GROUPS WHOSE NON-PERMUTABLE SUBGROUPS SATISFY CERTAIN CONDITIONS

by

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ABSTRACT

In this dissertation, we determine the structure of groups whose non-permutable subgroups satisfy certain conditions. In Chapter 1, we give the definitions and well-known results that we will use during the dissertation. In Chapter 2, we express our main result, which states that an infinite rank $\mathfrak{X}$-group with all proper subgroups permutable or of finite rank has all subgroups permutable. Before proving our main result in Chapter 4, we establish some preliminary results in Chapter 3 which are used in proving the main result.

In Final Chapter, we study the class of locally graded groups with all subgroups permutable or nilpotent of bounded class c. We prove that such groups are soluble of derived length bounded by a number depending on c. This chapter contains preliminary investigations into the problem of the structure of groups with all subgroups permutable or soluble.

NOTATION LIST

$\langle x \rangle$	Subgroup generated by x
$[u]$	Integer part of the real number u
$\mathrm{Aut}G$	the automorphism group of G
$C_G(H)$	Centralizer of the subgroup H in G
$G', [G,G]$	the derived subgroup of the group G
$G^{(i)}$	i-th term of the derived series of G
$\mathfrak{S}_d$	the class of soluble groups of derived length at most d
$\mathfrak{N}_c$	the class of nilpotent groups of class at most c
$T(G)$	the torsion subgroup of the group G
G_p	the p-primary component of the group G
$\underset{\lambda \in \Lambda}{\mathrm{Dr}} H_\lambda$	the direct product of the groups $\{H_\lambda \mid \lambda \in \Lambda\}$
$\prod_{\alpha \in \Lambda} N_\alpha$	the product of the groups $\{N_\alpha \mid \alpha \in \Lambda\}$
$r_p(G)$	the p-rank of the group G
$r_0(G)$	the torsion-free rank of the group G
$r(G)$	the rank of the group the group G
$\mathrm{HP}(\ G)$	Hirsch-Plotkin radical of the group G
$\mathrm{HP}_\alpha(G)$	α-th term of the upper Hirsch-Plotkin series of G

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CONTENTS

CHAPTER 1

INTRODUCTION

The structure of the groups all of whose subgroups have certain properties has been one of the important concepts in group theory. Many authors studied groups with certain conditions on their subgroups. One of the earliest examples of this type is the class of groups with all subgroups normal. Groups with this property are known to be *Dedekind groups* and their structure was determined by R. Dedekind and R. Baer by the following result.

PROPOSITION 1.1. [**20**, 5.3.7] *All the subgroups of a group G are normal if and only if G is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order.*

Another example is the class of all finite non-abelian groups in which every proper subgroup is abelian. These type of groups are called *Miller-Moreno groups*, and their classification is given in [**17**] by G.A. Miller and H. Moreno. Moreover, the structure of finite non-nilpotent groups with all proper subroups nilpotent was given by O. J. Schmidt in [**25**]. Also, for example, groups with all subgroups normal or abelian was studied by G.M. Romalis and N.F. Sesekin in [**21**], [**22**] and [**23**].

In [**24**], J.E. Roseblade studied the class of groups in which every subnormal subgroup is of bounded defect. Recently, in [**9**], M.J. Evans and Y. Kim studied groups in which every subnormal subgroup is of bounded defect or of finite rank, and they generalized the result of J.E. Roseblade. Moreover, in [**15**], L.A. Kurdachenko and H. Smith studied groups with all subgroups subnormal or of finite rank. The main results of the papers [**9**] and [**15**] will be given in Chapter 2. Also, in [**13**], K.

Iwasawa described the structure of the groups with all subgroups permutable. In a further section, we will give the main results of this paper, too.

The results in these papers motivated a desire to determine the structure of groups whose nonpermutable subgroups satisfy certain conditions. The main result is the structure of the class of groups in which every subgroup is permutable or of finite rank, which will be given in Chapter 2. Also, in Chapter 5, the solubility of the groups with all subgroups permutable or nilpotent of bounded class will be proved. Moreover, we will give a bound on the derived length of such a group.

In coming sections of this chapter, we will give the definitions and results which we will use in further chapters.

1.1. Generalized Nilpotent and Soluble Groups

In group theory, nilpotent and soluble groups play an important role. In this section, we will give some basic definitions and results about nilpotent and soluble groups, and their generalization.

DEFINITION 1.1. *A group G is said to be* soluble *(or* solvable*) if it has an abelian series, by which we mean a series $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ in which each factor G_{i+1}/G_i is abelian. If G is a soluble group, then the length of a shortest abelian series in G is called the* derived length *of G. The groups with derived length at most 1 are just the abelian groups. A soluble group with derived length at most 2 is said to be* metabelian.

The derived subgroup of a group G, denoted by G', is the subgroup generated by all commutators in G, hence $G' = [G, G]$. It is well-known that G' is a characteristic subgroup of G. We can generate a descending sequence of characteristic subgroups by repeatedly forming derived subgroups:

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \ldots$$

where $G^{(n+1)} = (G^{(n)})'$. This is called the *derived series* of G, although it need not reach 1 or even terminate. Clearly, all the factors $G^{(n)}/G^{(n+1)}$ are abelian. If G is soluble, it is well-known that the derived length of G is equal to the length of the derived series of G. Also, it is clear that a group G is metabelian if and only if $G'' = 1$.

We denote the class of soluble groups of derived length at most d by $\mathfrak{S}_d$. It is well-known that the class of soluble groups is closed with respect to the formation of subgroups, images and extension of its members. Here, we may give the following important result about finitely generated periodic soluble groups.

PROPOSITION 1.2. [**20**, 5.4.11] *A finitely generated soluble torsion group is finite.*

DEFINITION 1.2. *A group G is called* nilpotent *if it has a central series, that is, a normal series $1 = G_0 < G_1 < \ldots < G_n = G$ such that G_{i+1}/G_i is contained in the center of G/G_i for all i. The length of a shortest central series of G is the* nilpotent class *of G.*

We denote the class of nilpotent groups of class at most c by $\mathfrak{N}_c$. Nilpotent groups are obviously soluble. The symmetric group S_3 is an example of a soluble group which is not nilpotent. A finite p-group is nilpotent where p is a prime, and the class of nilpotent groups is closed under the formation of subgroups, images and finite direct product. Extension of a nilpotent group by a nilpotent group need not to be necessarily nilpotent.

Next, we can give the following important result which gives the relation between the class and the derived length of a nilpotent group.

PROPOSITION 1.3. [**20**, 5.1.12] *If the group G is nilpotent with positive class c, then its derived length is at most $[\log_2 c] + 1$.*

Now, we can continue with the generalization of nilpotent groups.

Definition 1.3. *A group G is called* locally nilpotent *if every finite subset of G is contained in a nilpotent subgroup of G.*

In a locally nilpotent group G, the elements of finite order form a characteristic subgroup of G. This subgroup is called the *torsion subgroup* of G and it is denoted by $T(G)$. Moreover, $T(G)$ is the direct product of its primary components. The following proposition is a consequence of this fact.

Proposition 1.4. *Let G be a peridic locally nilpotent group. Then G is the direct product of its primary components, hence we can write $G = \operatorname{Dr}_p G_p$.*

In a similar fashion, we can define *locally soluble, locally finite* and *locally (soluble-by-finite)* groups. Also, an extension of a locally soluble group by a finite group is called a *(locally soluble)-by-finite* group. The (locally soluble)-by-finite groups are also called *almost locally soluble* groups. By a result in [**4**, 1.1.5 Proposition], we know that periodic locally soluble groups are locally finite. Also, it is a well known fact from [**16**] that a simple locally soluble group is cyclic of prime order.

In a group G, the largest normal locally soluble subgroup of the group, if it exists, is called the *locally soluble radical* of the group. The following lemma gives us an idea about when the locally soluble radical exists.

Lemma 1.1. [**8**, Lemma 1] *Let G be a locally (soluble-by-finite) group. Then G has a locally soluble radical S and the locally soluble radical of G/S is trivial.*

So, if G is a (locally soluble)-by-finite group, then the locally soluble radical S of G exists, and $|G : S|$ is finite. Moreover, it is clear that the locally soluble radical of a group is characteristic if it exists.

1.2. Permutable Subgroups

Definition 1.4. *Let H be a subgroup of a group G. Then H is said to be* permutable *if it permutes with every subgroup of G, that is, $HK = KH$ for every subgroup K of G.*

We write H per G to denote that H is permutable in G. It is very clear that every normal subgroup is permutable, but the converse is not true. For example, consider the group $G = \left\langle x, y | x^{p^2} = 1, y^p = 1, y^{-1}xy = x^{p+1} \right\rangle$ of order p^3. We claim that the subgroup $\langle y \rangle$ is permutable in G, but it is not normal. Let H be the subgroup generated by elements of order p. Then, $x \notin H$, hence $H \neq G$. Moreover, $x^p, y \in H$ together imply that H has order p^2. So, H is elementary abelian. Now, any cyclic subgroup of H permutes with $\langle y \rangle$. Also, if we pick an element $a \in G \setminus H$ then $\langle a \rangle$ is of order p^2, hence it is maximal, so $\langle a \rangle$ is normal and permutes with $\langle y \rangle$. Any cyclic subgroup of G permutes with $\langle y \rangle$, hence $\langle y \rangle$ is permutable. However, $\langle y \rangle$ is not normal as $x^{-1}yx = yx^{-p}$. An alternate proof for the permutability of $\langle y \rangle$ can be given when p is an odd prime. If G is a finite p-group which is the product of two cyclic subgroups where p is an odd prime, B. Huppert showed in [**12**] that every subgroup of G is permutable. Hence, in our example, if p is odd, then every subgroup of G is permutable, hence $\langle y \rangle$ is permutable.

A subgroup H of G is called *subnormal* if there exist a positive integer n and a series of subgroups $\{H_i\}_{i=0}^n$ such that $H = H_0 \lhd H_1 \lhd \ldots \lhd H_{n-1} \lhd H_n = G$. Clearly, subnormality is another generalization of normality. We may look for a relation between permutable and subnormal subgroups. However, a subnormal subgroup does not need to be permutable. For example, consider the dihedral group of order 8, $D_8 = \langle x, y | x^2 = y^4 = 1, x^{-1}yx = y^{-1} \rangle$. Let $H = \langle x \rangle = \{1, x\}$ and $K = \langle xy \rangle = \{1, xy\}$. Note that $H = \{1, x\} \lhd \{1, x, y^2, xy^2\} \lhd D_8$ and $K = \{1, xy\} \lhd \{1, xy, y^2, xy^3\} \lhd D_8$, hence H and K are subnormal subgroups of D_8. However, $HK = \{1, x, xy, y\}$ and $KH = \{1, x, xy, y^3\}$, hence H and K are not permutable.

From the opposite point of view, we have the following result from [**27**] due to Stonehewer.

THEOREM 1.1. [**27**, Theorem B] *If H is a permutable subgroup of a finitely generated group G, then H is subnormal in G.*

Next, we continue with a basic definition which we will require in some future lemmas.

DEFINITION 1.5. *Let H be a subgroup of a group G. Then an element $g \in G$ said to have* infinite order modulo H *if $g^n \notin H$ for all nonzero integers n.*

Now, we can state the following well-known lemma about the normalizer of a permutable subgroup.

LEMMA 1.2. [**26**, 5.2.7 Lemma] *If M is a permutable subgroup of a group G and $g \in G$ is such that g has infinite order modulo M, then $g \in N_G(M)$.*

We will use Lemma 1.2 in the proof of the following useful lemma from [**10**].

LEMMA 1.3. [**10**, Lemma 2.6] *Let G be a group, and let H be a subgroup of G such that all subgroups of G containing H are permutable. If there exists an element of G having infinite order modulo H, then H is normal in G.*

PROOF. Let x and y be elements of G having finite order modulo H. As the subgroups H and $\langle H, x\rangle$ are permutable in G, the index

$$|\langle H, x, y\rangle : H| = |\langle H, x, y\rangle : \langle H, x\rangle| \cdot |\langle H, x\rangle : H|$$

is finite, and hence also the product xy has finite order modulo H. It follows that the set T of all elements of G having finite order modulo H is a proper subgroup of G.

Now, we claim that G is generated by $G \setminus T$. Let $g \in G$. If g has infinite order modulo H, then $g \in \langle G \setminus T \rangle$. Assume that g has finite order modulo H. Let $x \in G \setminus T$. Consider the elements gx^{-1} and x. If gx^{-1} has finite order modulo H then $x \in T$ since $T \leq G$, a contradiction. So, $gx^{-1}, x \in G \setminus T$. Also, $g = gx^{-1}x$, hence $g \in \langle G \setminus T \rangle$. Thus, $G = \langle G \setminus T \rangle$.

On the other hand, every element of $G \setminus T$ normalizes H by Lemma 1.2, and hence H is normal in G.

□

We say that a group G is *simple* if it has no proper non-trivial normal subgroups. The following well-known result in [**27**] shows that a simple group cannot have a permutable subgroup either.

PROPOSITION 1.5. [**27**, Corollary C2] *A simple group cannot have a proper, non-trivial permutable subgroup.*

LEMMA 1.4. *Let G be a group. Then, every subgroup of G is permutable if and only if $\langle g \rangle \langle h \rangle = \langle h \rangle \langle g \rangle$ for all $g, h \in G$.*

PROOF. If every subgroup of G is permutable, it is clear that $\langle g \rangle \langle h \rangle = \langle h \rangle \langle g \rangle$ for all $g, h \in G$. Assume now that $\langle g \rangle \langle h \rangle = \langle h \rangle \langle g \rangle$ for all $g, h \in G$. Let H, K be subgroups of G. Let $h^i k^j \in HK$ where $i, j \in \mathbb{Z}$. Then, by our assumption, $\langle h \rangle \langle k \rangle = \langle k \rangle \langle h \rangle$. Hence $h^i k^j = k^r h^s$ for some $r, s \in \mathbb{Z}$ and so $h^i k^j \in KH$. Thus, $HK \subseteq KH$. Similarly, if $k^i h^j \in KH$ where $i, j \in \mathbb{Z}$, then $\langle k \rangle \langle h \rangle = \langle h \rangle \langle k \rangle$ by our assumption, so $k^i h^j \in HK$ and $KH \subseteq HK$. Therefore $HK = KH$ which completes the proof.

□

Now, we can consider the groups with all subgroups permutable. Of course abelian groups and Dedekind groups are examples of such groups. The structure of groups all of whose subgroups are permutable was described by Iwasawa in [**13**]. To describe the structure of a group with all subgroups permutable, we need to understand power automorphisms.

An automorphism of a group G that leaves every subgroup invariant is called a *power automorphism*. Note that such an automorphism maps each element to one of its powers. Clearly, the set of power automorphisms of G is a subgroup of Aut$G$. The next lemma gives an example of a power automorphism of an abelian p-group for a prime p.

LEMMA 1.5. [**26**, 1.5.5 Lemma] *Let A be an abelian p-group and let r be a p-adic unit, $r = \sum_{i=0}^{\infty} r_i p^i$, say, where $r_i \in \mathbb{Z}, 0 < r_0 < p$ and $0 \leq r_i < p$ for $i \in \mathbb{N}$. For $a \in A$ with $o(a) = p^k (k \in \mathbb{N})$ we define $a^\sigma = a^{n_k}$ where $n_k = \sum_{i=0}^{k-1} r_i p^i$. Then σ is a power automorphism of A; we shall write a^r for a^σ.*

Next, we describe the structure of non-periodic groups with all subgroups permutable in the following two theorems. The structure of periodic groups with all subgroups permutable will be given in Section 1.3.

THEOREM 1.2. ([**13**], [**26**, 2.4.11 Theorem]) *Let G be a nonabelian group with all subgroups permutable. Assume that G has elements of infinite order. Then $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank one. If $G/T(G)$ is cyclic, then G is the semidirect product of $T(G)$ by an infinite cyclic group $\langle z \rangle$, and for every prime p there exists a p-adic unit $r(p)$ with $r(p) \equiv 1 \pmod{p}$ and $r(2) \equiv 1 \pmod{4}$ such that $a^z = a^{r(p)}$ for all $a \in T(G)_p$ where $a^{r(p)}$ is defined in Lemma 1.5. Conversely, for every prime p, let $r(p)$ be a p-adic unit with $r(p) \equiv 1 \pmod{p}$ and $r(2) \equiv 1 \pmod{4}$. If $\overline{G}$ is the semidirect product of an abelian torsion group T by an infinite cyclic group $\langle z \rangle$ with the automorphism given by $a^z = a^{r(p)}$ for all $a \in T_p$, then every subgroup of $\overline{G}$ is permutable and $\overline{G}/T(\overline{G})$ infinite cyclic.*

THEOREM 1.3. ([**13**], [**26**, 2.4.11 Theorem]) *Let G be a nonabelian group with all subgroups permutable. Assume that G has elements of infinite order. Then $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank one. Assume that $G/T(G)$ is not cyclic. Let* $\operatorname{Exp} T(G)_p = p^n$ *where* $n \in \mathbb{N} \cup \{0, \infty\}$. *Then there exist elements* $z_i \in G$, $a_i \in T(G)$, *primes* p_i *and p-adic units* $r_i(p)$ *such that* $G = \langle T(G), z_1, z_2, \ldots \rangle$ *and for all primes p and all* $i \in \mathbb{N}$,

(1) $r_i(p) \equiv 1 \pmod{p}$ *and* $r_i(2) \equiv 1 \pmod{4}$,

(2) $r_{i+1}(p)^{p_i} \equiv r_i(p) \pmod{p^n}$ *(that is* $r_{i+1}(p)^{p_i} = r_i(p)$ *if* $n = \infty$*)*,

(3) $o(z_i)$ *is infinite,*

(4) $a^{z_i} = a^{r_i(p)}$ *for all* $a \in T(G)_p$,

(5) $z_{i+1}^{p_i} = z_i a_i$, *and*

(6) $(z_i a_i)^{z_{i+1}} = z_i a_i$.

Conversely, let T be an abelian torsion group with $\operatorname{Exp} T_p = p^n$, *let* $a_i \in T$ *and* $r_i(p)$ *be p-adic units satisfying* (1) *and* (2) *for all primes p and all* $i \in \mathbb{N}$. *If we extend T successively by cyclic groups* $\langle z_i \rangle$ *according to the relations* (3)-(6), *we obtain a group* $\overline{G} = \langle T, z_1, z_2, \ldots \rangle$ *in which any two subgroups permute.*

1.3. Ascendant Subgroups

In this section, we start by introducing a useful generalization of subnormal subgroups.

DEFINITION 1.6. *An* ascending series *in a group G is a set of subgroups* $\{H_\alpha | \alpha < \beta\}$, *indexed by ordinals, such that:*

(a) $H_{\alpha_1} \leq H_{\alpha_2}$ *if* $\alpha_1 \leq \alpha_2$;

(b) $H_0 = 1$ *and* $G = \bigcup_{\alpha<\beta} H_\alpha$;

(c) $H_\alpha \lhd H_{\alpha+1}$;

(d) $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ *if λ is a limit ordinal.*

The H_α are called the terms *of the series, $H_{\alpha+1}/H_\alpha$ are called the* factors *of the series. A subgroup which occurs in some ascending series of a group G is called an* ascendant subgroup*.*

As the following lemma states, the intersection of two ascendant subgroups is again ascendant. This lemma is an easy consequence of Exercise 12.1.5 in [**20**].

LEMMA 1.6. *In a group, the intersection of two ascendant subgroups is an ascendant subgroup.*

PROOF. Let H, K be ascendant subgroups of G. Then we can find two ascending series:

$$H = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \dots H_\alpha = G$$

$$K = K_0 \triangleleft K_1 \triangleleft K_2 \triangleleft \dots K_\beta = G$$

where $H_\lambda = \bigcup_{\theta<\lambda} H_\theta$ if λ is a limit ordinal, and $K_\mu = \bigcup_{\theta<\mu} K_\theta$ if μ is a limit ordinal. Now consider the series

$$H \cap K = H \cap K_0 \leq H \cap K_1 \leq H \cap K_2 \leq \dots H \cap K_\beta = H \cap G = H = H_0 \leq H_1 \leq H_2 \leq \dots H_\alpha = G \qquad (1)$$

We know that $H_\alpha \triangleleft H_{\alpha+1}$ for all α, and $H_\lambda = \bigcup_{\theta<\lambda} H_\theta$ if λ is a limit ordinal. For any α, $(H \cap K_\alpha)^{(H \cap K_{\alpha+1})} = H \cap K_\alpha$ since $K_\alpha \triangleleft K_{\alpha+1}$. So we have $H \cap K_\alpha \triangleleft H \cap K_{\alpha+1}$ for any α. Also, $H \cap K_\lambda = H \cap (\cup_{\theta<\lambda} K_\theta) = \cup_{\theta<\lambda}(H \cap K_\theta)$ if λ is a limit ordinal. So (1) is an ascending series from $H \cap K$ to G and hence $H \cap K$ is an ascendant subgroup of G.

□

By Theorem 1.1, we know that a permutable subgroup of a finitely generated group is subnormal. So, it is convenient to ask about the relation between permutable and ascendant subgroups of a group. At this point, we have the following strong result due to Stonehewer.

THEOREM 1.4. [**27**, Theorem A] *If H is a permutable subgroup of a group G, then H is ascendant in G.*

Next, we can consider finitely generated subgroups of a group. By looking at the subnormality and ascendence of the finitely generated subgroups, we have the following definition.

DEFINITION 1.7. *A group is called a* Baer group *if every cyclic subgroup is subnormal. A group is called a* Gruenberg group *if every cyclic subgroup is ascendant.*

It is clear that any Baer group is also a Gruenberg group, and the following result shows that every Gruenberg group is locally nilpotent.

LEMMA 1.7. [**20**, 12.2.7] *If G is a Gruenberg group, then every finitely generated subgroup of G is ascendant and nilpotent.*

Now, we are ready to consider the class of groups with all subgroups permutable. The following result shows that such groups are locally nilpotent.

PROPOSITION 1.6. *Let G be a group in which all subgroups are permutable. Then G is Gruenberg, hence it is locally nilpotent. If G is periodic, then it is direct product of its primary components.*

PROOF. Let H be a cyclic subgroup of G. Then by our assumption, H is permutable, hence it is ascendant by Theorem 1.4. So, every cyclic subgroup of G is ascendant and it is a Gruenberg group. Thus, by Lemma 1.7, G is locally nilpotent. The last part follows from Proposition 1.4.

□

Note that this proposition also holds for groups with all subgroups ascendant since every permutable subgroup is ascendant by Theorem 1.4. In addition to Proposition 1.6, the following proposition shows that for periodic groups, the permutability of all subgroups is the same as the permutability of the subgroups of each primary component.

PROPOSITION 1.7. *Let G be a periodic locally nilpotent group. Then all the subgroups of G are permutable if and only if the subgroups of each primary component are permutable.*

PROOF. First of all, note that since G is a periodic locally nilpotent group, it is the direct product of its primary components G_p, hence we can write $G = \underset{p}{\mathrm{Dr}}\, G_p$. It is clear that if each subgroup of G is permutable, then each subgroup of G_p is permutable for all prime p. Now assume that each subgroup of each G_p is permutable in G_p. Let H, K be subgroups of $G = \underset{p}{\mathrm{Dr}}\, G_p$. Then clearly we can write $H = \underset{p}{\mathrm{Dr}}\, H_p$ and $K = \underset{p}{\mathrm{Dr}}\, K_p$ where $H_p, K_p \leq G_p$ for each prime p. By our assumption, each H_p and K_p permutes, hence $HK = KH$. Thus, each subgroup of G is permutable.

□

If G is a periodic group with all subgroups permutable, then by this proposition, it is clear that the structure of the primary components determines the structure of the group. Also, by [**4**, 1.1.5 Proposition], every periodic locally nilpotent group is locally finite, hence each primary component of a periodic locally nilpotent group is a locally finite p-group for some prime p. Hence, the following result due to K. Iwasawa gives the structure of periodic groups with all subgroups permutable.

THEOREM 1.5. ([**13**], [**26**, 2.4.14 Theorem]) *Let p be a prime. Let G be a non-abelian locally finite p-group with all subgroups permutable. Then, either*

(a) *G is a direct product of a quaternion group Q_8 of order 8 with an elementary abelian 2-group, or*

(b) *G contains an abelian normal subgroup A of exponent p^k with cyclic factor group G/A of order p^m where k and m are positive integers and, there exists an element $b \in G$ with $G = A\langle b\rangle$ and an integer s which is at least 2 in case $p = 2$ such that $s < k \leq s + m$ and $b^{-1}ab = a^{1+p^s}$ for all $a \in A$.*

By combining Theorem 1.2, Theorem 1.3 and Theorem 1.5, we have the following useful corollary.

Corollary 1.1. ([**13**], [**26**, 2.4.22 Theorem]) *If any two subgroups of the group G permute, then G is metabelian.*

1.4. Groups of Finite Rank

In this section, for most of the definitions, we use [**20**] and [**5**]. Firstly, in order to define the rank of an abelian group, we start with the following definition.

Definition 1.8. *A nonempty subset S of an abelian group A is called* linearly independent*, or simply* independent*, if $0 \notin S$ and, given distinct elements $s_1, \ldots, s_r$ of S and integers $m_1, \ldots, m_r$, the relation $m_1s_1 + \ldots + m_rs_r = 0$ implies that $m_is_i = 0$ for all i. If S is not independent, it is said to be* dependent*.*

After this definition, we can give the following well-known lemma.

Lemma 1.8. *An abelian group G is a direct sum of cyclic groups if and only if it is generated by an independent subset.*

By Zorn's Lemma, every independent subset of an abelian group G is contained in a maximal independent subset. Next, we have the following definition of rank for abelian p-groups and torsion-free abelian groups.

Definition 1.9. *If p is a prime and G an abelian group, the* p-rank *of G, denoted by $r_p(G)$, is defined as the cardinality of a maximal independent subset of elements of p-power order. Similarly the* 0-rank *or* torsion-free rank *of G, denoted by $r_0(G)$, is the cardinality of a maximal independent subset of elements of infinite order.*

It is well known that $r_0(G)$ and $r_p(G)$ only depend on the group G. This follows from the fact that for an abelian group G, two maximal independent subsets consisting of elements with order a power of the prime p have the same cardinality, and the same is true of maximal independent subsets consisting of elements of infinite order as atated in [**20**, 4.2.1].

Next, we can consider a well-known example of an abelian group. Let p be a prime. For each $i \geq 1$, let $G_i = \langle x_i \rangle$ be a cyclic group of order p^i. Let θ_i be the monomorphism $\theta_i : G_i \to G_{i+1}$ defined by $\theta_i(x_i) = x_{i+1}^p$. Now, we can think of G_i as a subgroup of G_{i+1}. So, we can form the group $G = \bigcup_{i \geq 1} G_i$. Clearly, G is the union of a chain of cyclic p-groups of orders $p, p^2, p^3, \ldots$. This infinite abelian p-group is called a *Prüfer group* of type p^∞ or *quasicyclic p-group*. It is denoted by C_{p^∞}. In terms of generators and relations we have

$$C_{p^\infty} = \left\langle x_i | x_{i+1}^p = x_i, x_1^p = 1; i = 1, 2, 3, \ldots \right\rangle$$

Now, we are ready to define Černikov groups which play a big role in group theory.

Definition 1.10. *A group which is an extension of a finite direct product of quasicyclic groups by a finite group is called a* Černikov group.

We say that a group G is *radicable* if we can solve equations of the form $x^n = g$ for arbitrary elements g and integers n. Radicable abelian groups are called *divisible* groups. C_{p^∞} is an example of a divisible group. The structure of divisible groups is well-known. The divisible p-groups are direct sums of Prüfer groups and torsion-free divisible groups are direct sums of copies of $\mathbb{Q}$. By the definition of Černikov groups, it is clear that a group G is Černikov if and only if it has a normal divisible abelian subgroup N of finite index, and N is a direct product of only finitely many quasicyclic groups. Now, we have the following definition about the largest divisible subgroup of a group.

DEFINITION 1.11. *Let G be a group. If G has a unique largest divisible subgroup R containing all other divisible subgroups, then we call R the* divisible radical *of G.*

A group need not have a divisible radical. If it exists, then clearly it is characteristic. The divisible radical of a group G is trivial if G has no normal divisible abelian subgroups. It is clear that every Černikov group G has a divisible radical and G is finite if and only if the divisible radical of G is trivial.

It makes sense now to define the rank of a group in general.

DEFINITION 1.12. *A group G is said to have* finite (Prüfer) rank *r if every finitely generated subgroup can be generated by r elements and r is the least such integer. If there is no such integer r, the group is said to have* infinite rank*. We denote the rank of G by $r(G)$.*

Like $r_0(G)$ and $r_p(G)$, $r(G)$ only depends on the group G. Clearly, the groups of rank 1 are locally cyclic groups. An example of a group of rank 1 is C_{p^∞}. A finite direct product of quasicyclic groups has finite rank, hence a Černikov group is of finite rank. Note that for an abelian p-group and a torsion-free abelian group, Definition 1.9 is consistent with Definition 1.12. If G is an abelian group, then we have $r(G) = r_0(G) + \max\limits_{p} r_p(G)$.

The following well-known lemma shows that subgroups and quotient groups of groups with finite rank also have finite rank and, the class of groups with finite rank is closed under forming extensions.

Lemma 1.9. *Let G be a group and let $N \leq G$.*

(i) *If G has finite rank r then N has rank at most r.*

(ii) *If $N \triangleleft G$ and G has rank r then G/N has rank at most r.*

(iii) *If $N \triangleleft G$ with N of rank r and G/N of rank s then G has rank at most $r + s$.*

Proof. The proofs of (i) and (ii) are obvious.

(iii) Suppose that $N \triangleleft G$ where N and G/N have finite ranks r and s respectively. Let $H = \langle a_1, a_2, \ldots a_k \rangle$ be a finitely generated subgroup of G. Then $H/H \cap N \cong HN/N$ is a k-generator group so there exist $b_1, b_2, \ldots, b_s$ such that $H/H \cap N = \langle b_1(H \cap N), b_2(H \cap N), \ldots, b_s(H \cap N) \rangle$. Then each $a_i = w_i(b_1, b_2, \ldots, b_s)c_i$ for some $c_i \in H \cap N$ and some word w_i in $b_1, b_2, \ldots, b_s$. Then $\langle c_1, c_2, \ldots, c_k \rangle$ can be generated by r elements, say $c_i = \widetilde{w}_i(d_1, d_2, \ldots, d_r)$, and hence $H = \langle b_1, b_2, \ldots, b_s, d_1, d_2, \ldots, d_r \rangle$. Thus G has rank at most $r + s$.

□

Next, we establish the following useful lemma.

Lemma 1.10. *Let G be an abelian group of infinite rank. Then G has a proper normal subgroup N of infinite rank which is a direct sum of cyclic subgroups.*

Proof. Let G be an abelian group of infinite rank. Then G has a linearly independent subset S of infinite cardinality. If $\langle S \rangle \neq G$, then $\langle S \rangle$ is a proper normal subgroup of G of infinite rank, and by Lemma 1.8, it is a direct sum of cyclic subgroups. Therefore we assume that $\langle S \rangle = G$. Let $x \in S$. Consider $T = S \setminus \{x\}$. Then, clearly $\langle T \rangle \subsetneqq \langle S \rangle = G$ and $\langle T \rangle$ has infinite rank. Thus, $\langle T \rangle$ is a proper normal subgroup of infinite rank. By Lemma 1.8, it is a direct sum of cyclic subgroups.

□

There is a vast literature concerned with groups of finite rank and groups with certain subgroups of finite rank. For example, the following lemma from [**7**] gives a strong result about locally soluble groups with all proper subgroups of finite rank.

Lemma 1.11. [**7**, Lemma 1] *Let G be a locally soluble group and suppose that every proper subgroup of G has finite rank. Then G has finite rank.*

Also, in [**6**], we have the following theorem about locally (soluble-by-finite) groups with all locally soluble subgroups of finite rank.

Theorem 1.6. [**6**, Theorem] *Let G be a locally (soluble-by-finite) group with all locally soluble subgroups of finite rank. Then G has finite rank and is almost locally soluble.*

Moreover, the following well-known and useful lemma gives the structure of a locally soluble group with finite rank.

Lemma 1.12. [**19**, Lemma 10.39] *Let G be a locally soluble group with finite rank r. Then there is a non-negative integer n depending only on r such that $G^{(n)}$ is a periodic hypercentral group with Černikov primary components.*

1.5. Radical Groups

It is well-known that the product of two normal nilpotent subgroups is again nilpotent which is Fitting's Theorem. The following well-known result due to Hirsch and Plotkin shows that the same result holds for normal locally nilpotent subgroups. It is known as The Hirsch-Plotkin Theorem.

Theorem 1.7. [**20**, 12.1.2] *Let H and K be normal locally nilpotent subgroups of a group. Then the product $J = HK$ is locally nilpotent.*

As a result of this theorem, we conclude that in any group G there is a unique maximal normal locally nilpotent subgroup containing all normal locally nilpotent subgroups of G, which is called the *Hirsch-Plotkin radical* of G. We denote it by $\mathrm{HP}(G)$ and it is a characteristic subgroup of G. Next, we can define the *upper Hirsch-Plotkin series* $\{HP_\alpha(G)\}$ of the group G by

$$\begin{aligned} \mathrm{HP}_0(G) &= 1 \\ \mathrm{HP}_{\alpha+1}(G)/\mathrm{HP}_\alpha(G) &= \mathrm{HP}(G/\mathrm{HP}_\alpha(G)) \text{ for ordinals } \alpha \\ \mathrm{HP}_\gamma(G) &= \bigcup_{\beta<\gamma} \mathrm{HP}_\beta(G) \text{ for limit ordinals } \gamma \end{aligned}$$

In Section 1.3, we defined an ascending series for a group. Clearly, $\{\mathrm{HP}_\alpha(G)\}$ is an ascending locally nilpotent series of charateristic subgroups. Now, we are ready to define an important class of groups.

Definition 1.13. *A group which has an ascending locally nilpotent series is said to be a* radical group.

The class of radical groups is an important class as it contains the locally nilpotent groups and the soluble groups. However, it does not contain the class of locally soluble groups. Note that a group G is radical if and only if its upper Hirsch-Plotkin series terminates at G. So, we conclude that a radical group has at least one ascending locally nilpotent series of characteristic subgroups.

Next, we state the following useful lemma which concludes that the product of normal radical subgroups is again radical.

Lemma 1.13. *Let G be a group and H be a subgroup of G. Assume that H is a product of proper normal radical subgroups. Then H is radical.*

Proof. Let H be a product of proper normal radical subgroups, that is, $H = \prod_\alpha N_\alpha$ where N_α is a proper normal radical subgroup of G for each α. Let N_α be any proper normal radical subgroup of G which occurs in the product $H = \prod_\alpha N_\alpha$. Then

N_α has an ascending series of characteristic subgroups $1 = N_{\alpha,0} \leq N_{\alpha,1} \leq N_{\alpha,2} \leq \ldots \leq N_{\alpha,\lambda_\alpha} = N_\alpha$ such that $N_{\alpha,\theta+1}/N_{\alpha,\theta}$ is locally nilpotent for any ordinal θ and $N_\beta = \cup_{\theta<\beta} N_{\alpha,\theta}$ if β is a limit ordinal. Note that each term of this series is a normal subgroup of G as N_α is normal in G. Consider the ascending series

$$1 = N_{1,0} \leq N_{1,1} \leq N_{1,2} \leq \ldots \leq N_{1,\lambda_1} = N_1 = N_1N_{2,0} \leq N_1N_{2,1} \leq \ldots \leq N_1N_{2,\lambda_2} = N_1N_2 = N_1N_2N_{3,0} \leq N_1N_2N_{3,1} \leq \ldots \leq \prod_\alpha N_\alpha = H.$$

It is clearly an ascending series of H and each factor of this series is locally nilpotent. □

Now, it is interesting to consider radical groups whose subgroups satisfy rank conditions. The following theorem gives information about the structure of a radical group whose Hirsch-Plotkin Radical has finite rank.

THEOREM 1.8. [**19**, Theorem 8.16] *Let G be a radical group whose Hirsch-Plotkin radical has finite rank r. Then G has a normal subgroup N such that N' is hypercentral and $|G : N|$ is finite and does not exceed a number depending only on r.*

Moreover, in [**1**], Baer and Heineken studied radical groups with finite rank. Then, they obtained a very important result about such groups which is now known as the Baer-Heineken Theorem. This shows, for example, the effect of the abelian subgroups on the structure of G.

THEOREM 1.9 (Baer-Heineken Theorem). [**1**] *Let G be a radical group. Then G has finite rank if and only if the abelian subgroups of G have finite rank.*

This theorem shows us that a radical group G of infinite rank contains an abelian subgroup A of infinite rank. By using the definition of rank, we conclude that A contains a subgroup of infinite rank which is a direct sum of cyclic groups.

1.6. $\mathfrak{X}$-groups

In this section, we will define an extensive and important class of groups which was introduced by N. S. Černikov in [**3**]. Firstly, we start with the following well-known definition.

Definition 1.14. *A group G is* locally graded *if every finitely generated non-trivial subgroup of G has a finite nontrivial image.*

The structure of locally graded groups with all proper subgroups of finite rank, and even the structure of locally graded groups of finite rank, is unknown, although well-researched in other classes of groups.

We can now describe the subclass of locally graded groups which we plan to discuss. Let Λ denote the set of closure operations $\{\mathbf{L}, \mathbf{R}, \acute{\mathbf{P}}, \grave{\mathbf{P}}\}$; these closure operations are the ones in standard use as defined in [**19**]. First of all, we define the closure operations $\mathbf{L}, \mathbf{R}, \acute{\mathbf{P}}$ and $\grave{\mathbf{P}}$. Let $\mathfrak{D}$ be a class of groups. Now, $\mathbf{L}\mathfrak{D}$ is the class of *locally $\mathfrak{D}$-groups*, consisting of all groups G such that every finite subset of G is contained in a $\mathfrak{D}$-subgroup. The use of the term "locally" is due to D. H. McLain. The class $\mathbf{R}\mathfrak{D}$ is called the class of *residually $\mathfrak{D}$-groups.* A group G is a residually $\mathfrak{D}$-group if and only if to each non-trivial element x of G there corresponds a normal subgroup $N(x)$ not containing x such that $G/N(x) \in \mathfrak{D}$. A group G is in the class $\acute{\mathbf{P}}\mathfrak{D}$ if and only if G has an ascending $\mathfrak{D}$-series, that is, an ascending series each of whose factors is a $\mathfrak{D}$-group. Similarly, a group G is in the class $\grave{\mathbf{P}}\mathfrak{D}$ if and only if G has a descending $\mathfrak{D}$-series. So, a class $\mathfrak{D}$ of groups is Λ-closed if and only if it is closed under the formation of local systems, subcartesien products and both ascending and descending normal series. In [**3**], Černikov defined a class $\mathfrak{X}$ of groups by taking the Λ-closure of the class of periodic locally graded groups. The class $\mathfrak{X}$ is an extensive class of groups containing, in particular, the classes of locally (soluble-by-finite) groups, radical groups and residually finite groups. We remark that the class $\mathfrak{X}$ is subgroup closed and closed under taking factor groups.

The following result shows that every $\mathfrak{X}$-group is locally graded, however it is unknown if every locally graded group is an $\mathfrak{X}$-group. For the proof of the following result, we refer the reader to [**14**].

PROPOSITION 1.8. *Every $\mathfrak{X}$-group is locally graded.*

Now we may inquire as to the structure of $\mathfrak{X}$-groups of finite rank. The following result is due to N. S. Černikov and gives a strong result about such groups.

THEOREM 1.10 (N. S. Černikov). [**3**] *An $\mathfrak{X}$-group of finite rank is (locally soluble)-by-finite.*

This theorem is very useful in the study of groups of finite rank. For example, if we consider an $\mathfrak{X}$-group G, we already know that the subgroups of G of finite rank are (locally soluble)-by-finite.

CHAPTER 2

MAIN RESULT

In this chapter, we will go over the main result of this research. Our aim is to determine the structure of the $\mathfrak{X}$-groups with all subgroups permutable or of finite rank and we prove the following theorem.

MAIN THEOREM. *Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then every subgroup of G is permutable.*

By combining the Main Theorem, Theorem 1.2, Theorem 1.3 and , Theorem 1.5 we can describe the structure of $\mathfrak{X}$-groups in which every subgroup is permutable or of finite rank.

This research was motivated by the papers [**9**] and [**15**]. These papers are concerned with the structure of groups in which every subgroup is either subnormal or of finite rank. In [**9**], the authors consider $\mathfrak{X}$-groups in which the subgroups of infinite rank are subnormal of defect at most d, for some fixed positive integer d and they obtained the following strong result concerning the structure of such groups.

THEOREM 2.1. [**9**, Theorem B] *Let G be an $\mathfrak{X}$- group in which every subgroup of infinite rank is subnormal of bounded defect at most d. Then G is of finite rank or nilpotent of class at most $f(d)$ where $f(d)$ is a positive integer depending only on d.*

Hence, this theorem generalizes a theorem of Roseblade [**24**] which states that if G is a group in which every subgroup is subnormal of bounded defect at most d, then G is nilpotent of class at most $f(d)$ where $f(d)$ is a positive integer depending only on d. In [**15**], no such bound on the defects is assumed and the authors obtain various results. For example, in Theorem 2 of [**15**], the authors consider locally (soluble-by-finite) groups of infinite rank, with all subgroups subnormal or of finite rank and they obtained the following result.

Theorem 2.2. [**15**, Theorem 2] *Let G be a locally soluble-by-finite group in which every subgroup of infinite rank is subnormal. If G has infinite rank then G is soluble, hence a Baer group.*

Moreover, in Theorem 3 of [**15**], torsion-free such groups are considered and the authors obtained the following result.

Theorem 2.3. [**15**, Theorem 3] *Let G be a torsion-free locally soluble-by-finite group in which every subgroup of infinite rank is subnormal. If G has infinite rank then G is nilpotent.*

Note that in the last two theorems, the class of locally soluble-by-finite groups are considered instead of an extensive class of groups such as locally graded groups. Recall that even the structure of locally graded groups of finite rank is unknown, hence it can be one reason why the authors did not consider this class instead of locally soluble-by-finite groups.

CHAPTER 3

PRELIMINARY RESULTS

We obtain several preliminary results which deal with certain special cases that arise in the proofs. Throughout we shall use the notation $A_1 \times A_2 \times A_3 \times \dots$ to denote the restricted direct product of the groups A_i.

LEMMA 3.1. *Let G be a group of infinite rank in which every subgroup of infinite rank is permutable. If G has a torsion-free abelian subgroup of infinite rank, then G is abelian.*

PROOF. By hypothesis G contains a permutable subgroup A of infinite rank and of the form $A = A_1 \times A_2 \times A_3 \times \dots$ where each $A_i = \langle a_i \rangle \cong \mathbb{Z}$ for all i.

We claim first that $A_i \trianglelefteq G$ for all i. For each j, define $B_j = A_1 \times A_2 \times \dots \times A_{j-1} \times A_{j+1} \times \dots$. Note that, for any j, $a_j^k \notin B_j$ for all nonzero $k \in \mathbb{Z}$ since $A_j \cap B_j = 1$. Also, every subgroup that contains B_j is permutable. Hence, by Lemma 1.3, $B_j \trianglelefteq G$ for all j. Now, for all i, $A_i = \cap_{j \neq i} B_j$ and hence $A_i \trianglelefteq G$.

Let $C = A_1 \times A_3 \times A_5 \times \dots$ and let $D = A_2 \times A_4 \times A_6 \times \dots$. Since C has infinite rank it follows that all subgroups of G/C are permutable. Now Theorem 1.2 implies that a non-abelian group with elements of infinite order and all subgroups permutable has torsion-free rank 1. Since G/C has infinite torsion-free rank it follows that G/C is abelian. In the same way G/D is also abelian. However G embeds in $G/C \times G/D$, since $C \cap D = 1$, and hence G is abelian, as required.

□

Later, we show that $\mathfrak{X}$-groups with all subgroups permutable or of finite rank are locally nilpotent. For the locally nilpotent groups, we have the following result.

Lemma 3.2. *Let G be a locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of $T(G)$ permute.*

Proof. We may assume that G is not abelian. Since G is radical of infinite rank it has an abelian subgroup of infinite rank, by Theorem 1.9. Hence, by Lemma 3.1, G has an abelian subgroup of the form $B = B_1 \times B_2 \times B_3 \times \ldots$ where, for all i, $B_i \cong \mathbb{Z}_{p_i^{n_i}}$ for some prime p_i and for some positive integer n_i. For each positive integer i define $A_i = B_i \times B_{i+1} \times B_{i+2} \times \ldots$. Then, we have an infinite sequence $A_1, A_2, A_3, \ldots$ of abelian subgroups of infinite rank such that $A_1 \gneqq A_2 \gneqq A_3 \gneqq \ldots$. Clearly $\cap_{i\geq 1} A_i = 1$.

Let g and h be elements in the torsion subgroup $T(G)$ of G. We claim that $\langle g \rangle \langle h \rangle = \cap_{i\geq 1} A_i \langle g \rangle \langle h \rangle$. Note that $A_i \langle g \rangle \langle h \rangle$ is a subgroup of G since A_i is permutable for all i. Furthermore, it is clear that $\langle g \rangle \langle h \rangle \subseteq \cap_{i\geq 1} A_i \langle g \rangle \langle h \rangle$.

Conversely, let x be an element of $\cap_{i\geq 1} A_i \langle g \rangle \langle h \rangle$. Then $x \in A_i \langle g \rangle \langle h \rangle$ for all i. So we have infinitely many equations

$$x = a_1 g^{m_1} h^{n_1} = a_2 g^{m_2} h^{n_2} = a_3 g^{m_3} h^{n_3} = a_4 g^{m_4} h^{n_4} = \ldots \qquad (1)$$

where $a_i \in A_i$ and m_i, n_i are non-negative integers for all i. Now, g and h have finite order, hence we can find a subset $\{k_1, k_2, k_3, \ldots\}$ of $\{1, 2, 3, \ldots\}$ such that

$$m_0 = m_{k_1} = m_{k_2} = m_{k_3} = \ldots \text{ and } n_0 = n_{k_1} = n_{k_2} = n_{k_3} = \ldots$$

where $m_0, n_0 \in \mathbb{Z}$. Then (1) implies the equations

$$x = a_{k_1} g^{m_0} h^{n_0} = a_{k_2} g^{m_0} h^{n_0} = a_{k_3} g^{m_0} h^{n_0} = \ldots \qquad (2)$$

where $m_0, n_0 \in \mathbb{Z}$ and $a_{k_i} \in A_{k_i}$ for all i. By (2), we conclude that $a_{k_1} = a_{k_2} = a_{k_3} = \ldots$. Thus $a_{k_1} \in \cap_{i\geq 1} A_{k_i} = 1$ and $x = g^{m_0} h^{n_0}$. Hence $x \in \langle g \rangle \langle h \rangle$ so $\langle g \rangle \langle h \rangle = \cap_{i\geq 1} A_i \langle g \rangle \langle h \rangle$. It follows that $\langle g \rangle \langle h \rangle$ is a subgroup and hence $\langle g \rangle \langle h \rangle = \langle h \rangle \langle g \rangle$. Consequently, any two subgroups of $T(G)$ permute by Lemma 1.4.

$\square$

The following corollary is now easy to establish.

Corollary 3.1. *Let G be a periodic locally nilpotent group of infinite rank in which every subgroup of infinite rank is permutable. Then any two subgroups of G permute. Furthermore, G has a proper normal subgroup of infinite rank.*

Proof. The first part is immediate from Lemma 3.2. To prove the second part, note that G is a periodic locally nilpotent group, hence it is direct product of its primary components, and we can write $G = \operatorname{Dr}_p G_p$. Assume that $G_p \neq 1$ for some prime p and that G is not a p-group. If G_p has infinite rank, then G_p is a proper normal subgroup of G of infinite rank. Therefore we may assume that G_p has finite rank for all primes p. Then the normal subgroup $N = \operatorname{Dr}_{q \neq p} G_q$ has infinite rank as G has infinite rank. So, G has a proper normal subgroup of infinite rank if it is not a p-group.

If G is an abelian p-group the result follows from Lemma 1.10. If G is a nonabelian p-group, we know that a periodic locally soluble group is locally finite from [**4**, 1.1.5 Proposition], hence the result follows using Theorem 1.5. □

According to Proposition 1.5, a simple group cannot have a proper nontrivial permutable subgroup. On the other hand, simple $\mathfrak{X}$-groups with all proper subgroups of finite rank are finite, as we show next.

Lemma 3.3. *Let G be a simple $\mathfrak{X}$-group with all proper subgroups of finite rank. Then G is finite.*

Proof. If G is finitely generated, then since it is locally graded, it contains a proper normal subgroup of finite index. Since G is simple it follows that G is finite. So suppose G is not finitely generated. If H is a finitely generated subgroup of G then H is a proper subgroup so has finite rank. Hence H is an $\mathfrak{X}$-group of finite rank and, by Theorem 1.10, H is (locally soluble)-by-finite. Since H is finitely generated it is soluble-by-finite and it follows that G is locally (soluble-by-finite). Assume that G is locally soluble. By a result due to Mal'cev([**16**]), we know that a simple locally soluble group is cyclic of prime order. On the other hand if G is not locally soluble then every locally soluble subgroup of G has finite rank and, by Theorem 1.6, G has finite rank. By Theorem 1.10 again, G must be (locally soluble)-by-finite and hence is finite. □

We next prove a very easy result which has several applications in our proofs.

LEMMA 3.4. *Let G be a group of infinite rank and suppose that M, N are normal subgroups of G such that $M \leq N$ and N/M is finite. Then either $C_G(N/M)$ is a proper normal subgroup of infinite rank or N/M is abelian. In the latter case if M is locally soluble then N is also locally soluble.*

PROOF. We have an embedding

$$G/C_G(N/M) \hookrightarrow \mathrm{Aut}(N/M)$$

and clearly $\mathrm{Aut}(N/M)$ is finite, so $G/C_G(N/M)$ is finite. Since G has infinite rank, $C_G(N/M)$ has infinite rank also. $G \neq C_G(N/M)$, $C_G(N/M)$ is a proper normal subgroup of infinite rank. If $G = C_G(N/M)$ then N/M is abelian. To prove that N is locally soluble when N/M is abelian, let K be a finitely generated subgroup of N. Then, $KM/M \cong K/K \cap M$ is finite abelian since N/M is finite. But since K is finitely generated and $K/K \cap M$ is finite, we have $K \cap M$ is finitely generated. Since M is locally soluble, we then have $K \cap M$ is soluble. Also, an extension of a soluble group by a soluble group is soluble, hence K is soluble. So, N is locally soluble. □

One feature of the proof of the main results in [**9**] that is important is the presence of proper normal subgroups of infinite rank. Our groups also exhibit this feature.

PROPOSITION 3.1. *Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G has a proper normal subgroup of infinite rank.*

Proof. We suppose, for a contradiction, that all proper normal subgroups of G have finite rank. If N is a proper normal subgroup of G then N is an $\mathfrak{X}$-group of finite rank and hence is (locally soluble)-by-finite by Theorem 1.10. If M is the locally soluble radical of N then $M \lhd G$ since M is a characteristic subgroup of N. Now, by Lemma 3.4, either $C_G(N/M)$ is a proper normal subgroup of infinite rank or N/M is abelian. However, by our assumption all proper normal subgroups of G have finite rank, so N/M is abelian. By the second part of Lemma 3.4, N is locally soluble. Thus, all proper normal subgroups of G are locally soluble. Let J be the product of the proper normal subgroups of G. Clearly, $J \lhd G$.

Suppose first that $G \neq J$. Then J has finite rank, by assumption and G/J is simple. However, by Proposition 1.5, a simple group cannot have a proper nontrivial permutable subgroup, so all the proper subgroups of G/J have finite rank. By Lemma 3.3, G/J is finite so G has finite rank, contrary to our hypothesis. Thus G is the product of its proper normal subgroups.

If N is a proper normal subgroup of G then, by Lemma 1.12, there is a positive integer k such that $N^{(k)}$ is a periodic hypercentral group. Thus, N is hyperabelian and hence N is radical. Since G is a product of radical groups, it is radical by Lemma 1.13. If G is not locally nilpotent then $\mathrm{HP}(G)$, the Hirsch-Plotkin radical of G, has finite rank and, by Theorem 1.8, G has a normal subgroup M such that M' is hypercentral and $|G : M|$ is finite. If $M \neq G$ then M has finite rank and hence G has finite rank. Thus $M = G$ and $M' = G'$ is hypercentral. If $G = G'$ then G is a perfect hypercentral group which contradicts Grün's Lemma, [**11**, Satz 4]. If $G' \neq G$, then G' has finite rank. If also G/G' has finite rank then G has finite rank, contrary to our assumption. If the abelian group G/G' has infinite rank, then it has a proper normal subgroup N/G' of infinite rank by Lemma 1.10, so N is a proper normal subgroup of G of infinite rank.

If G is locally nilpotent then Corollary 3.1 shows that G is not periodic. Thus $T(G) \neq G$ so $T(G)$ has finite rank and $G/T(G)$ has infinite rank, so is abelian by Lemma 3.1. Hence G has a proper normal subgroup of infinite rank in this case also, yielding a final contradiction.

□

The proof of the following theorem is based on [**9**, Lemma 2].

Theorem 3.1. *Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is soluble.*

Proof. By Proposition 3.1, G contains a proper normal subgroup of infinite rank, say N. Then by Corollary 1.1, G/N is metabelian since every subgroup of G/N is permutable. So, $G^{(2)} \leq N$. If $G^{(2)}$ has infinite rank then $G^{(2)}$ is an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Hence, by Proposition 3.1, $G^{(2)}$ has a proper normal subgroup N_0 of infinite rank. Again by Corollary 1.1, $G^{(2)}/N_0$ is metabelian. Hence $G^{(4)} \leq N_0$. If $G^{(4)}$ has infinite rank then $G/G^{(4)}$ is metabelian and we have $G^{(4)} \leq N_0 \lneqq G^{(2)} = G^{(4)}$, which is a contradiction. We have established that, in any case, $K = G^{(4)}$ has finite rank. Since K is an $\mathfrak{X}$-group of finite rank, by Theorem 1.10, K is (locally soluble)-by-finite. If S is the locally soluble radical of K, then S is a characteristic subgroup of K, hence it is normal in G. Also it is clear that K/S is finite. Then by Lemma 3.4, either $C_G(K/S)$ is a proper normal subgroup of infinite rank or K/S is abelian. If K/S is abelian, then K is locally soluble by the second part of Lemma 3.4. If $C_G(K/S)$ has infinite rank, then $G/C_G(K/S)$ is metabelian by Corollary 1.1. Then, $K \leq G^{(2)} \leq C_G(K/S)$, so K/S is abelian and hence K is locally soluble of finite rank, by Lemma 3.4. So, in any case, K is locally soluble of finite rank.

We next establish that K is soluble. By Lemma 1.12, there exists a positive integer n such that $K^{(n)}$ is a periodic hypercentral group with Černikov p-primary components. Let R denote the divisible radical of $K^{(n)}$. Clearly, $R \triangleleft G$ and R is a divisible abelian group by its definition. Note that R contains all the divisible normal subgroups of each p-primary component of $K^{(n)}$. Since each primary component of $K^{(n)}$ is Černikov, it is an extension of its divisible radical by a finite group. Hence, $K^{(n)}/R$ has finite p-primary components. Since R is abelian, it is soluble. Thus, in order to prove that K is soluble, we may assume that $R = 1$ since extension of a soluble group by a soluble group is again soluble. So, each p-primary component of $K^{(n)}$ is finite.

Let p be an arbitrary prime, and let P be the p-primary component of $K^{(n)}$. Since P char $K^{(n)}$, we have $P \triangleleft G$ and $C_G(P)$ has infinite rank by Lemma 3.4. Again by Corollary 1.1, $G/C_G(P)$ is metabelian and we have $P \leq G^{(2)} \leq C_G(P)$. Therefore P is abelian and $K^{(n)}$ is also abelian. Thus K is soluble and hence G is soluble. This completes the proof. □

THEOREM 3.2. *Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Then G is a Gruenberg group and hence is locally nilpotent.*

PROOF. By Theorem 3.1, G is soluble and hence is a radical group. We may assume that G is non-abelian. By Theorem 1.9, G has an abelian subgroup T of infinite rank and, by Lemma 3.1, G contains a subgroup A of infinite rank of the form $A = A_1 \times A_2 \times A_3 \times \ldots$ where each $A_i \cong \mathbb{Z}_{p_i^{n_i}}$ for certain primes p_i and certain positive integers n_i. Clearly A is permutable. We claim that A can be written in the form $A = B \times C$ where B, C are proper infinite rank subgroups of A. If A has infinite p-rank for some prime p, then we can write $A = L \times K_1 \times K_2 \times K_3 \times \ldots$ where $K_1 \times K_2 \times K_3 \times \ldots$ is the p-primary component of A, hence each K_i is a cyclic p-group. Then we can choose $B = L \times K_1 \times K_3 \times K_5 \times \ldots$ and $C = K_2 \times K_4 \times K_6 \times \ldots$. Clearly, B and C are proper infinite rank subgroups of A, $B \cap C = 1$ and $A = B \times C$.

If all the primary components of A have finite but unbounded ranks, then we can write $A = M_1 \times M_2 \times M_3 \times \ldots$, where M_i is a q_i-group, q_i's are distinct primes, rank $r(M_i)$ of each M_i is finite, $r(M_i) \leq r(M_j)$ if $i \leq j$ and $r(M_i)$'s are unbounded. Let $B = M_1 \times M_3 \times M_5 \times \ldots$ and $C = M_2 \times M_4 \times M_6 \times \ldots$. Then, B and C are proper infinite rank subgroups of A, $B \cap C = 1$ and $A = B \times C$. So, in any case A can be written in the form $A = B \times C$ where B, C are proper infinite rank subgroups of A. Now, we have to prove that if $g \in G$, then $\langle g \rangle$ is an ascendant subgroup of G.

Suppose first that g is an element of infinite order. Then $g^n \notin A$ for all $n \neq 0$. Clearly $\langle g \rangle B$ and $\langle g \rangle C$ are permutable subgroups of G, since they each have infinite rank, and hence they are ascendant in G, by Theorem 1.4. By Lemma 1.6, $\langle g \rangle B \cap \langle g \rangle C$ is ascendant in G. We claim that $\langle g \rangle = \langle g \rangle B \cap \langle g \rangle C$, from which it will follow that $\langle g \rangle$ is ascendant in G. Clearly, $\langle g \rangle \leq \langle g \rangle B \cap \langle g \rangle C$.

Let $x = bg^i = cg^j \in \langle g \rangle B \cap \langle g \rangle C$ where $b \in B$, $c \in C$ and $i, j \in \mathbb{Z}$. Then we have, $c^{-1}b = g^{j-i} \in A$ and it follows that $j - i = 0$. Hence $b = c = 1$ since $B \cap C = 1$. So, $x = g^i = g^j \in \langle g \rangle$, and we obtain $\langle g \rangle B \cap \langle g \rangle C \leq \langle g \rangle$. Therefore, $\langle g \rangle = \langle g \rangle B \cap \langle g \rangle C$.

Now we suppose that g has finite order k and first assume that $g^r \notin A$ for all r such that $0 < r < k$. As above, we can write $A = B \times C$ where both B and C have infinite rank. Now, $\langle g \rangle B$ and $\langle g \rangle C$ have infinite rank, so they are ascendant and hence their intersection is ascendant. We claim that $\langle g \rangle = \langle g \rangle B \cap \langle g \rangle C$, then clearly $\langle g \rangle$ will be ascendant. Clearly, $\langle g \rangle \leq \langle g \rangle B \cap \langle g \rangle C$. Conversely, let $x = bg^i = cg^j \in \langle g \rangle B \cap \langle g \rangle C$ where $b \in B$, $c \in C$ and i, j some positive integers such that $0 \leq i, j < k$ where without loss of generality $j \leq i$. Then we have $c^{-1}b = g^{j-i} \in A$ and by our assumption we obtain $j - i = 0$. Hence, $b = c = 1$ since $B \cap C = 1$. Thus, $x = g^i = g^j \in \langle g \rangle$, and we have $\langle g \rangle B \cap \langle g \rangle C \leq \langle g \rangle$. Therefore, $\langle g \rangle = \langle g \rangle B \cap \langle g \rangle C$ and it follows that $\langle g \rangle$ is an ascendant subgroup of G.

Assume now that $g^{s_1} \in A$ for some positive integer s_1 such that $0 < s_1 < k$ and assume, for a contradiction, that $\langle g \rangle$ is not ascendant in G. Let $X_1 = A$ and let $g^{s_1} = (a_1, a_2, \ldots, a_{t_1}, 1, 1, \ldots)$ where $t_1 \in \mathbb{N}$ and $a_i \in A_i$ for all i. Then, $g^{s_1} \in A_1 \times A_2 \times \ldots \times A_{t_1}$ but $g^{s_1} \notin A_{t_1+1} \times A_{t_1+2} \times A_{t_1+3} \ldots$. Set $X_2 = A_{t_1+1} \times A_{t_1+2} \times A_{t_1+3} \ldots$, so $g^{s_1} \in X_1 \setminus X_2$ and $X_1 \gneqq X_2$.

By the previous argument, with X_2 replacing A we see that there is a positive integer s_2 such that $0 < s_2 < k$ and $g^{s_2} \in X_2$. Then, as above, X_2 contains a proper subgroup X_3 of infinite rank such that $g^{s_2} \in X_2 \setminus X_3$.

Continuing in this way we construct an infinite chain of infinite rank abelian subgroups $X_1, X_2, X_3, \ldots$ and an infinite sequence $s_1, s_2, s_3, \ldots$ of positive integers such that $X_1 \gneqq X_2 \gneqq X_3 \gneqq \ldots$, $0 < s_i < k$ and $g^{s_i} \in X_i \setminus X_{i+1}$ for all i. However this yields a contradiction as follows. Since $|g| = k$ is finite, there exist positive integers l and m such that $g^{s_l} = g^{s_m}$, where we may assume that $l < m$. Note that $g^{s_l} \in X_l \setminus X_m$ since $l < m$, but $g^{s_m} \in X_m$, which is clearly a contradiction as $g^{s_l} = g^{s_m}$.

Hence $\langle g \rangle$ is an ascendant subgroup of G, G is a Gruenberg group and, by Lemma 1.7, G is locally nilpotent.

□

LEMMA 3.5. *Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup of infinite rank is permutable. Let g be an element of infinite order and h be an element of finite order. Then $\langle g \rangle \langle h \rangle$ is a subgroup of G and hence $\langle g \rangle$ and $\langle h \rangle$ permute.*

PROOF. We may assume that G is not abelian. By Theorem 3.2, G is locally nilpotent. We will follow a similar argument to that of Lemma 3.2. As in Lemma 3.2, G has a subgroup of type $B = B_1 \times B_2 \times B_3 \times \ldots$ where for any i, $B_i \cong \mathbb{Z}_{p_i^{n_i}}$ for certain primes p_i and certain positive integers n_i. For each positive integer i, define $A_i = B_i \times B_{i+1} \times B_{i+2} \times \ldots$. Then A_i has infinite rank, $A_1 \gneqq A_2 \gneqq A_3 \gneqq \ldots$ and $\cap_{i \geq 1} A_i = 1$.

We claim that $\langle g \rangle \langle h \rangle = \cap_{i \geq 1} A_i \langle g \rangle \langle h \rangle$. Clearly $\langle g \rangle \langle h \rangle \subseteq \cap_{i \geq 1} A_i \langle g \rangle \langle h \rangle$.

Conversely, let x be an element of $\cap_{i\geq 1} A_i \langle g\rangle \langle h\rangle$. Then $x \in A_i \langle g\rangle \langle h\rangle$ for all i and we have infinitely many equations

$$x = a_1 g^{m_1} h^{n_1} = a_2 g^{m_2} h^{n_2} = a_3 g^{m_3} h^{n_3} = a_4 g^{m_4} h^{n_4} = \dots \qquad (1)$$

where $a_i \in A_i$, $m_i \in \mathbb{Z}$ and $n_i \in \mathbb{Z}$ for all i. Since h has finite order, we can find a subset $\{k_1, k_2, k_3, \dots\}$ of $\{1, 2, 3, \dots\}$ such that

$$n_{k_1} = n_{k_2} = n_{k_3} = \dots$$

Let $n_0 \in \mathbb{Z}$ be the common value. Then (1) implies the equations

$$x = a_{k_1} g^{m_{k_1}} h^{n_0} = a_{k_2} g^{m_{k_2}} h^{n_0} = a_{k_3} g^{m_{k_3}} h^{n_0} = \dots \qquad (2)$$

where $n_0 \in \mathbb{Z}$, $m_{k_1}, m_{k_2}, \dots \in \mathbb{Z}$ and $a_{k_i} \in A_{k_i}$ for all i. So by (2), we have the equations

$$a_{k_1} g^{m_{k_1}} = a_{k_2} g^{m_{k_2}} = a_{k_3} g^{m_{k_3}} = \dots \qquad (3)$$

where $m_{k_i} \in \mathbb{Z}$ and $a_{k_i} \in A_{k_i}$ for all i.

It is clear that $g^{m_{k_i}}(g^{m_{k_j}})^{-1} = (a_{k_i})^{-1} a_{k_j} \in B \cap \langle g\rangle$ for all i, j. Since $B \cap \langle g\rangle = 1$ we have

$$m_{k_1} = m_{k_2} = m_{k_3} = \dots \text{ and } a_{k_1} = a_{k_2} = a_{k_3} = \dots$$

Let $m_0 \in \mathbb{Z}$ be the common value. Thus $a_{k_1} \in \cap_{i\geq 1} A_{k_i} = 1$ and $x = g^{m_0} h^{n_0}$ by (2). Hence $x \in \langle g\rangle \langle h\rangle$ and $\cap_{i\geq 1} A_i \langle g\rangle \langle h\rangle \subseteq \langle g\rangle \langle h\rangle$. It follows that $\langle g\rangle \langle h\rangle = \cap_{i\geq 1} A_i \langle g\rangle \langle h\rangle$ and we deduce that $\langle g\rangle$ and $\langle h\rangle$ permute.

□

Lemma 3.6. *Let G be an $\mathfrak{X}$-group of infinite rank in which every subgroup is permutable or of finite rank. If G has elements of infinite order, then every subgroup of $T(G)$ is normal in G.*

Proof. Let $x \in T(G)$ and let y be an element of infinite order. Then, by Lemma 3.5, $\langle x\rangle \langle y\rangle$ is a subgroup of G. Since $T(G)$ is a characteristic subgroup of G, we have $\langle x\rangle \langle y\rangle \cap T(G) \trianglelefteq \langle x\rangle \langle y\rangle$. Since $\langle x\rangle \leq T(G)$ we have

$$\langle x\rangle \langle y\rangle \cap T(G) = \langle x\rangle (\langle y\rangle \cap T(G)) = \langle x\rangle$$

by the Dedekind Modular Law. Hence, $\langle x\rangle \trianglelefteq \langle x\rangle \langle y\rangle$ and $y \in N_G(\langle x\rangle)$. But y is arbitrary, so every element of infinite order normalizes $\langle x\rangle$. Since G is locally nilpotent, by Theorem 3.2, G is generated by its elements of infinite order and hence $\langle x\rangle \trianglelefteq G$. Since x is arbitrary, we conclude that for any $x \in T(G)$, $\langle x\rangle \trianglelefteq G$. Thus, every subgroup of $T(G)$ is normal in G.

□

CHAPTER 4

PROOF OF THE MAIN THEOREM

Now we can prove the Main Theorem. The proof makes heavy use of Theorem 1.2 and Theorem 1.3.

Proof of the Main Theorem. By Theorem 3.2 G is locally nilpotent and clearly we may assume that G is nonabelian. It is enough to prove that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for all $x, y \in G$, by Lemma 1.4, and by Lemmas 3.2 and 3.5 we may assume that x and y have infinite order.

Since G has infinite rank it has an abelian subgroup T of infinite rank by Theorem 1.9. By Lemma 3.1 we may assume that the torsion-free rank of T is finite and it then follows that G has a permutable subgroup A of infinite rank such that $A = A_1 \times A_2 \times A_3 \times \ldots$ where $A_i = \langle a_i\rangle \cong \mathbb{Z}_{p_i^{n_i}}$, p_i is a prime and n_i is a positive integer for all i.

First we prove that $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank one. Since A has infinite rank, we can write $A = B \times C$ where both B and C have infinite rank, and $B \cap C = 1$. Now every subgroup of $T(G)$ is normal by Lemma 3.6 so B and C are normal subgroups of G. Note that all subgroups of G/B and G/C are permutable and hence, by [**26**, 2.4.8 Lemma], both $T(G/B)$ and $T(G/C)$ are abelian. Clearly $T(G/B) = T(G)/B$ and $T(G/C) = T(G)/C$. Since $T(G)/B \times T(G)/C$ is abelian the embedding

$$T(G) \hookrightarrow T(G)/B \times T(G)/C\ .$$

implies that $T(G)$ is abelian.

If G/B and G/C are both abelian then, the embedding

$$G \hookrightarrow G/B \times G/C \ .$$

yields that G is abelian, contrary to our assumption. Thus one of G/B or G/C is nonabelian, say G/B is nonabelian. Then by Theorem 1.2, $(G/B)/T(G/B)$ is a torsion-free abelian group of rank one. But $(G/B)/T(G/B) \cong G/T(G)$, hence $G/T(G)$ is a torsion-free abelian group of rank one.

Now it is clear that $T(G)$ has infinite rank. To prove that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ we merely need to show that $\langle x\rangle\langle y\rangle T(G)$ is a group with all subgroups permutable. Since $\langle x, y\rangle T(G)/T(G)$ is finitely generated it is cyclic. Thus for the remainder of the proof we may assume that $G/T(G)$ is infinite cyclic.

We claim that G is the semidirect product of $T(G)$ by an infinite cyclic group $\langle z\rangle$, and that for every prime p there exists a p-adic unit $r(p)$ with $r(p) \equiv 1 \pmod p$ and $r(2) \equiv 1 \pmod 4$ such that $a^z = a^{r(p)}$ for all $a \in T(G)_p$, the p-component of $T(G)$. Once the claim is established the Main Theorem will follow by Theorem 1.2.

Since $G/T(G)$ is cyclic, we can find an element z of infinite order such that $G/T(G) = \langle zT(G)\rangle$. Clearly, $T(G) \cap \langle z\rangle = 1$. Thus, G is the semidirect product of $T(G)$ by $\langle z\rangle$ and by Lemma 3.6, z induces a power automorphism in $T(G)$. Hence, for every prime p, by [**26**, 1.5.6 Lemma] there exists a p-adic unit $r(p)$ such that $a^z = a^{r(p)}$ for all $a \in T(G)_p$. If $T(G)_p = 1$, then let $r(p) = 1$ and, if $\mathrm{Exp}T(G)_2 \leq 2$, then let $r(2) = 1$. So assume that $T(G)_p \neq 1$ and $\mathrm{Exp}T(G)_2 \geq 4$.

Suppose that $T(G)_p \neq 1$. Let $b \in T(G)_p$ be an element of order p. As above there is a subgroup B of A such that B has infinite rank, G/B is non-abelian and $B \cap \langle b\rangle = 1$.

It follows from Theorem 1.2 that G/B is the semidirect product of $T(G/B)$ by an infinite cyclic group, say $\langle wB\rangle$. Furthermore, there exists a p-adic unit $\overline{r}(p)$ with $\overline{r}(p) \equiv 1 \pmod p$ and $\overline{r}(2) \equiv 1 \pmod 4$ such that $(aB)^{wB} = (aB)^{\overline{r}(p)}$ for all $aB \in T(G/B)_p$, the p-component of $T(G/B)$. Then $b^w = b^{\overline{r}(p)}c$ for some $c \in B$ and, by Lemma 3.6, $c = 1$. Since $G = T(G)\langle w\rangle = T(G)\langle z\rangle$ we have $r(p) \equiv \overline{r}(p) \equiv 1 \pmod{p}$.

The case when $\mathrm{Exp}T(G)_2 \geq 4$ can be handled in a similar way and we leave the details for the reader to deduce that $r(2) \equiv 1 \pmod{4}$.

$\square$

So, by the Main Theorem, it is clear that if a group G is a nonabelian $\mathfrak{X}$-group with all subgroups permutable or of finite rank, then it satisfies Theorem 1.2, Theorem 1.3 or Theorem 1.5.

CHAPTER 5

GROUPS WITH ALL SUBGROUPS PERMUTABLE OR NILPOTENT OF BOUNDED CLASS

In this final chapter, we consider another class of groups with certain properties on their non-permutable subgroups. This begins some preliminary investigations into the problem of the structure of groups with all subgroups permutable or soluble.

A group G is said to be a W*-group*, if every finitely generated non-nilpotent subgroup of G has a finite non-nilpotent image. The class of W-groups in which all proper subgroups are locally nilpotent or normal was described by B. Bruno and R.E. Phillips in detail in [**2**]. They considered the class of W-groups G such that G is not locally nilpotent and every subgroup of G is either locally nilpotent or normal. For example, in [**2**, Theorem B], they proved that such groups are soluble of derived length at most 3 and every locally nilpotent subgroup of such a group is nilpotent of class at most 2. Also, J. Otal and J. M. Peña studied locally graded groups in which every proper subgroup is normal or $\mathfrak{N}_c$ in [**18**]. They proved in [**18**, 2.1] that such a group G is $\mathfrak{N}_c$ or G' is finite. The research in this chapter was mainly motivated by the papers [**2**] and [**18**].

We shall prove the following theorem.

THEOREM 5.1. *Let G be a locally graded group with all subgroups permutable or nilpotent of class at most c. Then G is soluble of derived length at most $d = 4+[\log_2 c]$.*

Firstly, we note the following result from [**2**] for the proof of Theorem 5.1.

Proposition 5.1. [**2**, Proposition 2] *If G is locally graded and every subgroup of G is normal or nilpotent of class not exceeding c, then G is a W-group.*

Before proving Theorem 5.1, we establish the following useful result.

Lemma 5.1. *Let G be a finite group with all subgroups $\mathfrak{N}_c$ or permutable. Then G is soluble.*

Proof. Let G be a minimal counter example. Hence, G is not nilpotent. If all proper subgroups of G are nilpotent of class at most c, then by a result due to O.J. Schmidt in [**20**, 9.1.9], G is soluble. Hence, G has a proper permutable subgroup H. Hence by Proposition 1.5, G is not simple. Thus, G has a proper non-trivial normal subgroup N. By minimality of G, we have both N and G/N are soluble. Hence, G is soluble which is a contradiction and the result follows.

□

Now we can prove the Theorem 5.1.

Proof of Theorem 5.1. Firstly, we know that $\mathbf{L}\mathfrak{S}_d = \mathfrak{S}_d$. Hence if G is not finitely generated and if every finitely generated subgroup of G is soluble of derived length at most d, then G is $\mathbf{L}\mathfrak{S}_d$, hence G is soluble of derived length at most d. Thus, it is enough to prove the theorem when G is finitely generated.

Let G be a locally graded group with all subgroups permutable or nilpotent of class at most c. Assume that G is finitely generated and not soluble of derived length at most d, where $d = 4 + [\log_2 c]$. Hence, $G^{(n)} \neq 1$ for all positive integers $n \leq d$. Let $e = 1 + [\log_2 c]$, hence $d - e = 3$. Define $X = G^{(d-e)} = G^{(3)}$. Now, if $X \in \mathfrak{N}_c$, then by Proposition 1.3, X is soluble of derived length at most e, hence we have $X^{(e)} = G^{(d)} = 1$, a contradiction.

So, we have $X \notin \mathfrak{N}_c$. Thus, every subgroup of G/X is permutable and by Corollary 1.1, we have $G'' \leq X = G^{(3)}$, hence $X = G''$ is perfect. Now, G/X is a finitely generated soluble group. If G/X is periodic then by Proposition 1.2, it is finite. Since G is finitely generated, X is finitely generated. So, X is a finitely generated locally graded group, hence it has a proper normal subgroup U of finite index. Now, X/U is a finite group with all subgroups $\mathfrak{N}_c$ or permutable, hence by Lemma 5.1, we have X/U is soluble, a contradiction since X is perfect.

Hence, G/X is not periodic. Note that since every subgroup of G/X is permutable, it is locally nilpotent by Proposition 1.6. Hence, it is generated by its elements of infinite order. Now, our aim is to prove that every subgroup of X is $\mathfrak{N}_c$ or normal. Let $K \leq X$ and $K \notin \mathfrak{N}_c$. Then, $K^G \leq X$ as $X \triangleleft G$ and, $K^G \notin \mathfrak{N}_c$. Hence, all subgroups of G/K^G are permutable. So, by Corollary 1.1, G/K^G is metabelian. Thus, we have $X = G'' \leq K^G$ which implies $X = K^G$. So, we conclude that G/K^G is a non-periodic locally nilpotent group generated by its elements of infinite order. Note that $K \notin \mathfrak{N}_c$, hence every subgroup containing K is permutable. Also, G/K^G is non-periodic, so there exists $g \in G$ such that $g^n \notin K^G$ for every nonzero integer n. Now, since $K \leq K^G$, g has infinite order modulo K. Thus, we conclude that K is normal in G by Lemma 1.3.

Thus, we conclude that every subgroup of X is $\mathfrak{N}_c$ or normal. Hence, by Proposition 5.1, X is a W-group. Now, if X is not locally nilpotent, then by a theorem in [**2**, Theorem B], X is soluble, a contradiction since X is perfect. If X is locally nilpotent, then by a result in [**18**, 2.1], $X \in \mathfrak{N}_c$ or X' is finite. However, since X is perfect, X' is finite. But then, X is soluble by Lemma 5.1, which gives us the final contradiction. Hence, $G \in \mathfrak{S}_d$.

□

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