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Research Article

$(n - 1)$ -Step Derivations on n -Groupoids: The Case $n = 3$

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We define a ranked trigroupoid as a natural followup on the idea of a ranked bigroupoid. We consider the idea of a derivation on such a trigroupoid as representing a two-step process on a pair of ranked bigroupoids where the mapping d is a self-derivation at each step. Following up on this idea we obtain several results and conclusions of interest. We also discuss the notion of a couplet (D, d) on X , consisting of a two-step derivation d and its square $D = d \circ d$, for example, whose defining property leads to further observations on the underlying ranked trigroupoids also.

1. Introduction

The notion of derivations arising in analytic theory is extremely helpful in exploring the structures and properties of algebraic systems. Several authors [1, 2] studied derivations in rings and near rings. Jun and Xin [3] applied the notion of derivation in ring and near ring theory to BCI-algebras. In [4], the concept of derivation for lattices was introduced and some of its properties are investigated. For more details, the reader is referred to [3, 5–7].

Iséki and Tanaka introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [8, 9]. Neggers and Kim introduced the notion of d -algebras which is another useful generalization of BCK-algebras and then investigated several relations between d -algebras and BCK-algebras as well as several other relations between d -algebras and oriented digraphs [10]. Kim and Neggers [11] introduced the notion of $\text{Bin}(X)$ and obtained a semigroup structure. Bell and Kappe [1] studied rings in which derivations satisfy certain algebraic conditions. Alshehri [12] applied the notion of derivations in incline algebras.

The present authors [13] introduced the notion of ranked bigroupoids and discussed $(X, *, \omega)$ -self-(co)derivations. In addition, they defined rankomorphisms and $(X, *, \omega)$ -scalars for ranked bigroupoids and obtained some properties of these as well. Recently, Jun et al. [14] obtained further results

on derivations of ranked bigroupoids, and Jun et al. [15] introduced the notion of generalized coderivations in ranked bigroupoids and showed that new generalized coderivations of ranked bigroupoids are obtained by combining a generalized self-coderivation with a rankomorphism.

In this paper, we extend the theory of derivations on a ranked bigroupoid to that of a type of derivation on ranked trigroupoids, that is, two-step derivations on ranked trigroupoids $(X, *, \bullet, \diamond)$ considered as a couple of ranked bigroupoids $(X, *, \bullet)$ and (X, \bullet, \diamond) with $d : X \rightarrow X$ such a two-step derivation on $(X, *, \bullet, \diamond)$ if it is a self-derivation on both $(X, *, \bullet)$ and (X, \bullet, \diamond) . The role of the operation \bullet in this definition is the more interesting one since it acts as the minor operation in $(X, *, \bullet)$ and the major operation in (X, \bullet, \diamond) . From the results obtained below it is clear that it is indeed possible to obtain meaningful insights, especially via the notion of a couplet (D, d) on a ranked trigroupoid consisting of a pair of mappings $D, d : X \rightarrow X$ satisfying a natural condition (6) stated below which arises in a rather natural way from the context and is seen to be of interest in this study and presumably of any followup as well.

2. Preliminaries

An d -algebra [10] is a nonempty set X with a constant 0 and a binary operation “ $*$ ” satisfying the following axioms:

- (A) $x * x = 0$,
- (B) $0 * x = 0$,
- (C) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A BCK-algebra is a d -algebra X satisfying the following additional axioms:

- (D) $((x * y) * (x * z)) * (z * y) = 0$,
- (E) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Given a nonempty set X , we let $\text{Bin}(X)$ denote the collection of all groupoids $(X, *)$, where $*$: $X \times X \rightarrow X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and (X, \bullet) of $\text{Bin}(X)$, define a product “ \square ” on these groupoids as follows:

$$(X, *) \square (X, \bullet) = (X, \square), \tag{1}$$

where

$$x \square y = (x * y) \bullet (y * x) \tag{2}$$

for any $x, y \in X$. Using that notion, Kim and Neggers proved the following theorem.

Theorem 1 (see [11]). *$(\text{Bin}(X), \square)$ is a semigroup; that is, the operation “ \square ” as defined in general is associative. Furthermore, the left-zero-semigroup is the identity for this operation.*

A ranked bigroupoid is an algebraic system $(X, *, \bullet)$ where X is a nonempty set and “ $*$ ” and “ \bullet ” are binary operations defined on X . We may consider the first binary operation $*$ as the *major operation*, and the second binary operation \bullet as the *minor operation*.

Example 2 (see [16]). A K -algebra is defined as an algebraic system (G, \bullet, \odot) where (G, \bullet) is a group and where $x \odot y := x \bullet y^{-1}$. Hence every K -algebra is a ranked bigroupoid.

Example 3 (see [13]). We construct a ranked bigroupoid from any BCK-algebra. In fact, given a BCK-algebra $(X, *, 0)$, if we define a binary operation “ \wedge ” on X by $x \wedge y := x * (x * y)$ for any $x, y \in X$, then $(X, *, \wedge)$ is a ranked bigroupoid.

We introduce the notion of “ranked bigroupoids” to distinguish two bigroupoids $(X, *, \bullet)$ and $(X, \bullet, *)$. Even though $(X, *, \bullet) = (X, \bullet, *)$ in the sense of bigroupoids, the same is not true in the sense of ranked bigroupoids. This is analogous to the situation for sets, where $\{x, y\} = \{y, x\}$ but $\langle x, y \rangle \neq \langle y, x \rangle$ in general.

Given an element $(X, *) \in \text{Bin}(X)$, $(X, *)$ has a natural associated ranked bigroupoid $(X, *, *)$; that is, the major operation and the minor operation coincide.

Given a ranked bigroupoid $(X, *, \omega)$, a map $d : X \rightarrow X$ is called an $(X, *, \omega)$ -self-derivation if for all $x, y \in X$,

$$d(x * y) = (d(x) * y) \omega (x * d(y)). \tag{3}$$

In the same setting, a map $d : X \rightarrow X$ is called an $(X, *, \omega)$ -self-coderivation if for all $x, y \in X$,

$$d(x * y) = (x * d(y)) \omega (d(x) * y). \tag{4}$$

Note that if (X, ω) is a commutative groupoid, then $(X, *, \omega)$ -self-derivations are $(X, *, \omega)$ -self-coderivations. A map $d : X \rightarrow X$ is called an *abelian- $(X, *, \omega)$ -self-derivation* if it is both an $(X, *, \omega)$ -self-derivation and an $(X, *, \omega)$ -self-coderivation.

3. Two-Step Derivations and Couplets on Trigroupoids

An algebraic system $(X, *, \bullet, \diamond)$ is said to be a *ranked trigroupoid* if algebraic systems $(X, *, \bullet)$ and (X, \bullet, \diamond) are ranked bigroupoids. A *two-step derivation* on a ranked trigroupoid $(X, *, \bullet, \diamond)$ is a mapping $d : X \rightarrow X$ such that d is both an $(X, *, \bullet)$ -self-derivation and an (X, \bullet, \diamond) -self-derivation.

Obviously, if one considers ranked n -groupoids $(X, *_1, *_2, \dots, *_n)$, then one may consider $(n - 1)$ -step derivations $d : X \rightarrow X$ for which one has d as an $(X, *_k, *_k)$ -self-derivation for $k = 1, \dots, n - 1$.

In this paper we will mostly be interested in the case of two-step-derivations on ranked trigroupoids and some related pairs of maps (D, d) which we call *couplets*. For ranked n -groupoids where $n \geq 4$, we obtain triplets, quadruplets, and so forth, as the appropriate generalizations.

Let $d : X \rightarrow X$ be a two-step derivation on a ranked trigroupoid $(X, *, \bullet, \diamond)$. Then, for any $x, y \in X$, we have

$$\begin{aligned} d^2(x * y) &= d(d(x * y)) \\ &= d[(d(x) * y) \bullet (x * d(y))] \\ &= [d(d(x) * y) \bullet (x * d(y))] \\ &\quad \diamond [(d(x) * y) \bullet d(x * d(y))] \\ &= \{[(d^2(x) * y) \bullet (d(x) * d(y))] \bullet (x * d(y))\} \\ &\quad \diamond [(d(x) * y) \bullet [(d(x) * d(y)) \bullet (x * d^2(y))]]. \end{aligned} \tag{5}$$

If we let $D := d^2$, then it follows that

$$\begin{aligned} D(x * y) &= \{[(D(x) * y) \bullet (d(x) * d(y))] \bullet (x * d(y))\} \\ &\quad \diamond [(d(x) * y) \bullet [(d(x) * d(y)) \bullet (x * D(y))]]. \end{aligned} \tag{6}$$

We call (D, d) a *couplet* on a ranked trigroupoid $(X, *, \bullet, \diamond)$ if it satisfies condition (6), and $D(0) = 0$ if X contains a constant 0.

Example 4. Let \mathbf{R} be the set of all real numbers and let $(\mathbf{R}, \cdot, +, -)$ be the ranked trigroupoid where $\cdot, +$, and $-$ are usual multiplication, addition, and subtraction, respectively.

If we let (D, d) be a couplet on the ranked trigroupoid $(\mathbf{R}, \cdot, +, -)$, then

$$\begin{aligned} D(x \cdot y) &= [D(x)y + d(x)d(y) + xdy] \\ &\quad - [d(x)y + d(x)d(y) + xD(y)] \quad (7) \\ &= [D(x)y - xD(y)] + [xd(y) - d(x)y]. \end{aligned}$$

If we let $y := x$ in (7), then

$$D(x^2) = [D(x)x - xD(x)] + [xd(x) - d(x)x] = 0. \quad (8)$$

Thus $D(x^2) = 0$ for all $x \in \mathbf{R}$, whence $D(t) = 0$ for all $t \geq 0$. If we let $y := 1$ in (7), then $D(x) = [D(x) - xD(1)] + [xd(1) - d(x)] = D(x) + xd(1) - d(x)$. It follows that, for any $x \in \mathbf{R}$,

$$d(x) = xd(1), \quad (9)$$

which implies that $xd(y) - d(x)y = xyd(1) - xd(1)y = 0$. Hence, by (7), we have $D(xy) = D(x)y - xD(y) + xd(y) - d(x)y = D(x)y - xD(y)$; that is,

$$D(x \cdot y) = D(x)y - xD(y). \quad (10)$$

If we let $x < 0$ and $y := -1$ in (9), then we have $0 = D(-x) = D(x) \cdot (-1) - xD(-1)$. Hence $D(x) = -xD(-1)$ for all $x < 0$.

In Example 4, if d is a two-step derivation on $(\mathbf{R}, \cdot, +, -)$, then $d(xy) = d(x)y + xd(y)$ and $d(x + y) = (d(x) + y) - (x + d(y))$ for any $x, y \in \mathbf{R}$. It follows that $d(2x) = d(x + x) = (d(x) + x) - (x + d(x)) = 0$, which proves that $d(x) = d(2 \cdot (x/2)) = 0$ for all $x \in \mathbf{R}$.

Example 5. Let $X := [0, \infty)$ and let $(X, \cdot, +, +)$ be a ranked trigroupoid where $\cdot, +$ are usual multiplication and addition, respectively. Define a map $\varphi : X \rightarrow X$ satisfying $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in X$. For $k > 0, k \neq 1, \lambda \in \mathbf{R}$, we define a map $D : X \rightarrow X$ by $D(x) := \varphi(a)k^{a+\lambda}$ where $x = k^a$ and $D(0) = 0$. Then D is a $(X, \cdot, +)$ -self-derivation. In fact, if $x = k^a, y = k^b$ for some $a, b \in X$, then

$$\begin{aligned} D(x \cdot y) &= D(k^a \cdot k^b) \\ &= \varphi(a + b)k^{a+b+\lambda} \\ &= [\varphi(a) + \varphi(b)]k^{a+b+\lambda} \quad (11) \\ &= [\varphi(a)k^{a+\lambda}]k^b + [\varphi(b)k^{b+\lambda}]k^a \\ &= D(x)y + xD(y). \end{aligned}$$

Assume that (D, d) is a couplet on the ranked trigroupoid $(X, \cdot, +, +)$ for some self- $(X, +, +)$ -derivation d . Then,

$$\begin{aligned} D(x \cdot y) &= D(x)y + d(x)d(y) + xd(y) + d(x)y \\ &\quad + d(x)d(y) + xD(y). \quad (12) \end{aligned}$$

Since $D(x \cdot y) = D(x)y + xD(y)$, we obtain

$$0 = d(x)[y + d(y)] + [x + d(x)]d(y). \quad (13)$$

Proposition 6. *There is no two-step derivation on the ranked trigroupoid $(\mathbf{R}, \cdot, -, +)$.*

Proof. Assume that $d : \mathbf{R} \rightarrow \mathbf{R}$ is a two-step derivation on $(\mathbf{R}, \cdot, -, +)$. Then, for any $x, y \in \mathbf{R}$,

$$d(x \cdot y) = d(x)y - xd(y), \quad (14)$$

$$d(x - y) = (d(x) - y) + (x - d(y)). \quad (15)$$

If we let $y := -x$ in (15), then

$$\begin{aligned} d(2x) &= (d(x) - (-x)) + (x - d(-x)) \\ &= 2x + (d(x) - d(-x)), \quad (16) \end{aligned}$$

$$\begin{aligned} d(-x^2) &= d(x) \cdot (-x) - x \cdot d(-x) \\ &= -x \cdot d(x) - x \cdot d(-x). \quad (17) \end{aligned}$$

If we let $x := 0$ in (16), then $d(0) = d(2 \cdot 0) = 2 \cdot 0 + (d(0) - d(-0)) = 0$. On the other hand, if we let $x := 1$ and $y := 0$ in (15), then $d(1 - 0) = (d(1) - 0) + (1 - d(0))$, proving that $d(0) = 1$ is a contradiction. Hence there is no two-step derivation on the ranked trigroupoid $(\mathbf{R}, \cdot, -, +)$. \square

Proposition 7. *There is no two-step derivation on the ranked trigroupoid $(\mathbf{R}, \cdot, +, +)$.*

Proof. Assume that $d : \mathbf{R} \rightarrow \mathbf{R}$ is a two-step derivation on $(\mathbf{R}, \cdot, +, +)$. Then, for any $x, y \in \mathbf{R}$,

$$d(x \cdot y) = d(x)y + xd(y), \quad (18)$$

$$d(x + y) = (d(x) + y) + (x + d(y)). \quad (19)$$

If we let $y := 0$ in (19), then $d(x + 0) = d(x) + 0 + x + d(0)$ and hence $d(0) = -x$ for all $x \in \mathbf{R}$. If we let $y := 0, x := 0$ in (18), then $-x = d(0 \cdot 0) = d(0) \cdot 0 + 0 \cdot d(0) = 0$, for any $x \in \mathbf{R}$, is a contradiction. \square

Proposition 8. *Let (D, d) be a couplet on a ranked trigroupoid $(\mathbf{R}, \cdot, -, +)$. If $d(1) \neq 1/2$ and $\lambda := (D(1) - d(1))/(2d(1) - 1)$, then $D(xy) = D(x)y + xD(y) - 2\lambda^2xy$ for all $x, y \in \mathbf{R}$. In particular, if $D(1) = d(1)$, then D is a $(\mathbf{R}, \cdot, +)$ -self-derivation.*

Proof. If (D, d) is a couplet on a ranked trigroupoid $(\mathbf{R}, \cdot, -, +)$, then, for any $x, y \in \mathbf{R}$, we have

$$\begin{aligned} D(xy) &= (D(x)y - d(x)d(y)) - xd(y) + d(x)y \\ &\quad - (d(x)d(y) - xD(y)) \quad (20) \\ &= D(x)y + xD(y) - 2d(x)d(y) \\ &\quad + (d(x)y - xd(y)). \end{aligned}$$

Since $d(1) \neq 1/2$, if we let $y := 1$ in (20), then $D(x) = D(x) + xD(1) - 2d(x)d(1) + d(x) - xd(1)$. It follows that

$$\begin{aligned} 0 &= xD(1) - 2d(x)d(1) + d(x) - xd(1) \\ &= x[D(1) - d(1)] - [2d(1) - 1]d(x). \quad (21) \end{aligned}$$

Hence $d(x) = (D(1) - d(1)) / (2d(1) - 1)x = \lambda x$ for any $x \in \mathbf{R}$. If we change $d(x)$ into λx for any $x \in \mathbf{R}$, then we obtain

$$\begin{aligned} D(xy) &= D(x)y + xD(y) - 2(\lambda x)(\lambda y) + (\lambda x)y - x(\lambda y) \\ &= D(x)y + xD(y) - 2\lambda^2 xy \end{aligned} \tag{22}$$

for all $x, y \in \mathbf{R}$. In particular, if $D(1) = d(1)$, then $\lambda = 0$ and hence $D(xy) = D(x)y + xD(y)$; that is, D is a $(\mathbf{R}, \cdot, +)$ -self-derivation. \square

4. Frame Algebras and $fr(3)$ -Algebras

A groupoid $(X, *, 0)$ is said to be a *frame algebra* if it satisfies the axioms (A), (B), and

$$(F) \quad x * 0 = x, \text{ for all } x \in X.$$

Example 9. (1) Every BCK-algebra is a frame algebra. (2) Every lattice implication algebra (see [17, 18]) is a frame algebra.

The collection of frame algebras includes the collection of BCK-algebras and it is a variety. In a frame algebra the element 0 is unique. Indeed, if 0_1 and 0_2 are both zeros, then $x \neq 0_1, 0_2$ yields $0_1 = x * x = 0_2$.

Proposition 10. *The collection of all frame algebras $(X, *, 0)$ forms a subsemigroup of the semigroup $(\text{Bin}(X), \square)$.*

Proof. Given frame algebras $(X, *, 0), (X, \bullet, 0)$, if we let $(X, \square) := (X, *) \square (X, \bullet)$, then $x \square y = (x * y) \bullet (y * x)$ for all $x, y \in X$. It follows that $x \square x = (x * x) \bullet (x * x) = 0 \bullet 0 = 0, 0 \square x = (0 * x) \bullet (x * 0) = 0 \bullet x = 0$, and $x \square 0 = (x * 0) \bullet (0 * x) = x \bullet 0 = x$, proving that (X, \square) is a frame algebra. This proves the proposition. \square

Given groupoids $(X, *), (X, \bullet) \in \text{Bin}(X)$, we define

$$(X, *)S(X, \bullet) := \{(X, \nabla) \mid \forall x, y \in X, x \nabla y \in \{x * y, x \bullet y\}\}. \tag{23}$$

Proposition 11. *Let $(X, *, 0)$ and $(X, \bullet, 0)$ be frame algebras. If $(X, \nabla) \in (X, *)S(X, \bullet)$, then (X, ∇) is a frame algebra.*

Proof. If $(X, \nabla) \in (X, *)S(X, \bullet)$, then $x \nabla y \in \{x * y, x \bullet y\}$ for all $x, y \in X$. It follows that $x \nabla x \in \{x * x, x \bullet x\} = \{0\}$ implies that $x \nabla x = 0$. Moreover, $0 \nabla x \in \{0 * x, 0 \bullet x\} = \{0\}$ shows that $0 \nabla x = 0$, and $x \nabla 0 \in \{x * 0, x \bullet 0\} = \{x\}$ shows that $x \nabla 0 = x$, proving that (X, ∇) is a frame algebra. \square

A ranked trigroupoid $(X, *, \bullet, \diamond)$ is called an *$fr(3)$ -algebra* if

$$(G) \quad (X, *, 0_1), (X, \bullet, 0_2), (X, \diamond, 0_3) \text{ are frame algebras,}$$

$$(H) \quad 0_1 = 0_2 = 0_3.$$

Theorem 12. *Let $(X, *, \bullet, \diamond)$ be an $fr(3)$ -algebra. If $d : X \rightarrow X$ is a two-step derivation on X , then*

- (i) $d(0) = 0,$
- (ii) $d(x) = d(x) \bullet x = d(x) \diamond x$ for all $x \in X.$

Proof. (i) If d is a two-step derivation on $(X, *, \bullet, \diamond)$, then for any $x, y \in X$, we have

$$d(x * y) = (d(x) * y) \bullet (x * d(y)), \tag{24}$$

$$d(x \bullet y) = (d(x) \bullet y) \diamond (x \bullet d(y)). \tag{25}$$

It follows that $d(0) = d(0 * y) = (d(0) * y) \bullet (0 * d(y)) = (d(0) * y) \bullet 0 = d(0) * y$; that is, $d(0) = d(0) * y$, for all $y \in X$. If we let $y := d(0)$, then by applying (A) we obtain $d(0) = d(0) * d(0) = 0$.

(ii) Given $x \in X$, we have $d(x) = d(x * 0) = (d(x) * 0) \bullet (x * d(0)) = d(x) \bullet x$ and $d(x) = d(x \bullet 0) = (d(x) \bullet 0) \diamond (x \bullet d(0)) = d(x) \diamond x$. \square

Given a two-step derivation d on a trigroupoid $(X, *, \bullet, \diamond)$, we denote its kernel by $\text{Ker}(d) := \{x \in X \mid d(x) = 0\}$.

Proposition 13. *Let $(X, *, \bullet, \diamond)$ be an $fr(3)$ -algebra. If $d : X \rightarrow X$ is a two-step derivation on X , then*

- (i) $x * d(x), x \bullet d(x) \in \text{Ker}(d),$
- (ii) $x \in \text{Ker}(d)$ implies that $x * y, x \bullet y \in \text{Ker}(d),$
- (iii) $\text{Ker}(d) \subseteq \text{Ker}(d^2),$

for all $x, y \in X$.

Proof. (i) If we let $y := d(x)$ in (24) and (25), respectively, then $d(x * d(x)) = (d(x) * d(x)) \bullet (x * d(d(x))) = 0 \bullet (x * d^2(x)) = 0$ and $d(x \bullet d(x)) = (d(x) \bullet d(x)) \diamond (x \bullet d(d(x))) = 0 \diamond (x \bullet d^2(x)) = 0$, proving that $x * d(x), x \bullet d(x) \in \text{Ker}(d)$ for any $x \in X$.

(ii) If $x \in \text{Ker}(d)$, then $d(x * y) = (d(x) * y) \bullet (x * d(y)) = (0 * y) \bullet (x * d(y)) = 0$ and $d(x \bullet y) = (d(x) \bullet y) \diamond (x \bullet d(y)) = 0 \diamond (x \bullet d(y)) = 0$ for any $y \in X$, proving that $x * y, x \bullet y \in \text{Ker}(d)$.

(iii) If $x \in \text{Ker}(d)$, then $d^2(x) = d(d(x)) = d(0) = 0$ by Theorem 12, which shows that $x \in \text{Ker}(d^2)$. \square

Proposition 14. *Let $(X, *, \bullet, \diamond)$ be an $fr(3)$ -algebra. If $d : X \rightarrow X$ is a two-step derivation on X , then*

$$x \in \text{Ker}(d^2) \text{ implies } x * y \in \text{Ker}(d^2) \tag{26}$$

for all $y \in X$.

Proof. If $x \in \text{Ker}(d^2)$, then $d(d(x)) = 0$ and hence

$$\begin{aligned} d^2(x * y) &= d(d(x * y)) = d[(d(x) * y) \bullet (x * d(y))] \\ &= [d(d(x) * y) \bullet (x * d(y))] \\ &\quad \diamond [(d(x) * y) \bullet d(x * d(y))] \\ &= [d(d(x) * y) \bullet (d(x) * d(y))] \bullet (x * d(y)) \end{aligned}$$

$$\begin{aligned} & \diamond [(d(x) * y) \bullet d(x * d(y))] \\ &= 0 \diamond [(d(x) * y) \bullet d(x * d(y))] = 0, \end{aligned} \tag{27}$$

for all $y \in X$, which proves that $x * y \in \text{Ker}(d^2)$. \square

Note that $d^2(x \bullet y)$ may not be computable unless the behavior of $d(u \diamond v)$ is specified, since $d^2(x \bullet y) = d((d(x) \bullet y) \diamond (x \bullet d(y))) = d(u \diamond v)$ for some $u, v \in X$.

Proposition 15. *Let $(X, *, \bullet, \diamond)$ be an $fr(3)$ -algebra. If (D, d) is a couplet of $(X, *, \bullet, \diamond)$, then $\text{Ker}(D) * X \subseteq \text{Ker}(D)$.*

Proof. If (D, d) is a couplet of $(X, *, \bullet, \diamond)$ and if $x \in \text{Ker}(D)$, then $D(x) * y = 0 * y = 0$ for any $y \in X$. It follows from (6) that $D(x * y) = 0 \diamond [(d(x) * y) \bullet ((d(x) * d(y)) \bullet (x * D(y)))] = 0$, proving that $x * y \in \text{Ker}(D)$. \square

Let (X, \leq) be a poset with minimal element 0. Define a binary operation “ $*$ ” on X by

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x, & \text{otherwise.} \end{cases} \tag{28}$$

Then $(X, *, 0)$ is a BCK-algebra, called a *standard BCK-algebra* inherited from the poset (X, \leq) .

Proposition 16. *Let $(X, *, 0)$ be a standard BCK-algebra. Let $(X, *, \bullet, \diamond)$ be an $fr(3)$ -algebra and let (D, d) be a couplet of $(X, *, \bullet, \diamond)$. If $D(x * y) \neq 0$ for some $x, y \in X$, then $D(x * y) = D(x) * y = D(x)$ and $x * y = x$.*

Proof. Let $D(x * y) \neq 0$. We claim that $D(x) * y \neq 0$. Suppose that $D(x) * y = 0$. Since $(X, *, \bullet, \diamond)$ is a $fr(3)$ -algebra, by applying (6), we obtain that $D(x * y) = 0$ is a contradiction. Since $(X, *, 0)$ is a standard BCK-algebra, we obtain $D(x) * y = D(x)$. We claim that $x * y = x$. If $x * y = 0$, then $D(x * y) = D(0) = 0$ is a contradiction. It follows that $D(x * y) = D(x) = D(x) * y$, proving the proposition. \square

5. Classification of Linear Ranked Trigroupoids

Let $X := \mathbf{R}$ be the real field and let $(X, *, \bullet, \diamond)$ be a ranked trigroupoid, where $(X, *)$, (X, \bullet) , and (X, \diamond) are linear groupoids; that is, $x * y := A + Bx + Cy$, $x \bullet y := a + bx + cy$, and $x \diamond y := \alpha + \beta x + \gamma y$ for any $x, y \in X$, where $A, B, C, a, b, c, \alpha, \beta, \gamma \in X$ (fixed).

Thus, if $d : X \rightarrow X$ is a two-step derivation on $(X, *, \bullet, \diamond)$, then for any $x, y \in X$, we have

$$\begin{aligned} d(x * y) &= (d(x) * d(y)) \bullet (x * d(y)) \\ &= (A + Bd(x) + Cy) \bullet (A + Bx + Cd(y)) \\ &= a + (b + c)A + B(cx + bd(x)) + C(by + cd(y)) \end{aligned} \tag{29}$$

and also in a similar manner we obtain

$$\begin{aligned} d(x \bullet y) &= \alpha + (\beta + \gamma)a + b(\gamma x + \beta d(x)) \\ &\quad + c(\beta y + \gamma d(y)). \end{aligned} \tag{30}$$

If $d(x) = 0$ for all $x \in X$, then (29) and (30) become

$$\begin{aligned} a + (b + c)A + Bcx + Cby &= 0, \\ \alpha + (\beta + \gamma)a + b\gamma x + c\beta y &= 0 \end{aligned} \tag{31}$$

for any $x, y \in X$. If we let $x := 0, y := 0$ in (31), then we obtain $a + (b + c)A = 0, \alpha + (\beta + \gamma)a = 0$. Hence, (31) reduces to

$$Bcx + Cby = 0, \tag{32}$$

$$b\gamma x + c\beta y = 0 \tag{33}$$

for all $x, y \in X$. If we let $x := 1, y := 0$ and let $x := 0, y := 1$ in (32) and (33), respectively, then we obtain

$$Bc = 0, \quad Cb = 0, \quad b\gamma = 0, \quad c\beta = 0. \tag{34}$$

From this information, we obtain the following propositions.

Proposition 17. *Let $X := \mathbf{R}$ be the real field and let $(X, *, \bullet, \diamond)$ be a ranked trigroupoid, where $(X, *)$, (X, \bullet) , (X, \diamond) are linear groupoids; that is, $x * y := A + Bx + Cy$, $x \bullet y := a + bx + cy$, and $x \diamond y := \alpha + \beta x + \gamma y$ for any $x, y \in X$, where $A, B, C, a, b, c, \alpha, \beta, \gamma \in X$ (fixed). Let $d : X \rightarrow X$ be a two-step derivation such that $d(x) = 0$ for all $x \in X$.*

- (1) *If $bc \neq 0$ and $b + c \neq 0$, then $x * y = -a/(b + c)$, $x \bullet y := a + bx + cy$, $x \diamond y = 0$;*
- (2) *If $bc \neq 0$ and $b + c = 0$, then $x * y = A$, $x \bullet y = b(x - y)$, $x \diamond y = 0$.*

Proof. (i) If $bc \neq 0$, then it follows from (34) that we obtain $B = C = 0, \beta = \gamma = \alpha = 0$. If $b + c \neq 0$, then we have $x * y = -a/(b + c)$ and $x \diamond y = 0$.

(ii) If $bc \neq 0$ and $b + c = 0$, then $0 = a + (b + c)A = a$ and A is arbitrary, and hence we obtain the result. \square

Proposition 18. *Let $X := \mathbf{R}$ be the real field and let $(X, *, \bullet, \diamond)$ be a ranked trigroupoid, where $(X, *)$, (X, \bullet) , (X, \diamond) are linear groupoids; that is, $x * y := A + Bx + Cy$, $x \bullet y := a + bx + cy$, and $x \diamond y := \alpha + \beta x + \gamma y$ for any $x, y \in X$, where $A, B, C, a, b, c, \alpha, \beta, \gamma \in X$ (fixed). Let $d : X \rightarrow X$ be a two-step derivation such that $d(x) = 0$ for all $x \in X$:*

- (i) *if $b = c = 0$ and $b + c \neq 0$, then $x * y = A + Bx + Cy$, $x \bullet y := 0$, $x \diamond y = \beta x + \gamma y$;*
- (ii) *if $b \neq 0, c = \beta = 0$, then $x * y = -a/b + Bx$, $x \bullet y = a + bx$, $x \diamond y = 0$;*
- (iii) *if $b \neq 0, c = 0, \beta \neq 0$, then $x * y = \alpha/b\beta + Bx$, $x \bullet y = -\alpha/\beta + bx$, $x \diamond y = \alpha + \beta y$;*
- (iv) *if $b = 0, c \neq 0, \gamma = 0$, then $x * y = -a/c + Cy$, $x \bullet y = a + cy$, $x \diamond y = 0$;*
- (v) *if $b = 0, c \neq 0, \gamma \neq 0$, then $x * y = \alpha/c\gamma + Cy$, $x \bullet y = -\alpha/\gamma + cy$, $x \diamond y = \alpha + \gamma y$.*

Proof. The proof is similar to Proposition 17 and we omit it. \square

In Propositions 17 and 18, we observe that there are 6 different types of linear ranked trigroupoids in the special case of $d(x) = 0$ for all $x \in X$, and most of them are classified by the properties of b, c in $x \bullet y = a + bx + cy$.

6. Conclusion

The notion of two-step derivations is a generalization of derivations. This leads to the study of trigroupoids, and we explore some relations with several algebras, for example, BCK-algebras, frame algebras, and so forth. The classification of linear ranked trigroupoids then explains a number of concrete algebraic structures with derivations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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