\frac{2i}{\kappa} G_{n,\alpha} e^\alpha \right\}, \tag{33}$$

where e^α , $\alpha = 1, 2, 3$ are the generators of $SB(2, \mathbb{C})$. We choose the following representation for them and the $SU(2)$ generators which we denote by e_α :

$$e_\alpha = \frac{1}{2}\sigma_\alpha, \quad e^\alpha = \frac{1}{2}(i\sigma_\alpha + \epsilon_{\alpha\beta\gamma}\sigma_\beta).$$

In this representation e^α , and hence $\ell_n^{(-)}$, are lower triangular matrices,² and we can write $r = \frac{2}{\kappa}e^\alpha \otimes e_\alpha$ and $\tilde{r} = -\frac{2}{\kappa}e_\alpha \otimes e^\alpha$. We note here that the ordering of the $SU(2)$ and $SB(2, \mathbb{C})$ matrices in (30) appears to be important. Upon keeping $\ell_n^{(-)}$ a lower triangular matrix, we were unable to find a consistent Poisson structure for B_n and $\ell_n^{(\pm)}$ in the case where they appear in the reverse order in the definition (30) of $d_n^{(\pm)}$. On the other hand, the reverse order is necessary if instead we choose $\ell_n^{(-)}$ to be upper triangular matrices (simultaneously replacing e^α by upper triangular matrices).

Now by taking the limit $\kappa \rightarrow \infty$, the Poisson brackets (32) imply

$$\begin{aligned} \{G_{n,\alpha}, G_{m,\beta}\} &= \epsilon_{\alpha\beta\gamma}\delta_{n,m}G_{n,\gamma} + O\left(\frac{1}{\kappa}\right), \\ \{G_{n,\alpha}, B_m\} &= -iB_m e_\alpha \delta_{n,m} + i e_\alpha B_m \delta_{n,m-1} + O\left(\frac{1}{\kappa}\right), \\ \{B_n, B_m\} &= O\left(\frac{1}{\kappa}\right). \end{aligned} \tag{34}$$

By defining $G_n = -i\chi^2 G_{n,\alpha}\sigma_\alpha$, χ , the zeroth-order terms in (34) can be written according to (8). We have thus recovered the algebra of the non-abelian Toda lattice.

We note that the continuum limit of the non-abelian Toda lattice cannot be described in terms of the currents \tilde{I}_α and \tilde{J}_α as the $SL(2, \mathbb{C})$ algebra (7) is ill-defined when $\kappa \rightarrow \infty$. On the other hand, when κ is finite we cannot identify $\mathcal{I}_\alpha(x)$ and $\mathcal{J}_\alpha(x)$ appearing in (9) with the currents $I_\alpha(x)$ and $J_\alpha(x)$ as we did for the non-abelian Toda lattice. From (6) and (23), $d_n^{(-)}$ can be written

$$d_n^{(-)} = \exp \left\{ a e_\alpha \frac{\frac{1}{\chi^2\kappa} I_\alpha(x_n) - i J_\alpha(x_n)}{1 + \frac{1}{\chi^4\kappa^2}} \right\}, \tag{35}$$

while (30), (33) and (9) imply

$$d_n^{(-)} = \exp \{-i a e_\alpha \mathcal{J}_\alpha(x_n)\} \exp \left\{ -i \frac{a}{\chi^2\kappa} e^\alpha \mathcal{I}_\alpha(x_n) \right\}. \tag{36}$$

By comparing (36) and (35) we in general get a complicated expression for the continuum variables I_α and J_α in terms of \mathcal{I}_α and \mathcal{J}_α .

² It can be checked that the Poisson brackets (32) are consistent with setting the matrix element $[\ell_n^{(-)}]_{12} = 0$.

5. Lie–Poisson symmetry

We now discuss the symmetry properties for the above Poisson structure, i.e. the lattice algebra defined by the Poisson brackets (14), (19) and (22). When we took two different limits ($a \rightarrow 0$ and $\kappa \rightarrow \infty$) of this algebra we obtained two different Poisson bracket algebras (7) and (8) both of which, as we saw, admit canonical transformations. They were given by (10) for the case of the non-abelian Toda theory and (29) for the case of the Kac–Moody algebra. Although the $a \rightarrow 0$ and $\kappa \rightarrow \infty$ limits produce systems with canonical symmetries, no such canonical symmetries exist for this algebra before taking these limits. The Poisson bracket algebra defined by (14), (19) and (22) instead admits Lie–Poisson transformations. They are the classical analogues of quantum group transformations, and are defined as follows [14]:

Let \mathcal{S} denote the space of symmetries. Unlike theories containing canonical symmetries, \mathcal{S} carries a Poisson structure $\{ , \}_\mathcal{S}$. Let \mathcal{O} denote the space of classical observables, which for us is spanned by the matrices $d_n^{(\pm)}$, whose Poisson structure is $\{ , \}_\mathcal{O}$. \mathcal{S} acting on \mathcal{O} defines a map, $\sigma : \mathcal{S} \times \mathcal{O} \rightarrow \mathcal{O}$. \mathcal{S} induces a Lie–Poisson action on \mathcal{O} if σ is a Poisson map, which means that if f_1 and f_2 are functions on \mathcal{O} , then

$$\sigma \circ \{f_1, f_2\}_\mathcal{O} = \{ \sigma \circ f_1, \sigma \circ f_2 \}_{\mathcal{O} \times \mathcal{S}}, \tag{37}$$

where the product Poisson structure is assumed on $\mathcal{O} \times \mathcal{S}$, which means that the symmetry variables have zero Poisson brackets with the classical observables. (For convenience of notation we shall drop the subscripts \mathcal{S} and \mathcal{O} on the Poisson brackets in the discussion below.)

For the symmetry transformations of the classical observables $d_n^{(-)}$ we take

$$d_n^{(-)} \rightarrow d_n^{(-)'} = w_n^{(-)} d_n^{(-)} v_n^{(-)}, \quad w_n^{(-)}, v_n^{(-)} \in SL(2, \mathbb{C}). \tag{38}$$

Thus $w_n^{(-)}$ and $v_n^{(-)}$ span \mathcal{S} and (38) defines the map σ . The matrices $w_n^{(-)}$ and $v_n^{(-)}$ have a non-trivial Poisson structure. If we choose

$$\begin{aligned} \{ v_{n-1}^{(-)}, v_n^{(-)} \} &= [r, v_{n-1}^{(-)} v_n^{(-)}] \delta_{n,m}, \\ \{ w_{n-1}^{(-)}, w_n^{(-)} \} &= -[\tilde{r}, w_{n-1}^{(-)} w_n^{(-)}] \delta_{n,m}, \\ \{ v_{n-1}^{(-)}, w_n^{(-)} \} &= (v_{n-1}^{(-)} r w_n^{(-)} - w_n^{(-)} r v_{n-1}^{(-)}) \delta_{n,m-1}, \end{aligned} \tag{39}$$

then, as we show below, (38) defines a Poisson map, and hence a Lie–Poisson symmetry. Here we assume that $w_n^{(-)}$ and $v_n^{(-)}$ have zero Poisson brackets with the observables $d_n^{(\pm)}$, so that we get a product Poisson structure on $\mathcal{O} \times \mathcal{S}$. The first Poisson bracket in (39) is compatible with group multiplication, which implies that $\{v_n\}$ for every n defines a Lie–Poisson group. The remaining Poisson brackets in (39) can be obtained from the first upon setting

$$w_n^{(-)} = v_{n-1}^{(-)-1}. \tag{40}$$

After making such a restriction we see that the $d_n^{(-)}$ variables transform analogously to the Toda lattice variables B_n (10) (only here the symmetry parameters $v_n^{(-)}$ and the observables $d_n^{(-)}$ span $SL(2, \mathbb{C})$ rather than $SU(2)$, and the transformation is Lie-Poisson rather than canonical). Since we can associate a different group element $v_n^{(-)}$ with each link on the lattice, (38) correspond to gauge transformations.

To check that (38) is a Poisson map we note that the left-hand side of (14) transforms to

$$\begin{aligned} \{d_{n1}^{(-)'} , d_{m2}^{(-)'}\} &= \{w_{n1}^{(-)} d_{n1}^{(-)} v_{n1}^{(-)} , w_{m2}^{(-)} d_{m2}^{(-)} v_{m2}^{(-)}\} \\ &= \{w_{n1}^{(-)} , w_{m2}^{(-)}\} d_{n1}^{(-)} d_{m2}^{(-)} v_{n1}^{(-)} v_{m2}^{(-)} \\ &\quad + w_{n1}^{(-)} w_{m2}^{(-)} d_{n1}^{(-)} d_{m2}^{(-)} \{v_{n1}^{(-)} , v_{m2}^{(-)}\} \\ &\quad + w_{n1}^{(-)} d_{n1}^{(-)} \{v_{n1}^{(-)} , w_{m2}^{(-)}\} d_{m2}^{(-)} v_{m2}^{(-)} \\ &\quad + w_{m2}^{(-)} d_{m2}^{(-)} \{w_{n1}^{(-)} , v_{m2}^{(-)}\} d_{n1}^{(-)} v_{n1}^{(-)} \\ &\quad + w_{n1}^{(-)} w_{m2}^{(-)} \{d_{n1}^{(-)} , d_{m2}^{(-)}\} v_{n1}^{(-)} v_{m2}^{(-)}. \end{aligned} \tag{41}$$

Using (14) and (39) we then obtain

$$-\left(d_{n1}^{(-)'} d_{m2}^{(-)'} r + \tilde{r} d_{n1}^{(-)'} d_{m2}^{(-)'}\right) \delta_{n,m} + d_{n1}^{(-)'} r d_{m2}^{(-)'} \delta_{n,m-1} + d_{m2}^{(-)'} \tilde{r} d_{n1}^{(-)'} \delta_{n,m+1} \tag{42}$$

which is how the right-hand side of (14) transforms under (38). Hence (38) is a Poisson map.

From the * operation we note that the $d_n^{(+)}$ variables transform according to

$$d_n^{(+)} \rightarrow d_n^{(+)'} = w_n^{(+)} d_n^{(+)} v_n^{(+)}, \tag{43}$$

where the (+) symmetry parameters (i.e. $w_n^{(+)}$ and $v_n^{(+)}$) are obtained from the (-) symmetry parameters (i.e. $w_n^{(-)}$ and $v_n^{(-)}$) via the operations of inverse and conjugation, i.e. $w_n^{(+)} = w_n^{(-)\dagger^{-1}}$ and $v_n^{(+)} = v_n^{(-)\dagger^{-1}}$. We can deduce the Poisson structure for all of the variables $w_n^{(\pm)}$ and $v_n^{(\pm)}$ by again demanding that the transformation is a Poisson map. For this we examine how the left- and right-hand sides of (19) transform under (38) and (43). The appropriate Poisson brackets between the (-) and (+) symmetry parameters are

$$\begin{aligned}
 \{v_n^{(-)}, v_m^{(+)}\} &= [r, v_n^{(-)} v_m^{(+)}] \delta_{n,m}, \\
 \{w_n^{(-)}, w_m^{(+)}\} &= -[r, w_n^{(-)} w_m^{(+)}] \delta_{n,m}, \\
 \{v_n^{(-)}, w_m^{(+)}\} &= (v_n^{(-)} r w_m^{(+)} - w_m^{(+)} r v_n^{(-)}) \delta_{n,m-1}, \\
 \{w_n^{(-)}, v_m^{(+)}\} &= (v_m^{(+)} r w_n^{(-)} - w_n^{(-)} r v_m^{(+)}) \delta_{n,m+1},
 \end{aligned}
 \tag{44}$$

while the Poisson brackets between the (+) and (+) symmetry parameters are obtained by taking the conjugate inverse of (39) and assuming the property (18). The last three equations in (44) follow from the first if we once again apply (40) (which then also implies $w_n^{(+)} = v_{n-1}^{(+)-1}$).

Finally, we remark about the generators of the Lie–Poisson transformations. It is in general known that the charges associated with such transformations are group-valued [23]. If we limit our discussion to $SU(2)$ transformations, then we can set $v_n^{(-)} = v_n^{(+)} = v_n \in SU(2)$, and $d_n^{(\pm)}$ transforms according to

$$d_n^{(\pm)} \rightarrow d_n^{(\pm)'} = v_{n-1}^{-1} d_n^{(\pm)} v_n, \quad v_n \in SU(2).
 \tag{45}$$

The generators of these transformations are the set of all $SB(2, \mathbb{C})$ matrices $\ell_n^{(-)}$. To see this we can compute their Poisson brackets with the variables $d_n^{(\pm)}$. We find

$$\begin{aligned}
 \ell_n^{(-)-1} \{ \ell_n^{(-)}, d_m^{(\pm)} \} &= r d_m^{(\pm)} \delta_{n,m-1} - d_m^{(\pm)} r \delta_{n,m} \\
 &= \frac{2}{\kappa} e^\alpha \otimes [e_\alpha d_m^{(\pm)} \delta_{n,m-1} - d_m^{(\pm)} e_\alpha \delta_{n,m}].
 \end{aligned}
 \tag{46}$$

From the brackets [] on the right-hand side of (46) we can construct infinitesimal $SU(2)$ gauge transformations analogous to (45). In this way $\ell_n^{(-)}$ generate the Lie–Poisson transformations. We note using (33), that $\ell_n^{(-)}$ contain the generators G_n of the canonical transformations (10) at first order in the expansion parameter $1/\kappa$. Thus the Lie–Poisson transformations (45) correspond to a deformation of the canonical symmetries of the non-abelian Toda lattice.

Just as $SU(2)$ transformations are generated by the $SB(2, \mathbb{C})$ matrices, $SB(2, \mathbb{C})$ transformations analogous to (45) are generated by the $SU(2)$ matrices B_n .

6. Lattice dynamics

It remains to write down the lattice Hamiltonian associated with the Poisson brackets (14), (19) and (22). It should give $H_{1/\xi}$ in the continuum limit, so that we recover the alternative formulation of the chiral model. We shall also require it to reduce to the

non-abelian Toda lattice Hamiltonian (11) when $\xi \rightarrow \infty$. Both of these requirements are satisfied for

$$H_{1/\xi}^{\text{lat}} = -\frac{1}{4a\chi^2} \sum_n \left((\xi^2 + 1) \text{Tr} d_n^{(-)} d_n^{(-)\dagger} - 2 \text{Tr} (d_n^{(-)} + d_n^{(-)\dagger}) \right). \tag{47}$$

The Toda Hamiltonian (11) is recovered using

$$\begin{aligned} d_n^{(-)} &\rightarrow B_n, \\ \text{Tr} d_n^{(-)} d_n^{(-)\dagger} &\rightarrow -\frac{2}{\kappa^2 \chi^4} \text{Tr} G_n^2 + 2, \quad \text{as } \kappa \rightarrow \infty, \end{aligned} \tag{48}$$

while the continuum Hamiltonian (4) is recovered using

$$\begin{aligned} \text{Tr} (d_n^{(-)} + d_n^{(-)\dagger}) &\rightarrow \frac{2a^2}{\kappa^2} \left(\tilde{I}_\alpha(x_n) \tilde{I}_\alpha(x_n) - \tilde{J}_\alpha(x_n) \tilde{J}_\alpha(x_n) \right), \\ \text{Tr} d_n^{(-)} d_n^{(-)\dagger} &\rightarrow \frac{4a^2}{\kappa^2} \tilde{I}_\alpha(x_n) \tilde{I}_\alpha(x_n), \quad \text{as } a \rightarrow 0. \end{aligned} \tag{49}$$

In Section 3, we saw that the Toda lattice Hamiltonian, H_0^{lat} was not invariant under the most general canonical transformation (10). Similarly, $H_{1/\xi}^{\text{lat}}$ is not invariant under the general Lie–Poisson transformations (38) (where we are assuming (40)). Rather, the dynamics is preserved only under the action of the global $SU(2)$ subgroup. Invariance of the linear terms in $d_n^{(-)}$ implies that $v_1^{(\pm)} = v_2^{(\pm)} = \dots = v_N^{(\pm)} = v^{(\pm)}$, N being the total number of links, while invariance of the quadratic terms further restricts $v^{(-)} = v^{(+)} = v \in SU(2)$. On the other hand, the term $\text{Tr} d_n^{(-)} d_n^{(-)\dagger}$ alone is invariant under local $SU(2)$ transformations generated by $\ell_m^{(-)}$. Therefore, $\ell_m^{(-)}$ has zero Poisson brackets with the quadratic term in the Hamiltonian. Its Hamilton equations of motion are then determined by the linear terms in (47) using the Poisson brackets (46). We find

$$a\chi^2 \kappa \dot{\ell}_n^{(-)} \ell_n^{(-)-1} = e^\alpha \text{Tr} e_\alpha (d_{n+1}^{(-)} - d_n^{(-)} - d_{n+1}^{(-)\dagger} + d_n^{(-)\dagger}). \tag{50}$$

From this equation we are unable to construct the conserved charges associated with the global symmetry purely from the $\ell_n^{(-)}$ matrices. Thus we do not have the analogue of the conserved generators G of the non-abelian Toda lattice.

The equations of motion for $d_n^{(-)}$ are a bit more complicated. From the Poisson brackets (14) and (19) we find

$$\begin{aligned} \left\{ d_n^{(-)}, \sum_m \text{Tr} d_m^{(-)} \right\} &= \frac{2}{\kappa} \left\{ d_n^{(-)} e^\alpha \text{Tr} e_\alpha (d_{n+1}^{(-)} - d_n^{(-)}) \right. \\ &\quad \left. + e^\alpha d_n^{(-)} \text{Tr} e^\alpha (d_n^{(-)} - d_{n-1}^{(-)}) \right\}, \\ \left\{ d_n^{(-)}, \sum_m \text{Tr} d_m^{(-)\dagger} \right\} &= \frac{2}{\kappa} \left\{ -d_n^{(-)} e^\alpha \text{Tr} e_\alpha (d_{n+1}^{(-)} - d_n^{(-)})^\dagger \right. \\ &\quad \left. + e^\alpha d_n^{(-)} \text{Tr} e_\alpha (d_n^{(-)} - d_{n-1}^{(-)})^\dagger \right\}, \end{aligned}$$

$$\{d_n^{(-)}, \sum_m \text{Tr} d_m^{(-)} d_m^{(-)\dagger}\} = \frac{2i}{\kappa} [d_n^{(-)} d_n^{(-)\dagger} - d_{n-1}^{(-)\dagger} d_{n-1}^{(-)}]_{t\ell} d_n^{(-)}. \quad (51)$$

The equations of motion, which follows from the Hamiltonian (47), are then

$$\begin{aligned} \alpha \chi^2 \kappa d_n^{(-)} d_n^{(-)-1} &= -\frac{i}{2} (1 + \chi^4 \kappa^2) [d_n^{(-)} d_n^{(-)\dagger} - d_{n-1}^{(-)\dagger} d_{n-1}^{(-)}]_{t\ell} \\ &+ e_\alpha \text{Tr} e^\alpha (d_n^{(-)} - d_{n-1}^{(-)}) + e^\alpha \text{Tr} e_\alpha (d_n^{(-)} - d_{n-1}^{(-)})^\dagger \\ &+ d_n^{(-)} e^\alpha d_n^{(-)-1} \text{Tr} e_\alpha (d_{n+1}^{(-)} - d_n^{(-)} - d_{n+1}^{(-)\dagger} + d_n^{(-)\dagger}). \end{aligned} \quad (52)$$

7. Application to two-dimensional lattice gauge theories

We saw above that the lattice descriptions of the chiral model do not fully utilize the symmetries (be they canonical or Lie-Poisson) of the Poisson brackets, as these are gauge symmetries. The symmetry breaking was due to the presence of linear terms in the two different Hamiltonians (11) and (47). On the other hand, if we only keep quadratic-like terms in the Hamiltonians, we can construct lattice gauge theories, which is the purpose of this section. For the two resulting systems, the relevant gauge group will be $SU(2)$, although it is implemented as canonical transformations in one system and Lie-Poisson transformations in the other.

The restoration of the gauge symmetry (10) or (38) will mean that we will be left with a trivial theory because we can eliminate all but a few degrees of freedom. With regard to the canonically invariant theory defined by the Poisson brackets (8), recall that G_n appearing in (11) are the generators of the gauge symmetries which here are implemented as canonical transformations (10). We must therefore impose the Gauss law constraints

$$G_n = 0, \quad (53)$$

at all lattice sites. Furthermore, we can use the gauge symmetry (10) to eliminate the degrees of freedom in B_n . If we again assume the periodic boundary conditions $B_{n+N} = B_n$ then this can be done everywhere except at one lattice site.

Because of the constraints (53) the quadratic term in the Hamiltonian (11) vanishes. To recover two-dimensional lattice QCD we need to reexpress the non-abelian Toda lattice variables in terms of another set of variables. These new variables define a local algebra, specifically $[T^*SU(2)]^{\otimes N}$. Here we associate a cotangent bundle of $SU(2)$ with each point on the lattice. The new variables are a set of traceless antihermitian matrices E_n which play the role of the electric field along the links of the lattice and generate right $SU(2)$ transformations at lattice site n . Their Poisson brackets can be written

$$\frac{2}{\chi^2} \{E_n, E_m\} = [C, E_m] \delta_{n,m}, \quad \frac{2}{\chi^2} \{E_n, B_m\} = -B_m C \delta_{n,m}. \quad (54)$$

These equations along with the last equation in (8) define the cotangent bundle of $SU(2)$ at each lattice site n . For a given n , E_n and B_n span the six-dimensional phase space of the rigid rotor. To recover the first two equations in (8) from (54) we can set

$$G_n = E_n - B_{n+1} E_{n+1} B_{n+1}^\dagger. \tag{55}$$

The electric fields E_n undergo the following canonical transformations:

$$E_n \rightarrow E'_n = v_n^{-1} E_n v_n, \tag{56}$$

which together with (10) preserve the Poisson brackets (54). From the definition (55) of G_n , the Poisson brackets of G_n with E_m are given by

$$\frac{2}{\chi^2} \{ E_n, G_m \} = [C, E_m] \delta_{n,m}, \tag{57}$$

from which it again follows that G_n are the generators of the canonical transformations.

The electric fields E_n are subject to the constraints (53) (where G_n is expressed in terms of E_n using (55)). The two-dimensional version of the Kogut–Susskind formulation [24] of lattice QCD is recovered by choosing the Hamiltonian to be

$$H_{KS} = \sum_{n=1}^N \text{Tr } E_n^2. \tag{58}$$

With regard to the Lie–Poisson invariant theory defined by Poisson brackets (14), (19) and (22), if we restrict the gauge group to $SU(2)$, then the analog of the Gauss law constraint is

$$\ell_n^{(-)} = \mathbf{1}, \tag{59}$$

at all lattice sites. Using (33) we recover (53) from (59) in the limit $\kappa \rightarrow \infty$. The remaining degrees of freedom B_n in $d_n^{(-)}$ can be eliminated using the gauge transformation (45) (except at one lattice site, if we assume the periodic boundary conditions $d_{N+n}^{(-)} = d_n^{(-)}$).

In defining the Hamiltonian for this system, if we want to make a connection with two-dimensional QCD, quadratic terms like those appearing in (47) are unsuitable for this purpose. This is because they can be expressed solely in terms of the generators $\ell_n^{(-)}$ of the Lie–Poisson transformation (45). For this we note that

$$\sum_{n=1}^N \text{Tr } d_n^{(-)} d_n^{(-)\dagger} = \sum_{n=1}^N \text{Tr } \ell_n^{(-)} \ell_n^{(-)\dagger}. \tag{60}$$

Consequently, such terms are trivial due to (59). As was true in the Kogut–Susskind theory, we can express the dynamical variables, here $\ell_n^{(-)}$ and B_n , in terms of new variables which span a local algebra. We can also write the Hamiltonian in terms of these

variables. The resulting system is a Lie–Poisson deformation of the Kogut–Susskind formulation of gauge theories and it has been examined previously in Refs. [16,17].

In analogy to (55), we set

$$\ell_n^{(-)} = k_n^{(-)} \tilde{k}_{n+1}^{(-)}, \tag{61}$$

where $k_n^{(-)}$ and $\tilde{k}_n^{(-)}$ are $SB(2, \mathbb{C})$ matrices, which are analogous to the electric fields E_n in the Kogut–Susskind system. The algebra (32) for $\ell_n^{(-)}$ and B_n is now recovered from the local algebra

$$\begin{aligned} \{ k_{n1}^{(-)}, k_{m2}^{(-)} \} &= [r, k_{n1}^{(-)} k_{m2}^{(-)}] \delta_{n,m}, \\ \{ \tilde{k}_{n1}^{(-)}, \tilde{k}_{m2}^{(-)} \} &= [r, \tilde{k}_{n1}^{(-)} \tilde{k}_{m2}^{(-)}] \delta_{n,m}, \\ \{ k_{n1}^{(-)}, \tilde{k}_{m2}^{(-)} \} &= 0, \\ \{ k_{n1}^{(-)}, B_{m2} \} &= -B_{m2} r k_{n1}^{(-)} \delta_{n,m}, \\ \{ \tilde{k}_{n1}^{(-)}, B_{m2} \} &= \tilde{k}_{n1}^{(-)} r B_{m2} \delta_{n,m}. \end{aligned} \tag{62}$$

If we also assume the Poisson brackets

$$\{ k_{n1}^{(+)}, k_{m2}^{(-)} \} = [\tilde{r}, k_{n1}^{(+)} k_{m2}^{(-)}] \delta_{n,m}, \tag{63}$$

where $k_n^{(+)} = k_n^{(-)\dagger^{-1}}$, then we can show that

$$\sum_{n=1}^N \text{Tr} k_n^{(-)} k_n^{(-)\dagger}$$

is gauge invariant, i.e. it has zero Poisson brackets with the gauge generators $\ell_n^{(-)}$. Unlike (60), it is not trivial due to the constraints and it can be taken to be the Hamiltonian of the system. If we further set

$$k_n^{(-)} = \exp \left\{ -\frac{2i}{\kappa} E_{n,\alpha} e^\alpha \right\}, \tag{64}$$

we can recover (up to factors and an infinite additive constant) the Kogut–Susskind Hamiltonian H_{KS} in the $\kappa \rightarrow \infty$ limit. The resulting deformation of the Kogut–Susskind formulation of lattice gauge theories was examined previously (in two, three and four dimensions) in Refs. [16,17].

Above we have seen that by writing $d_n^{(-)} = B_n k_n^{(-)} \tilde{k}_{n+1}^{(-)}$, the non-local algebra of [7] can be expressed entirely in terms of a local algebra. The latter is given by (62),

along with the first Poisson bracket in (32). From [21], we have that $(B_n, k_n^{(-)})$ and $(B_n, \tilde{k}_n^{(-)})$ define two different parametrizations of the classical double algebra (for each n), one where the double variable is written as the product $B_n k_n^{(-)}$, and the other where the double variable is written as the product $\tilde{k}_n^{(-)} B_n$.

8. Conclusion

We have shown how to apply the current algebra of Ref. [7] in order to get a new lattice description of the chiral model. As this current algebra admits Lie–Poisson symmetries, so does the new lattice description of the chiral model. The Lie–Poisson symmetries get promoted to quantum group symmetries upon quantization. The quantum mechanical commutation relations for the operators analogous to $d_n^{(-)}$ are known [7], while those for the operators analogous to $d_n^{(+)}$ are readily obtained from their Poisson brackets. To get the quantum mechanical Hamiltonian one basically only needs to replace the traces in (47) with deformed traces. (See for example Ref. [21].)

The above lattice description of the chiral model is possible because the continuum limit of the lattice current algebra is the same as the algebra appearing in the alternative Hamiltonian description of Ref. [11], i.e. it is the $SL(2, \mathbb{C})$ Kac–Moody algebra. In Refs. [12,6] this alternative Hamiltonian description was generalized to the case of the chiral model with a Wess–Zumino term (whose coefficient was arbitrary). The latter also relied upon the $SL(2, \mathbb{C})$ Kac–Moody algebra, only here, unlike in (7), both central terms were required. To include the Wess–Zumino term, we need to generalize (7) to³

$$\begin{aligned} \{\tilde{I}_\alpha(x), \tilde{I}_\beta(y)\} &= -\{\tilde{J}_\alpha(x), \tilde{J}_\beta(y)\} = \epsilon_{\alpha\beta\gamma} \tilde{I}_\gamma(x) \delta(x-y) + \xi' \epsilon_{\alpha\beta\gamma} \tilde{J}_\gamma(x) \delta(x-y), \\ \{\tilde{I}_\alpha(x), \tilde{J}_\beta(y)\} &= \epsilon_{\alpha\beta\gamma} \tilde{J}_\gamma(x) \delta(x-y) - \xi' \epsilon_{\alpha\beta\gamma} \tilde{I}_\gamma(x) \delta(x-y) + \frac{\xi}{2\chi^2} \delta_{\alpha\beta} \partial_x \delta(x-y), \end{aligned} \tag{65}$$

where ξ' is real. If we now define the complex current $j(x)$ according to

$$j(x) = \frac{\tilde{I}_\alpha(x) \sigma_\alpha + i \tilde{J}_\alpha(x) \sigma_\alpha}{1 - i \xi'}, \tag{66}$$

instead of (28), we recover the algebra given by (25) and (27) where κ is now complex:

$$\kappa = \frac{\xi}{\chi^2} (1 - i \xi'). \tag{67}$$

This is the most general $SL(2, \mathbb{C})$ Kac–Moody algebra. If we want to obtain it as the continuum limit of a lattice current algebra we need κ in (14) to be complex. Thus far, we have not been successful in finding a consistent algebra for this case, i.e. one that

³ Now in comparing with Ref. [12], the parameters ρ and τ of that reference are given by $\rho = -\xi \xi'$ and $\tau^2 = -\xi'^{-2}$.

satisfies (18), and we therefore have been unable to generalize our system in order to get a new lattice description of the chiral model with a Wess–Zumino term.

Of course another concern is the question of integrability. The conserved charges for the two-dimensional chiral model are well known. Upon going to the lattice, a Lax pair construction can be made using the Toda model description if one works with the general linear group, rather than say $SU(2)$ [2]. However, this construction is not readily adaptable to other cases. It may be that neither of the models presented here are integrable for arbitrary groups. However, the Hamiltonian systems on the lattice are not unique, and further study may yield solvable models.

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References

- [1] J.M. Maillet, Phys. Lett. B 162 (1985) 137; B 167 (1986) 402; Nucl. Phys. B 269 (1986) 54.
- [2] V.E. Korepin, J. Sov. Math. 23 (1983) 2429;
L. Freidel and J.M. Maillet, Phys. Lett. B 262 (1991) 278; B 263 (1991) 403.
- [3] L. Hlavaty, J. Math. Phys. 36 (1995) 4883;
L. Hlavaty and A. Kundu, Int. J. Mod. Phys. A 11 (1996) 2143.
- [4] M.A. Semenov-Tian-Shansky and A. Sevostyanov, Classical and Quantum Non-ultralocal Systems on the Lattice, hep/th-9509029.
- [5] O. Babelon and L. Bonora, Phys. Lett. B 253 (1991) 365.
- [6] S.G. Rajeev, A. Stern and P. Vitale, Phys. Lett. B 388 (1996) 769.
- [7] A. Alekseev, L. Faddeev and M. Semenov-Tian-Shansky, Commun. Math. Phys. 149 (1992) 335.
- [8] M.A. Semenov-Tian-Shansky, Publ. RIMS, Kyoto University 21, no. 6 (1985) 1237; Theor. Math. Phys. 93 (1992) 312 (in Russian).
- [9] N. Reshetikhin and M.A. Semenov-Tian-Shansky, Lett. in Math. Phys. 19 (1990) 133.
- [10] L. Faddeev and A. Volkov, Shift Operator for Non-abelian Lattice Current Algebra, hep-th/9606088.
- [11] S.G. Rajeev, Phys. Lett. B 217 (1989) 123.
- [12] S.G. Rajeev, G. Sparano and P. Vitale, Int. J. Mod. Phys. A 31 (1994) 5469.
- [13] A. Sevostyanov, Int. J. Mod. Phys. A 11 (1996) 4241.
- [14] For a review, see Chari and Pressley, A Guide to Quantum Groups (Cambridge Univ. Press, 1994).
- [15] A. Alekseev and S. Shatashvili, Commun. Math. Phys. 133 (1990) 353.
- [16] S.A. Frolov, Mod. Phys. Lett. A 10 (1995) 2619; 2885; Physical phase space of lattice Yang–Mills theory and the moduli space of flat connections on a Riemann surface, Munich preprint LMU-TPW 95-17.
- [17] G. Bimonte, A. Stern and P. Vitale, Phys. Rev. D 54 (1996) 1054.
- [18] V.G. Drinfel'd, Sov. Math. Doklady 27 (1983) 68; in Proc. Int. Congr. Math. (Berkeley), vol. 1 (Academic Press, New York, 1986).
- [19] S. Majid, Pacific J. Math. 141 (1990) 311.
- [20] A. Yu. Alekseev and A.Z. Malkin, Paris preprint PAR- LPTHE 93-08.
- [21] G. Marmo, A. Simoni and A. Stern, Int. J. Mod. Phys. A 10 (1995) 99;
A. Stern and I. Yakushin, Mod. Phys. Lett. A 10 (1995) 399.
- [22] A. Yu. Alekseev and L.D. Faddeev, Commun. Math. Phys. 162 (1993) 147.
- [23] J. Lu and A. Weinstein, J. Diff. Geom. 31 (1990) 501.
- [24] J. Kogut and L. Susskind, Phys. Rev. D 11 (1975) 395.