A DIFFERENCE OF COMPOSITION OPERATORS ON BERGMAN SPACE

by

SOUMYADIP ACHARYYA

KABE MOEN, COMMITTEE CHAIR
ZHIJIAN WU, CO-CHAIR
ROBERT L. MOORE
YANG XIAO
JAMES WANG

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ABSTRACT

Let us consider the difference operator $C_\varphi - C_\psi$ acting on A^2_α . In an effort to characterize Hilbert-Schmidtness of $C_\varphi - C_\psi$ in terms of integrability condition involving σ, the pseudohyperbolic distance between the inducing functions φ and ψ, Choe, Hosokawa and Koo [**CHK**] proved that:

$$\|C_\varphi - C_\psi\|^2_{\mathrm{HS}} \asymp \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA_\alpha(z)}{(1-|\varphi(z)|^2)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA_\alpha(z)}{(1-|\psi(z)|^2)^{2+\alpha}}\,.$$

At the very beginning of this dissertation, we give a simpler proof of this result using a change of variable method.

Let $0 < p \leq q < \infty$. Saukko [**S**] , in an effort to characterize the boundedness and compactness of the difference operator $C_\varphi - C_\psi$ from A^p_α to A^q_β , proved the following estimates.

$$\text{(i) } \|C_\varphi - C_\psi\| \asymp \sup_{a\in\mathbb{D}} \|\,(C_\varphi - C_\psi)\,k_a\|_{q,\beta} \asymp \max\{\|\sigma C_\varphi\|, \|\sigma C_\psi\|\}\,,$$

where the comparability constants will depend only on α, β, p and q.

(ii) Let $1 < p \leq q < \infty$. Assume $C_\varphi - C_\psi$ is bounded from A^p_α to A^q_β. Then

$$\|C_\varphi - C_\psi\|_e \asymp \overline{\lim_{|a|\to 1}}\,\|\,(C_\varphi - C_\psi)\,k_a\|_{q,\beta} \asymp \max\{\|\sigma C_\varphi\|_e\,, \|\sigma C_\psi\|_e\}\,,$$

where the comparability constants will depend only on α, β, p and q.

We generalize Saukko's results taking $L^q(\mu)$ as our target function space where μ can be any non-negative Borel measure on $\mathbb{D}$.

Finally, we will give a complete characterization of the Hilbert-Schmidtness, boundedness and compactness of another class of difference operators.

DEDICATION

This dissertation is dedicated to all those who supported me directly and indirectly throughout this journey.

LIST OF SYMBOLS

$\mathbb{C}$	The complex plane
$\mathbb{D}$	The open unit disk in the complex plane $\mathbb{C}$
$H(\mathbb{D})$	The space of all analytic functions on $\mathbb{D}$
A^p_α	Standard weighted Bergman space with $0 < p < \infty,\ \alpha > -1$
k_a	Normalized Bergman kernel
$L^p(\mu)$	Lebesgue space over $\mathbb{D}$ with respect to a non-negative Borel measure μ, $p > 0$
$\sigma_a(b)$	$\dfrac{a-b}{1-\overline{a}b}$, $a\,, b \in \mathbb{D}$
$\rho\ (a,b)$	$\lvert\sigma_a(b)\rvert = \left\lvert\dfrac{a-b}{1-\overline{a}b}\right\rvert$
φ, ψ	Analytic maps : $\mathbb{D} \to \mathbb{D}$
$\sigma(z)$	$\sigma_{\varphi(z)}(\psi(z)) = \dfrac{\varphi(z)-\psi(z)}{1-\overline{\varphi(z)}\psi(z)}$, $z \in \mathbb{D}$
C_φ	The composition operator with symbol φ
u	A measurable function on $\mathbb{D}$
uC_φ	The weighted composition operator with weight u and symbol φ
$H^\infty(\mathbb{D})$	The space of all bounded analytic functions on $\mathbb{D}$
$T^\star$	The adjoint of a bounded operator T
$\lVert T\rVert_e$	The essential norm of a bounded operator T
$\lVert T\rVert_{HS}$	The Hilbert-Schmidt norm of a bounded operator T
$\asymp$	Comparability notation

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"Education is the manifestation of the perfection already in man."
– Swami Vivekananda.

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CONTENTS

CHAPTER 1

INTRODUCTION

Let φ be a holomorphic self-map on $\mathbb{D}$. Then

$$C_{\varphi}(f) = f \circ \varphi \,,\ f \in H(\mathbb{D})$$

clearly defines a linear operator on the vector space $H(\mathbb{D})$. This operator is called **The composition operator with symbol** φ.

An important generalization of composition operator is weighted composition operator. If u is a measurable function on $\mathbb{D}$, then **The weighted composition operator** uC_{φ} is defined in $H(\mathbb{D})$ by

$$(uC_{\varphi})(f)(z) = u(z)\, f \circ \varphi(z) \,,\ f \in H(\mathbb{D}), z \in \mathbb{D}.$$

The main purpose of studying composition operators is to describe the operator theoretic properties(for example, boundedness, compactness, schatten-class membership etc.) of C_{φ} in terms of the function theoretic properties of φ. The domain of a composition operator is usually taken to be some Banach space consisting of holomorphic functions: for example, Bergman space, Hardy space, and the Dirichlet space. For the purpose of this work, We will limit our analysis mostly to composition operators whose domain are standard weighted Bergman spaces A^p_{α}. Occasionally we will also refer to the H^p spaces. Definitions, along with all necessary background material, will follow in chapter 2, but we will first review some earlier history and then some relatively recent results that motivate our current work.

The question of boundedness of composition operators is a basic one. Even among some very well-known Holomorphic spaces, it may happen that some natural composition operators are unbounded because the space is "too small" or because it is "too big". On the spaces of "medium size", C_φ is usually bounded. For example, it is a well-known application of Littlewood subordination principle (See [**CM**]) that composition operators are always bounded on H^p spaces with the following norm estimates.

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{1}{p}} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{p}}.$$

Using similar techniques , as indicated in [**CM**] , it can also be shown that composition operators are always bounded on weighted Bergman spaces A^p_α with the following norm estimates.

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{2+\alpha}{p}} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{2+\alpha}{p}}.$$

Along with boundedness results, much effort has also been expended on characterizing those analytic maps φ which induce compact or Hilbert-Schmidt composition operators. For a detailed history, we refer to [**ST**], [**MS**], [**SH**1] and [**CM**].

Another area of particular interest is the topological structure of the space of all composition operators. When X is a Banach space of analytic functions , let $\mathcal{C}(X)$ denote the space of all composition operators on X with topology induced by the operator norm metric. Investigation on the topological structure of $\mathcal{C}(X)$ started in 1981, with Berkson's isolation result for $\mathcal{C}(H^p)$, $1 \leq p < \infty$:

Berkson's Isolation Theorem: If φ has radial limits of modulus 1 on a set $E \subseteq \partial\mathbb{D}$ of positive measure, then for every other holomorphic self map ψ of $\mathbb{D}$,

$$\|C_\varphi - C_\psi\| \geq \sqrt{\frac{|E|}{2}}.$$

The above clearly implies that the composition operator C_φ is isolated (i.e. singleton component) in the topological space $\mathcal{C}(H^p)$, $1 \leq p < \infty$. This is how the study of the component structure of $\mathcal{C}(H^p)$ got associated with studying the difference operator $C_\varphi - C_\psi$. Berkson's result has been further generalized by Barbara D.MacCluer [**M2**] and, at about the same time, by J.H.Shapiro and Carl Sundberg [**SS**]. Components in $\mathcal{C}(H^2)$ have been also studied in [**M2**] and [**SS**]. Based on some support that came from [**M2**] and [**CM**] , Shapiro and Sundberg , in [**SS**] , made the following conjecture:

Two composition operators may belong to the same component of $\mathcal{C}(H^2)$ *iff they differ by a compact operator.*

The conjecture remained open for 11 years, until Moorhouse and Toews [**MT**] found a counter-example in 2001. Two years later, Paul Bourdon [**B**] produced a whole class of counter-examples.

This is how the effort to characterize compact difference of composition operators began for Hardy and Bergman spaces. In 2005 , Moorhouse [**M1**] proved that the difference operator $C_\varphi - C_\psi$ is compact on A^2_α iff both

$$\lim_{|z|\to 1} |\sigma(z)| \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0 \text{ and } \lim_{|z|\to 1} |\sigma(z)| \frac{1-|z|^2}{1-|\psi(z)|^2} = 0 \, .$$

Moorhouse , in the same paper , also proved that if u is a bounded analytic function on $\mathbb{D}$, then the weighted composition operator uC_φ is compact on A^2_α iff

$$\lim_{|z|\to 1} |u(z)| \frac{1-|z|^2}{1-|\psi(z)|^2} = 0 \, .$$

Since σ is bounded on $\mathbb{D}$, the above two theorems motivated Erno Saukko[**S**] , in 2011 , to suspect that there might be some strong connection between the difference operator $C_\varphi - C_\psi$ and the weighted composition operators σC_φ and σC_ψ . In fact , in [**S**] , he proved there is such a connection. More precisely,

(**Boundedness of** $C_\varphi - C_\psi$): Suppose $0 < p \leq q < \infty$ and $\alpha > -1$. Then the difference operator $C_\varphi - C_\psi$ is bounded from A^p_α into A^q_β iff each of the two weighted composition operators σC_φ and σC_ψ is bounded from A^p_α into L^q_β . Furthermore,

$$\|C_\varphi - C_\psi\| \asymp \sup_{a\in\mathbb{D}} \|\left(C_\varphi - C_\psi\right) k_a\|_{q,\beta} \asymp \max\left\{\|\sigma C_\varphi\|, \|\sigma C_\psi\|\right\},$$

where the comparability constants will depend only on α, β, p and q.

(**Compactness of** $C_\varphi - C_\psi$): Suppose $1 < p \leq q < \infty$ and $\alpha > -1$. Assume that $C_\varphi - C_\psi$ is bounded from A^p_α to A^q_β . Then

$$\|C_\varphi - C_\psi\|_e \asymp \overline{\lim_{|a|\to 1}} \|\left(C_\varphi - C_\psi\right) k_a\|_{q,\beta} \asymp \max\left\{\|\sigma C_\varphi\|_e, \|\sigma C_\psi\|_e\right\},$$

where the opearators σC_φ and σC_ψ both map A^p_α into L^q_β and the comparability constants will depend only on α, β, p and q. In particular, $C_\varphi - C_\psi$ is compact iff both σC_φ and σC_ψ are compact.

In our work, we have generalized Saukko's boundedness and compactness criteria for $C_\varphi - C_\psi$ in the case when target Banach space is $L^q\left(\mu\right)$ with μ being any non-negative Borel measure on $\mathbb{D}$.

In a different context , Moorhouse's theorems also motivated Choe , Hosokawa and Koo [**CHK**] to think that like the essential norm of $C_\varphi - C_\psi$, the Hilbert-Schmidt norm of $C_\varphi - C_\psi$ should also involve the cancellation factor σ. In fact, they proved

$$\|C_\varphi - C_\psi\|^2_{\mathrm{HS}} \asymp \int_{\mathbb{D}} \frac{|\,\sigma^2\left(z\right)|\, dA_\alpha\left(z\right)}{\left(1-|\varphi\left(z\right)|^2\right)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\,\sigma^2\left(z\right)|\, dA_\alpha\left(z\right)}{\left(1-|\psi\left(z\right)|^2\right)^{2+\alpha}},$$

where $C_\varphi - C_\psi$ is considered as a bounded operator on the Hilbert space A^2_α .

In this dissertation, we give a simpler proof of the above result using a change of variable method. In fact, as shown in chapter 4 , this method works well to estimate the Hilbert-Schmidt norm of another class of difference operators , too.

CHAPTER 2

PRELIMINARIES

2.1. Lebesgue space and Weighted Bergman space

Definitions. Let μ be a non-negative Borel measure on $\mathbb{D}$ and $p > 0$. Then $L^p(\mu)$, *the Lebesgue space over* $\mathbb{D}$ *with respect to the measure* μ , consists of all functions f defined on $\mathbb{D}$ for which

$$\|f\|_{p,\mu} = \left[\int_{\mathbb{D}} |f(z)|^p \, d\mu(z)\right]^{\frac{1}{p}} < \infty.$$

When $p \geq 1, \|.\|_{p,\mu}$ defines a norm and $L^p(\mu)$ becomes a Banach space.
Let $\alpha > -1$ and A denote the normalized Lebesgue area measure on $\mathbb{D}$ i.e., in terms of real (rectangular and polar) coordinates , we have

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta \, , \, z = x + iy = re^{i\theta}.$$

Now we define the normed area measure A_α on $\mathbb{D}$ by

$$A_\alpha(E) = (\alpha + 1) \int_E \left(1 - |z|^2\right)^\alpha dA(z)$$

for every Lebesgue-measurable set $E \subseteq \mathbb{D}$. For the sake of simplicity, we will denote $L^p(A_\alpha)$ by L^p_α and $\|.\|_{p,A_\alpha}$ by $\|.\|_{p,\alpha}$. In terms of these notations , *the weighted Bergman space* A^p_α is the subspace of L^p_α that consists of analytic functions on $\mathbb{D}$.

Taking $p = 2$ and $\alpha = 0$, we get the well-known *Bergman space* which is denoted by A^2.

For $p > 0$ and $\alpha > -1$, we define the *Bergman kernel function for* A^p_α by

$$K^{p,\alpha}_a = \left(\frac{1}{(1-\overline{a}z)^2}\right)^{\frac{\alpha+2}{p}} .$$

Then the *normalized Bergman kernel function for* A^p_α is given by

$$k^{p,\alpha}_a = \left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)^{\frac{\alpha+2}{p}} .$$

For the sake of notational simplicity , when dealing with the space A^p_α , we will write $K_a = K^{p,\alpha}_a$ and $k_a = k^{p,\alpha}_a$.

Some standard properties of Bergman kernel function will be used in this work from time to time. Some good references for this section are [**DS**] and [**HKZ**].

2.2. Bounded, Compact and Hilbert-Schmidt operator

Definitions. Let H and K be normed linear spaces. A linear transformation T from H to K is *Bounded* if there exists some $M > 0$ such that

$$\|Tf\|_K \leq M\|f\|_H \ \forall f \in H.$$

The smallest such M is defined to be the *operator norm of* T, denoted by $\|T\|$.

It is easy to check that $B(H,K)$, the set of all bounded linear transformations from H to K, is a normed linear space.

T from H to K is *Compact* if T (closed unit ball in H) has a compact closure in K.

It is a standard result that $B_0(H,K)$, *the set of all compact linear transformations from* H *to* K is a closed subspace of $B(H,K)$. For $T \in B(H,K)$, $\|T\|_e$, *the essential norm of* T, is defined to be the distance of T from $B_0(H,K)$. Hence, T is compact iff $\|T\|_e = 0$.

A bounded linear operator T on a separable Hilbert space X with an orthonormal basis $\{e_j\}$ is called *Hilbert-Schmidt* if

$$\|T\|_{HS(X)} = \left\{ \sum_{j=0}^{\infty} \|Te_j\|^2 \right\}^{\frac{1}{2}} < \infty.$$

For the sake of notation, when dealing with $\|T\|_{HS(X)}$ for some T and X , we will simply write $\|T\|_{HS(X)} = \|T\|_{HS}$. $\|T\|_{HS}$ is called the *Hilbert-Schmidt norm* of T. As is well known, the above sum (possibly ∞) is independent of the choice of the orthonormal basis ; See [**Z**] for details. It is also well known that every Hilbert-Schmidt operator is compact ; See [**W**, section 6.2] for details.

2.3. Comparability of quantities

Let A and B be two positive quantities. Then A *and* B *are comparable*, denoted by $A \asymp B$, if $\dfrac{A}{B}$ is bounded above and below by positive constants in the associated limiting process.

The following three lemmas are obvious.

Lemma 2.3.1. *If* $A \asymp B$ *and* $C \asymp D$, *then* $A + C \asymp B + D$

Lemma 2.3.2. *If* $A \asymp B$ *and* $C \leq B$, *then* $A + C \asymp B$

Lemma 2.3.3. *If* $A \asymp B$ *and* $C \geq 0$, *then* $A + C \asymp B + C$

2.4. Carleson measure

2.4.1. Definition and A basic Lemma. Let $0 < p, q < \infty$ and $\alpha > -1$. A non-negative Borel measure μ on $\mathbb{D}$ is a (A^p_α, q) - Carleson measure if the inclusion map $I : A^p_\alpha \to L^q(\mu)$ is bounded i.e., there exists a constant $M > 0$, independent of the choice of $f \in A^p_\alpha$ such that $\|f\|_{q,\mu} \leq M\|f\|_{p,\alpha}$ $\forall f \in A^p_\alpha$.

The characterization of (A^p_α, q) - Carleson measure depends on whether $p \leq q$ or $p > q$. In this dissertation, we focus only on the case where $p \leq q$. Among several versions of Carleson measure theorem , the following is the most relevant to our work.

LEMMA 2.4.1. *Let $p \leq q$ and fix r such that $0 < r < 1$. Then TFAE:*

(i) *The measure μ is a (A^p_α, q) -Carleson measure.*

(ii) $\|\mu\|_{\frac{q}{p},\alpha,r} = \sup\limits_{a\in\mathbb{D}} \dfrac{\mu(\Delta(a,r))}{\left(1-|a|^2\right)^{(\alpha+2)\frac{q}{p}}} < \infty.$

(iii) $\sup\limits_{a\in\mathbb{D}} \|k_a\|^q_{q,\mu} = \sup\limits_{a\in\mathbb{D}} \displaystyle\int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{(\alpha+2)\frac{q}{p}} d\mu(z) < \infty.$

Furthermore, $\|I\|^q$ and the quantities in (ii) and (iii) are all comparable with comparability constants depending only on α, p, q, μ and r.

PROOF. The equivalence of (i) and (ii) is proved in [**L**] . A method to prove the equivalence between (ii) and (iii) can be found for example in the proof of Theorem 7.4 in [**Z**] . The comparability of the quantities follows from the proofs. □

REMARK 1. *We will usually apply Lemma 2.4.1 with $r = 1/2$. In that case we will simply write $\|\mu\|_{\frac{q}{p},\alpha,\frac{1}{2}} = \|\mu\|_{\frac{q}{p},\alpha}$. For the sake of simplicity, the quantity $\|\mu\|_{\frac{q}{p},\alpha}$ will further be denoted by $\|\mu\|$.*

2.4.2. Weighted composition operator and Carleson measure. Suppose $u : \mathbb{D} \to \mathbb{C}$ is a measurable function , φ is an analytic map from $\mathbb{D}$ to $\mathbb{D}$ and μ is a non-negative Borel measure on $\mathbb{D}$. We now define a measure $\mu^q_{u,\varphi}$ on $\mathbb{D}$ by

$$\mu^q_{u,\varphi}(E) = \int_{\varphi^{-1}(E)} |u(z)|^q \, d\mu(z)$$

for all Borel sets $E \subseteq \mathbb{D}$. Then a standard measure - theoretic change of variables shows

$$\|(uC_\varphi) f\|_{q,\mu} = \|f\|_{q,\mu^q_{u,\varphi}} \ \forall f \in H(\mathbb{D}). \tag{2.4.1}$$

(see [**H1**] for details.)

The following Lemma is an easy consequence of (2.4.1) and Lemma 2.4.1. It connects the boundedness of a weighted composition operator and a Carleson measure condition.

LEMMA 2.4.2. *Suppose* $p \leq q$ *and* $0 < r < 1$. *Then TFAE:*

(i) uC_φ *boundedly maps* A^p_α *into* $L^q(\mu)$.

(ii) $\|\mu^q_{u,\varphi}\|_{\frac{q}{p},\alpha,r} < \infty$.

(iii) $\sup_{a\in\mathbb{D}} \|(uC_\varphi) k_a\|^q_{q,\mu} < \infty$.

Furthermore, $\|uC_\varphi\|^q$ and the quantities in (ii) and (iii) are all comparable with comparability constants depending only on α, μ, p, q and r.

2.4.3. A different version of Carleson measure theorem. Except for Lemma 2.4.1, we will need another equivalent version of Carleson measure theorem which , too , is well-known in literature.

Definition. Let I be an arc in the unit circle $\partial\mathbb{D}$. Then *the Carleson square on* I is defined by

$$S(I) = \left\{ z \in \mathbb{D} : |z| \geq 1 - |I|, \frac{z}{|z|} \in I \right\}.$$

The following version of Carleson measure theorem is well-known.

LEMMA 2.4.3. *Let $p \leq q$. Then TFAE:*

(i) *There is a constant $C_1 > 0$ such that, for any $f \in A^p_\alpha$, $\int_{\mathbb{D}} |f(z)|^q \, d\mu(z) \leq C_1 \|f\|^q_{p,\alpha}$.*

(ii) *There is a constant $C_2 > 0$ such that, for any arc $I \in \partial\mathbb{D}$, $\mu(S(I)) \leq C_2 \, |I|^{\frac{(2+\alpha)\,q}{p}}$.*

(iii) *There is a constant $C_3 > 0$ such that, for every $a \in \mathbb{D}$, $\int_{\mathbb{D}} |\sigma'_a(z)|^{\frac{(2+\alpha)\,q}{p}} \, d\mu(z) \leq C_3$.*

In fact, if we define

$$\|\mu\| = \sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{\frac{(2+\alpha)\,q}{p}}},$$

then $\|\mu\|$ and the above constants are comparable.

PROOF. For a detailed proof, we refer to [**ASX**], [**H2**] and [**L**]. □

The following Lemma is important.

LEMMA 2.4.4. *Let $p \leq q$. Also fix $0 < r < 1$. Let*

$$N_r^* = \sup_{|a| \geq r} \int_{\mathbb{D}} |\sigma'_a(z)|^{\frac{(2+\alpha)q}{p}} \, d\mu(z).$$

If μ is an $(A^p_\alpha, q)-$ Carleson measure then so is $\mu_r = \mu \restriction_{(\mathbb{D} \setminus \mathbb{D}_r)}$, where $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$.

Moreover, $\|\mu_r\| \leq M N_r^*$, where M is an absolute constant.

PROOF. The proof is the same as the proof of Lemma 1 and Lemma 2 in [**ČZ1**].

□

2.5. The $\{R_n\}$ sequence and its properties

We first need to recall the well-known definition of the sequence $\{R_n\}$ of operators on A^p_α. Then we discuss some significant properties of $\{R_n\}$ which will be particularly useful in proving our estimates in the next chapter.

Definition. Let $n \in \mathbb{N}$. The partial sum operator S_n on $H(\mathbb{D})$ is defined by

$$S_n\left(\sum_{j=0}^{\infty} f_j z^j\right) = \sum_{j=0}^{n} f_j z^j.$$

Clearly, S_n is finite-rank and therefore compact on A^p_α for each $p > 0$. Let $R_n = I - S_n$. Then it is a standard result that $\{R_n\}$ is uniformly bounded on A^p_α when $p > 1$. For details , see [**DS**].

LEMMA 2.5.1. *If T is a bounded linear transformation from A^p_α to $L^q(\mu)$, then*

$$\|T\|_e \leq \overline{\lim_{n\to\infty}} \|TR_n\|.$$

PROOF. For each n , $R_n + S_n = I$, the identity operator.

Hence $T = TR_n + TS_n$.

Then $\|T\|_e = \|TR_n + TS_n\|_e = \|TR_n\|_e$, since S_n is compact and hence so is TS_n .

So $\|T\|_e \leq \|TR_n\| \ \forall\, n$, which clearly proves the Lemma. □

Notation. By *U.C.C* , we mean *uniformly convergent on each compact subset.*

We would like to give a detailed proof of the following Lemma . A sketch of its proof has its root in Proposition 3 , Page 9 in [**ČZ1**].

LEMMA 2.5.2. (i) $R_n \xrightarrow{U.C.C} 0$ *on* A_α^p *and* $R_n^* \xrightarrow{U.C.C} 0$ *on* $(A_\alpha^p)^*$ *for each* $p > 1$.

(ii) *Let* $1 < p \leq q$. *If* T *is a bounded linear transformation from* A_α^p *to* $L^q(\mu)$, *then there exists a positive constant* λ *satisfying*

$$\lambda \overline{\lim_{n\to\infty}} \|TR_n\| \leq \|T\|_e \leq \underline{\lim}_{n\to\infty} \|TR_n\|.$$

PROOF. (i) It is a standard fact that $R_n \to 0$ pointwise if $p > 1$ and is therefore uniformly bounded by the Uniform Boundedness principle. Hence $\{R_n\}$ is equicontinuous and therefore $R_n \xrightarrow{U.C.C} 0$ on A_α^p.

To prove the second part of (i) , we first consider the integral pairing that identifies $(A_\alpha^p)^*$ and $A_\alpha^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z).$$

Then,

CLAIM 1. $\langle R_n f, g\rangle_\alpha = \langle f, R_n g\rangle_\alpha$ $\forall n \in \mathbb{N}, f \in A_\alpha^p$ *and* $g \in A_\alpha^{p'}$.

PROOF. Let us consider the power series representations

$$f(z) = \sum_{j=0}^{\infty} f_j z^j \text{ and } g(z) = \sum_{k=0}^{\infty} g_k z^k.$$

Then,

$$\langle R_n f, g\rangle_\alpha = \int_{\mathbb{D}} R_n f(z)\overline{g(z)}\, dA_\alpha(z)$$

$$= \int_{\mathbb{D}} \left(\sum_{j=n+1}^{\infty} f_j z^j \right) \left(\sum_{k=0}^{\infty} \overline{g}_k \overline{z}^k \right) dA_\alpha (z)$$

$$= \int_{\mathbb{D}} \left(\sum_{j=n+1,k=0}^{\infty} f_j \overline{g}_k z^j \overline{z}^k \right) dA_\alpha (z)$$

$$= \sum_{j=n+1,k=0}^{\infty} f_j \overline{g}_k \int_0^{2\pi} \int_0^1 r^{j+k+1} \left(1 - r^2\right)^\alpha e^{(j-k)i\theta} dr d\theta \text{ where } z = re^{i\theta}.$$

But,

$$\int_0^{2\pi} e^{(j-k)i\theta} d\theta = 0 \text{ iff } j \neq k.$$

Hence,

$$\langle R_n f, g \rangle_\alpha = \sum_{j=n+1}^{\infty} f_j \overline{g}_j \int_0^1 r^{2j+1} \left(1 - r^2\right)^\alpha dr.$$

Similarly,

$$\langle f, R_n g \rangle_\alpha = \sum_{j=n+1}^{\infty} f_j \overline{g}_j \int_0^1 r^{2j+1} \left(1 - r^2\right)^\alpha dr.$$

Hence the claim is proved. □

Now by Riesz Representation Theorem, $\forall F \in (A_\alpha^p)^*$, there exists $\varphi_F \in A_\alpha^{p'}$ such that

$$F(f) = \langle f, \varphi_F \rangle_\alpha \ \forall f \in A_\alpha^p.$$

Hence,

$$R_n^* F(f) = F(R_n f) = \langle R_n f, \varphi_F \rangle_\alpha = \langle f, R_n \varphi_F \rangle_\alpha \ \forall f \in A_\alpha^p.$$

Hence applying Cauchy-Schwarz inequality,

$$\|R_n^* F\| \leq \|R_n \phi_F\|_{p', \alpha}.$$

Now let S be a compact subset of $(A_\alpha^p)^*$. Since the identification : $(A_\alpha^p)^* \to A_\alpha^{p'}$ defined by $F \to \varphi_F$ is continuous, therefore $\{\varphi_F : F \in S\}$ is compact in $A_\alpha^{p'}$. Now since $p > 1$, therefore $p' > 1$. Hence by first part of (i) , $R_n \xrightarrow{U.C.C} 0$ on $A_\alpha^{p'}$. This implies $\sup_{F \in S} \|R_n \varphi_F\|_{p', \alpha} \to 0$ as $n \to \infty$.

Consequently, $\sup\limits_{F\in S}\|R_n^*F\| \leq \sup\limits_{F\in S}\|R_n\varphi_F\|_{p',\alpha} \to 0$ as $n \to \infty$. This proves the second part of (i).

(ii) It is already known that $\{R_n\}$ is uniformly bounded on A_α^p. Hence there exists a positive constant M such that $\|R_n\| \leq M\ \forall\, n$.

So for any compact operator K from A_α^p to $L^q(\mu)$, we have ,

$$\|T-K\| = \tfrac{1}{M}\, M\, \|T-K\|$$

$$\geq \tfrac{1}{M}\, \|R_n\|\, \|T-K\|$$

$$\geq \tfrac{1}{M}\, \|\,(T-K)\, R_n\|$$

$$= \tfrac{1}{M}\, \|TR_n - KR_n\| \geq \tfrac{1}{M}\, \left(\|TR_n\| - \|KR_n\|\right)$$

Now, $\|KR_n\| = \|\,(KR_n)^*\,\| = \|R_n^*K^*\| = \sup\limits_{\|F\|_{(L^q(\mu))^*}\leq 1}\|R_n^*K^*F\|.$

Since K is compact from A_α^p to $L^q(\mu)$, therefore K^* is compact from $(L^q(\mu))^*$ to $(A_\alpha^p)^*$. Also by Lemma 2.5.2(i),

$$R_n^* \xrightarrow{U.C.C} 0 \text{ on } (A_\alpha^p)^*.$$

Consequently,

$$\sup_{\|F\|_{(L^q(\mu))^*}\leq 1}\|R_n^*K^*F\| \to 0 \text{ as } n \to \infty$$

i.e.,

$$\lim_{n\to\infty}\|KR_n\| = 0.$$

Hence,

$$\|T-K\| \geq \tfrac{1}{M}\,\overline{\lim_{n\to\infty}}\,\|TR_n\| = \lambda\overline{\lim_{n\to\infty}}\,\|TR_n\|,$$

where $\lambda = \frac{1}{M}$.

Since the above is true for an arbitrary compact operator K from A_α^p to $L^q(\mu)$, therefore ,

$$\|T\|_e \geq \lambda \overline{\lim_{n\to\infty}} \|TR_n\|.$$

On the other hand, we have already proved in Lemma 2.5.1 that

$$\|T\|_e \leq \|TR_n\| \;\forall\, n.$$

Hence,

$$\|T\|_e \leq \underline{\lim_{n\to\infty}} \|TR_n\|.$$

This completes the proof of the Lemma. □

2.6. Pseudohyperbolic metric and its properties

Definition. We define the pseudohyperbolic metric $\rho : \mathbb{D} \times \mathbb{D} \to [0, 1)$ by

$$\rho(a, b) = |\sigma_a(b)| = \left|\frac{a - b}{1 - \overline{a}b}\right|.$$

The pseudohyperbolic metric obeys the following so-called strong form of triangle inequality:

$$\rho(z, w) \leq \frac{\rho(z, a) + \rho(a, w)}{1 + \rho(z, a)\,\rho(a, w)} \;\forall a, z, w \in \mathbb{D}. \tag{2.6.1}$$

Furthermore, if we fix r such that $0 < r < 1$, then there exist constants A, B and C depending only on r such that whenever $z, w \in \mathbb{D}$ with $\rho(z, w) < r$,

$$A^{-1} \leq \frac{1 - |z|^2}{1 - |w|^2} \leq A, \tag{2.6.2}$$

$$B^{-1} \leq \left|\frac{1 - \overline{\xi} z}{1 - \overline{\xi} w}\right| \leq B, \;\forall\, \xi \in \mathbb{D}, \tag{2.6.3}$$

and

$$C^{-1} \leq \frac{1 - |w|^2}{|1 - \overline{w} z|} \leq C. \tag{2.6.4}$$

The above estimates are all elementary and for detailed proofs, see [**DS**].

Notation The pseudohyperbolic open disk centered at a with radius r will be denoted by $\Delta(a, r)$.

CHAPTER 3

THREE OPERATOR PROPERTIES OF $C_\varphi - C_\psi$

In this chapter, we will discuss conditions under which $C_\varphi - C_\psi$ is Hilbert-Schmidt, bounded and compact respectively.

First, let us recall the following result from [**CHK**], mentioned in the Introduction.

THEOREM A. *Let us consider the difference operator $C_\varphi - C_\psi$ acting boundedly on A^2_α . Then,*

$$\|C_\varphi - C_\psi\|^2_{HS} \asymp \int_{\mathbb{D}} \frac{|\,\sigma^2\,(z)\,|\,dA_\alpha\,(z)}{(1-|\varphi\,(z)\,|^2)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\,\sigma^2\,(z)\,|\,dA_\alpha\,(z)}{(1-|\psi\,(z)\,|^2)^{2+\alpha}}\,.$$

In this chapter, we will first give an alternative proof of **Theorem** *A* using a change of variable technique.

We also recall, from Introduction, that Saukko [**S**] proved the following estimates:

THEOREM B. *Suppose $0 < p \le q < \infty$ and $\alpha > -1$. Then the difference operator $C_\varphi - C_\psi$ is bounded from A^p_α into A^q_β if and only if each of the two weighted composition operators σC_φ and σC_ψ is bounded from A^p_α into L^q_β . Furthermore,*

$$\|C_\varphi - C_\psi\| \asymp \sup_{a\in\mathbb{D}} \|\,(C_\varphi - C_\psi)\,k_a\|_{q,\,\beta} \asymp \max\left\{\|\sigma C_\varphi\|, \|\sigma C_\psi\|\right\}\,,$$

where the comparability constants will depend only on α, β, p and q.

Theorem C. *Suppose* $1 < p \leq q < \infty$ *and* $\alpha > -1$. *Assume that* $C_\varphi - C_\psi$ *is bounded from* A^p_α *to* A^q_β. *Then,*

$$\|C_\varphi - C_\psi\|_e \asymp \overline{\lim_{|a|\to 1}} \| \left(C_\varphi - C_\psi\right) k_a\|_{q,\,\beta} \asymp \max\left\{\|\sigma C_\varphi\|_e\,, \|\sigma C_\psi\|_e\right\}\,,$$

where the opearators σC_φ *and* σC_ψ *both map* A^p_α *into* L^q_β *and the comparability constants will depend only on* α, β, p *and* q. *In particular,* $C_\varphi - C_\psi$ *is compact if and only if both* σC_φ *and* σC_ψ *are compact.*

We will generalize Saukko's boundedness and compactness criteraia for $C_\varphi - C_\psi$ taking $L^q(\mu)$ as the target Banach space where μ can be any non-negative Borel measure on $\mathbb{D}$.

3.1. An Integral Expression for Hilbert-Schmidt Norm

In Preliminary, we have defined a Hilbert-Schmidt operator on a Hilbert space H. Let us assume u, v be both analytic maps from $\mathbb{D}$ to $\mathbb{C}$ such that $uC_\varphi - vC_\psi$ is bounded on the Hilbert space A^2_α. Then by definition,

$$\|uC_\varphi - vC_\psi\|^2_{\mathrm{HS}} = \sum_{j=0}^{\infty} \| \left(uC_\varphi - vC_\psi\right) e_j\|^2,$$

where $e_j(z) = \sqrt{\dfrac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}}\, z^j$ for each $z \in \mathbb{D}$, the standard orthonormal basis for A^2_α.

Now for each j,

$$\| \left(uC_\varphi - vC_\psi\right) e_j\|^2 = \langle \left(uC_\varphi - vC_\psi\right) e_j, \left(uC_\varphi - vC_\psi\right) e_j\rangle$$

$$= \int_{\mathbb{D}} |u(z)|^2 |e_j \circ \varphi(z)|^2 dA_\alpha(z) + \int_{\mathbb{D}} |v(z)|^2 |e_j \circ \psi(z)|^2 dA_\alpha(z)$$
$$-2\,\Re \int_{\mathbb{D}} u(z)\,\overline{v(z)}\, e_j \circ \varphi(z)\, \overline{e_j \circ \psi(z)}\, dA_\alpha(z)$$

$$= \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)} \int_{\mathbb{D}} I(u,v,\varphi,\psi,j)(z)\, dA_\alpha(z)\ ,$$

where,

$$I(u,v,\varphi,\psi,j)(z) = |u(z)|^2|\varphi(z)|^{2j} + |v(z)|^2|\psi(z)|^{2j} - 2\,\Re\left(u(z)\,\overline{v(z)}\,(\varphi(z))^j\ \overline{\psi(z)}^{\,j}\right).$$

Hence,

$$\|uC_\varphi - vC_\psi\|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)} \int_{\mathbb{D}} I(u,v,\varphi,\psi,j)(z)\, dA_\alpha(z)$$

$$= \int_{\mathbb{D}} \underbrace{\sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)} I(u,v,\varphi,\psi,j)(z)}_{S(z)}\, dA_\alpha(z)\,.$$

The interchange of integration and summation is justified, because for each fixed $z \in \mathbb{D}$, each of the three series

$$\sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}|\varphi(z)|^{2j}\,,\ \sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}|\psi(z)|^{2j}\,,\ \sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}|\,(\varphi(z))^j\ \overline{\psi(z)}^{\,j}$$

converges uniformly in $z \in \mathbb{D}$.

Therefore,

$$S(z) = |u(z)|^2\left(\sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}|\varphi(z)|^{2j}\right) + |v(z)|^2\left(\sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}|\psi(z)|^{2j}\right)$$
$$-2\,\Re\left(u(z)\,\overline{v(z)}\,\sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}\,(\varphi(z))^j\ \overline{\psi(z)}^{\,j}\right).$$

It is an elementary fact that $\sum_{j=0}^{\infty} \frac{\Gamma(j+\beta)}{\Gamma(\beta)\, j!} x^j = \frac{1}{(1-x)^{\beta}}$ for $\beta > 0$ and $|x| < 1$.

Taking $\beta = 2+\alpha$, we obtain $\sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{\Gamma(2+\alpha)\, j!} x^j = \frac{1}{(1-x)^{2+\alpha}}$ whenever $|x| < 1$.

If we take $x = |\varphi(z)|^2$ in above formula, we get that

$$\sum_{j=0}^{\infty}\frac{\Gamma(j+2+\alpha)}{\Gamma(2+\alpha)\,j!}|\,\varphi(z)\,|^{2j}=\frac{1}{(1-|\varphi(z)\,|^2)^{2+\alpha}}.$$

Similarly,

$$\sum_{j=0}^{\infty}\frac{\Gamma(j+2+\alpha)}{\Gamma(2+\alpha)\,j!}|\,\psi(z)\,|^{2j}=\frac{1}{(1-|\psi(z)\,|^2)^{2+\alpha}},$$

$$\sum_{j=0}^{\infty}\frac{\Gamma(j+2+\alpha)}{\Gamma(2+\alpha)\,j!}(\varphi(z))^j\;\overline{\psi(z)}^{\,j}=\frac{1}{\left(1-\varphi(z)\,\overline{\psi(z)}\right)^{2+\alpha}}.$$

Hence,

$$S(z)=\frac{|\,u(z)\,|^2}{(1-|\varphi(z)\,|^2)^{2+\alpha}}+\frac{|\,v(z)\,|^2}{(1-|\psi(z)\,|^2)^{2+\alpha}}-2\,\Re\left(\frac{u(z)\,\overline{v(z)}}{\left(1-\varphi(z)\,\overline{\psi(z)}\right)^{2+\alpha}}\right).$$

In particular, taking $u=v\equiv 1$, we have,

$$\|C_\varphi-C_\psi\|_{\mathrm{HS}}^2=\int_{\mathbb{D}}S(z)\,dA_\alpha(z)$$

$$=\int_{\mathbb{D}}\left[\frac{1}{(1-|\varphi(z)\,|^2)^{2+\alpha}}+\frac{1}{(1-|\psi(z)\,|^2)^{2+\alpha}}-2\,\Re\left(\frac{1}{\left(1-\varphi(z)\,\overline{\psi(z)}\right)^{2+\alpha}}\right)\right]dA_\alpha(z).$$

3.2. The Alternative Proof of Theorem *A*

From the previous section,

$\|C_\varphi-C_\psi\|_{\mathrm{HS}}^2$

$$=\int_{\mathbb{D}}\left[\frac{1}{(1-|\varphi(z)\,|^2)^{2+\alpha}}+\frac{1}{(1-|\psi(z)\,|^2)^{2+\alpha}}-2\,\Re\left(\frac{1}{\left(1-\varphi(z)\,\overline{\psi(z)}\right)^{2+\alpha}}\right)\right]dA_\alpha(z).$$

We notice that the integrand is of the form

$$L(a,b)=\frac{1}{(1-|a|^2)^{2+\alpha}}+\frac{1}{(1-|b|^2)^{2+\alpha}}-2\,\Re\left(\frac{1}{\left(1-a\bar{b}\right)^{2+\alpha}}\right),$$

where, $a = \varphi(z)$ and $b = \psi(z)$.

To prove **Theorem** A, we clearly need to factor out $\sigma(z) = \dfrac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\psi(z)}$ from the above integrand. So, we make a change of variable: $b = \sigma_a(w) = \dfrac{a - w}{1 - \overline{a}w}$. Then $w = \sigma_a(b)$. And, $1 - |b|^2 = \dfrac{(1 - |a|^2)(1 - |w|^2)}{|1 - \overline{a}w|^2}$. Substituting in $L(a, b)$, we get,

$$L(a,b) = \frac{1}{(1 - |a|^2)^{2+\alpha}} + \frac{|1 - \overline{a}w|^{4+2\alpha}}{(1 - |a|^2)^{2+\alpha}(1 - |w|^2)^{2+\alpha}} - 2\,\Re\left(\frac{(1 - a\overline{w})^{2+\alpha}}{(1 - |a|^2)^{2+\alpha}}\right).$$

$$= \frac{1}{(1 - |a|^2)^{2+\alpha}}\left[1 + \frac{|1 - \overline{a}w|^{4+2\alpha}}{(1 - |w|^2)^{2+\alpha}} - 2\,\Re\,(1 - \overline{a}w)^{2+\alpha}\right]$$

$$= \frac{1}{(1 - |a|^2)^{2+\alpha}}\left[\frac{|1 - \overline{a}w|^{4+2\alpha}}{(1 - |w|^2)^{2+\alpha}} + \left|1 - (1 - \overline{a}w)^{2+\alpha}\right|^2 - |1 - \overline{a}w|^{4+2\alpha}\right]$$

$$= \frac{1}{(1 - |a|^2)^{2+\alpha}}\left[\frac{|1 - \overline{a}w|^{4+2\alpha}}{(1 - |w|^2)^{2+\alpha}}\left(1 - \left(1 - |w|^2\right)^{2+\alpha}\right) + \left|1 - (1 - \overline{a}w)^{2+\alpha}\right|^2\right]$$

$$= \frac{|w|^2}{(1 - |b|^2)^{2+\alpha}}\left[\frac{1 - (1 - |w|^2)^{2+\alpha}}{|w|^2}\right] + \frac{|a|^2|w|^2}{(1 - |a|^2)^{2+\alpha}}\left|\frac{1 - (1 - \overline{a}w)^{2+\alpha}}{\overline{a}w}\right|^2.$$

Now since $\|C_\varphi - C_\psi\|^2_{\text{HS}} = \|C_\psi - C_\varphi\|^2_{\text{HS}}$, therefore

$$\|C_\varphi - C_\psi\|^2_{\text{HS}} = \frac{1}{2}\int_{\mathbb{D}} (L(a,b) + L(b,a))\, dA_\alpha(z),$$

where, $a = \varphi(z)$ and $b = \psi(z)$. Also, $|\eta| = |w|$ where $\eta = \sigma_b(a)$. Hence,

$$\|C_\varphi - C_\psi\|^2_{\text{HS}} = \frac{1}{2}\int_{\mathbb{D}} J(a,b,w)\, dA_\alpha(z),$$

where,

$$J(a,b,w)=\underbrace{\frac{|w|^2}{\left(1-|b|^2\right)^{2+\alpha}}\left(\frac{1-\left(1-|w|^2\right)^{2+\alpha}}{|w|^2}\right)}_{\text{Term 1}}+\underbrace{\frac{|a|^2|w|^2}{\left(1-|a|^2\right)^{2+\alpha}}\left|\frac{1-\left(1-\overline{a}w\right)^{2+\alpha}}{\overline{a}w}\right|^2}_{\text{Term 2}}$$

$$+\underbrace{\frac{|w|^2}{\left(1-|a|^2\right)^{2+\alpha}}\left(\frac{1-\left(1-|w|^2\right)^{2+\alpha}}{|w|^2}\right)}_{\text{Term 3}}+\underbrace{\frac{|b|^2|w|^2}{\left(1-|b|^2\right)^{2+\alpha}}\left|\frac{1-\left(1-\overline{b}\eta\right)^{2+\alpha}}{\overline{b}\eta}\right|^2}_{\text{Term 4}}.$$

Now it is an elementary fact that the real-valued function $f(x)=\dfrac{1-(1-x)^{2+\alpha}}{x}$ has a removable discontinuity of positive limiting value at $x=0$ and hence has both a positive supremum and a positive infimum on $[0,1)$. (**Fact 1**)

Similarly, the complex-valued function g from $\mathbb{D}$ to $\mathbb{C}$ defined by $g(z)=\dfrac{1-(1-z)^{2+\alpha}}{z}$ has a removable singularity of positive limiting value at $z=0$ and hence $|g(z)|$ has a positive supremum on $\mathbb{D}$. (**Fact 2**)

We now find an upper bound and a lower bound for $J(a,b,w)$.

Upper Bound. Appying **Fact 1** on Term 1 and Term 3 and **Fact 2** on Term 2 and Term 4, we get that,

$$J(a,b,w)\lesssim\frac{|w|^2}{\left(1-|b|^2\right)^{2+\alpha}}+\frac{|a|^2|w|^2}{\left(1-|a|^2\right)^{2+\alpha}}+\frac{|w|^2}{\left(1-|a|^2\right)^{2+\alpha}}+\frac{|b|^2|w|^2}{\left(1-|b|^2\right)^{2+\alpha}}$$

$$=\frac{|w|^2\left(1+|b|^2\right)}{\left(1-|b|^2\right)^{2+\alpha}}+\frac{|w|^2\left(|a|^2+1\right)}{\left(1-|a|^2\right)^{2+\alpha}}<\frac{2|w|^2}{\left(1-|b|^2\right)^{2+\alpha}}+\frac{2|w|^2}{\left(1-|a|^2\right)^{2+\alpha}}.$$

Lower Bound. Since Term 2 and Term 4 are both non-negative, therefore,

$$J(a,b,w)\geq\frac{|w|^2}{\left(1-|b|^2\right)^{2+\alpha}}\left(\frac{1-\left(1-|w|^2\right)^{2+\alpha}}{|w|^2}\right)+\frac{|w|^2}{\left(1-|a|^2\right)^{2+\alpha}}\left(\frac{1-\left(1-|w|^2\right)^{2+\alpha}}{|w|^2}\right)$$

$$\gtrsim \frac{|w|^2}{(1-|b|^2)^{2+\alpha}} + \frac{|w|^2}{(1-|a|^2)^{2+\alpha}}\text{, applying } \textbf{Fact 1}$$

Therefore,

$$J(a,b,w) \asymp \frac{|w|^2}{\left(1-|a|^2\right)^{2+\alpha}} + \frac{|w|^2}{\left(1-|b|^2\right)^{2+\alpha}} \ ,$$

where,

$$a=\varphi(z)\,, b=\psi(z) \text{ and } w=\sigma(z)=\frac{\varphi(z)-\psi(z)}{1-\overline{\varphi(z)}\psi(z)}.$$

Hence,

$$\|C_\varphi - C_\psi\|_{\mathrm{HS}}^2 \asymp \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA_\alpha(z)}{(1-|\varphi(z)\,|^2)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA_\alpha(z)}{(1-|\psi(z)\,|^2)^{2+\alpha}}\,.$$

This was the statement in **Theorem** ***A*.**

COROLLARY 3.2.1. *Let us consider the difference operator $C_\varphi - C_\psi$ acting boundedly on A_α^2. Then $C_\varphi - C_\psi$ is Hilbert-Schmidt iff*

$$\int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA_\alpha(z)}{(1-|\varphi(z)\,|^2)^{2+\alpha}} < \infty \textit{ and } \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA_\alpha(z)}{(1-|\psi(z)\,|^2)^{2+\alpha}} < \infty.$$

3.3. Boundedness and Compactness of $C_\varphi - C_\psi$

We generalize **Theorem** *B* and **Theorem** *C* by considering the difference operator $C_\varphi - C_\psi$ acting from A_α^p into $L^q(\mu)$, where μ can be any non-negative Borel measure on $\mathbb{D}$ and $0 < p \leq q < \infty$.

THEOREM 1. *Suppose $0 < p \leq q < \infty$ and $\alpha > -1$. Then the difference operator $C_\varphi - C_\psi$ is bounded from A_α^p to $L^q(\mu)$ if and only if each of the two weighted composition operators σC_φ and σC_ψ is bounded from A_α^p to $L^q(\mu)$. Furthermore,*

$$\|C_\varphi - C_\psi\| \asymp \sup_{a\in\mathbb{D}} \|\,(C_\varphi - C_\psi)\,k_a\|_{q,\,\mu} \asymp \max\left\{\|\sigma C_\varphi\|, \|\sigma C_\psi\|\right\},$$

where the comparability constants will depend only on α, μ, p and q.

Theorem 2. *Suppose* $1 < p \leq q < \infty$ *and* $\alpha > -1$. *Assume that* $C_\varphi - C_\psi$ *is bounded from* A^p_α *to* $L^q(\mu)$. *Then*

$$\|C_\varphi - C_\psi\|_e \asymp \overline{\lim_{|a|\to 1}} \|(C_\varphi - C_\psi) k_a\|_{q,\mu} \asymp \max\{\|\sigma C_\varphi\|_e, \|\sigma C_\psi\|_e\},$$

where the opearators σC_φ *and* σC_ψ *both map* A^p_α *into* $L^q(\mu)$ *and the comparability constants will depend only on* α, μ, p *and* q. *In particular,* $C_\varphi - C_\psi$ *is compact if and only if both* σC_φ *and* σC_ψ *are compact.*

We will break the discussion of our proofs into three sections. The first section discusses some asymptotically equivalent expressions for the essential norm of a weighted composition operator uC_φ acting from A^p_α to $L^q(\mu)$. The second and the third section discuss the upper and lower estimates, respectively, for the operator norm and the essential norm of the difference operator.

3.4. Essential Norm of a Weighted Composition operator

Theorem 3. *Suppose* $1 < p \leq q < \infty$ *and let* $u : \mathbb{D} \to \mathbb{C}$ *be a measuarable function such that* uC_φ *from* A^p_α *to* $L^q(\mu)$ *is bounded. Then*

$$\|uC_\varphi\|_e^q \asymp \lim_{s\to 1^-} \|\left(\mu^q_{u,\varphi}\right)_s\| \asymp \underline{\lim_{n\to\infty}} \|(uC_\varphi) R_n\|^q_{A^p_\alpha \to L^q(\mu)}$$

$$\asymp \overline{\lim_{n\to\infty}} \|(uC_\varphi) R_n\|^q_{A^p_\alpha \to L^q(\mu)} \asymp \overline{\lim_{|a|\to 1}} \|(uC_\varphi) k_a\|^q_{q,\mu},$$

where the comparability constants depend only on α, μ, p *and* q.

Proof. Saukko, in [**S**], has proposed a special case of this theorem when $\mu = A_\beta$. He suggested a proof, refering to Lemma 1, 2 and proof of Theorem 2 in [ČZ2]. We applied essentially the same technique to prove this theorem, with some necessary modifications.

If we apply 2nd part of Lemma 2.5.2 , taking $T = uC_\varphi$, then the comparability of the first, 3rd and fourth quantities follow immediately. Then, to prove the theorem, it is sufficient to prove that the first, 2nd and 5th quantities are all comparable.
To prove it, Let us first fix $s \in (0,1)$. Since uC_φ from A^p_α to $L^q(\mu)$ is bounded, therefore by applying Lemma 2.5.1 we arrive at

$$\|uC_\varphi\|_e^q \leq \overline{\lim_{n\to\infty}} \|(uC_\varphi) R_n\|^q_{A^p_\alpha \to L^q(\mu)}. \tag{3.4.1}$$

Now, for each fixed n,

$$\|(uC_\varphi) R_n\|^q_{A^p_\alpha \to L^q(\mu)} = \sup_{\|f\|_{p,\alpha}\leq 1} \|uC_\varphi R_n(f)\|^q_{q,\mu}.$$

And, for $f \in A^p_\alpha$,

$$\|uC_\varphi R_n(f)\|^q_{q,\mu} = \int_{\mathbb{D}} |u(z)|^q |R_n f(\varphi(z))|^q d\mu(z)$$

$$= \int_{\mathbb{D}} |R_n f(w)|^q d\mu^q_{u,\varphi}(w) = \underbrace{\int_{\mathbb{D}\setminus\mathbb{D}_s} |R_n f(w)|^q d\mu^q_{u,\varphi}(w)}_{I_1(f,n)} + \underbrace{\int_{\mathbb{D}_s} |R_n f(w)|^q d\mu^q_{u,\varphi}(w)}_{I_2(f,n)}.$$

Hence,

$$\sup_{\|f\|_{p,\alpha}\leq 1} \|uC_\varphi R_n(f)\|^q_{q\,\mu} \leq \sup_{\|f\|_{p,\alpha}\leq 1} I_1(f,n) + \sup_{\|f\|_{p,\alpha}\leq 1} I_2(f,n),$$

i.e.,

$$\|(uC_\varphi) R_n\|^q_{A^p_\alpha \to L^q(\mu)} \leq \sup_{\|f\|_{p,\alpha}\leq 1} I_1(f,n) + \sup_{\|f\|_{p,\alpha}\leq 1} I_2(f,n).$$

Taking limit superior,

$$\overline{\lim_{n\to\infty}} \|(uC_\varphi) R_n\|^q_{A^p_\alpha \to L^q(\mu)} \leq \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\alpha}\leq 1} I_1(f,n) + \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\alpha}\leq 1} I_2(f,n). \tag{3.4.2}$$

Combining (3.4.1) and (3.4.2), we obtain

$$\|uC_\varphi\|_e^q \leq \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} I_1\left(f,n\right) + \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} I_2\left(f,n\right). \tag{3.4.3}$$

CLAIM 2.

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} I_2\left(f,n\right) = 0.$$

PROOF OF THE CLAIM. To prove the above claim, let us first fix $n \in \mathbb{N}$.

Let us fix $w \in \mathbb{D}$. Then it is a well-known fact that for any $f \in A_\alpha^2$,

$$R_n f\left(w\right) = \int_{\mathbb{D}} \frac{R_n f\left(z\right) dA_\alpha\left(z\right)}{\left(1-\overline{z}w\right)^{2+\alpha}}. \tag{3.4.4}$$

We note that the left hand side of (3.4.4) represents the composition of the point-evaluation functional K_w in A_α^p and the R_n operator, each of which is a continuous function on A_α^p. Hence left hand side of (3.4.4) defines a continuous linear functional on A_α^p.

Since w is fixed, therefore $\dfrac{1}{\left(1-\overline{z}w\right)^{2+\alpha}}$ is a bounded function of z. Hence the right hand side of (3.4.4), too, defines a continuous linear functional on A_α^p.

Since (3.4.4) holds for any $f \in A_\alpha^2$, it holds when f is any polynomial, but polynomials are dense in A_α^p. So (3.4.4) shows that two continuous linear functionals on A_α^p agree on a dense subset of A_α^p. Hence they agree on A_α^p, i.e.,

$$R_n f\left(w\right) = \int_{\mathbb{D}} \frac{R_n f\left(z\right) dA_\alpha\left(z\right)}{\left(1-\overline{z}w\right)^{2+\alpha}}\ \forall f \in A_\alpha^p,$$

i.e.,

$$R_n f\left(w\right) = \left\langle R_n f, K_w\right\rangle_\alpha,$$

where,

$$K_w(z) = \frac{1}{(1-\overline{w}z)^{2+\alpha}} = \sum_{j=0}^{\infty} \frac{\Gamma(2\,+\,\alpha+j)}{\Gamma(2\,+\,\alpha)\,j!}(\overline{w}z)^j \ .$$

Then,

$$\begin{aligned}
|R_n f(w)| &= |\langle R_n f, K_w\rangle_\alpha| \\
&= |\langle f, R_n K_w\rangle_\alpha|\,, \\
&= \left|\int_{\mathbb{D}} f(z)\,\overline{R_n K_w(z)}\;dA_\alpha(z)\right| \\
&\le \|f\|_{p,\,\alpha}\,\|R_n K_w\|_{p',\,\alpha}\,, \text{using Hölder's inequality.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_2(f,n) &= \int_{\mathbb{D}_s} |R_n f(w)|^q\, d\mu^q_{u,\varphi}(w) \\
&\le \|f\|^q_{p,\,\alpha} \int_{\mathbb{D}_s} \|R_n K_w\|^q_{p',\,\alpha}\, d\mu^q_{u,\varphi}(w)\ ,
\end{aligned}$$

which implies

$$\sup_{\|f\|_{p,\,\alpha}\le 1} I_2(f,n) \le \int_{\mathbb{D}_s} \|R_n K_w\|^q_{p',\,\alpha}\, d\mu^q_{u,\varphi}(w)\ .$$

So, to prove Claim 2, it is enough to show that $\overline{\lim_{n\to\infty}} \int_{\mathbb{D}_s} \|R_n K_w\|^q_{p',\,\alpha}\, d\mu^q_{u,\varphi}(w) = 0$.

To show that, we will estimate $R_n K_w$ pointwise.

For $z \in \mathbb{D}$,

$$\begin{aligned}
|R_n K_w(z)| &= \left|\sum_{j=n+1}^{\infty} \frac{\Gamma(2+\alpha+j)}{\Gamma(2+\alpha)\,j!}(\overline{w}z)^j\right| \\
&\le \sum_{j=n+1}^{\infty} \frac{\Gamma(2+\alpha+j)}{\Gamma(2+\alpha)\,j!}|w|^j\,|z|^j\ .
\end{aligned}$$

So, if $w \in \mathbb{D}_s$, then,

$$|R_n K_w(z)| \leq \sum_{j=n+1}^{\infty} \frac{\Gamma(2+\alpha+j)}{\Gamma(2+\alpha)\, j!} s^j = c_n(s), \text{ say.}$$

Integrating, we obtain,

$$\|R_n K_w\|_{p',\alpha}^q = \left(\int_{\mathbb{D}} |R_n K_w(z)|^{p'} dA_\alpha(z)\right)^{\frac{q}{p'}} \leq (c_n(s))^q \left(\int_{\mathbb{D}} dA_\alpha(z)\right)^{\frac{q}{p'}} = (\alpha+1)^{-\frac{q}{p'}} (c_n(s))^q .$$

Hence,

$$\begin{aligned}
\int_{\mathbb{D}_s} \|R_n K_w\|_{p',\alpha}^q \, d\mu_{u,\varphi}^q(w) &\leq (\alpha+1)^{-\frac{q}{p'}} (c_n(s))^q \mu_{u,\varphi}^q(\mathbb{D}_s) \\
&= (\alpha+1)^{-\frac{q}{p'}} (c_n(s))^q \left(\frac{\mu_{u,\varphi}^q(\Delta(0,s))}{\left(1-|0|^2\right)^{\frac{(2+\alpha)q}{p}}} \right) \\
&\leq (\alpha+1)^{-\frac{q}{p'}} (c_n(s))^q \sup_{a\in\mathbb{D}} \left(\frac{\mu_{u,\varphi}^q(\Delta(a,s))}{\left(1-|a|^2\right)^{\frac{(2+\alpha)q}{p}}} \right) \\
&= (\alpha+1)^{-\frac{q}{p'}} (c_n(s))^q \|\mu_{u,\varphi}^q\|_{\frac{q}{p},\alpha,s} \\
&\lesssim (\alpha+1)^{-\frac{q}{p'}} (c_n(s))^q \|\mu_{u,\varphi}^q\|_{\frac{q}{p},\alpha} , \text{ using Lemma 2.4.1 and 2.4.2.}
\end{aligned}$$

The associated comparability constant in the above line depends only on α, μ, p, q and s. Now $c_n(s)$ is the tail of the convergent series $\sum_{i=0}^{\infty} \frac{\Gamma(2+\alpha+i)}{\Gamma(2+\alpha)\, i!} s^i = \frac{1}{(1-s)^{2+\alpha}}$. Therefore for fixed s, $\lim_{n\to\infty} c_n(s) = 0$. This completes the proof of Claim 2. □

Hence (3.4.3) implies

$$\|uC_\varphi\|_e^q \leq \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\alpha}\leq 1} I_1(f,n) . \tag{3.4.5}$$

We will now estimate $\sup_{\|f\|_{p,\alpha}\leq 1} I_1(f,n)$.

$$I_1(f,n) = \int_{\mathbb{D}\setminus\mathbb{D}_s} |R_n f(w)|^q \, d\mu_{u,\varphi}^q(w) = \int_{\mathbb{D}\setminus\mathbb{D}_s} |R_n f(w)|^q \, d\left(\mu_{u,\varphi}^q\right)_s(w) \leq \int_{\mathbb{D}} |R_n f(w)|^q \, d\left(\mu_{u,\varphi}^q\right)_s(w) .$$

Since $\mu_{u,\varphi}^{q}$ is a $(A_{\alpha}^{p}, q)-$ Carleson measure, therefore by Lemma 2.4.4, so is $\left(\mu_{u,\varphi}^{q}\right)_{s}$. Hence applying Lemma 2.4.3 and the fact that $\{R_n\}$ is uniformly bounded on A_{α}^{p},

$$\int_{\mathbb{D}} |R_n f(w)|^q \, d\left(\mu_{u,\varphi}^{q}\right)_{s}(w) \lesssim \| \left(\mu_{u,\varphi}^{q}\right)_{s} \| \, \|R_n f\|_{p,\alpha}^{q} \lesssim \| \left(\mu_{u,\varphi}^{q}\right)_{s} \| \, \|f\|_{p,\alpha}^{q} .$$

Consequently,

$$I_1(f,n) \lesssim \| \left(\mu_{u,\varphi}^{q}\right)_{s} \| \, \|f\|_{p,\alpha}^{q} ,$$

which implies

$$\sup_{\|f\|_{p,\alpha} \leq 1} I_1(f,n) \lesssim \| \left(\mu_{u,\varphi}^{q}\right)_{s} \| \text{ for any } n.$$

So,

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\alpha} \leq 1} I_1(f,n) \lesssim \| \left(\mu_{u,\varphi}^{q}\right)_{s} \| .$$

Hence from (3.4.5),

$$\|uC_{\varphi}\|_{e}^{q} \lesssim \| \left(\mu_{u,\varphi}^{q}\right)_{s} \| .$$

The above holds for any $s \in (0,1)$. Hence ,

$$\|uC_{\varphi}\|_{e}^{q} \lesssim \lim_{s\to 1^{-}} \| \left(\mu_{u,\varphi}^{q}\right)_{s} \| . \tag{3.4.6}$$

The following Lemma is well-known.

LEMMA 3.4.1. *Let X and Y be Banach spaces and T from X to Y is a bounded linear transformation. Let $\{k_n\}$ be a sequence in X such that $\|k_n\|_X = 1$ for every n and $k_n \to 0$ weakly when $n \to \infty$. Then*

$$\|T\|_e \geq \overline{\lim_{n\to\infty}} \|T\,k_n\|_Y .$$

It is a standard result that, in A_{α}^{p}, where $p > 1$, $k_a \to 0$ weakly when $|a| \to 1$. Hence applying Lemma 3.4.1, we obtain that

$$\|uC_{\varphi}\|_{e}^{q} \geq \overline{\lim_{|a|\to 1}} \|uC_{\varphi}(k_a)\|_{q,\mu}^{q} . \tag{3.4.7}$$

Now, combining (3.4.6) and (3.4.7),

$$\overline{\lim_{|a|\to 1}} \|uC_\varphi(k_a)\|_{q,\mu}^q \leq \|uC_\varphi\|_e^q \lesssim \lim_{s\to 1^-} \|\left(\mu_{u,\varphi}^q\right)_s\| . \tag{3.4.8}$$

In view of (3.4.8), it only remains to show that

$$\lim_{s\to 1^-} \|\left(\mu_{u,\varphi}^q\right)_s\| \lesssim \overline{\lim_{|a|\to 1}} \|uC_\varphi(k_a)\|_{q,\mu}^q . \tag{3.4.9}$$

By Lemma 2.4.4,

$$\lim_{s\to 1^-} \|\left(\mu_{u,\varphi}^q\right)_s\| \lesssim \lim_{s\to 1^-} N_s^* ,$$

where,

$$N_s^* = \sup_{|a|\geq s} \int_{\mathbb{D}} |\sigma'_a(w)|^{\frac{(2+\alpha)q}{p}} d\mu_{u,\varphi}^q(w) .$$

Now,

$$\lim_{s\to 1^-} N_s^* = \overline{\lim_{|a|\to 1}} \int_{\mathbb{D}} |\sigma'_a(w)|^{\frac{(2+\alpha)q}{p}} d\mu_{u,\varphi}^q(w) = \overline{\lim_{|a|\to 1}} \int_{\mathbb{D}} \left(|\sigma'_a(\varphi(z))|^{\frac{2+\alpha}{p}}\right)^q |u(z)|^q d\mu(z)$$

$$= \overline{\lim_{|a|\to 1}} \|uC_\varphi(k_a)\|_{q,\mu}^q .$$

Therefore, (3.4.9) is established and that completes the proof of Theorem 3. □

3.5. Upper bounds for the Difference operator

In this section we will give upper estimates for the norm and the essential norm of the difference operator $C_\varphi - C_\psi$ acting from A_α^p to $L^q(\mu)$. We will need the following important and well - known lemma.

LEMMA 3.5.1. *Let $0 < p \leq q$. Then there exists a constant $C = C(\alpha, p, q)$ such that for all $a \in \mathbb{D}$, $z \in \Delta\left(a, \frac{2-\sqrt{3}}{2}\right)$ and $f \in A_\alpha^p$ with $\|f\|_{p,\alpha} \leq 1$, we have*

$$|f(z) - f(a)|^q \leq C |\sigma_z(a)|^q \frac{\int_{\Delta\left(a,\frac{1}{2}\right)} |f(w)|^p dA_\alpha(w)}{\left(1-|a|^2\right)^{\frac{(2+\alpha)q}{p}}} .$$

Proof. A detailed proof can be done using the same technique as in Lemma 4.3.1 in [**Z**]. □

Theorem 3.5.1. *Let $0 < p \leq q < \infty$ and suppose both operators σC_φ and σC_ψ boundedly map A_α^p into $L^q(\mu)$. Then the following holds:*

(i) *The difference operator $C_\varphi - C_\psi$ boundedly maps A_α^p into $L^q(\mu)$ and*

$$\|C_\varphi - C_\psi\|^q \lesssim \max\left\{\|\mu_{\sigma,\varphi}^q\|, \|\mu_{\sigma,\psi}^q\|\right\}.$$

(ii) *If $p > 1$, then*

$$\|C_\varphi - C_\psi\|_e^q \lesssim \max\left\{\lim_{r\to 1^-} \|\left(\mu_{\sigma,\varphi}^q\right)_r\|, \lim_{r\to 1^-} \|\left(\mu_{\sigma,\psi}^q\right)_r\|\right\}.$$

In each of the above statements, The comparability constant depends only on α, μ, p and q.

Proof. The technique to prove (i) is quite similar to the proof of (ii) and it will be outlined after the proof of (ii). So, for a moment we suppose that (i) holds. Thus the difference operator is assumed to be bounded.

Before beginning to prove (ii), we refer to the paper [**S**] again. Our proof technique has its root in [**S**]. However, it seems to us that Saukko's proof is not correct in that case when $|\sigma|$ is sufficiently small. In that particular case, our approach is very different.

We are now ready to prove (ii).

By hypothesis, $C_\varphi - C_\psi$ is bounded from A^p_α to $L^q(\mu)$. Hence, by Lemma 2.5.1,

$$\|C_\varphi - C_\psi\|_e^q \leq \overline{\lim_{n\to\infty}} \|(C_\phi - C_\psi) R_n\|^q = \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{\mathbb{D}} |(C_\phi - C_\psi) \circ R_n f(z)|^q \, d\mu(z). \tag{3.5.1}$$

We now break the disk $\mathbb{D}$ into E and E' where $E = \left\{ z \in \mathbb{D} : |\sigma(z)| \geq \frac{2-\sqrt{3}}{2} \right\}$ and $E' = \mathbb{D} \setminus E$. For a fixed $n \in \mathbb{N}$ and $f \in A^p_\alpha$, let us define

$$I_n(f) = \int_E |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z) \ ,$$

and,

$$J_n(f) = \int_{E'} |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z).$$

Then,

$$\int_{\mathbb{D}} |(C_\phi - C_\psi) \circ R_n f(z)|^q \, d\mu(z) = I_n(f) + J_n(f).$$

Consequently,

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{\mathbb{D}} |(C_\phi - C_\psi) \circ R_n f(z)|^q \, d\mu(z) \ \leq \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} I_n(f) + \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} J_n(f).$$

Applying (3.5.1),

$$\|C_\varphi - C_\psi\|_e^q \leq \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} I_n(f) \ + \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} J_n(f). \tag{3.5.2}$$

Let us now estimate $I_n(f)$.

$$I_n(f) = \int_E \frac{1}{|\sigma(z)|^q} |(\sigma C_\varphi - \sigma C_\psi) \circ R_n f(z)|^q \, d\mu(z)$$

$$\leq \left(\frac{2}{2-\sqrt{3}} \right)^q \int_E |(\sigma C_\varphi - \sigma C_\psi) \circ R_n f(z)|^q \, d\mu(z)$$

$$\leq \left(\frac{2}{2-\sqrt{3}} \right)^q 2^q \left(\int_E |(\sigma C_\varphi) \circ R_n f(z)|^q \, d\mu(z) + \int_E |(\sigma C_\psi) \circ R_n f(z)|^q \, d\mu(z) \right).$$

It follows that

$$I_n(f) \lesssim \int_{\mathbb{D}} |(\sigma C_\varphi) \circ R_n f(z)|^q \, d\mu(z) + \int_{\mathbb{D}} |(\sigma C_\psi) \circ R_n f(z)|^q \, d\mu(z).$$

Taking supremum of both sides,

$$\sup_{\|f\|_{p,\,\alpha} \le 1} I_n(f) \lesssim \|(\sigma C_\varphi) R_n\|^q + \|(\sigma C_\psi) R_n\|^q.$$

Taking limit superior, we obtain,

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha} \le 1} I_n(f) \lesssim \overline{\lim_{n\to\infty}} \|(\sigma C_\varphi) R_n\|^q + \overline{\lim_{n\to\infty}} \|(\sigma C_\psi) R_n\|^q.$$

Theorem 3 implies

$$\overline{\lim_{n\to\infty}} \|(\sigma C_\varphi) R_n\|^q \asymp \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\varphi}\right)_r\| \text{ and } \overline{\lim_{n\to\infty}} \|(\sigma C_\psi) R_n\|^q \asymp \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\psi}\right)_r\|.$$

Hence,

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha} \le 1} I_n(f) \lesssim \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\varphi}\right)_r\| + \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\psi}\right)_r\|$$

$$\le 2 \max\left\{ \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\varphi}\right)_r\|, \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\psi}\right)_r\| \right\}. \tag{3.5.3}$$

In view of $(3.5.2)$ and $(3.5.3)$, it is now enough to show that

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha} \le 1} J_n(f) \lesssim \max\left\{ \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\varphi}\right)_r\|, \lim_{r\to 1^-} \|\left(\mu^q_{\sigma,\psi}\right)_r\| \right\}. \tag{3.5.4}$$

We will now estimate $J_n(f)$ by breaking the disk $\mathbb{D}$ in the following way:

Let us fix $r \in (0,1)$. Then,

$$J_n(f) = \int_{E'} |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z)$$

$$= \int_{E' \cap \varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z) + \int_F |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z),$$

where $F = E^{'} \cap \varphi^{-1}(\mathbb{D} \setminus \mathbb{D}_r)$. Then,

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} J_n(f) \leq \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{E^{'}\cap\varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi)\circ R_n f(z)|^q \, d\mu(z)$$

$$+ \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_F |(C_\varphi - C_\psi)\circ R_n f(z)|^q \, d\mu(z). \tag{3.5.5}$$

Let us now consider the term $\int_F |(C_\varphi - C_\psi)\circ R_n f(z)|^q \, d\mu(z)$.

We note that $F \subseteq E^{'} = \left\{z : |\sigma(z)| < \frac{2-\sqrt{3}}{2}\right\}$. Also since $\{R_n\}$ is uniformly bounded on A^p_α, therefore when $\|f\|_{p,\alpha} \leq 1$, we can assume, without loss of generality, that $\|R_n f\|_{p,\alpha} \leq 1$ for each $n \in \mathbb{N}$. Therefore we can apply Lemma 3.5.1 on $R_n f$ to estimate the above integral.

So, for each $n \in \mathbb{N}$ and each $f \in A^p_\alpha$ satisfying $\|f\|_{p,\alpha} \leq 1$, let us apply Lemma 3.5.1, taking

$$f = R_n(f),\; z = R_n f(\psi(z)),\; a = R_n f(\varphi(z)),$$

which gives us ,

$$\int_F |(C_\varphi - C_\psi)\circ R_n f(z)|^q \, d\mu(z) = \int_F |R_n f(\psi(z)) - R_n f(\varphi(z))|^q \, d\mu(z)$$

$$\lesssim \int_F |\sigma(z)|^q \; \frac{\int_{\Delta\left(\varphi(z),\,\frac{1}{2}\right)} |R_n f(w)|^p \, dA_\alpha(w)}{\left(1 - |\varphi(z)|^2\right)^{\frac{(2+\alpha)\,q}{p}}} \; d\mu(z)$$

$$= \int_F |\sigma(z)|^q \left[\int_{\Delta\left(\varphi(z),\,\frac{1}{2}\right)} \frac{|R_n f(w)|^p}{\left(1 - |\varphi(z)|^2\right)^{\frac{(2+\alpha)\,q}{p}}} \; dA_\alpha(w)\right] \; d\mu(z).$$

We note that when $w \in \Delta\left(\varphi(z), \frac{1}{2}\right)$, then $1 - |w|^2 \asymp 1 - |\varphi(z)|^2$, by (2.6.2) in Preliminaries. Therefore,

$$\int_F |\sigma(z)|^q \left[\int_{\Delta\left(\varphi(z),\frac{1}{2}\right)} \frac{|R_n f(w)|^p}{\left(1-|\varphi(z)|^2\right)^{\frac{(2+\alpha)q}{p}}} \, dA_\alpha(w)\right] d\mu(z)$$

$$\lesssim \int_F |\sigma(z)|^q \int_{\Delta\left(\varphi(z),\frac{1}{2}\right)} \frac{|R_n f(w)|^p}{\left(1-|w|^2\right)^{\frac{(2+\alpha)q}{p}}} dA_\alpha(w)\, d\mu(z)$$

$$= \int_F |\sigma(z)|^q \int_{\mathbb{D}} \frac{|R_n f(w)|^p}{\left(1-|w|^2\right)^{\frac{(2+\alpha)q}{p}}} \chi_{\Delta\left(\varphi(z),\frac{1}{2}\right)}(w)\, dA_\alpha(w)\, d\mu(z)$$

$$= \int_F |\sigma(z)|^q \int_{\mathbb{D}} \frac{|R_n f(w)|^p}{\left(1-|w|^2\right)^{\frac{(2+\alpha)q}{p}}} \chi_{\varphi^{-1}\left(\Delta\left(w,\frac{1}{2}\right)\right)}(z)\, dA_\alpha(w)\, d\mu(z)$$

$$= \int_{\mathbb{D}} \left[\int_F |\sigma(z)|^q \chi_{\varphi^{-1}\left(\Delta\left(w,\frac{1}{2}\right)\right)}(z)\, d\mu(z)\right] \frac{|R_n f(w)|^p}{\left(1-|w|^2\right)^{\frac{(2+\alpha)q}{p}}} dA_\alpha(w),$$

using the elementary fact that $\chi_{\Delta\left(\varphi(z),\frac{1}{2}\right)}(w) = \chi_{\varphi^{-1}\left(\Delta\left(w,\frac{1}{2}\right)\right)}(z)$ and applying Tonelli's theorem to switch integrals.

$$= \int_{\mathbb{D}} \left[\int_{F\cap\varphi^{-1}\left(\Delta\left(w,\frac{1}{2}\right)\right)} |\sigma(z)|^q \, d\mu(z)\right] \frac{|R_n f(w)|^p}{\left(1-|w|^2\right)^{\frac{(2+\alpha)q}{p}}} dA_\alpha(w)$$

$$\leq \int_{\mathbb{D}} |R_n f(w)|^p \underbrace{\left[\frac{\int_{\varphi^{-1}\left[\left(\Delta\left(w,\frac{1}{2}\right)\right)\cap(\mathbb{D}\setminus\mathbb{D}_r)\right]} |\sigma(z)|^q \, d\mu(z)}{\left(1-|w|^2\right)^{\frac{(2+\alpha)q}{p}}}\right]}_{G(w,r)} dA_\alpha(w)$$

Now by definition,

$$G(w,r) = \frac{\left(\mu^q_{\sigma,\varphi}\right)_r \left(\Delta\left(w,\, \frac{1}{2}\right)\right)}{\left(1-|w|^2\right)^{\frac{(2+\alpha)q}{p}}}$$

Since σC_φ is bounded from A^p_α to $L^q(\mu)$, therefore by Lemma 2.4.2 , $\|\mu^q_{\sigma,\varphi}\| < \infty$. Consequently, by Lemma 2.4.4 , $\| \left(\mu^q_{\sigma,\varphi}\right)_r \| < \infty$. So for any $w \in \mathbb{D}$,

$$G(w,r) \leq \| \left(\mu^q_{\sigma,\varphi}\right)_r \| < \infty.$$

Therefore the above double integral

$$\leq \int_{\mathbb{D}} |R_n f(w)|^p \, \| \left(\mu^q_{\sigma,\varphi}\right)_r \| \, dA_\alpha(w) \lesssim \|f\|^p_{p,\,\alpha} \, \| \left(\mu^q_{\sigma,\varphi}\right)_r \| ,$$

since $\{R_n\}$ is uniformly bounded on A^p_α. Therefore,

$$\int_F |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z) \lesssim \|f\|^p_{p,\,\alpha} \, \| \left(\mu^q_{\sigma,\varphi}\right)_r \| \; \forall\, n, \forall\, f.$$

Taking supremum and limit superior,

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_F |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z) \lesssim \| \left(\mu^q_{\sigma,\varphi}\right)_r \|.$$

This implies, applying (3.5.5),

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} J_n(f) \lesssim \overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z) + \| \left(\mu^q_{\sigma,\varphi}\right)_r \|. \tag{3.5.6}$$

We will now estimate $\int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z)$.

Since $E' \cap \varphi^{-1}(\mathbb{D}_r) \subseteq E'$, therefore we can apply Lemma 3.5.1 as previously, to obtain

$$\int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi) \circ R_n f(z)|^q \, d\mu(z)$$

$$\lesssim \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |\sigma(z)|^q \, \frac{\int_{\Delta\left(\varphi(z),\,\frac{1}{2}\right)} |R_n f(w)|^p \, dA_\alpha(w)}{\left(1 - |\varphi(z)|^2\right)^{\frac{(2+\alpha)\,q}{p}}} \, d\mu(z).$$

We note that $\dfrac{1}{1-|\varphi(z)|^2} \leq \dfrac{1}{1-r^2}$ whenever $z \in \varphi^{-1}(\mathbb{D}_r)$. Therefore, for fixed r, the above double integral

$$\lesssim \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |\sigma(z)|^q \int_{\Delta\left(\varphi(z),\,\frac{1}{2}\right)} |R_n f(w)|^p \, dA_\alpha(w)\, d\mu(z)$$

$$= \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |\sigma(z)|^q \int_{\mathbb{D}} |R_n f(w)|^p \, \chi_{\Delta\left(\varphi(z),\,\frac{1}{2}\right)}(w)\; dA_\alpha(w)\; d\mu(z)$$

$$= \int_{\mathbb{D}} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)\cap\varphi^{-1}\left(\Delta\left(w,\,\frac{1}{2}\right)\right)} |\sigma(z)|^q \; d\mu(z)\, |R_n f(w)|^p \; dA_\alpha(w).$$

We now observe that whenever $z \in \varphi^{-1}(\mathbb{D}_r) \cap \varphi^{-1}\left(\Delta\left(w,\,\frac{1}{2}\right)\right)$,

$$|\varphi(z)| < r \text{ and } \rho(w,\varphi(z)) < \frac{1}{2}.$$

An elementary computation shows that the function $f(x,y) = \dfrac{x+y}{1+xy}$ attains a maximum value in the rectangle $[0,l]\times[0,m]$ at the point (l,m).

Then,

$$|w| = \rho(w,0) \leq \frac{\rho(w,\,\varphi(z)) + \rho(\varphi(z),0)}{1+\rho(w,\varphi(z))\,\rho(\varphi(z),0)}, \text{ by strong form of triangle inequality}$$

$$\leq \frac{\frac{1}{2}+r}{1+\frac{r}{2}} = \frac{1+2r}{2+r} = R, \text{ say.}$$

Then clearly $R<1$ and $w \in \mathbb{D}_R$. Hence,

$$\int_{\mathbb{D}} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)\cap\varphi^{-1}\left(\Delta\left(w,\,\frac{1}{2}\right)\right)} |\sigma(z)|^q \; d\mu(z)\, |R_n f(w)|^p \; dA_\alpha(w)$$

$$= \int_{\mathbb{D}_R} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)\cap\varphi^{-1}\left(\Delta\left(w,\,\frac{1}{2}\right)\right)} |\sigma(z)|^q \; d\mu(z)\, |R_n f(w)|^p \; dA_\alpha(w)$$

$$\leq \int_{\mathbb{D}_R} \left[\int_{\varphi^{-1}(\mathbb{D}_r)} |\sigma(z)|^q \; d\mu(z)\right] |R_n f(w)|^p \; dA_\alpha(w)$$

$$= \int_{\mathbb{D}_R} \left[\frac{\mu^q_{\sigma,\varphi}\,(\Delta\,(0,r))}{\left(1-|0|^2\right)^{\frac{(2+\alpha)\,q}{p}}} \right] |R_n f\,(w)|^p\, dA_\alpha\,(w)$$

$$\leq \|\mu^q_{\sigma,\varphi}\|_{\frac{q}{p},\,\alpha\,,\,r} \int_{\mathbb{D}_R} |R_n f\,(w)|^p\, dA_\alpha\,(w)\,.$$

Therefore,

$$\int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi)\circ R_n f\,(z)|^q\, d\mu\,(z) \leq \|\mu^q_{\sigma,\varphi}\|_{\frac{q}{p},\,\alpha\,,\,r} \int_{\mathbb{D}_R} |R_n f\,(w)|^p\, dA_\alpha\,(w)\,.$$

Taking supremum and limit superior, we obtain

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi)\circ R_n f\,(z)|^q\, d\mu\,(z)$$

$$\leq \left(\|\mu^q_{\sigma,\varphi}\|_{\frac{q}{p},\,\alpha\,,\,r}\right) \left[\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{\mathbb{D}_R} |R_n f\,(w)|^p\, dA_\alpha\,(w) \right]. \tag{3.5.7}$$

Claim 3.

$$\sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{\mathbb{D}_R} |R_n f\,(w)|^p\, dA_\alpha\,(w) \to 0 \;\; as\; n\to\infty$$

Proof sketch. The proof of the above claim is very similar to the proof of claim 2 in section 3.4. We provide a sketch of the proof below:

We have shown in section 3.4 that

$$R_n f\,(w) = \int_{\mathbb{D}} \frac{R_n f\,(z)\, dA_\alpha\,(z)}{(1-\overline{z}w)^{2+\alpha}}, \forall f \in A^p_\alpha\,,$$

i.e.,

$$R_n f\,(w) = \langle R_n f, K_w\rangle_\alpha\,,$$

where,

$$K_w(z) = \frac{1}{(1-\overline{w}z)^{2+\alpha}} = \sum_{j=0}^{\infty} \frac{\Gamma(2\,+\,\alpha+j)}{\Gamma(2\,+\,\alpha)\,j!} (\overline{w}z)^j \,.$$

Then,

$$\begin{aligned} |R_n f(w)| &= |\langle R_n f, K_w\rangle_\alpha| \\ &= |\langle f, R_n K_w\rangle_\alpha| \,, \text{ using Claim 1 in section 2.5.} \\ &= \left| \int_{\mathbb{D}} f(z) \overline{R_n K_w(z)}\; dA_\alpha(z) \right| \\ &\leq \|f\|_{p,\,\alpha} \|R_n K_w\|_{p',\,\alpha} \,, \text{using Hölder's inequality.} \end{aligned}$$

Therefore,

$$\int_{\mathbb{D}_R} |R_n f(w)|^p \, dA_\alpha(w) \leq \|f\|_{p,\,\alpha}^p \int_{\mathbb{D}_R} \|R_n K_w\|_{p',\,\alpha}^p \, dA_\alpha(w) \,,$$

which implies

$$\sup_{\|f\|_{p,\,\alpha}\leq 1} \int_{\mathbb{D}_R} |R_n f(w)|^p \, dA_\alpha(w) \leq \int_{\mathbb{D}_R} \|R_n K_w\|_{p',\,\alpha}^p \, dA_\alpha(w)$$

So to prove our claim, it is enough to show that $\lim\limits_{n\to\infty} \int_{\mathbb{D}_R} \|R_n K_w\|_{p',\,\alpha}^p \, dA_\alpha(w) = 0$.

In section 3.4, we have shown for any $z \in \mathbb{D}$,

$$|R_n K_w(z)| \leq \sum_{j=n+1}^{\infty} \frac{\Gamma(2+\alpha+j)}{\Gamma(2+\alpha)\,j!} |w|^j\, |z|^j \leq \sum_{j=n+1}^{\infty} \frac{\Gamma(2+\alpha+j)}{\Gamma(2+\alpha)\,j!} |w|^j \,.$$

Hence for any $w \in \mathbb{D}_R$,

$$|R_n K_w(z)| \leq \sum_{j=n+1}^{\infty} \frac{\Gamma(2+\alpha+j)}{\Gamma(2+\alpha)\,j!} R^j = c_n(R)\,, \text{ say}\,.$$

Integrating , we get that

$$\|R_nK_w\|_{p',\alpha}^{p} = \left(\int_{\mathbb{D}} |R_nK_w(z)|^{p'}\, dA_\alpha(z)\right)^{\frac{p}{p'}} \leq (c_n(R))^p \left(\int_{\mathbb{D}} dA_\alpha(z)\right)^{\frac{p}{p'}},$$

which implies

$$\|R_nK_w\|_{p',\alpha}^{p} \leq (\alpha+1)^{-\frac{p}{p'}} (c_n(R))^p .$$

Hence,

$$\int_{\mathbb{D}_R} \|R_nK_w\|_{p',\alpha}^{p}\, dA_\alpha(w) \;\leq\; (\alpha+1)^{-\frac{p}{p'}} (c_n(R))^p A_\alpha(\mathbb{D}_R) \;\leq\; (\alpha+1)^{-\frac{p}{p'}} (c_n(R))^p (\alpha+1)^{-1} . \tag{3.5.8}$$

Since $c_n(R)$ is the tail of the convergent series $\sum_{j=0}^{\infty} \frac{\Gamma(2+\alpha+j)}{\Gamma(2+\alpha)\, j!} R^j = \frac{1}{(1-R)^{2+\alpha}}$, therefore $\lim_{n\to\infty} c_n(R) = 0$. Consequently, applying (3.5.8), $\lim_{n\to\infty} \int_{\mathbb{D}_R} \|R_nK_w\|_{p',\alpha}^{p}\, dA_\alpha(w) = 0$. This proves the claim. □

Since Claim 3 is established, therefore by (3.5.7),

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\alpha}\leq 1} \int_{E'\cap\varphi^{-1}(\mathbb{D}_r)} |(C_\varphi - C_\psi)\circ R_nf(z)|^q\, d\mu(z) = 0.$$

Hence, applying (3.5.6),

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\alpha}\leq 1} J_n(f) \lesssim 0 + \| \left(\mu_{\sigma,\varphi}^{q}\right)_r \| = \| \left(\mu_{\sigma,\varphi}^{q}\right)_r \|.$$

Since the above is true for any $r \in (0,1)$, therefore,

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_{p,\alpha}\leq 1} J_n(f) \lesssim \lim_{r\to 1^-} \| \left(\mu_{\sigma,\varphi}^{q}\right)_r \|,$$

which clearly proves our desired inequality (3.5.4). Hence proof of (ii) is complete.

We will now prove (i). By definition ,

$$\|C_\varphi - C_\psi\|^q = \sup_{\|f\|_{p,\,\alpha}\le 1} \int_{\mathbb{D}} |C_\varphi f(z) - C_\psi f(z)|^q \, d\mu(z).$$

Breaking the disk $\mathbb{D}$ exactly as in the proof of (ii) ,

$$\|C_\varphi - C_\psi\|^q \le \underbrace{\sup_{\|f\|_{p,\,\alpha}\le 1} \int_E |f(\varphi(z)) - f(\psi(z))|^q \, d\mu(z)}_{I_1}$$

$$+ \underbrace{\sup_{\|f\|_{p,\,\alpha}\le 1} \int_{E'} |f(\varphi(z)) - f(\psi(z))|^q \, d\mu(z).}_{I_2} \tag{3.5.9}$$

Exactly as in the proof of (ii) , we can show that $I_1 \lesssim \|\sigma C_\varphi\|^q + \|\sigma C_\psi\|^q$. Hence by Lemma 2.4.2,

$$I_1 \lesssim \|\mu^q_{\sigma,\varphi}\| + \|\mu^q_{\sigma,\psi}\| \le 2\, \max\left\{\|\mu^q_{\sigma,\varphi}\|,\, \|\mu^q_{\sigma,\psi}\|\right\}. \tag{3.5.10}$$

Also, similar to the proof of (ii) , we can apply Lemma 3.5.1 on f, use Tonelli's theorem and (2.6.2) , to obtain

$$I_2 \lesssim \|\mu^q_{\sigma,\varphi}\|. \tag{3.5.11}$$

Combining (3.5.9), (3.5.10) and (3.5.11), the proof of (i) is now complete. □

3.6. Lower bounds for the Difference operator

THEOREM 3.6.1. *Let $0 < p \le q < \infty$. Suppose the difference operator $C_\varphi - C_\psi$ maps A^p_α boundedly into $L^q(\mu)$. Then*

(i) *The operators σC_φ and σC_ψ both map A^p_α into $L^q(\mu)$ and*

$$\sup_{a\in\mathbb{D}} \|(C_\varphi - C_\psi)\, k_a\|^q_{q,\,\mu} \gtrsim \max\left\{\|\mu^q_{\sigma,\varphi}\|,\, \|\mu^q_{\sigma,\psi}\|\right\}.$$

$$\text{(ii)} \ \overline{\lim_{|a|\to 1}} \| (C_\varphi - C_\psi)\, k_a\|_{q,\,\mu}^q \gtrsim \max\left\{ \lim_{r\to 1^-} \| \left(\mu_{\sigma,\varphi}^q\right)_r \| \,,\ \lim_{r\to 1^-} \| \left(\mu_{\sigma,\psi}^q\right)_r \| \right\}.$$

The above comparability constants depend only on α, μ, p *and* q.

Proof. In our proof, we had to use the measures $\mu_{\sigma,\,\varphi}^q$ and $\mu_{\sigma,\,\psi}^q$ instead of $\mu_{\sigma,\,\varphi}^{q,\,\beta}$ and $\mu_{\sigma,\,\psi}^{q,\,\beta}$, used in [**S**]. So, our proof is essentially an elementary modification of the technique used to prove Theorem 4.5 in [**S**]. Details omitted. □

Applying Lemma 3.4.1,

$$\|C_\varphi - C_\psi\|_e^q \ \geq \ \overline{\lim_{|a|\to 1}} \| (C_\varphi - C_\psi)\, k_a\|_{q,\,\mu}^q \,. \tag{3.6.1}$$

Hence, combining Theorem 3.5.1(ii), (3.6.1) and Theorem 3.6.1(ii),

$$\max\left\{ \lim_{r\to 1^-} \| \left(\mu_{\sigma,\varphi}^q\right)_r \| \,,\ \lim_{r\to 1^-} \| \left(\mu_{\sigma,\psi}^q\right)_r \| \right\} \gtrsim \|C_\varphi - C_\psi\|_e^q \gtrsim \overline{\lim_{|a|\to 1}} \| (C_\varphi - C_\psi)\, k_a\|_{q,\,\mu}^q$$

$$\gtrsim \max\left\{ \lim_{r\to 1^-} \| \left(\mu_{\sigma,\varphi}^q\right)_r \| \,,\ \lim_{r\to 1^-} \| \left(\mu_{\sigma,\psi}^q\right)_r \| \right\} ,$$

which clearly proves Theorem 2. Similarly, combining Theorem 3.5.1(i) and Theorem 3.6.1(i), Theorem 1 is proved.

CHAPTER 4

A GENERALIZED DIFFERENCE OPERATOR

We recall from chapter 3 that if we assume $uC_\varphi - vC_\psi$ is bounded on the Hilbert space A^2_α , then

$$\|uC_\varphi - vC_\psi\|^2_{\mathrm{HS}} = \int_{\mathbb{D}} S(z)\, dA_\alpha(z)\,,$$

where,

$$S(z) = \frac{|\,u(z)\,|^2}{(1-|\varphi(z)\,|^2)^{2+\alpha}} + \frac{|\,v(z)\,|^2}{(1-|\psi(z)\,|^2)^{2+\alpha}} - 2\,\Re\left(\frac{u(z)\,\overline{v(z)}}{\left(1-\varphi(z)\,\overline{\psi(z)}\right)^{2+\alpha}}\right). \quad (4.0.2)$$

Then taking $u = v \equiv 1$ in (4.0.2) , in particular, and applying a change of variable method alongwith some elementary estimations , we proved

$$\|C_\varphi - C_\psi\|^2_{\mathrm{HS}} \asymp \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\, dA_\alpha(z)}{(1-|\varphi(z)\,|^2)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\, dA_\alpha(z)}{(1-|\psi(z)\,|^2)^{2+\alpha}}\,. \quad (4.0.3)$$

We would like to remember that our change of variable method is very helpful in factoring out the cancelation factor σ involved in the above integral expression for $\|C_\varphi - C_\psi\|^2_{\mathrm{HS}}$.

Based on this, we asked a natural question.

Question. Let us assume $uC_\varphi - vC_\psi$ is bounded on A^2_α where u and v are two bounded analytic weights. Then using a similar change of variable method, can we find an asymptotically equivalent expression for $\|uC_\varphi - vC_\psi\|^2_{\mathrm{HS}}$, similar to (4.0.3) , with some suitable cancelation factor involving all of u, v and σ? If yes, then what is that cancelation factor?

The answer is generally no. In fact, the technique was unable to determine the appropriate cancelation factor for $\|uC_\varphi - vC_\psi\|_{\mathrm{HS}}^2$, if exists at all.

Then we considered a special case where $u = \varphi$ and $v = \psi$ and $\alpha = 0$. That gives us the operator $\varphi C_\varphi - \psi C_\psi$ acting boundedly on A^2. Notice that it becomes the Zero operator if $\varphi = \psi$. Clearly $\varphi = \psi$ iff $\sigma = 0$. Based on this observation, we guessed that the appropriate cancelation factor for $\varphi C_\varphi - \psi C_\psi$ is also σ. In fact our guess is true. Interestingly, we proved that $\|\varphi C_\varphi - \psi C_\psi\|_{\mathrm{HS}}^2$ has the same asymptotically equivalent expression as $\|C_\varphi - C_\psi\|_{\mathrm{HS}}^2$.

4.1. Hilbert-Schmidt norm of $\varphi C_\varphi - \psi C_\psi$

THEOREM 4. *Let us consider the operator $\varphi C_\varphi - \psi C_\psi$ acting boundedly on A^2. Then,*

$$\|\varphi C_\varphi - \psi C_\psi\|_{HS}^2 \asymp \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA(z)}{(1-|\varphi(z)|^2)^2} + \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA(z)}{(1-|\psi(z)|^2)^2}.$$

PROOF. Taking $u = \varphi$, $v = \psi$ in (4.0.2), and $\alpha = 0$, we have,

$$\|\varphi C_\varphi - \psi C_\psi\|_{\mathrm{HS}}^2 = \int_{\mathbb{D}} \left[\frac{|\varphi(z)|^2}{(1-|\varphi(z)|^2)^2} + \frac{|\psi(z)|^2}{(1-|\psi(z)|^2)^2} - 2\,\Re\left(\frac{\varphi(z)\,\overline{\psi(z)}}{\left(1-\varphi(z)\,\overline{\psi(z)}\right)^2} \right) \right] dA(z).$$

So , the integrand is of the form $F\ (a,b)$ where,

$$F\ (a,b) = \frac{|a|^2}{\left(1-|a|^2\right)^2} + \frac{|b|^2}{\left(1-|b|^2\right)^2} - 2\,\Re\left(\frac{a\bar{b}}{\left(1-a\bar{b}\right)^2}\right), \qquad (4.1.1)$$

with $a = \varphi(z)$ and $b = \psi(z)$.

We will now apply the same change of variable method that we have used previously on $C_\varphi - C_\psi$.

let $b = \sigma_a(w)$. Then $w = \sigma_a(b)$ and $1-|b|^2 = \dfrac{\left(1-|a|^2\right)\left(1-|w|^2\right)}{|1-\overline{a}w|^2}$. This implies

$$\frac{|b|^2}{\left(1-|b|^2\right)^2} = \frac{|a-w|^2}{|1-\overline{a}w|^2} \times \frac{|1-\overline{a}w|^4}{\left(1-|a|^2\right)^2\left(1-|w|^2\right)^2} = \frac{|a-w|^2\,|1-\overline{a}w|^2}{\left(1-|a|^2\right)^2\left(1-|w|^2\right)^2},$$

and,

$$\frac{a\overline{b}}{\left(1-a\overline{b}\right)^2} = \frac{a\left(\dfrac{\overline{a}-\overline{w}}{1-a\overline{w}}\right)}{\left[1-\dfrac{|a|^2-a\overline{w}}{1-a\overline{w}}\right]^2} = \frac{a\left(\overline{a}-\overline{w}\right)}{1-a\overline{w}} \times \frac{\left(1-a\overline{w}\right)^2}{\left(1-|a|^2\right)^2} = \frac{a\left(\overline{a}-\overline{w}\right)\left(1-a\overline{w}\right)}{\left(1-|a|^2\right)^2}.$$

Substituting in (4.1.1), $F\ (a,b) = \dfrac{1}{\left(1-|a|^2\right)^2}\left(|a|^2 + \dfrac{|a-w|^2\,|1-\overline{a}w|^2}{\left(1-|w|^2\right)^2} - 2\,\Re\, a\left(\overline{a}-\overline{w}\right)\left(1-a\overline{w}\right)\right)$

$$= \frac{1}{\left(1-|a|^2\right)^2}\left(|a-(a-w)\left(1-\overline{a}w\right)|^2 + |a-w|^2|1-\overline{a}w|^2\left\{\frac{1}{\left(1-|w|^2\right)^2}-1\right\}\right)$$

$$= \frac{|w|^2}{\left(1-|a|^2\right)^2}\left(\left||a|^2+1-\overline{a}w\right|^2 + \frac{|a-w|^2\,|1-\overline{a}w|^2\left(2-|w|^2\right)}{\left(1-|w|^2\right)^2}\right).$$

By symmetry ,

$$\|\varphi C_\varphi - \psi C_\psi\|^2_{\text{HS}} = \|\psi C_\psi - \varphi C_\varphi\|^2_{\text{HS}}.$$

Therefore,

$$\|\varphi C_\varphi - \psi C_\psi\|^2_{\text{HS}} = \frac{1}{2}\int_{\mathbb{D}} \left[F\ (a,b) + F\ (b,a)\right], \text{ where } a = \varphi(z),\ b = \psi(z)\ .$$

Hence,

$$\|\varphi C_\varphi - \psi C_\psi\|^2_{\text{HS}} = \frac{|w|^2}{2}\int_{\mathbb{D}} I\,(a,b,w)\ dA\,(z)\ , \tag{4.1.2}$$

where,

$$I\left(a,b,w\right)=\frac{\left||a|^{2}+1-\bar{a}w\right|^{2}}{\left(1-|a|^{2}\right)^{2}}+\frac{\left||b|^{2}+1-\bar{b}\eta\right|^{2}}{\left(1-|b|^{2}\right)^{2}}$$

$$+\frac{|a-w|^{2}\,|1-\bar{a}w|^{2}\left(2-|w|^{2}\right)}{\left(1-|a|^{2}\right)^{2}\left(1-|w|^{2}\right)^{2}}+\frac{|b-\eta|^{2}\left|1-\bar{b}\eta\right|^{2}\left(2-|w|^{2}\right)}{\left(1-|b|^{2}\right)^{2}\left(1-|w|^{2}\right)^{2}}\,. \qquad (4.1.3)$$

Here $\eta=\sigma_b\left(a\right)$ and we have used the elementary fact that $|w|=|\eta|$. We will now estimate each term in the right hand side of (4.1.3) . A routine computation shows that ,

$$\frac{|a-w|^{2}\,|1-\bar{a}w|^{2}\left(2-|w|^{2}\right)}{\left(1-|a|^{2}\right)^{2}\left(1-|w|^{2}\right)^{2}}=\frac{|b|^{2}\left(2-|w|^{2}\right)}{\left(1-|b|^{2}\right)^{2}}\asymp\frac{|b|^{2}}{\left(1-|b|^{2}\right)^{2}}\ ,\ \text{since } 1\leq 2-|w|^{2}\leq 2\,.$$

Similarly ,

$$\frac{|b-\eta|^{2}\left|1-\bar{b}\eta\right|^{2}\left(2-|w|^{2}\right)}{\left(1-|b|^{2}\right)^{2}\left(1-|w|^{2}\right)^{2}}\asymp\frac{|a|^{2}}{\left(1-|a|^{2}\right)^{2}}\,.$$

Let us now consider the first and second terms in the right hand side of (4.1.3).

Estimating the first term : Applying triangle inequality in bothways,

$$3\geq\left||a|^{2}+1-\bar{a}w\right|\geq|a|^{2}+1-|\bar{a}w|\geq|a|^{2}+1-|a|\ . \qquad (4.1.4)$$

So, our goal is now to find a lower bound for $|a|^{2}+1-|a|$. Therefore, we consider the function $f\left(x\right)=x^{2}+1-x$ in the interval $[0,1]$. Then $f'\left(x\right)=2x-1$. Setting $f'\left(x\right)=0$ will give us $x=\frac{1}{2}$. Now , $f\left(\frac{1}{2}\right)=\frac{3}{4}$ and $f\left(0\right)=f\left(1\right)=1$. So clearly $\frac{3}{4}$ will be a required lower bound for the function $f\left(x\right)=x^{2}+1-x$ in the interval $[0,1]$. Therefore, applying (4.1.4), $3\geq\left||a|^{2}+1-\bar{a}w\right|\geq\frac{3}{4}$. Hence it follows that,

$$\frac{\left||a|^{2}+1-\bar{a}w\right|^{2}}{\left(1-|a|^{2}\right)^{2}}\asymp\frac{1}{\left(1-|a|^{2}\right)^{2}}\,.$$

Estimating the 2nd term : Exactly as for the first term, we can determine the following estimate for the second term.

$$\frac{\left||b|^2+1-\bar{b}\eta\right|^2}{\left(1-|b|^2\right)^2} \asymp \frac{1}{\left(1-|b|^2\right)^2}.$$

Substituting all these in (4.1.3),

$$I(a,b,w) \asymp \frac{1}{\left(1-|a|^2\right)^2}+\frac{1}{\left(1-|b|^2\right)^2}+\frac{|b|^2}{\left(1-|b|^2\right)^2}+\frac{|a|^2}{\left(1-|a|^2\right)^2}$$

$$=\frac{1+|a|^2}{\left(1-|a|^2\right)^2}+\frac{1+|b|^2}{\left(1-|b|^2\right)^2} \asymp \frac{1}{\left(1-|a|^2\right)^2}+\frac{1}{\left(1-|b|^2\right)^2}.$$

$a=\varphi(z)$, $b=\psi(z)$ and $w=\sigma(z)$. Hence from (4.1.2) and (4.1.3) , we arrive at

$$\|\varphi C_\varphi-\psi C_\psi\|_{\mathrm{HS}}^2 \asymp \int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA(z)}{(1-|\varphi(z)\,|^2)^2}+\int_{\mathbb{D}} \frac{|\,\sigma^2(z)\,|\,dA(z)}{(1-|\psi(z)\,|^2)^2}.$$

This completes the proof. $\square$

Corollary 4.1.1. *Let us consider the operator $\varphi C_\varphi-\psi C_\psi$ acting boundedly on A^2. Then $\varphi C_\varphi-\psi C_\psi$ is Hilbert-Schmidt iff $C_\varphi-C_\psi$ is Hilbert-Schmidt.*

4.2. A generalization of corollary 4.1.1

Looking at Corollary 4.1.1 , we can raise a natural question. Does the result hold for A_α^2 for a general α ? The answer, in fact, is yes. It can even be strengthened. For example , we have been able to generalize Corollary 4.1.1 in the case when the difference operator is of the form $\varphi^n C_\varphi-\psi^n C_\psi$, acting on a weighted Bergman space A_α^2 . For the sake of clarity, f^n , where f is a complex-valued function , will stand for pointwise multiplication of f n times.

Theorem 4.2.1. *Let us fix $n \in \mathbb{N}$. We consider the operator $\varphi^n C_\varphi-\psi^n C_\psi$ acting on A_α^2. Then $\varphi^n C_\varphi-\psi^n C_\psi$ is Hilbert-Schmidt iff $C_\varphi-C_\psi$ is Hilbert-Schmidt .*

PROOF. Before proving this theorem, we will first introduce a notation. Let $f \in H(\mathbb{D})$. For $m \in \mathbb{N} \cup \{0\}$, let us define the function $\widehat{f^m} \in H(\mathbb{D})$ as follows:

$$\widehat{f^m}(z) = z^m f(z), z \in \mathbb{D}.$$

For $m \in \mathbb{N} \cup \{0\}$, it can be easily checked that

$$(\varphi^m C_\varphi - \psi^m C_\psi)(f) = (C_\varphi - C_\psi)\left(\widehat{f^m}\right). \tag{4.2.1}$$

We now start proving the theorem. By definition,

$$\|\varphi^n C_\varphi - \psi^n C_\psi\|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \|(\varphi^n C_\varphi - \psi^n C_\psi) e_j\|^2,$$

where,

$$e_j(z) = \sqrt{\frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}}\, z^j, z \in \mathbb{D}.$$

Then by $(4.2.1)$,

$$\|\varphi^n C_\varphi - \psi^n C_\psi\|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \|(C_\varphi - C_\psi)\widehat{e_j^n}\|^2.$$

For each $z \in \mathbb{D}$,

$$\widehat{e_j^n}(z) = z^n e_j(z) = z^n \sqrt{\frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}}\, z^j = \sqrt{\frac{\Gamma(j+2+\alpha)}{j!\,\Gamma(2+\alpha)}}\, z^{n+j}$$

$$= \underbrace{\left(\sqrt{\frac{(n+j)!}{j!}\frac{\Gamma(j+2+\alpha)}{\Gamma(n+j+2+\alpha)}}\right)}_{A(n,j,\alpha)} \left(\sqrt{\frac{\Gamma(n+j+2+\alpha)}{(n+j)!\,\Gamma(2+\alpha)}}\, z^{n+j}\right) = A(n,j,\alpha)\; e_{n+j}(z).$$

Hence,

$$\|\varphi^n C_\varphi - \psi^n C_\psi\|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} (A(n,j,\alpha))^2 \|(C_\varphi - C_\psi) e_{n+j}\|^2. \tag{4.2.2}$$

Claim 4. *$A(n,j,\alpha)$ is bounded above and below by constants that depend only on n and α.*

Proof of the claim. For each j,

$$(n+j)! = (n+j)(n-1+j)\ldots.(j+1)\,j!\,.$$

Applying an elementary property of Gamma function ,

$$\Gamma(n+j+2+\alpha) = (n-1+j+2+\alpha)(n-2+j+2+\alpha)\ldots..(j+2+\alpha)\,\Gamma(j+2+\alpha)\ .$$

Hence,

$$A(n,j,\alpha) = \sqrt{\frac{(n+j)(n-1+j)\ldots.(j+1)}{(n-1+j+2+\alpha)(n-2+j+2+\alpha)\ldots..(j+2+\alpha)}}$$

$$= \sqrt{\left(\frac{n+j}{n-1+j+2+\alpha}\right)\left(\frac{n-1+j}{n-2+j+2+\alpha}\right)\cdots\left(\frac{j+1}{j+2+\alpha}\right)}\,. \tag{4.2.3}$$

We notice that each factor inside the above square-root is of the form $\dfrac{n-k+1+j}{n-k+j+2+\alpha}$, where $k = 1,2,3,\ldots,n$. Since $\alpha > -1$, therefore for each k , $\dfrac{n-k+1+j}{n-k+j+2+\alpha} < 1$. Consequently ,

$$A(n,j,\alpha) < 1\ \forall\ j\,. \tag{4.2.4}$$

To find a lower bound for $A(n,j,\alpha)$, we divide into cases.

Case I. Let $j \leq n$. Then for each k ,

$$\frac{n-k+1+j}{n-k+j+2+\alpha} \geq \frac{n-k+1}{2n-k+2+\alpha} \geq \frac{1}{2n-1+2+\alpha} = \frac{1}{2n+1+\alpha}\,.$$

Case II. Let $j > n$. We observe that for each k ,

$$\frac{n-k+1+j}{n-k+j+2+\alpha} = \frac{n-k+1+j}{(n-k+1+j)+1+\alpha}\,.$$

Since $\alpha > -1$, therefore $1+\alpha > 0$. We now use an elementary fact that the function $f(x) = \dfrac{x}{x+c}$, $c > 0$ is increasing. Therefore when $j > n$,

$$\frac{n-k+1+j}{n-k+j+2+\alpha} \geq \frac{n-k+1+n}{n-k+n+2+\alpha} = \frac{2n-k+1}{2n-k+2+\alpha} \geq \frac{2n-n+1}{2n-1+2+\alpha} = \frac{n+1}{2n+1+\alpha}\,.$$

Combining Case I and Case II, we have, for each j and for each $k = 1, 2, 3, \ldots\ldots, n$,

$$\frac{n-k+1+j}{n-k+j+2+\alpha} \;\geq\; \frac{1}{2n+1+\alpha}\,.$$

Hence from (4.2.3) ,

$$A(n, j, \alpha) \;\geq\; \frac{1}{(2n+1+\alpha)^{\frac{n}{2}}} \;\;\forall\; j. \tag{4.2.5}$$

(4.2.4) and (4.2.5) together prove Claim 4. □

Applying Claim 4 and (4.2.2) ,

$$\|\varphi^n C_\varphi - \psi^n C_\psi\|_{\mathrm{HS}}^2 \;\asymp \sum_{j=0}^{\infty} \|\,(C_\varphi - C_\psi)\, e_{n+j}\|^2 < \infty\,,$$

where the comparability constants depend only on n and α . Clearly

$$\sum_{j=0}^{\infty} \|\,(C_\varphi - C_\psi)\, e_{n+j}\|^2 = \sum_{j=n}^{\infty} \|\,(C_\varphi - C_\psi)\, e_j\|^2\,.$$

So,

$$\|\varphi^n C_\varphi - \psi^n C_\psi\|_{\mathrm{HS}}^2 \;\asymp \sum_{j=n}^{\infty} \|\,(C_\varphi - C_\psi)\, e_j\|^2\,.$$

Therefore $\varphi^n C_\varphi - \psi^n C_\psi$ is Hilbert-Schmidt iff $\sum\limits_{j=n}^{\infty} \|\,(C_\varphi - C_\psi)\, e_j\|^2 < \infty$. However, $\sum\limits_{j=n}^{\infty} \|\,(C_\varphi - C_\psi)\, e_j\|^2 < \infty$ iff $\sum\limits_{j=0}^{\infty} \|\,(C_\varphi - C_\psi)\, e_j\|^2 < \infty$, which is equivalent to the

Hilbert-Schmidtness of $C_\varphi - C_\psi$. Hence , $\varphi^n C_\varphi - \psi^n C_\psi$ is Hilbert-Schmidt iff $C_\varphi - C_\psi$ is Hilbert-Schmidt. This completes the proof of the Theorem 4.2.1 . □

4.3. Hilbert-Schmidt norm of another difference operator

Reversing the order of the weights , we now consider the operator $\psi^n C_\varphi - \varphi^n C_\psi$ acting boundedly on A_α^2 . Notice that it becomes zero operator if $\varphi = \psi$ or equivalently $\sigma = 0$. So, again we expect σ to be involved in the Hilbert-Schmidt norm of $\psi^n C_\varphi - \varphi^n C_\psi$ as the cancellation factor. In fact, this hope of ours is actually true. For the sake of simplicity , we will prove it taking $\alpha = 0$. The case when $\alpha \neq 0$ can be handled by a slight modification of the same proof technique.

THEOREM 5. *Let us fix $n \in \mathbb{N}$ and consider the operator $\psi^n C_\varphi - \varphi^n C_\psi$ acting boundedly on A^2. Then*

$$\|\psi^n C_\varphi - \varphi^n C_\psi\|_{HS}^2 \asymp \sum_{k=0}^{n-1} (k+1) \int_{\mathbb{D}} |\left(\psi(z)\right)^n \left(\varphi(z)\right)^k - \left(\varphi(z)\right)^n \left(\psi(z)\right)^k|^2 \, dA(z)$$

$$+ \int_{\mathbb{D}} \frac{|\varphi(z)|^n |\psi(z)|^n |\sigma^2(z)| \, dA(z)}{\left(1 - |\varphi(z)|^2\right)^2} + \int_{\mathbb{D}} \frac{|\varphi(z)|^n |\psi(z)|^n |\sigma^2(z)| \, dA(z)}{\left(1 - |\psi(z)|^2\right)^2} .$$

PROOF. We will first introduce a notation . For $f \in H(\mathbb{D})$ and $n \in \mathbb{N} \cup \{0\}$, let us define the function $\widehat{f_n} \in H(\mathbb{D})$ as follows:

$$\widehat{f_n}(z) = \frac{R_{n-1} f(z)}{z^n}, \ z \in \mathbb{D}.$$

Equivalently,

$$\widehat{f_n}(z) = \sum_{j=0}^{\infty} f_{n+j} \, z^j , \ z \in \mathbb{D}.$$

Then the following Lemma is obvious .

LEMMA 4.3.1. *For each $n \in \mathbb{N} \cup \{0\}$, $f \in H(\mathbb{D})$,*

$$\left(\psi^n C_\varphi - \varphi^n C_\psi\right)(f) = \left(\psi^n C_\varphi - \varphi^n C_\psi\right)\left(S_{n-1} f\right) + \varphi^n \psi^n \left(C_\varphi - C_\psi\right)\left(\widehat{f_n}\right).$$

Proof of the Lemma. Let $z \in \mathbb{D}$. Then

$$(\psi^n C_\varphi - \varphi^n C_\psi)(f)\ (z) = (\psi^n C_\varphi - \varphi^n C_\psi)(S_{n-1} f + R_{n-1} f)\ (z)$$

$$= (\psi^n C_\varphi - \varphi^n C_\psi)(S_{n-1} f)\ (z) + (\psi^n C_\varphi - \varphi^n C_\psi)(R_{n-1} f)\ (z)\,.$$

But,

$$(\psi^n C_\varphi - \varphi^n C_\psi)(R_{n-1} f)\ (z) = (\psi(z))^n\ R_{n-1} f(\varphi(z)) - (\varphi(z))^n\ R_{n-1} f(\psi(z))$$

$$= (\psi(z))^n\ (\varphi(z))^n \left[\frac{R_{n-1} f(\varphi(z))}{(\varphi(z))^n}\right] - \ (\varphi(z))^n\ (\psi(z))^n \left[\frac{R_{n-1} f(\psi(z))}{(\psi(z))^n}\right]$$

$$= (\psi(z))^n\ (\varphi(z))^n\ \widehat{f_n}(\varphi(z)) - \ (\varphi(z))^n\ (\psi(z))^n\ \widehat{f_n}(\psi(z))$$

$$= \varphi^n \psi^n (C_\varphi - C_\psi)\left(\widehat{f_n}\right)(z)\,.$$

This completes the proof of this Lemma. □

Applying Lemma 4.3.1 for $f = e_k$, where e_k is the $k-$th element of the standard orthonormal basis of A^2, we get

$$(\psi^n C_\varphi - \varphi^n C_\psi)(e_k) = (\psi^n C_\varphi - \varphi^n C_\psi)(S_{n-1} e_k) + \ \varphi^n \psi^n (C_\varphi - C_\psi)\left(\widehat{(e_k)_n}\right)\,. \tag{4.3.1}$$

Case I. Let $k \leq n-1$. $e_k(z) = \sqrt{k+1}\, z^k$. Therefore $R_{n-1} e_k = 0$ and consequently $\widehat{(e_k)_n} = 0$. Also,

$$S_{n-1}\, e_k(z) = \sqrt{k+1}\, z^k\,.$$

Therefore , from (4.3.1) ,

$$(\psi^n C_\varphi - \varphi^n C_\psi)(e_k)(z) = \sqrt{k+1}\ \left[(\psi(z))^n (\varphi(z))^k - (\varphi(z))^n (\psi(z))^k\right]\,.$$

Case II. Let $k > n-1$. Then $S_{n-1} e_k(z) = 0$. Also,

$$\widehat{(e_k)_n}(z) = \sqrt{k+1}\, z^{k-n}\,.$$

Consequently , from (4.3.1) ,

$$\left(\psi^n C_\varphi - \varphi^n C_\psi\right)(e_k)(z) = \sqrt{k+1}\ (\varphi(z))^n (\psi(z))^n \left[(\varphi(z))^{k-n} - (\psi(z))^{k-n}\right] .$$

Combining the above cases, we obtain the following Lemma.

LEMMA 4.3.2. *Let us fix n. Then*

$$\left(\psi^n C_\varphi - \varphi^n C_\psi\right)(e_k)(z) = \begin{cases} \sqrt{k+1}\ \left[(\psi(z))^n (\varphi(z))^k - (\varphi(z))^n (\psi(z))^k\right], \textit{if } k \leq n-1. \\ \sqrt{k+1}\ (\varphi(z))^n (\psi(z))^n \left[(\varphi(z))^{k-n} - (\psi(z))^{k-n}\right], \textit{if } k > n-1. \end{cases}$$

We will now start proving Theorem 5. By definition,

$$\|\psi^n C_\varphi - \varphi^n C_\psi\|_{\mathrm{HS}}^2 = \sum_{k=0}^{\infty} \|\left(\psi^n C_\varphi - \varphi^n C_\psi\right) e_k\|^2.$$

Now , applying Lemma 4.3.2 ,

$$\begin{aligned} \sum_{k=0}^{\infty} \|\left(\psi^n C_\varphi - \varphi^n C_\psi\right) e_k\|^2 &= \sum_{k=0}^{n-1} \|\left(\psi^n C_\varphi - \varphi^n C_\psi\right) e_k\|^2 + \sum_{k=n}^{\infty} \|\left(\psi^n C_\varphi - \varphi^n C_\psi\right) e_k\|^2 \\ &= \sum_{k=0}^{n-1} (k+1) \int_{\mathbb{D}} |(\psi(z))^n (\varphi(z))^k - (\varphi(z))^n (\psi(z))^k|^2 \, dA(z) \\ &+ \sum_{k=n}^{\infty} (k+1) \int_{\mathbb{D}} |\varphi(z)|^{2n} |\psi(z)|^{2n} |(\varphi(z))^{k-n} - (\psi(z))^{k-n}|^2 \, dA(z) . \end{aligned}$$

The above shows that to prove Theorem 5 , we have already obtained the correct first term on the right hand side. We will now estimate the 2nd term. The 2nd term

$$= \int_{\mathbb{D}} |\varphi(z)|^{2n} |\psi(z)|^{2n} F(n, \varphi, \psi) \, dA(z) \ ,$$

where,

$$F(n, \varphi, \psi) = \sum_{k=n}^{\infty} (k+1) \ |(\varphi(z))^{k-n} - (\psi(z))^{k-n}|^2.$$

An elementary complex algebra shows that

$$F\left(n,\varphi,\psi\right)=\sum_{k=n}^{\infty}\left(k+1\right)\left|\varphi\left(z\right)\right|^{2k-2n}+\sum_{k=n}^{\infty}\left(k+1\right)\left|\psi\left(z\right)\right|^{2k-2n}$$

$$-2\,\Re\left(\sum_{k=n}^{\infty}\left(k+1\right)\left(\varphi\left(z\right)\overline{\psi\left(z\right)}\right)^{k-n}\right).$$

We notice that each of the above three series is of the form $\sum_{k=n}^{\infty}\left(k+1\right)x^{k-n}$ where $|x|<1$. Now , for $|x|<1$,

$$\sum_{k=n}^{\infty}\left(k+1\right)x^{k-n}=\sum_{k=0}^{\infty}\left(k+n+1\right)x^{k}=\sum_{k=0}^{\infty}\left(k+1\right)x^{k}+n\sum_{k=0}^{\infty}x^{k}$$

$$=\frac{1}{\left(1-x\right)^{2}}+\frac{n}{1-x}=\frac{1+n\left(1-x\right)}{\left(1-x\right)^{2}}\,. \qquad (4.3.2)$$

We will apply (4.3.2) , taking $x=\left|\varphi\left(z\right)\right|^{2},\left|\psi\left(z\right)\right|^{2}$ and $\varphi\left(z\right)\overline{\psi\left(z\right)}$ to obtain

$$F\left(n,\varphi,\psi\right)=\frac{1+n\left(1-\left|\varphi\left(z\right)\right|^{2}\right)}{\left(1-\left|\varphi\left(z\right)\right|^{2}\right)^{2}}+\frac{1+n\left(1-\left|\psi\left(z\right)\right|^{2}\right)}{\left(1-\left|\psi\left(z\right)\right|^{2}\right)^{2}}-2\,\Re\left(\frac{1+n\left(1-\varphi\left(z\right)\overline{\psi\left(z\right)}\right)}{\left(1-\varphi\left(z\right)\overline{\psi\left(z\right)}\right)^{2}}\right)$$

$$=\frac{1+n\left(1-|a|^{2}\right)}{\left(1-|a|^{2}\right)^{2}}+\frac{1+n\left(1-|b|^{2}\right)}{\left(1-|b|^{2}\right)^{2}}-2\,\Re\left(\frac{1+n\left(1-a\bar{b}\right)}{\left(1-a\bar{b}\right)^{2}}\right), \qquad (4.3.3)$$

where $a=\varphi\left(z\right)$ and $b=\psi\left(z\right)$.

Now let us make the same change of variable as used previously:

$$b=\sigma_{a}\left(w\right)=\frac{a-w}{1-\overline{a}w}\,.$$

Then $w=\sigma_{a}\left(b\right)$. Also,

$$1-|b|^{2}=\frac{\left(1-|a|^{2}\right)\left(1-|w|^{2}\right)}{|1-\overline{a}w|^{2}},$$

and,

$$1 - a\bar{b} = \frac{\left(1 - |a|^2\right)}{1 - a\overline{w}} \ .$$

Substituting in (4.3.3) ,

$$F\left(n, \varphi, \psi\right) = \frac{1 + n\left(1 - |a|^2\right)}{\left(1 - |a|^2\right)^2} + \frac{1 + \dfrac{n\left(1 - |a|^2\right)\left(1 - |w|^2\right)}{|1 - \overline{a}w|^2}}{\dfrac{\left(1 - |a|^2\right)^2\left(1 - |w|^2\right)^2}{|1 - \overline{a}w|^4}} - 2\,\Re\,\frac{1 + \dfrac{n\left(1 - |a|^2\right)}{1 - a\overline{w}}}{\dfrac{\left(1 - |a|^2\right)^2}{\left(1 - a\overline{w}\right)^2}}$$

$$= \frac{1 + n\left(1 - |a|^2\right)}{\left(1 - |a|^2\right)^2} + \left\{\frac{|1 - \overline{a}w|^2 + n\left(1 - |a|^2\right)\left(1 - |w|^2\right)}{\left(1 - |a|^2\right)^2\left(1 - |w|^2\right)^2}\right\}|1 - \overline{a}w|^2$$

$$- 2\,\Re\left(\frac{\left(1 - a\overline{w}\right) + n\left(1 - |a|^2\right)}{\left(1 - |a|^2\right)^2}\right)\left(1 - a\overline{w}\right)$$

$$= \frac{1}{\left(1 - |a|^2\right)^2}\ G\left(n, \varphi, \psi\right),$$

where ,

$$G\left(n, \varphi, \psi\right) = 1 + n\left(1 - |a|^2\right) + \frac{|1 - \overline{a}w|^4}{\left(1 - |w|^2\right)^2} + \frac{n\left(1 - |a|^2\right)|1 - \overline{a}w|^2}{1 - |w|^2}$$

$$-2\,\Re\,\left(1 - a\overline{w}\right)^2 - 2n\left(1 - |a|^2\right)\,\Re\left(1 - a\overline{w}\right).$$

Regrouping terms , we have

$$G\left(n, \varphi, \psi\right) = \left[1 - 2\Re\left(1 - a\overline{w}\right)^2 + |1 - a\overline{w}|^4\right] + \frac{|1 - a\overline{w}|^4}{\left(1 - |w|^2\right)^2} - |1 - a\overline{w}|^4$$

$$+n\left(1 - |a|^2\right)\left[1 + \frac{|1 - \overline{a}w|^2}{1 - |w|^2} - 2\,\Re\left(1 - a\overline{w}\right)\right]$$

$$= \left|1-(1-a\overline{w})^2\right|^2 + |1-a\overline{w}|^4 \left\{ \frac{1}{\left(1-|w|^2\right)^2} - 1 \right\}$$

$$+n\left(1-|a|^2\right)\left[1-2\Re(1-a\overline{w}) + |1-a\overline{w}|^2 + |1-a\overline{w}|^2 \left\{ \frac{1}{1-|w|^2} - 1 \right\}\right]$$

$$= |2-a\overline{w}|^2 |a|^2 |w|^2 + \frac{|1-a\overline{w}|^4 \left(2-|w|^2\right)|w|^2}{\left(1-|w|^2\right)^2} + n\left(1-|a|^2\right)\left[|1-(1-a\overline{w})|^2 + \frac{|1-a\overline{w}|^2 |w|^2}{1-|w|^2}\right]$$

$$= |2-a\overline{w}|^2 |a|^2 |w|^2 + \frac{|1-a\overline{w}|^4 \left(2-|w|^2\right)|w|^2}{\left(1-|w|^2\right)^2} + n\left(1-|a|^2\right)\left[|a|^2|w|^2 + \frac{|1-a\overline{w}|^2 |w|^2}{1-|w|^2}\right].$$

It is an elementary observation that $\dfrac{|1-a\overline{w}|^2}{1-|w|^2} = \dfrac{1-|a|^2}{1-|b|^2}$. Hence ,

$$G(n,\varphi,\psi) = |2-a\overline{w}|^2 |a|^2 |w|^2$$

$$+ \frac{\left(1-|a|^2\right)^2\left(2-|w|^2\right)|w|^2}{\left(1-|b|^2\right)^2} + n\left(1-|a|^2\right)\left[|a|^2|w|^2 + \frac{\left(1-|a|^2\right)|w|^2}{1-|b|^2}\right]$$

$$= |w|^2\left[|2-a\overline{w}|^2|a|^2 + \frac{\left(1-|a|^2\right)^2\left(2-|w|^2\right)}{\left(1-|b|^2\right)^2} + n\left(1-|a|^2\right)|a|^2 + \frac{n\left(1-|a|^2\right)^2}{1-|b|^2}\right]$$

$$= |w|^2\left[|a|^2\left\{|2-a\overline{w}|^2 + n\left(1-|a|^2\right)\right\} + \frac{\left(1-|a|^2\right)^2}{\left(1-|b|^2\right)^2}\left(2-|w|^2\right) + \frac{n\left(1-|a|^2\right)^2}{1-|b|^2}\right].$$

Applying triangular inequality in bothways,

$$1 \leq |2-a\overline{w}|^2 \leq 9.$$

Also,

$$0 \leq n\left(1-|a|^2\right) \leq n \text{ and } 1 \leq 2-|w|^2 \leq 2.$$

Hence ,

$$G(n,\varphi,\psi) \asymp |w|^2\left[|a|^2 + \frac{\left(1-|a|^2\right)^2}{\left(1-|b|^2\right)^2} + \frac{n\left(1-|a|^2\right)^2}{1-|b|^2}\right].$$

This implies that

$$F(n,\varphi,\psi) \asymp \frac{|w|^2}{\left(1-|a|^2\right)^2}\left[|a|^2 + \frac{\left(1-|a|^2\right)^2}{\left(1-|b|^2\right)^2} + \frac{n\left(1-|a|^2\right)^2}{1-|b|^2}\right]$$

$$= \frac{|w|^2|a|^2}{\left(1-|a|^2\right)^2} + \frac{|w|^2}{\left(1-|b|^2\right)^2} + \frac{n|w|^2}{1-|b|^2}. \tag{4.3.4}$$

Clearly from definition, $F(n,\varphi,\psi) = F(n,\psi,\varphi)$. Hence interchanging a and b, it also follows that

$$F(n,\varphi,\psi) \asymp \frac{|w|^2|b|^2}{\left(1-|b|^2\right)^2} + \frac{|w|^2}{\left(1-|a|^2\right)^2} + \frac{n|w|^2}{1-|a|^2}. \tag{4.3.5}$$

Adding (4.3.4) and (4.3.5) , we obtain

$$F(n,\varphi,\psi) \asymp \frac{|w|^2|a|^2}{\left(1-|a|^2\right)^2} + \frac{|w|^2}{\left(1-|b|^2\right)^2} + \frac{n|w|^2}{1-|b|^2} + \frac{|w|^2|b|^2}{\left(1-|b|^2\right)^2} + \frac{|w|^2}{\left(1-|a|^2\right)^2} + \frac{n|w|^2}{1-|a|^2}$$

$$= \left[\frac{|w|^2|a|^2}{\left(1-|a|^2\right)^2} + \frac{|w|^2}{\left(1-|a|^2\right)^2} + \frac{n|w|^2}{1-|a|^2}\right] + \left[\frac{|w|^2|b|^2}{\left(1-|b|^2\right)^2} + \frac{|w|^2}{\left(1-|b|^2\right)^2} + \frac{n|w|^2}{1-|b|^2}\right]$$

$$= \left[\frac{|w|^2\left(|a|^2+1\right)}{\left(1-|a|^2\right)^2} + \frac{n|w|^2}{1-|a|^2}\right] + \left[\frac{|w|^2\left(|b|^2+1\right)}{\left(1-|b|^2\right)^2} + \frac{n|w|^2}{1-|b|^2}\right]$$

$$\asymp \left[\frac{|w|^2}{\left(1-|a|^2\right)^2} + \frac{n|w|^2}{1-|a|^2}\right] + \left[\frac{|w|^2}{\left(1-|b|^2\right)^2} + \frac{n|w|^2}{1-|b|^2}\right],$$

since $1 \le |a|^2+1, |b|^2+1 \le 2$. Then,

$$F(n,\varphi,\psi) \asymp \frac{|w|^2}{\left(1-|a|^2\right)^2}\left[1+n\left(1-|a|^2\right)\right] + \frac{|w|^2}{\left(1-|b|^2\right)^2}\left[1+n\left(1-|b|^2\right)\right].$$

Clearly, $1 \leq 1+n\left(1-|a|^2\right), 1+n\left(1-|b|^2\right) \leq 1+n$. Hence,

$$F(n,\varphi,\psi) \asymp \frac{|w|^2}{\left(1-|a|^2\right)^2}+\frac{|w|^2}{\left(1-|b|^2\right)^2}=\frac{|\sigma^2(z)|}{\left(1-|\varphi(z)|^2\right)^2}+\frac{|\sigma^2(z)|}{\left(1-|\psi(z)|^2\right)^2} .$$

This completes the proof of Theorem 5 . □

Corollary 4.3.1. *Let us fix $n \in \mathbb{N}$ and consider the operator $\psi^n C_\varphi - \varphi^n C_\psi$ acting boundedly on A^2. Then $\psi^n C_\varphi - \varphi^n C_\psi$ is Hilbert-Schmidt iff*

$$\int_{\mathbb{D}} \frac{|\varphi(z)|^n|\psi(z)|^n|\,\sigma^2(z)\,|\,dA(z)}{(1-|\varphi(z)|^2)^2} < \infty \text{ and } \int_{\mathbb{D}} \frac{|\varphi(z)|^n|\psi(z)|^n|\,\sigma^2(z)\,|\,dA(z)}{(1-|\psi(z)|^2)^2} < \infty.$$

4.4. A more general difference operator

As the progress suggests , we have been able to obtain a complete characterization of the Hilbert-Schmidtness of $uC_\varphi - vC_\psi$ in terms of integrability conditions on φ and ψ where u and v are of the form φ^n and ψ^n respectively or vice-versa. In an even more general setting , we may consider a difference operator of the form $\varphi^k\psi^{n-k}C_\varphi - \varphi^{n-k}\psi^k C_\psi$. Then it is easy to check that

$$\varphi^k\psi^{n-k}C_\varphi - \varphi^{n-k}\psi^k C_\psi = \begin{cases} \varphi^k\psi^k\left(\psi^{n-2k}C_\varphi - \varphi^{n-2k}C_\psi\right) \text{if } k \leq n-k. \\ \varphi^{n-k}\psi^{n-k}\left(\varphi^{2k-n}C_\varphi - \psi^{2k-n}C_\psi\right) \text{if } n-k \leq k. \end{cases}$$

Clearly, the external weights can be absorbed within the weighted area measure on $\mathbb{D}$ while integrating a function in A^2_α . Therefore , in view of Theorem 5 and Theorem 4.2.1 , the problem of obtaining a complete characterization of the Hilbert-Schmidtness of the operator $\varphi^k\psi^{n-k}C_\varphi - \varphi^{n-k}\psi^k C_\psi$ is also solved.

4.5. Boundedness and compactness of the new difference operators

In section 4.4 , we have introduced a new type of difference operator and have discussed its Hilbert-Schmidtness. In this section , we discuss its boundedness and compactness.

THEOREM 4.5.1. *Let* $0 < p \leq q < \infty$. *Then* $\varphi^n C_\varphi - \psi^n C_\psi$ *is bounded (compact) from* A^p_α *to* $L^q(\mu)$ *iff* $C_\varphi - C_\psi$ *is bounded (compact) from* A^p_α *to* $L^q(\mu)$.

PROOF. We recall from (4.2.1) that for each $n \in \mathbb{N} \cup \{0\}$ and each $f \in H(\mathbb{D})$,

$$(\varphi^n C_\varphi - \psi^n C_\psi)(f) = (C_\varphi - C_\psi)\left(\widehat{f^n}\right).$$

Therefore to prove this Theorem, it is sufficient to prove the following claim.

CLAIM 5. *Let* $f \in H(\mathbb{D})$. *Then* $f \in A^p_\alpha$ *iff* $\widehat{f^n} \in A^p_\alpha$.

PROOF OF THE CLAIM. It is obvious that if $f \in A^p_\alpha$ then $\widehat{f^n} \in A^p_\alpha$. To prove the converse, let $\widehat{f^n} \in A^p_\alpha$. Then,

$$\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) = \underbrace{\int_{\mathbb{D}_{\frac{1}{2}}} |f(z)|^p \, dA_\alpha(z)}_{I_1} + \underbrace{\int_{\mathbb{D}\setminus\mathbb{D}_{\frac{1}{2}}} |f(z)|^p \, dA_\alpha(z)}_{I_2}$$

Since $f \in H(\mathbb{D})$, therefore I_1 is clearly finite. Let us now estimate I_2 .

$$I_2 = \int_{\mathbb{D}\setminus\mathbb{D}_{\frac{1}{2}}} |f(z)|^p \, dA_\alpha(z) = \frac{1}{|z|^{np}} \int_{\mathbb{D}\setminus\mathbb{D}_{\frac{1}{2}}} |z|^{np} \, |f(z)|^p \, dA_\alpha(z) \leq 2^{np} \|z^n f\| < \infty$$

Consequently , $\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty$ or equivalenly $f \in A^p_\alpha$. ☐

This completes the proof of the Theorem. ☐

Theorem 4.5.2. *$\psi^n C_\varphi - \varphi^n C_\psi$ is bounded (compact) from A^p_α to $L^q(\mu)$ iff $C_\varphi - C_\psi$ is bounded (compact) from A^p_α to $L^q(\varphi^n \psi^n \mu)$.*

Proof. Lemma 4.3.1 gives us that for each $n \in \mathbb{N} \cup \{0\}$ and each $f \in H(\mathbb{D})$,

$$(\psi^n C_\varphi - \varphi^n C_\psi)(f) = (\psi^n C_\varphi - \varphi^n C_\psi)(S_{n-1} f) + \varphi^n \psi^n (C_\varphi - C_\psi)\left(\widehat{f_n}\right).$$

Since S_{n-1} is a finite-rank operator on A^p_α, therefore $(\psi^n C_\varphi - \varphi^n C_\psi) S_{n-1}$ is a finite-rank operator from A^p_α to $L^q(\mu)$. Every finite-rank operator is bounded and compact. Also by Claim 5 , $f \in A^p_\alpha$ iff $\widehat{f_n} \in A^p_\alpha$. Hence the result follows. □

Remark 2. *The boundedness and compactness criteria for $C_\varphi - C_\psi$ have been established for a target space $L^q(\mu)$ where μ can be any non-negative Borel measure on $\mathbb{D}$. Therefore, in view of the operator-theoretic equation in section* 4.4 *, Theorem* 4.5.1 *and Theorem* 4.5.2 *, the problem of complete characterization of boundedness and compactness of the operator $\varphi^k \psi^{n-k} C_\varphi - \varphi^{n-k} \psi^k C_\psi$ is now solved .*

CHAPTER 5

FUTURE RESEARCH PLANS

Our ultimate goal is to characterize boundedness, compactness and Hilbert-Schmidtness of an operator of the form $uC_\varphi - vC_\psi$ where u and v are any pair of bounded analytic functions on $\mathbb{D}$. In chapter 3 and 4, we solved those problems for a very special class of weights viz when $u = \varphi^k\psi^{n-k}$ and $v = \psi^k\varphi^{n-k}$ for some integers n and k. Our immediate next plans are highlighted in the following two sections.

5.1. Other analytic weights u and v

In [**CHK**], the authors established integrability conditions on φ and ψ , involving the cancelation factor σ, to characterize Hilbert-Schmidtness of $C_\varphi - C_\psi$ on A^2_α. Using a change of variable method, we have been able to obtain similar characterizations for a more general operator $\varphi^k\psi^{n-k}C_\varphi - \psi^k\varphi^{n-k}C_\psi$, involving the same cancelation factor σ. So the next thing we want to try is to investigate the Hilbert-Schmidtness of $uC_\varphi - vC_\psi$ when u and v are other types of combinations of φ and ψ. We suspect that even when u and v are combinations of φ and ψ , the cancellation factor is not necessarily σ. In those cases, it may be a great challenge to determine the right cancellation factor. We hope some modification of our change of variable method will help us to determine that cancellation factor. Once we know what is our cancellation factor, a natural expectation is to be able to involve the same factor in the boundedness and compactness problem associated with our operator.

5.2. Compactness of $C_\varphi - C_\psi$ from A_α^1 to $L^q(\mu)$

We recall from chapter 3 that assuming $p > 1$ is essential to prove the compactness criterion of $C_\varphi - C_\psi$. Because unless $p > 1$, the sequence $\{R_n\}$ is not necessarily uniformly bounded, a tool that has been used in several places to estimate the essential norms. So, it is also one of our future plans to develop a method that will help us investigate the compactness of $C_\varphi - C_\psi$ acting from A_α^1 to $L^q(\mu)$.

5.3. Schatten-p membership of $C_\varphi - C_\psi$

At the very beginning of our research, we investigated schatten-p membership of $C_\varphi - C_\psi$ acting on A_α^2. For $p = 4$, we found a nice integrability condition involving a double integral. However , we were unable to detect a cancellation factor that would separate φ and ψ in terms of integrals . In future, we want to continue this effort.

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