

MANAGING RISK WITH SHORT TERM FUTURES CONTRACTS

by

CHUNHUI YU

A DISSERTATION

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ABSTRACT

In this dissertation, we search for an optimal strategy to reduce the running risk in hedging a long-term supply commitment with short-term futures contracts under a certain constraint on the terminal risk, which leads to a class of intrinsic optimization problems. Motivated by a simple model initially discussed in [1] Culp and Miller, [8] Mello and Parsons, [4] Glasserman and a best strategy model by [5] Larcher and Leobacher, we studied an optimization problem as following:

Under the condition that

$$\int_0^1 (1 - g(s))^2 ds \leq x,$$

where $x \in [0, \frac{1}{6\sqrt{3}}e^{-\pi}2\sqrt{3}]$. Which measurable function $g : [0, 1] \rightarrow R$ minimizes the value of

$$\sup_{t \in [0, 1]} \int_0^t (t - g(s))^2 ds ?$$

We will show that a unique solution to this general problem always exists. By solving it numerically, we obtain a general dynamic solution. Furthermore, we will discuss properties of the solution and give analytic solutions in some special cases.

DEDICATION

This dissertation is dedicated to everyone who helped me and guided me through the trials and tribulations of creating this manuscript. In particular, my family and close friends who stood by me throughout the time taken to complete this work.

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It has been a long journey for me from the childhood days in a small village in China to the present day. Along the journey I have faced challenging and joyful moments. I have also met many people who have not only upheld faith in my abilities but also were able to inspire me to a greater goal. I dedicate this dissertation to my mother, father and sister; without their support it would have been impossible for me to continue my studies.

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CHAPTER 1

PRELIMINARIES

1.1. Futures and Hedging with futures

Some good references for this section is [7]John C. Hull's *Options, Futures, and Other Derivatives* and [13]Marek Capinski and Tomasz Zastawniak's *Mathematics for Finance-An Introduction to Financial Engineering*.

1.1.1. Forward Contracts. A **forward contract** is an agreement to buy or sell an asset on a fixed date in the future, called the *delivery time*, for a price specified in advance, called the *forward price*. The party to the contract who agrees to sell the asset is said to be taking a *short forward position*. The other party, obliged to buy the asset at delivery, is said to have a *long forward position*.

Let us denote the time when the forward contract is exchanged by 0, the delivery time by T , and the forward price by $F(0, T)$. The time t market price of the underlying asset will be denoted by $S(t)$. No payment is made by either party at time 0, when the forward contract is exchanged. At delivery the party with a long forward position will benefit if $F(0, T) < S(T)$, making an profit of $S(T) - F(0, T)$. Meanwhile, the party holding a short forward position will suffer a loss of $S(T) - F(0, T)$. If $F(0, T) > S(T)$, the situation will be reversed.

1.1.2. Futures Contracts. One of the two parties to a forward contract will be losing money. There is always a risk of default by the party suffering a loss. Futures contracts are designed to eliminate or minimize such risk.

A **futures contract** is a standardized agreement between two parties that commits to buy or sell an asset at a set price on a given date in the future. Typically,

it involves an underlying asset, a specified delivery time T , the delivery price $f(0, T)$ and a random cash flow referred to marking to market: the holder of a long futures position will receive the amount

$$f(n\tau, T) - f(n\tau - \tau, T)$$

at each time step $n = 1, 2, \dots, N$. Here $N\tau = T$ and $f(n\tau, T)$, called *futures price*, is dictated by the market, except $f(0, T)$ which is known at time $t = 0$. The following two conditions are imposed:

1. The futures price at delivery is $f(T, T) = S(T)$;
2. At each time step, the value of a futures position is zero after marking to market.

Certain practical regulations are enforced to ensure that the obligations involved in a futures position are fulfilled. Each party entering into a futures contract has to pay a deposit, called the *initial margin*, which is kept by the clearing house as collateral. At each time step n , the amount $f(n\tau, T) - f(n\tau - \tau, T)$ is added to the deposit account of the long futures position, and $f(n\tau - \tau, T) - f(n\tau, T)$ to the deposit account of the short futures position. Any excess amount above the initial margin can be withdrawn. On the other hand, if the deposit account balance drops below a certain level, called the *maintenance margin*, the clearing house will issue a *margin call*, requesting the account owner to make a payment and restore the deposit to the level of the initial margin. The futures position will be closed immediately by the clearing house if the account owner fails to respond to a margin call. In general, a futures position can be closed at any time, in which case the deposit will be returned to the account owner.

1.1.3. Hedging. The aim of **Hedging** is to use futures markets to reduce or eliminate a particular risk. This risk might be related to the price of oil, a foreign

exchange rate, the level of the stock market, or some other variable. A perfect hedge is one that completely eliminates the risk. In practice, perfect hedges are rare. For the most part, therefore, a study of hedging using futures contracts is a study of the ways in which hedges can be constructed so that they perform as close to perfect as possible. Hedges that involve taking a long position in a futures contract are known as long hedges. A long hedge is appropriate when a company knows it will have to purchase a certain asset in the future and wants to lock in a price now.

A short hedge is a hedge that involves a short position in futures contracts. A short hedge is appropriate when the hedger already owns an asset and expects to sell it at some time in the future.

The arguments in favor of hedging are so obvious that they hardly need to be stated. However, in practice, many risks are left unhedged.

1.1.4. Spot risk and running risk. Assume that a firm seeks to hedge expected cashflows from its delivery contract throughout the life of the contract and not just at the terminal date. In particular, we suppose that the firm hedges to try to prevent the actual cash balance from falling short of the expected cash balance by the amount $a(t)$, which we take to be large. Write A_t for the actual cash balance at time t under an arbitrary hedging strategy, and say that a shortfall occurs when $A_t \leq E[A_t] - a(t)$. Small shortfalls are unlikely to have a significant impact on the firm, so we are primarily interested in large $a(t)$.

By the *spot risk* at time t we mean

$$P(A_t - E[A_t] < -a(t)),$$

the probability of a shortfall at time t . Another more relevant measure is

$$P\left(\min_{0 \leq s \leq t} (A_s - E[A_s]) < -a(s)\right),$$

the probability of a shortfall any time up to t , which we call the *running risk* to t . Generally, these two probabilities are hard to capture. However, as in our setting, if the cash balance is Gaussian, we can use spot variance $Var[A_t]$ to measure the spot risk and use $\sup_{0 \leq s \leq t} Var[A_s]$ to measure the running risk.

Remark Throughout this dissertation, we will assume that the following are true for some market participants:

1. The market participants are subject to no transactions costs when they trade.
2. The market participants are subject to the same tax rate on all net trading profits.
3. The market participants can borrow money at the same risk-free rate of interest as they can lend money.
4. The market participants take advantage of arbitrage opportunities as they occur.

The following notation will be used throughout this dissertation:

T : Time until delivery date in a future contract

S_0 : Present price of the asset underlying of the futures contract.

F_0 : Present futures price

r : Risk-free rate of interest per annum, expressed with continuous compounding.

1.2. Stochastic Process and Brownian Motion

A good reference for this section is [12] Bernt Øksendal's *Stochastic Differential Equations—An Introduction with Applications*.

1.2.1. Stochastic Process. A stochastic process is the counterpart of a deterministic process considered in probability theory. Instead of dealing only with one possible outcome of how the process might evolve under time (as it is the case, for example, for solutions of an ordinary differential equation), in a random process there is some indeterminacy in its future evolution described by probability distributions.

This means that even if the initial condition (or starting point) is known, there are more possibilities the process might go to, but some paths are more probable and others less.

DEFINITION 1.2.1. *A stochastic process is a parameterized collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space (Ω, \mathcal{F}, P) and assuming values in \mathbf{R}^n .

Note that for each $t \in T$ fixed, we have a random variable

$$\omega \rightarrow X_t(\omega); \omega \in \Omega$$

On the other hand, fixing $\omega \in \Omega$, we can consider the function

$$t \rightarrow X_t(\omega); t \in T$$

which is called a *path* of X_t .

Intuitively, we can think of t as “time” and each ω as an individual “particle” or “experiment”. With this picture $X_t(\omega)$ would represent the position (or result) at time t of the particle ω . Sometimes it’s convenient to write $X(t, \omega)$ instead of $X_t(\omega)$. Thus we may also regard the process as a function of two variables

$$(t, \omega) \rightarrow X(t, \omega)$$

from $T \times \Omega$ into \mathbf{R}^n . This is often a natural point of view in stochastic analysis.

1.2.2. Normal Distribution(Gaussian). A **Normal Distribution** with mean μ and variance σ^2 is a random variable X , denoted as $X \sim N(\mu, \sigma^2)$ with density function

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

on the domain $x \in (-\infty, \infty)$. A Normal distribution is also called a Gaussian distribution. The so-called standard normal distribution is the normal distribution with $\mu = 0$ and $\sigma = 1$. The standard normal distribution has the following density function

$$P(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}}$$

1.2.3. The Brownian Motion. The Brownian Motion is the most fundamental continuous time stochastic process. R. Brown in 1826 observed the irregular motion of pollen particles spelling in water. He and others noted that the path of a given particle is very irregular, having a tangent at no point, and the motions of two distinct particles appear to be independent. In 1900 L. Bachelier attempted to describe fluctuations in stock prices mathematically and essentially discovered first certain results later extended by A. Einstein in 1905.

To describe the motion mathematically, it is natural to use the concept of a stochastic process $B_t(\omega)$, interpreted as the position at time t of the pollen grain ω .

To construct $\{B_t\}_{t \geq 0}$ it suffices to specify a family $\{v_{t_1, \dots, t_k}\}$ of probability measures. These measures will be chosen so that they agree with our observations of the pollen grain behavior:

Fix $x \in \mathbf{R}^n$ and define

$$p(t, x, y) = (2\pi t)^{-n/2} \cdot \exp\left(-\frac{|x - y|^2}{2t}\right) \quad \text{for } y \in \mathbf{R}^n, t > 0 \quad (1.2.1)$$

If $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$, define a measure v_{t_1, \dots, t_k} on \mathbf{R}^{nk} by

$$\begin{aligned} v_{t_1, \dots, t_k}(F_1 \times \dots \times F_k) &= \\ &= \int_{F_1 \times \dots \times F_k} p(t_1, x, x_1) p(t_2 - t_1, x_1, x_2) \dots p(t_k - t_{k-1}, x_{k-1}, x_k) d_{x_1} \dots d_{x_k} \end{aligned} \quad (1.2.2)$$

where we use the notation $d_y = d_{y_1} \dots d_{y_k}$ for Lebesgue measure and the convention that $p(0, x, y) d_y = \delta_x(y)$, the unit point mass at x .

We can extend this definition to all finite sequences of t'_i s, since $\int_{\mathbf{R}^n} p(t, x, y) dy = 1$ for all $t \geq 0$. By Kolmogorov's theorem, there exists a probability space $(\Omega, \mathcal{F}, P^x)$ and a stochastic process $\{B_t\}_{t \geq 0}$ on Ω such that the finite-dimensional distribution of B_t are

$$\begin{aligned} & P^x(B_{t_1} \in F_1, \dots, B_{t_k} \in F_k) = \\ &= \int_{F_1 \times \dots \times F_k} p(t_1, x, x_1) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k) d_{x_1} \cdots d_{x_k} \end{aligned} \quad (1.2.3)$$

DEFINITION 1.2.2. *Such a process is called (a version of) Brownian motion starting at x (observe that $P^x(B_0 = x) = 1$).*

We state some basic properties of Brownian motion:

(i) B_t is a *Gaussian process*, i.e. for all $0 \leq t_1 \leq \dots \leq t_k$ the random variable $Z = (B_{t_1}, \dots, B_{t_k}) \in \mathbf{R}^{nk}$ has a *(multi) normal distribution*. This means that there exists a vector $M \in \mathbf{R}^{nk}$ and a non-negative definite matrix $C = [c_{jm}] \in \mathbf{R}^{nk \times nk}$ (the set of all $nk \times nk$ -matrices with real entries) such that

$$E^x \left[\exp \left(i \sum_{j=1}^{nk} u_j Z_j \right) \right] = \exp \left(-\frac{1}{2} \sum_{j,m} u_j c_{jm} u_m + i \sum_j u_j M_j \right) \quad (1.2.4)$$

for all $u = (u_1, \dots, u_{nk}) \in \mathbf{R}^{nk}$, where $i = \sqrt{-1}$ is the imaginary unit and E^x denotes expectation with respect to P^x . Moreover,

$$M = E^x [Z] = (x, x, \dots, x) \in \mathbf{R}^{nk} \quad (1.2.5)$$

is the mean value of Z ,

and

$$c_{jm} = E^x [(Z_j - M_j) (Z_m - M_m)] \quad (1.2.6)$$

is the covariance matrix of Z .

Hence,

$$E^x[B_t] = x \quad \text{for all } t \geq 0$$

and

$$E^x[(B_t - x)^2] = nt, \quad E^x[(B_t - x)(B_s - x)] = n \cdot \min(s, t)$$

(ii) B_t has *independent increments*, i.e.

$$\begin{aligned} B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_k} - B_{t_{k-1}} \text{ are independent} \\ \text{for all } 0 \leq t_1 < t_2 \cdots < t_k \end{aligned} \tag{1.2.7}$$

(iii) For almost every ω , the sample path $t \rightarrow B_t(\omega)$ is continuous on $[0, T]$, i.e.

$$P(\{\omega : B_t(\omega) \text{ is a continuous path}\}) = 1$$

A stochastic process $\{B_t\}_{t \geq 0}$ is said to be a **Standard Brownian motion** process if: (i) $B_0 = 0$; (ii) $\{B_t\}_{t \geq 0}$ has stationary independent increments; (iii) for every $t > 0$, B_t is normally distributed with mean 0 and variance t .

1.2.4. Geometric Brownian motion. A **Geometric Brownian motion** (GBM) (occasionally, exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion. A stochastic process S_t is said to follow a GBM if it satisfies the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where B_t is a standard Brownian motion and μ ('the percentage drift') and σ ('the percentage volatility') are constants.

1.2.5. Stochastic differential equation. A *stochastic differential equation (SDE)* is a differential equation in which one or more of the terms is a stochastic process, thus resulting in a solution which is itself a stochastic process.

The general notation of stochastic differential equation used in financial mathematics is

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad (1.2.8)$$

where B_t denotes a Standard Brownian motion, $b(X_t, t)$ and $\sigma(X_t, t)$ are continuous stochastic functions with respect to random variable X_t . The equation above characterizes the behavior of the continuous time stochastic process X_t as the sum of an ordinary Lebesgue integral and an Itô integral.

Now, let's rewrite equation (1.2.8) in a discrete form with a replacement of B_t by a proper stochastic process W_t . Let $0 = t_0 < t_1 < \dots < t_m$ and consider a discrete version:

$$X_{k+1} - X_k = b(t_k, x_k)\Delta t_k + \sigma(t_k, X_k)W_k\Delta t_k,$$

where

$$X_{t_j} = X(t_j), \quad W_k = W_{t_k}, \quad \Delta t_k = t_{k+1} - t_k.$$

Replace $W_k\Delta t_k$ by $\Delta V_k = V_{t_{k+1}} - V_{t_k}$, where $\{V_t\}_{t \geq 0}$ is some suitable stochastic process. It suggests that V_t should have *stationary independent increments with mean 0*. It turns out that the only such process with continuous paths is the Brownian motion B_t . Thus we put $V_t = B_t$ and obtain from the last equation:

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j \quad (1.2.9)$$

In the next section, we will show that it's possible to prove the limit of the right hand side of (1.2.9) exists, in some sense, when $\Delta t_j \rightarrow 0$. Then by applying the usual

integration notation, we should obtain

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \quad (1.2.10)$$

1.3. Itô Integral and Itô Formula

1.3.1. Itô Integral. Suppose $0 \leq S \leq T$ and $f(t, \omega)$ is given. We want to define

$$\int_S^T f(t, \omega)dB_t(\omega),$$

where $B_t(\omega)$ is 1-dimensional Brownian motion starting at the origin, for a wide class of functions $f : [0, \infty) \times \Omega \rightarrow \mathbf{R}$.

DEFINITION 1.3.1. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions

$$f(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbf{R}$$

such that

- (i) $(t, \omega) \rightarrow f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ -measurable, where \mathcal{B} denotes the Borel σ -algebra on $[0, \infty)$.
- (ii) $f(t, \omega)$ is \mathcal{F}_t -adapted.
- (iii) $E[\int_S^T f(t, \omega)^2 dt] < \infty$

DEFINITION 1.3.2 (The Itô Integral). Let $f \in \mathcal{V}(S, T)$. Then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t, \omega)dB_t(\omega) = \lim_{n \rightarrow \infty} \int_S^T \phi_n(t, \omega)dB_t(\omega) \quad \text{limit in } L^2(P) \quad (1.3.1)$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E \left[\int_S^T (f(t, \omega) - \phi_n(t, \omega))^2 dt \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (1.3.2)$$

Furthermore, from the definition above, we get the following important

COROLLARY 1.3.1 (The Itô Isometry).

$$E \left[\left(\int_S^T f(t, \omega) dB_t \right)^2 \right] = E \left[\int_S^T f^2(t, \omega) dt \right] \quad \text{for all } f \in \mathcal{V}(S, T) \quad (1.3.3)$$

COROLLARY 1.3.2. If $f(t, \omega) \in \mathcal{V}(S, T)$ and $f_n(t, \omega) \in \mathcal{V}(S, T)$ for $n = 1, 2, \dots$ and $E \left[\int_S^T (f_n(t, \omega) - f(t, \omega))^2 dt \right] \rightarrow 0$ as $n \rightarrow \infty$, then

$$\int_S^T f_n(t, \omega) dB_t(\omega) \rightarrow \int_S^T f(t, \omega) dB_t(\omega) \quad \text{in } L^2(P) \text{ as } n \rightarrow \infty$$

Sketch of the construction of Itô integral: The idea is natural: First we define Itô integral $\mathcal{I}[\phi]$ for a simple class of functions ϕ . Then we show that each measurable \mathcal{F}_t -adapted function $f(t, \omega)$ can be approximated (in an appropriate sense) by such ϕ 's and we use this to define $\int f dB_t$ as the limit of $\int \phi dB_t$ as $\phi \rightarrow f$.

A function ϕ is called *elementary* if it has the form

$$\phi(t, \omega) = \sum_j e_j(\omega) \cdot \mathcal{X}_{[t_j, t_{j+1})}(t) \quad (1.3.4)$$

Note that each function e_j must be \mathcal{F}_{t_j} -measurable.

For elementary functions $\phi(t, \omega)$ we define the integral:

$$\int_S^T \phi(t, \omega) dB_t(\omega) = \sum_{j \geq 0} e_j(\omega) [B_{t_{j+1}} - B_{t_j}] (\omega) \quad (1.3.5)$$

The idea is to extend the definition from elementary functions to any \mathcal{F}_t -measurable function $f(t, \omega)$. □

We illustrate this integral with an example: Assume $B_0 = 0$. Then

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t$$

Put $\phi_n(s, \omega) = \sum B_j(\omega) \cdot \mathcal{X}_{[t_j, t_{j+1})}(s)$, where $B_j = B_{t_j}$. Then

$$\begin{aligned} E \left[\int_0^t (\phi_n - B_s)^2 ds \right] &= E \left[\sum_j \int_{t_j}^{t_{j+1}} (B_j - B_s)^2 ds \right] \\ &= \sum_j \int_{t_j}^{t_{j+1}} (s - t_j) ds = \sum_j \frac{1}{2} (t_{j+1} - t_j)^2 \rightarrow 0 \quad \text{as } \Delta t_j \rightarrow 0 \end{aligned}$$

So by Corollary 1.3.2

$$\int_0^t B_s dB_s = \lim_{\Delta t_j \rightarrow 0} \int_0^t \phi_n dB_s = \lim_{\Delta t_j \rightarrow 0} \sum_j B_j \Delta B_j$$

Now

$$\begin{aligned} \Delta(B_j^2) &= B_{j+1}^2 - B_j^2 \\ &= (B_{j+1} - B_j)^2 + 2B_j(B_{j+1} - B_j) \\ &= (\Delta B_j)^2 + 2B_j \Delta B_j, \end{aligned}$$

and therefore, since $B_0 = 0$,

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (\Delta B_j)^2 + 2 \sum_j B_j \Delta B_j$$

or

$$\sum_j B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2} \sum_j (\Delta B_j)^2.$$

since $\sum_j (\Delta B_j)^2 \rightarrow t$ in $L^2(P)$ as $\Delta t_j \rightarrow 0$, the result follows.

1.3.2. Itô Formula. The basic definition of Itô integrals is not very useful when we try to evaluate a given integral. This is similar to the situation for ordinary Riemann integrals, where we do not use the basic definition but rather the fundamental theorem of calculus plus the chain rule in the explicit calculations.

In this context, it turns out that it is possible to establish an Itô integral version of the chain rule, called the Itô formula. The Itô formula is very useful for evaluating

Itô integrals.

Here, we introduce *Itô processes* (also called *stochastic integrals*) as sums of a dB_s and a ds integral. Thus we define

DEFINITION 1.3.3 (1-dimensional Itô process). *Let B_t be 1-dimensional Brownian motion on (Ω, \mathcal{F}, P) . A (1-dimensional) Itô process (or stochastic integral) is a stochastic process X_t on (Ω, \mathcal{F}, P) of the form*

$$X_t = X_0 + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) dB_s, \quad (1.3.6)$$

where $v \in \mathcal{W}_{\mathcal{H}}$, so that

$$P \left[\int_0^t v(s, \omega)^2 ds < \infty \text{ for all } t \geq 0 \right] = 1 \quad (1.3.7)$$

We also assume that u is \mathcal{H}_t -adapted and

$$P \left[\int_0^t |u(s, \omega)| ds < \infty \text{ for all } t \geq 0 \right] = 1 \quad (1.3.8)$$

If X_t is an Itô process of the form (1.3.6), then the equation (1.3.6) is sometimes written in the shorter differential form

$$dX_t = udt + vdB_t \quad (1.3.9)$$

THEOREM 1 (The 1-dimensional Itô formula). *Let X_t be an Itô process given by*

$$dX_t = udt + vdB_t$$

Let $g(t, x) \in C^2([(0, \infty)] \times \mathbf{R})$ (i.e. g is twice continuously differentiable on $[(0, \infty)] \times \mathbf{R}$). Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2, \quad (1.3.10)$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt \quad (1.3.11)$$

More generally,

THEOREM 2 (Integration by parts). *suppose $f(s, \omega)$ is continuous and of bounded variation with respect to $s \in [0, t]$, then*

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s \quad (1.3.12)$$

Sketch of proof of the Itô formula: First observe that if we substitute

$$dX_t = udt + vdB_t$$

in (1.3.10) and use (1.3.11) we get the equivalent expression

$$\begin{aligned} g(t, X_t) &= g(0, X_0) + \int_0^t \left(\frac{\partial g}{\partial s}(s, X_s) + u_s \frac{\partial g}{\partial x}(s, X_s) + \frac{1}{2} v_s^2 \cdot \frac{\partial^2 g}{\partial x^2}(s, X_s) \right) ds \\ &\quad + \int_0^t v_s \cdot \frac{\partial g}{\partial x}(s, X_s) dB_s \quad \text{where } u_s = u(s, \omega), v_s = v(s, \omega) \end{aligned} \quad (1.3.13)$$

Note that (1.3.13) is an Itô process.

We may assume that $g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$ are bounded, for if (1.3.13) is proved in this case we obtain the general case by approximating by C^2 functions $g_n, \frac{\partial g_n}{\partial t}, \frac{\partial g_n}{\partial x}$ and $\frac{\partial^2 g_n}{\partial x^2}$ are bounded for each n and converge uniformly on compact subset of $[0, \infty) \times \mathbf{R}$ to $g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$, respectively. Moreover, we see that we may assume that $u(t, \omega)$

and $v(t, \omega)$ are elementary functions. Using Taylor's theorem we get

$$\begin{aligned}
g(t, X_t) &= g(0, X_0) + \sum_j \Delta g(t_j, X_j) \\
&= g(0, X_0) + \sum_j \frac{\partial g}{\partial t} \Delta t_j + \sum_j \frac{\partial g}{\partial x} \Delta X_j + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial t^2} (\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial t \partial x} (\Delta t_j) (\Delta X_j) \\
&\quad + \frac{1}{2} \sum_j \frac{\partial^2 g}{\partial x^2} (\Delta X_j)^2 + \sum_j R_j,
\end{aligned}$$

where $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$ etc. are evaluated at the points (t_j, X_{t_j}) ,

$$\Delta t_j = t_{j+1} - t_j, \Delta X_j = X_{t_{j+1}} - X_{t_j}, \Delta g(t_j, X_j) = g(t_{j+1}, X_{j+1}) - g(t_j, X_j)$$

and $R_j = o(|\Delta t_j|^2 + |\Delta X_j|^2)$ for all j .

If $\Delta t_j \rightarrow 0$, then

$$\sum_j \frac{\partial g}{\partial t} \Delta t_j = \sum_j \frac{\partial g}{\partial t}(t_j, X_j) \Delta t_j \rightarrow \int_0^t \frac{\partial g}{\partial t}(s, X_s) ds \quad (1.3.14)$$

$$\sum_j \frac{\partial g}{\partial x} \Delta X_j = \sum_j \frac{\partial g}{\partial x}(t_j, X_j) \Delta X_j \rightarrow \int_0^t \frac{\partial g}{\partial x}(s, X_s) ds \quad (1.3.15)$$

Moreover, since u and v are elementary we get

$$\begin{aligned}
\sum_j \frac{\partial^2 g}{\partial x^2} (\Delta X_j)^2 &= \sum_j \frac{\partial^2 g}{\partial x^2} u_j^2 (\Delta t_j)^2 + 2 \sum_j \frac{\partial^2 g}{\partial x^2} u_j v_j (\Delta t_j) (\Delta B_j) \\
&\quad + \sum_j \frac{\partial^2 g}{\partial x^2} v_j^2 (\Delta B_j)^2, \text{ where } u_j = u(t_j, \omega), v_j = v(t_j, \omega)
\end{aligned} \quad (1.3.16)$$

The first two terms here tend to 0 as $\Delta t_j \rightarrow 0$. For example,

$$E \left[\left(\sum_j \frac{\partial^2 g}{\partial x^2} u_j v_j (\Delta t_j) (\Delta B_j) \right)^2 \right] = \sum_j E \left[\left(\frac{\partial^2 g}{\partial x^2} u_j v_j \right)^2 \right] (\Delta t_j)^3 \rightarrow 0 \text{ as } \Delta t_j \rightarrow 0$$

We claim that the last term tends to

$$\int_0^t \frac{\partial^2 g}{\partial x^2} v^2 ds \quad \text{in } L^2(P), \text{ as } \Delta t_j \rightarrow 0 \quad (1.3.17)$$

To prove this, put $a(t) = \frac{\partial^2 g}{\partial x^2}(t, X_t)v^2(t, \omega)$, $a(t) = a(t_j)$ and consider

$$E \left[\left(\sum_j a_j (\Delta B_j)^2 - \sum_j a_j \Delta t_j \right)^2 \right] = \sum_{i,j} E [a_i a_j ((\Delta B_i)^2 - \Delta t_i) ((\Delta B_j)^2 - \Delta t_j)]$$

If $i < j$ then $a_i a_j ((\Delta B_i)^2 - \Delta t_j)$ and $((\Delta B_j)^2 - \Delta t_j)$ are independent so the terms vanish in this case, and similarly if $i > j$. So we are left with

$$\begin{aligned} \sum_j E [a_j^2 ((\Delta B_j)^2 - \Delta t_j)^2] &= \sum_j E[a_j^2] \cdot E [(\Delta B_j)^4 - 2(\Delta B_j)^2 \Delta t_j + (\Delta t_j)^2] \\ &= \sum_j E[a_j^2] \cdot (3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2) \\ &= 2 \sum_j E[a_j^2] \cdot (\Delta t_j)^2 \\ &\rightarrow 0 \quad \text{as } \Delta t_j \rightarrow 0 \end{aligned}$$

In other words, we have established that

$$\sum_j a_j (\Delta B_j)^2 \rightarrow \int_0^t a(s) ds \quad \text{in } L^2(P) \text{ as } \Delta t_j \rightarrow 0$$

and this is often expressed shortly by the formula

$$(dB_t)^2 = dt \tag{1.3.18}$$

The argument above also proves that $\sum R_j \rightarrow 0$ as $\Delta t_j \rightarrow 0$. That completes the proof of the Itô formula. \square

Let us apply the Itô Formula to the integral $I = \int_0^t B_s dB_s$ that we calculated in the last section.

Choose $X_t = B_t$ and $g(t, x) = \frac{1}{2}x^2$. Then

$$Y_t = g(t, B_t) = \frac{1}{2}B_t^2.$$

Then by Itô's formula,

$$\begin{aligned} dY_t &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dB_t)^2 \\ &= B_t dB_t + \frac{1}{2}(dB_t)^2 \\ &= B_t dB_t + \frac{1}{2}dt. \end{aligned}$$

Hence

$$d\left(\frac{1}{2}B_t^2\right) = B_t dB_t + \frac{1}{2}dt.$$

In other words,

$$\frac{1}{2}B_t^2 = \int_0^t B_s dB_s + \frac{1}{2}t.$$

COROLLARY 1.3.3 (Itô Product Rule). *Let X_t and Y_t be two different Itô processes. Define $Z_t = X_t \cdot Y_t$, then Z_t is also an Itô process. and*

$$dZ_t = d(X_t Y_t) = dX_t Y_t + X_t dY_t + dX_t dY_t \tag{1.3.19}$$

Itô Formula is crucial in deriving differential equations for the value of derivative securities such as stock options and is essential in the derivation of Black and Scholes Equation.

1.3.3. Jensen's Inequality. Let (Ω, μ) be a measurable space, such that $\mu(\Omega) = 1$. If g is a real-valued function that is μ -integrable, and if ϕ is a convex function on the real axis, then:

$$\phi\left(\int_{\Omega} g d\mu\right) \leq \int_{\Omega} \phi \circ g d\mu.$$

The same result can be equivalently stated in a probability theory setting, by a simple change of notation. Let (Ω, P) be a probability space, X an integrable real-valued random variable and ϕ a convex function. Then:

$$\phi[E[X]] \leq E[\phi[X]].$$

CHAPTER 2

A class of Hedging problems

In this chapter, we will introduce a class of hedging problems based on a simple model for the underlying asset's future price. We will also introduce some previous work on these problems and discuss how to find the optimal strategy to minimize the risk. Some good references for this chapter are [5]G. Larcher and G. Leobacher's *An optimal strategy for hedging with short-term futures contracts* and [4]P. Glasserman's *Shortfall risk in long-term hedging with short-term futures contracts*

2.1. Background of a Hedging problem

A firm has a commitment to deliver a fixed quantity q of a commodity at a forward price a_n at dates $n = 1, \dots, N$. By doing so the firm is taking the risk of the underlying commodities' future price movements. The firm will have to pay the market price for the underlying commodities to deliver the contract which might result in loss of cash if the spot price of the underlying commodity rises abruptly.

To reduce the risk of losses due to possible price fluctuations, the firm might enter into a sequence of short-dated futures contracts to protect itself from the effects of large price fluctuations. Assume that the market price of the underlying commodities is given by a simple stochastic differential equation, i.e.

$$dS_t = \mu dt + \sigma dB_t, \tag{2.1.1}$$

where B_t is standard Brownian motion on $[0, 1]$, μ and σ are constants, $\sigma > 0$. We further assume that the interest rate $r = 0$.

The firm commits to supplying at each time t in the interval $[0, T]$ a commodity at rate q and the deterministic price a_t . There is no loss in assuming $T = 1$ and $q = 1$. The unhedged cumulative cash flow from this contract is

$$C_t = \int_0^t a_s - S_s ds$$

Since $S_t = S_0 + \int_0^t dS_s$, we can get

$$C_t = \int_0^t (a_s - S_0) ds + \int_0^t (s - t) dS_s \quad (2.1.2)$$

Consider now at time t a short term future with maturity $t + \delta$ and futures price $F_{t,t+\delta}$. The payoff of such a contract is

$$S_{t+\delta} - F_{t,t+\delta}.$$

If we write $F_{t,t+\delta} = S_t + b_{t,t+\delta}\delta$, then we get for the payoff

$$S_{t+\delta} - S_t - b_{t,t+\delta}\delta$$

where $b_{t,t+\delta}\delta$ is the *basis* of the future, i.e. the deviation from the “natural” price S_t .

The payoff from a hedging strategy holding $G_{n\delta}$ futures at time $n\delta$ is therefore

$$H_{k\delta} := \sum_{n=0}^{k-1} G_{n\delta}(S_{t+\delta} - S_t) - \sum_{n=0}^{k-1} G_{n\delta}b_{t,t+\delta}\delta$$

Letting $\delta \rightarrow 0$ gives the continuous form

$$H_t = \int_0^t G_s dS_s - \int_0^t G_s b_s ds$$

where we assume that $b_t := \lim_{\delta \rightarrow 0} b_{t,t+\delta}$ exists and is regular enough to guarantee existence of the integral. Since $dS_t = \mu dt + \sigma dB_t$, this means that

$$H_t = \int_0^t G_s \sigma dB_s + \int_0^t G_s (\mu - b_s) ds.$$

Note that if $G_s(\mu - b_s)$ is deterministic for almost all s , the cash flow H from a hedging strategy G satisfies

$$H_t - E[H_t] = \int_0^t G_s \sigma dB_s$$

The hedged cumulative cash flow from this contract is $C_t + H_t$. Let $V[C_t + H_t]$ be variance of this cash flow.

$$C_t - E[C_t] + H_t - E[H_t] = \sigma \int_0^t (s - t + G_s) dB_s$$

By Itô isometry we have

$$\begin{aligned} V[C_t + H_t] &= E[(\sigma \int_0^t (s - t + G_s) dB_s)^2] \\ &= \sigma^2 \int_0^t E[(s - t + G_s)^2] ds \\ &= \sigma^2 \int_0^t E[(t - s - G_s)^2] ds. \end{aligned}$$

From Jensen's inequality it follows that

$$\int_0^t E[(t - s - G_s)^2] ds \geq \int_0^t (E[t - s - G_s])^2 ds$$

with equality if and only if G_s is the deterministic for almost all $s \in [0, 1]$. So we may restrict our search to deterministic strategies. If we write $g(s) := s + G_s$, then $V[C_t + H_t] = \sigma^2 \int_0^t (t - g(s))^2 ds$.

Question: Now what strategy G should we use?

2.2. Previous works

Observe that if the cash flow is unhedged, i.e. $G_s = 0$, and $g(s) = s$, then $V[C_t + H_t] = \frac{1}{3}\sigma^2 t^3$. One policy is to require that the risk at the terminal date

(terminal risk) equals to zero, i.e.

$$V[C_1 + H_1] = 0,$$

which is called *rolling stack* hedging strategy. Based on our model, we can easily get $g(s) = 1, a.e.$ and $V[C_t + H_t] = \sigma^2 t(t - 1)^2$. So,

$$\max_{t \in [0,1]} V[C_t + H_t] := \begin{cases} \frac{1}{3} \approx 0.3333 & \text{when } t = 1 \text{ for no hedge} \\ \frac{4}{27} \approx 0.1481 & \text{when } t = \frac{1}{3} \text{ for rolling stack} \end{cases}$$

A primary objective for hedging is to protect the firm from the effects of large price fluctuation. We will examine how effectively the *rolling stack* accomplishes this.

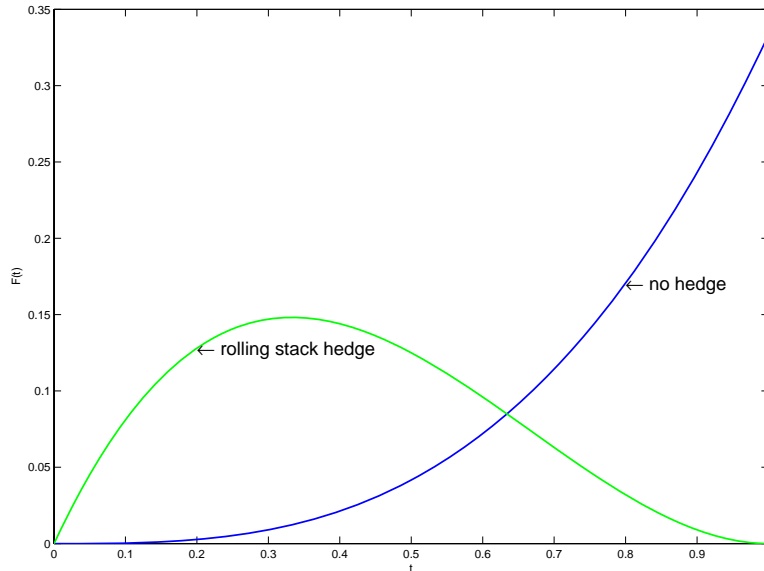


FIGURE 2.1. The spot risks of no hedge and rolling stack hedge

Figure 2.1 displays the comparison of the spot risk of unhedged and that of rolling stack hedged cash balance over the life of the contract. Clearly, in this simple model we study, the rolling stack eliminates the terminal risk completely, however, it also

increases the spot risk in the early life of this hedge (as noted by [8]Mello and Parsons 1995).

In order to make the hedge more effectively, [4]P. Glassman(2001) pointed out that we need to minimize the maximum of the running spot risk, i.e. we want to find a strategy G for which the maximum of the running risk gets its minimum. We denote it as:

$$\sup_{t \in [0,1]} V[C_t + H_t] = \sup_{t \in [0,1]} E[(C_t + H_t - E[C_t + H_t])^2] \rightarrow \min$$

Based on the model of rolling stack hedge, P. Glassman gave an improved model, holding $(\tau - s)$ futures contracts at time s , rather than $1 - s$; i.e., by the strategy

$$g_\tau(s) := \begin{cases} \tau & \text{if } s \in [0, \tau], \\ s & \text{if } s \in (\tau, 1]. \end{cases}$$

which is called the fixed-horizon hedge. This hedge will make the spot risk at τ zero and remaining unhedged in $[\tau, 1]$. For any τ , we can evaluate the spot risk under g_τ . The maximal spot risk occurs either at $t = \frac{\tau}{3}$ with $V_t = \frac{4}{27}\delta^2\tau^3$ or at $t = 1$ with $V_t = \delta^2(\frac{2}{3}\tau^3 - \tau^2 + \frac{1}{3})$. The optimal τ , which minimizes the maximal spot risk, will makes these two equal. It implies that

$$\frac{4}{27}\delta^2\tau^3 = \delta^2\left(\frac{2}{3}\tau^3 - \tau^2 + \frac{1}{3}\right).$$

Numerically, we can get $\tau \approx 0.733$ and the corresponding maximal spot risk is about 0.0583.

Figure 2.2 displays the spot risk of Optimal fixed-horizon hedge over the life of the contract along with that for the rolling stack hedge. Clearly, regard to reducing the running risk, the optimal fixed-horizon hedge is more effective than rolling stack hedge.

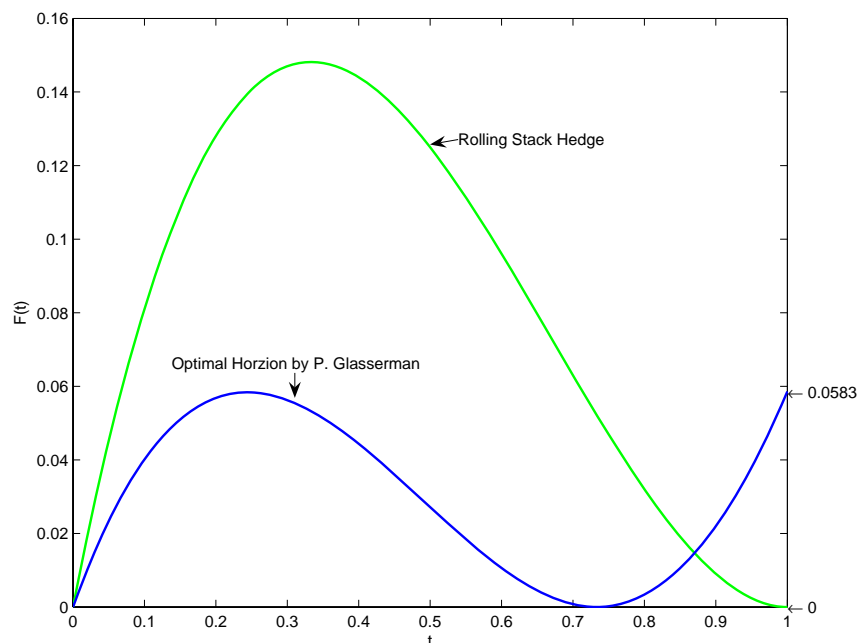


FIGURE 2.2. The spot risk of Rolling Stack and Optimal Horizon

Following P. Glassman's idea, [5]G.Larcher and G.Leobacher(2003) found a unique optimal strategy that provides the minimal running risk $\sup_{t \in [0,1]} V[C_t + H_t]$. They proved the following theorem:

THEOREM 3. *There is a unique continuous function $g : [0, 1] \rightarrow R$ that satisfies the condition*

$$\begin{aligned}
 F : &= \max_{t \in [0,1]} \int_0^t (t - g(s))^2 ds \\
 &= \inf_{\substack{h: [0,1] \rightarrow R \\ h \text{ integrable}}} \sup_{t \in [0,1]} \int_0^t (t - h(s))^2 ds
 \end{aligned}$$

g is given by the formula

$$g(s) := \begin{cases} 3t_0 & \text{for } s \in [0, t_0], \\ e^{-\frac{\eta}{2}} \cos\left(\frac{\sqrt{3}\eta}{2}\right) & \text{for } s = \frac{1}{\sqrt{3}}e^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \frac{\pi}{6}\right) \text{ with } \eta \in [0, \frac{\pi}{3\sqrt{3}}], \\ 1 & \text{if } s \in [1/2, 1]. \end{cases}$$

where $t_0 = \frac{1}{2\sqrt{3}}e^{-\frac{\pi}{6\sqrt{3}}}$. For the maximal risk we have $F = F_g(1) = \frac{1}{6\sqrt{3}}e^{\frac{-\pi}{2\sqrt{3}}} = 0.0388532$

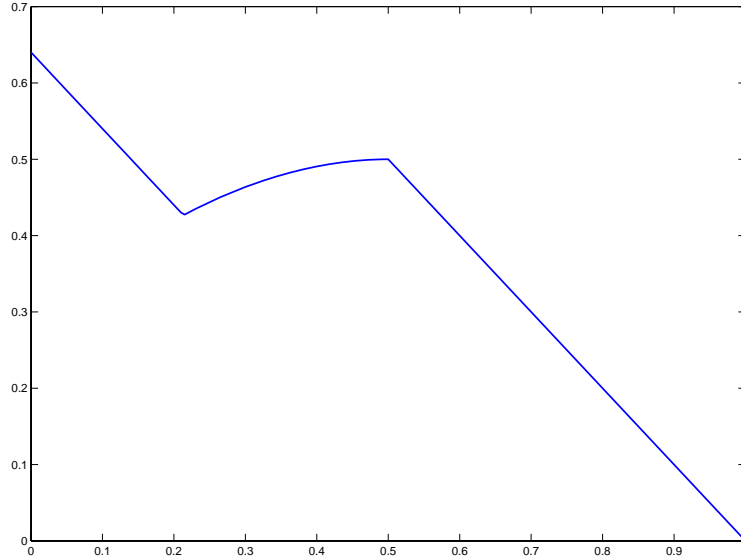


FIGURE 2.3. Strategy function of Optimal Hedge

Figure 2.3 and Figure 2.4 displays the graph of optimal hedge strategy function $G_s = g(s) - s$ and that of the corresponding variance function respectively.

Figure 2.5 displays a comparison of spot risks under the above hedge strategies. Regarding to minimizing the running risk, the optimal hedge $g(s)$ gives the most effective result.

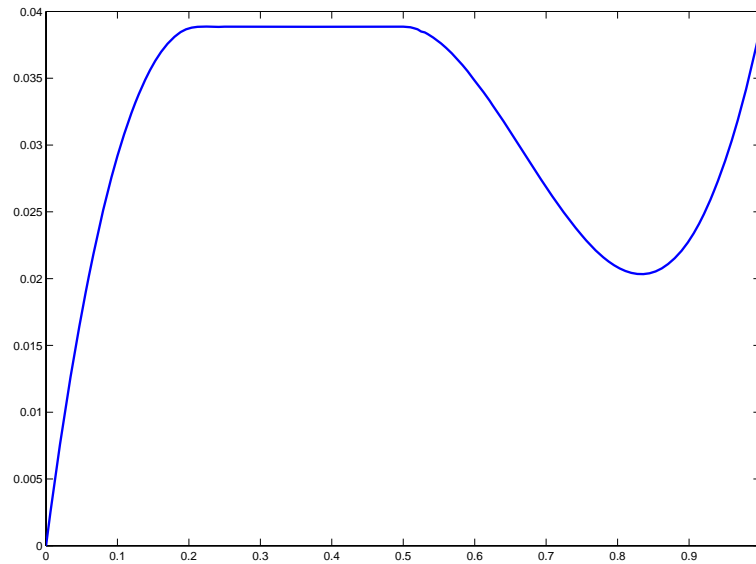


FIGURE 2.4. Spot risk of Optimal Hedge

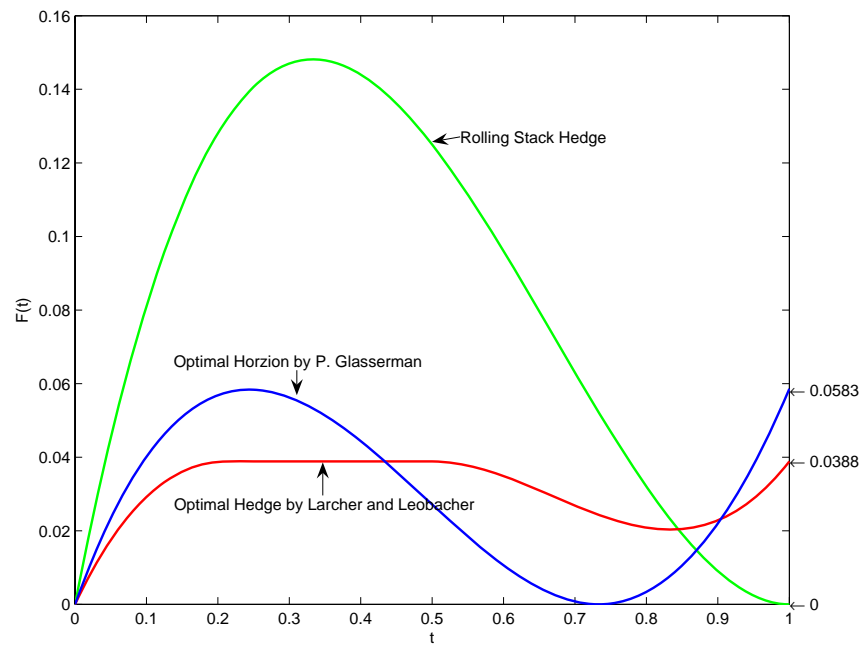


FIGURE 2.5. Comparison of spot risk under different hedging strategies

CHAPTER 3

A generalized optimization problem

In this chapter, based on previous work, we will introduce a new optimization problem with a consideration of restricting the terminal risk. We will prove the existence of the solution to this optimization problem and also provide a numerical solution. A good reference for this chapter is [5]G. Larcher and G. Leobacher's *An optimal strategy for hedging with short-term futures contracts*

3.1. Introduce a new problem

In Chapter 2, we discussed how the rolling stack hedge could eliminate the terminal risk completely but increases the spot risk in the early life. On the other hand, the optimal hedge given by G. Larcher and G. Leobacher could minimize the maximum of the running spot risk. However, in G. Larcher and G. Leobacher's solution, the terminal risk equals to the maximal running risk, and quantitatively, comparing to the rolling stack hedge, the terminal risk increases from 0 to $\frac{1}{6\sqrt{3}}e^{\frac{-\pi}{2\sqrt{3}}}$.

Since terminal date is very important for a contract, a relatively high terminal risk is certainly not what customers prefer. Therefore, a firm may want to restrict the terminal risk when it tries to find the best hedging strategy. Based on the simple model given in Chapter 2, we will discuss our new problem—what strategy should we use to minimize the maximum running spot risk if there is a constraint on the terminal risk?

Briefly, we want to study the following problem: Under the constraint that

$$\int_0^1 (1 - g(s))^2 ds \leq x,$$

for which measurable function g , if any, does the functional

$$\sup_{t \in [0,1]} \int_0^t (t - g(s))^2 ds$$

attain its minimum? Once we get the optimal function $g(s)$, we can get the optimal strategy function $G(s)$ by $G(s) = g(s) - s$. From G. Larcher and G. Leobachers' work, it is reasonable to assume that $0 \leq x \leq \frac{1}{6\sqrt{3}} e^{\frac{-\pi}{2\sqrt{3}}}$.

We introduce some notation: Let $R[0, 1]$ denote the space of all real-valued, Riemann-integrable functions on $[0, 1]$. For a function $h \in R[0, 1]$ we write

$$F_h(t) := \int_0^t (t - h(s))^2 ds,$$

and we let $F := \inf_{h \in R[0,1]} \sup_{t \in [0,1]} \{F_h(t) : F_h(1) \leq x\}$.

Main Theorem I: For $0 \leq x \leq \frac{1}{6\sqrt{3}} e^{\frac{-\pi}{2\sqrt{3}}}$, there exists a non-decreasing function $g \in R[0, 1]$ such that

$$F = \max_{t \in [0,1]} \{F_g(t) : F_g(1) \leq x\}.$$

Moreover, this function g is unique (up to a set of measure 0).

Main Theorem II: For $0 \leq x \leq \frac{1}{6\sqrt{3}} e^{\frac{-\pi}{2\sqrt{3}}}$, let $u(x)$ be the unique real solution of

$$\frac{1}{2} \left(\frac{1}{3} + u^2 \right)^{\frac{3}{2}} e^{-\sqrt{3}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u))} - \frac{1}{4}(1 + u)^2 u = x,$$

which can be solved numerically, then the unique optimal function $g(s)$ is defined by

$$g(s, x) := \begin{cases} 3t_0(x) & \text{for } s \in [0, t_0(x)], \\ \sqrt{3}B(x)e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}\eta}{2} + \phi(x)) & \text{for } s = B(x)e^{-\frac{1}{2}\eta} \cos(\frac{\sqrt{3}}{2}\eta + \phi(x) + \frac{\pi}{6}), \\ & \text{with } \eta \in [0, \eta_0(x)]. \\ 1 & \text{if } s \in [t_1(x), 1]. \end{cases}$$

where $t_1(x) = \frac{1-u(x)}{2}$, $B(x) = \sqrt{\frac{1}{3} + u(x)^2}$, $\phi = \tan^{-1}(\sqrt{3}u(x))$, $\eta_0(x) = \frac{2}{\sqrt{3}}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u(x)))$ and $t_0(x) = \frac{1}{2}\sqrt{\frac{1}{3} + u(x)^2}e^{-\frac{1}{\sqrt{3}}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u(x)))}$.

And the corresponding maximal risk is

$$F = \frac{1}{4}(1 + u(x))^2u(x) + x.$$

3.2. Proof of the Main Theorem I

In this section, we prove the Main Theorem I and related properties. [5]G. Larcher and G. Leobacher proved the special case of Main Theorem I when there is no constraint on $F_g(1)$. For the general case, our proof approaches are similar to theirs'. For reader's convenience, I still write the proof down. In Lemma 3.2.4, I use a different method to make the proof process work more smoothly.

To prove the Main Theorem I, several steps are required. First we restate this problem: Let $R[0, 1]$ denote the space of all real-valued, Riemann-integrable functions on $[0,1]$. Determine

$$F := \inf_{h \in R[0,1]} \sup_{t \in [0,1]} \{F_h(s) : F_h(1) \leq x\} \quad (3.2.1)$$

where the infimum is taken over all possible $h \in R[0, 1]$ and the function g for which the infimum is attained. If such g exists, then it gives the optimal strategy $G(s) = g(s) - s, s \in [0, 1]$.

For an arbitrary $h \in R[0, 1]$ define

$$h_1(s) := \begin{cases} 0 & \text{if } h(s) < 0, \\ h(s) & \text{if } 0 \leq h(s) \leq 1, \\ 1 & \text{if } h(s) > 1. \end{cases}$$

Then $h_1(s) \in R[0, 1]$. It is easy to verify that $F_{h_1}(t) \leq F_h(t)$ for all $t \in [0, 1]$. Hence we can restrict attention to functions h with values in $[0, 1]$ and for such functions F_h is continuous.

For some $h \in R[0, 1]$ we define

$$h_2(s) := \begin{cases} h(s) & \text{if } h(s) \geq s, \\ s & \text{if } h(s) < s. \end{cases}$$

Then h_2 is in $R[0, 1]$ and $F_{h_2}(t) \leq F_h(t)$ for all $t \in [0, 1]$. Hence we can restrict ourselves to function h satisfying $h(s) \geq s$ for all $s \in [0, 1]$.

LEMMA 3.2.1. *If $\{f_i\}_{i \geq 1}$ is a sequence of nondecreasing functions, $f_i : [0, 1] \rightarrow [0, 1]$, then there is a nondecreasing function $f : [0, 1] \rightarrow [0, 1]$ and a subsequence $\{f_{i_k}\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} f_{i_k} = f$ almost everywhere.*

PROOF. By Helly's first Theorem[See P.254 Theorem 6.4 in [10]'An introduction to probability theory' by By P.A.P.Moran], there exists a nondecreasing function f and a subsequence $\{f_{i_k}\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} f_{i_k} = f$ at all continuity points in $[0, 1]$ of the function f . Since f is bounded and non-decreasing, it has countable discontinuous points. Therefore, $\lim_{k \rightarrow \infty} f_{i_k} = f$ almost everywhere. \square

From Monotone Convergence Theorem, it follows that $\lim_{k \rightarrow \infty} f_{i_k} = f$ and $\lim_{k \rightarrow \infty} f_{i_k}^2 = f^2$ in $L^1[0, 1]$.

Therefore, we have:

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, 1]} \int_0^t (f_{i_k}(s) - f(s)) ds = 0$$

and

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, 1]} \int_0^t (f_{i_k}(s)^2 - f(s)^2) ds = 0$$

LEMMA 3.2.2. *Let h be a non-negative and integrable function, then given $\epsilon > 0$, we can find a step function ϕ such that $|h - \phi| < \epsilon$ except on a set of measure less than ϵ ; i.e., $m\{s : |h(s) - \phi(s)| \geq \epsilon\} < \epsilon$.*

PROOF. See Proposition 3.22 in [11]‘Real Analysis’ 3rd Edition by H.L. Royden

□

Lemma 3.2.2 implies that we can find a sequence of step functions $\{\phi_n\}$ that converges to h in measure.

LEMMA 3.2.3. *Given $h \in L^p$, $1 \leq p \leq \infty$ and $\epsilon > 0$, there is a step function ϕ such that $\|h - \phi\|_p < \epsilon$*

PROOF. See Proposition 6.8 in [11]‘Real Analysis’ 3rd Edition by H.L. Royden □

Theorem of Existence *For $0 \leq x \leq \frac{1}{6\sqrt{3}}e^{\frac{-\pi}{2\sqrt{3}}}$, there exists a nondecreasing function $g : [0, 1] \rightarrow [0, 1]$ such that $g(s) \geq s$, $s \in [0, 1]$, and*

$$F = \max_{t \in [0,1]} \{F_g(t) : F_g(1) \leq x\} \tag{3.2.2}$$

Remark: [5]G. Larcher and G. Leobacher proved the case of no constraint on $F_g(1)$ in Proposition 2.1.

PROOF. For all $\epsilon > 0$, by the definition of infimum, there exists an integrable function h such that $\max_{t \in [0,1]} F_h(t) < F + \epsilon$.

From the lemma 3.2.3, for given ϵ there exist a step function $\phi : [0, 1] \rightarrow [0, 1]$, such that $\int_0^1 |h(s) - \phi(s)| ds < \epsilon$.

Since $|(h(s) - \phi(s))| < 2$ and $|(t - h(s))| < 2$,

$$\begin{aligned}
& \int_0^t (t - \phi(s))^2 ds \\
&= \int_0^t (t - h(s) + h(s) - \phi(s))^2 ds \\
&\leq \int_0^t ((t - h(s))^2 + (h(s) - \phi(s))^2 + 2|t - h(s)||h(s) - \phi(s)|) ds \\
&\leq \int_0^t (t - h(s))^2 ds + \max_{s \in [0, t]} (|h(s) - \phi(s)|) \int_0^t |h(s) - \phi(s)| ds \\
&\quad + 2 \max_{s \in [0, t]} (|t - h(s)|) \int_0^t |h(s) - \phi(s)| ds \\
&\leq \int_0^t (t - h(s))^2 ds + 5 \int_0^1 |h(s) - \phi(s)| ds \\
&\leq \int_0^t (t - h(s))^2 ds + 5\epsilon
\end{aligned}$$

Therefore, we can claim that there exists a step function $\phi : [0, 1] \rightarrow [0, 1]$ such that $\max_{t \in [0, 1]} F_\phi(t) < F + \epsilon$. Furthermore, we can assume that $\phi : [0, 1] \rightarrow [0, 1]$ and $\phi(s) \geq s$.

For a measurable set $E \in [0, 1]$, we write

$$\lambda(E) = \int_E ds.$$

Now, we are going to prove that there exists an increasing step function $\phi_1(s)$, such that $\max_{t \in [0, 1]} F_{\phi_1}(t) < F + \epsilon$. Assume that there are real numbers $0 = t_0 < t_1 < \dots < t_N = 1$ and ξ_1, \dots, ξ_N , such that $\xi_i \neq \xi_{i+1}$, $1 \leq i \leq N - 1$, and $\phi(s) = \xi_i$ for $s \in (t_{i-1}, t_i]$, $1 \leq i \leq N$, $\phi(0) := 0$.

If $\xi_1 < \xi_2 < \dots < \xi_N$, we are done. Otherwise we choose $1 \leq i \leq N$ with $\xi_i > \xi_{i+1}$ and define

$$\phi_1(s) := \begin{cases} \phi(s) & \text{for } s \notin (t_{i-1}, t_{i+1}], \\ w(\xi_i, \xi_{i+1}) & \text{for } s \in (t_{i-1}, t_{i+1}]. \end{cases}$$

where

$$w(\xi_i, \xi_{i+1}) = \frac{(t_i - t_{i-1})\xi_i + (t_{i+1} - t_i)\xi_{i+1}}{(t_{i+1} - t_{i-1})}.$$

We can check that indeed $F_{\phi_1}(t) \leq F_{\phi(t)}$ holds for $t \in [0, 1]$. This can be done by considering the intervals $[0, t_{i-1}]$, $[t_{i-1}, t_{i+1}]$, $[t_{i+1}, 1]$.

The proof is as follows:

There are three cases to be considered.

When $t \leq t_{i-1}$, $\int_0^t (t - \phi_1(s))^2 ds = \int_0^t (t - \phi(s))^2 ds$.

When $t \in [t_{i-1}, t_{i+1}]$, note that

$$\int_0^t (t - \phi(s))^2 ds = \int_0^{t_{i-1}} (t - \phi(s))^2 ds + \int_{t_{i-1}}^t (t - \phi(s))^2 ds$$

It is sufficient to show that $\int_{t_{i-1}}^t (t - \phi(s))^2 ds \geq \int_{t_{i-1}}^t (t - \phi_1(s))^2 ds$

Since

$$\frac{1}{\lambda([t_{i-1}, t])} \int_{t_{i-1}}^t ds = 1$$

Let

$$E(f) = \frac{1}{\lambda([t_{i-1}, t])} \int_{t_{i-1}}^t f(s) ds$$

$$\begin{aligned} \frac{1}{\lambda([t_{i-1}, t])} \int_{t_{i-1}}^t (t - \phi(s))^2 ds &= E[(t - \phi(s))^2] \\ &\geq [E(t - \phi(s))]^2 \\ &= [t - E(\phi(s))]^2 \\ &= E[[t - E(\phi(s))]^2] \\ &= \frac{1}{\lambda([t_{i-1}, t])} \int_{t_{i-1}}^t (t - E(\phi(s)))^2 ds \\ &\geq \frac{1}{\lambda([t_{i-1}, t])} \int_{t_{i-1}}^t (t - \phi_1(s))^2 ds \end{aligned}$$

The last inequality satisfied since for $t \in [t_{i-1}, t_{i+1}]$ and $\xi_i > \xi_{i+1} > t$,

$$\begin{aligned}
E(\phi(s)) &= \frac{1}{\lambda([t_{i-1}, t])} \int_{t_{i-1}}^t \phi(s) ds \\
&= \frac{\lambda([t_{i-1}, t_i])\xi_i + \lambda([t_i, t])\xi_{i+1}}{\lambda([t_{i-1}, t])} \\
&\geq \frac{\lambda([t_{i-1}, t_i])\xi_i + \lambda([t_i, t_{i+1}])\xi_{i+1}}{\lambda([t_{i-1}, t_{i+1}])} \\
&= \phi_1(s) > \xi_{i+1} > t
\end{aligned}$$

Therefore, $\int_{t_{i-1}}^t (t - \phi(s))^2 ds > \int_{t_{i-1}}^t (t - \phi_1(s))^2 ds$ holds.

Similarly, we can show that when $t_{i+1} < t < 1$, $\int_0^t (t - \phi_1(s))^2 ds \leq \int_0^t (t - \phi(s))^2 ds$.

So, for $t \in [0, 1]$, $\int_0^t (t - \phi_1(s))^2 ds < \int_0^t (t - \phi(s))^2 ds$. We repeat this construction finitely often, ending up with a nondecreasing step function ϕ_k satisfying $F_{\phi_k}(t) \leq F + \epsilon$ for $t \in [0, 1]$.

Thus, the result follows by lemma 3.2.1. □

LEMMA 3.2.4. *We have $F_g(1) = x$.*

Remark: [5]G. Larcher and G. Leobacher proved the special case of $x = \frac{1}{6\sqrt{3}}e^{\frac{-\pi}{2\sqrt{3}}}$ in Lemma 2.1 by induction. We give a much simpler proof as follows, which is also good for their case.

PROOF. Let us suppose that $F_g(1) < x$. Then there exists a $\tau_0 \in (0, 1)$, such that $F_g(t) < x$ on $[\tau_0, 1]$. For $a > 1$ and $\tau > \tau_0$, we define

$$g_1(s) := \begin{cases} \frac{1}{a}g(as) & \text{if } s \in [0, \frac{\tau}{a}], \\ 1 & \text{if } s \in (\frac{\tau}{a}, 1]. \end{cases}$$

For $t \in [0, \frac{\tau}{a}]$,

$$\begin{aligned}
F_{g_1}(t) &= \int_0^t (t - g_1(s))^2 ds \\
&= \frac{1}{a} \int_0^t (t - \frac{1}{a}g(as))^2 ads \\
&= \frac{1}{a} \int_0^{at} (t - \frac{1}{a}g(s))^2 ds \\
&= \frac{1}{a^3} \int_0^{at} (at - g(s))^2 ds \\
&= \frac{1}{a^3} F_g(at) \\
&< F_g(at)
\end{aligned}$$

For $t > \frac{\tau}{a}$,

$$\begin{aligned}
F_{g_1}(t) &= \int_0^t (t - g_1(s))^2 ds \\
&= \int_0^{\frac{\tau}{a}} (t - g_1(s))^2 ds + \int_{\frac{\tau}{a}}^t (t - g_1(s))^2 ds \\
&= \int_0^{\frac{\tau}{a}} (t - \frac{\tau}{a} + \frac{\tau}{a} - g_1(s))^2 ds + \int_{\frac{\tau}{a}}^t (t - g_1(s))^2 ds \\
&= \int_0^{\frac{\tau}{a}} (t - \frac{\tau}{a})^2 ds + 2(t - \frac{\tau}{a}) \int_0^{\frac{\tau}{a}} (\frac{\tau}{a} - g_1(s)) ds + \int_0^{\frac{\tau}{a}} (\frac{\tau}{a} - g_1(s))^2 ds + \int_{\frac{\tau}{a}}^t (t - g_1(s))^2 ds \\
&= (t - \frac{\tau}{a})^2 \frac{\tau}{a} + 2(t - \frac{\tau}{a}) \int_0^{\frac{\tau}{a}} (\frac{\tau}{a} - g_1(s)) ds + F_{g_1}(\frac{\tau}{a}) + \int_{\frac{\tau}{a}}^t (t - g_1(s))^2 ds \\
&< (1 - \frac{\tau}{a})^2 \frac{\tau}{a} + 4(1 - \frac{\tau}{a}) \frac{\tau}{a} + F_g(\tau) + 4(1 - \frac{\tau}{a})
\end{aligned}$$

Without lost of generality, suppose $\tau = 1 - \delta$ and $a = 1 + \delta$, we can get:

$$F_{g_1}(t) < \left(\frac{2\delta}{1+\delta} \right)^2 \frac{1-\delta}{1+\delta} + 4 \left(\frac{2\delta}{1+\delta} \right) \frac{1-\delta}{1+\delta} + F_g(\tau) + 4 \frac{2\delta}{1+\delta}$$

Let

$$\rho(\delta) = \left(\frac{2\delta}{1+\delta}\right)^2 \frac{1-\delta}{1+\delta} + 4 \left(\frac{2\delta}{1+\delta}\right) \frac{1-\delta}{1+\delta} + 4 \frac{2\delta}{1+\delta}$$

Since $\rho(\delta)$ is continuous on $[0, 1)$ and $\rho(0) = 0$, for any positive $\epsilon < x - F_g(\tau)$, there exists a $\delta_0 \in (0, 1)$ such that $\rho(\delta_0) < \epsilon$. Therefore, with such a δ_0 , we have $F_{g_1}(t) < F$ for $t \in [0, \frac{\tau}{a}]$ and $F_{g_1}(t) < x < F$ for $t \in [\frac{\tau}{a}, 1]$. Which implies that we can find some g_1 , such that $F_{g_1}(t) < F$ for all $t \in [0, 1]$. This contradicts the optimality of g and we have $F_g(1) = x$. \square

Theorem of Uniqueness For $0 \leq x \leq \frac{1}{6\sqrt{3}}e^{\frac{-\pi}{2\sqrt{3}}}$, the function g which satisfies $F = \max_{t \in [0,1]} \{F_g(t) : F_g(1) \leq x\}$ is unique (up to a set of measure 0)

PROOF. Let g_1 be another measurable function with

$$\max_{t \in [0,1]} F_{g_1}(t) = F.$$

For $t \in [0, 1]$ we have

$$\begin{aligned} F_{\frac{1}{2}g_1 + \frac{1}{2}g}(t) &= \int_0^t \left(\frac{1}{2}(t - g_1(s)) + \frac{1}{2}(t - g(s)) \right)^2 ds \\ &\leq \int_0^t \left(\frac{1}{2}(t - g_1(s))^2 + \frac{1}{2}(t - g(s))^2 \right) ds \\ &= \frac{1}{2}F_{g_1}(t) + \frac{1}{2}F_g(t) \leq F \end{aligned}$$

with the first equality if and only if $g_1(s) = g(s)$ for a.e. $s \in [0, t]$. Since $F_{\frac{1}{2}g_1 + \frac{1}{2}g}(t) \leq F$ for all $t \in [0, 1]$ we have $F_{\frac{1}{2}g_1 + \frac{1}{2}g}(1) = x$ by Lemma 3.2.4, and therefore the first equality holds in the above inequality for $t = 1$, which means that g_1 equals g , a.e. \square

3.3. Properties of g and F

We first introduce some notation: Let $g : [0, 1] \rightarrow [0, 1]$ denote a nondecreasing function that solves our minimization problem $\max_{t \in [0,1]} \{F_g(t) : F_g(1) \leq x\} = F$. Let H

be the set where F_g attains its maximum; that is, $H := \{t \in [0, 1] : F_g(t) = F\}$. Since F_g is continuous, H is a closed subset of $[0, 1]$.

In this section, we will note some basic properties of H and g . Since most of those properties are independent of x , the approaches to get these properties are similar to what G. Larcher and G. Leobacher provide in their special case. Specifically, Lemma 3.3.1, Lemma 3.3.2, Lemma 3.3.3, Lemma 3.3.4, Lemma 3.3.6 and Lemma 3.3.7 are similar results as G. Larcher and G. Leobacher discussed in [5].

LEMMA 3.3.1. *g is constant on the connected components of $[0, 1] \setminus H$.*

PROOF. Suppose on the contrary that g is not constant on the interval $(\tau, \tau + \delta)$, where τ is in $[0, 1] \setminus H$. Since $F_g(\tau) < F$, we can choose $\delta > 0$ small enough to ensure that $F_g(t) < F$ for all $t \in [\tau, \tau + \delta]$. We define a function g_1 by

$$g_1(s) := \begin{cases} g(s) & \text{if } s \notin (\tau, \tau + \delta), \\ \frac{1}{\delta} \int_{\tau}^{\tau+\delta} g(\eta) d\eta & \text{if } s \in (\tau, \tau + \delta). \end{cases}$$

When $t \geq \tau + \delta$, we have

$$\begin{aligned} F_{g_1}(t) &= \int_0^t (t - g_1(s))^2 ds \\ &= \int_0^{\tau} (t - g(s))^2 ds \\ &\quad + \int_{\tau}^{\tau+\delta} \left(t - \frac{1}{\delta} \int_{\tau}^{\tau+\delta} g(\eta) ds(\eta) \right)^2 ds \\ &\quad + \int_{\tau+\delta}^t (t - g(s))^2 ds \\ &< \int_0^t (t - g(s))^2 ds = F_g(t), \end{aligned}$$

Since

$$\begin{aligned} \int_{\tau}^{\tau+\delta} \left(t - \frac{1}{\delta} \int_{\tau}^{\tau+\delta} g(\eta) d\eta \right)^2 ds &= \frac{1}{\delta} \left(\int_{\tau}^{\tau+\delta} (t - g(\eta)) d\eta \right)^2 \\ &< \int_{\tau}^{\tau+\delta} (t - g(\eta))^2 d\eta \end{aligned}$$

by the Cauchy-Schwarz inequality and our assumption that g is not a constant function on $(\tau, \tau + \delta)$. This provides the strictness of the inequality.

When $\tau \leq t < \tau + \delta$, for any given ϵ , we can choose τ small enough, such that $|F_{g_1}(t) - F_g(t)| < \epsilon < F - F_g(t)$, then $F_{g_1}(t) < F$. Thus we have found a function g_1 such that $F_{g_1}(\cdot) \leq F$ and $F_{g_1}(1) < x$, contradicting Lemma 3.2.4. Therefore, g is constant on the connected components of $[0, 1] \setminus H$. \square

LEMMA 3.3.2. *On $(0, 1)$ the left and right derivatives of F_g exist. Moreover, if $g_{\pm}(t) := \lim_{h \rightarrow \pm 0} g(t + h)$ denotes the left and right limits of g at t , respectively, we have*

$$(F_g)'_{\pm}(t) = (t - g_{\pm}(t))^2 + 2 \int_0^t (t - g(s)) ds \quad (3.3.1)$$

The proof is straightforward.

LEMMA 3.3.3. *g is continuous on $(0, 1)$ and F_g is continuously differentiable on $(0, 1)$.*

PROOF. If t is in the complement of H , then g is continuous at t by Lemma 3.3.1. Now let $t \in H \setminus 1$. From Lemma 3.3.2 and the fact that F_g has a maximum in t we conclude that

$$(F_g)'_{+}(t) = (t - g_{+}(t))^2 + 2 \int_0^t (t - g(s)) ds \leq 0 \quad (3.3.2)$$

and

$$(F_g)'_{-}(t) = (t - g_{-}(t))^2 + 2 \int_0^t (t - g(s)) ds \geq 0 \quad (3.3.3)$$

Therefore, we have $(t - g_-)^2 \geq (t - g_+)^2$, i.e. $(t - g_-)^2 \geq (t - g_+)^2$. On the other hand, $0 < t \leq g_- \leq g_+$ so that $(t - g_-)^2 \leq (t - g_+)^2$. Putting the above together we get $(t - g_-)^2 = (t - g_+)^2$, and so $g_- = g_+$. \square

From now on, we let $t_0 := \min H$ and $t_1 := \max H$ (H is a compact subset of $(0,1)$).

LEMMA 3.3.4. $g(t) = 3t_0$ for $t \in (0, t_0]$.

PROOF. From Lemmas 3.3.1, 3.3.2 and 3.3.3, and the fact that t_0 is a maximum of F_g , we have

$$\begin{aligned} 0 = (F_g)'(t_0) &= (t_0 - g(t_0))^2 + 2 \int_0^{t_0} (t_0 - g(s)) ds \\ &= (t_0 - g(t_0))^2 + 2t_0(t_0 - g(t_0)) \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 \neq F = F_g(t_0) &= \int_0^{t_0} (t_0 - g(t_0))^2 ds \\ &= t_0(t_0 - g(t_0))^2 \end{aligned}$$

so that $t_0 - g(t_0) \neq 0$ and therefore $0 = (t_0 - g(t_0)) + 2t_0$, which implies the result of this lemma. \square

LEMMA 3.3.5. $g(t) = 1$ for all $t \in [t_1, 1]$, where $(t_1 - 1)^2(2t_1 - 1) + F - x = 0$.

Remark: The case of when $x = F$ has been discussed in Lemma 2.6 in [5].

PROOF. From Lemma 3.3.2 and the fact that $g(1) = 1$ we have for the left derivative of F_g in 1.

$$\begin{aligned} (F_g)'_-(1) &= (1 - g(1))^2 + 2 \int_0^1 (1 - g(s)) ds \\ &= 2 \int_0^1 (1 - g(s)) ds > 0 \end{aligned}$$

Hence there exists a $t_1 \in (t_0, 1) \cap H$ such that the interval $(t_1, 1)$ is in the complement of H . Therefore g is constant on $[t_1, 1]$ and, since $g(1) = 1$, $g(s) = 1$ for all $s \in [t_1, 1]$.

From $F_g(1) = x$ and $F_g(t_1) = F$, we can get:

$$\begin{aligned} \left(\frac{x}{F}\right) \int_0^{t_1} (t_1 - g(s))^2 ds &= \int_0^1 (1 - g(s))^2 ds \\ &= \int_0^{t_1} (1 - g(s))^2 ds \\ &= (1 - t_1)^2 t_1 + 2(1 - t_1) \int_0^{t_1} (t_1 - g(s)) ds + \int_0^{t_1} (t_1 - g(s))^2 ds \end{aligned}$$

and therefore

$$(1 - t_1)^2 t_1 + 2(1 - t_1) \int_0^{t_1} (t_1 - g(s)) ds + F - x = 0.$$

On the other hand,

$$0 = (F_g)'(t_1) = (t_1 - g(t_1))^2 + 2 \int_0^{t_1} (t_1 - g(s)) ds$$

so that

$$(t_1 - 1)^2(2t_1 - 1) + F - x = 0.$$

This proves our claim. □

Since $F - x > 0$, there is only one real solution for the equation

$$(t_1 - 1)^2(2t_1 - 1) + F - x = 0.$$

LEMMA 3.3.6. F_g is constant on $[t_0, t_1]$.

PROOF. Suppose this were not the case. Then $[t_0, t_1]$ contains a connected component of the complement of H . (i.e. some open interval (τ_1, τ_2) with $\tau_1, \tau_2 \in H$ and $(\tau_1, \tau_2) \in [0, 1] \setminus H$). From lemma , we know that g is constant on (τ_1, τ_2) . Since for $t \in (\tau_1, \tau_2)$, $F_g(t) = \int_0^{t_0} (t - g(s))^2 ds + \int_{t_0}^{\tau_1} (t - g(s))^2 ds + \int_{\tau_1}^t (t - g(s))^2 ds$, $g(s)$ is constant

on $[\tau_1, \tau_2]$, it is easy to see that F_g coincides with a polynomial p on $[\tau_1, \tau_2]$, where p has degree 3 and satisfies the conditions $p(\tau_1) = p(\tau_2) = F$ and $p'(\tau_1) = p'(\tau_2) = 0$. That means there exist another $\tau_3 \in (\tau_1, \tau_2)$ and $p'(\tau_3) = 0$. However, p' is at most a polynomial of degree 2. Therefore, p' is constant. That implies F_g is constant on (τ_1, τ_2) , which contradicts our assumption. \square

LEMMA 3.3.7. g obeys the following ordinary differential equation on (t_0, t_1) :

$$g'(t) = \frac{3t - 2g}{t - g} \tag{3.3.4}$$

PROOF. Observe that $F_g(t) = F$ for all $t \in (t_0, t_1)$, such that F'_g and F''_g both vanish on (t_0, t_1) . Thus we get,

$$0 = F'_g(t) = (t - g(t))^2 + 2 \int_0^t (t - g(s)) ds$$

for all $t \in I$. Further, we have

$$0 = F''_g(t) = 2(t - g(t))(1 - g'(t)) + 2(2t - g(t))$$

for all $t \in (t_0, t_1)$. Eliminating the term involving the integral from these two equations and rearranging gives the ODE. \square

3.4. Proof of the Main Theorem II

From Lemma 3.3.4 and Lemma 3.3.5, we know that $g(t) = 3t_0$ for $t \in [0, 3t_0]$ and $g(t) = 1$ for $t \in [t_1, 1]$. We now investigate the function g on the interval $[t_0, t_1]$. From Lemma 3.3.7, we know the following ordinary differential equation holds for $t \in (t_0, t_1)$.

$$g'(t) = \frac{3t - 2g}{t - g}$$

which we write as

$$\frac{\frac{dg}{d\eta}}{\frac{dt}{d\eta}} = \frac{3t - 2g}{t - g}$$

with independent parameter η .

$$\frac{dg}{d\eta} = 3t - 2g, g(\eta = 0) = 1$$

$$\frac{dt}{d\eta} = t - g, t(\eta = 0) = t_1$$

The solution to this system is

$$g(\eta) = \sqrt{3}Be^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \phi\right)$$

and

$$t(\eta) = Be^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \phi + \frac{\pi}{6}\right)$$

where B and ϕ are some constant.

PROOF. Clearly,

$$\begin{aligned} \frac{d^2g}{d\eta^2} &= 3\frac{dt}{d\eta} - 2\frac{dg}{d\eta} \\ &= 3(t - g) - 2(3t - 2g) \\ &= -3t + 2g - g \\ &= -\frac{dg}{d\eta} - g \end{aligned}$$

So,

$$\frac{d^2g}{d\eta^2} + \frac{dg}{d\eta} + g = 0$$

Similarly,

$$\frac{d^2t}{d\eta^2} + \frac{dt}{d\eta} + t = 0$$

The characteristic equation is $r^2 + r + 1 = 0$. Suppose

$$g(\eta) = Ae^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \phi\right)$$

$$t(\eta) = Be^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \psi\right)$$

We can get:

$$\begin{aligned} \frac{dg}{d\eta} &= -\frac{1}{2}Ae^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \phi\right) - \frac{\sqrt{3}}{2}Ae^{-\frac{1}{2}\eta} \sin\left(\frac{\sqrt{3}}{2}\eta + \phi\right) \\ &= Be^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \psi\right) - 2Ae^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \phi\right) \end{aligned}$$

Then

$$\frac{\sqrt{3}}{2}A \cos\left(\frac{\sqrt{3}}{2}\eta + \phi\right) - \frac{1}{2}A \cos\left(\frac{\sqrt{3}}{2}\eta + \phi\right) = \sqrt{3}B \cos\left(\frac{\sqrt{3}}{2}\eta + \psi\right)$$

Therefore,

$$A \cos\left(\frac{\sqrt{3}}{2}\eta + \phi + \frac{\pi}{6}\right) = \sqrt{3}B \cos\left(\frac{\sqrt{3}}{2}\eta + \psi\right)$$

which requires that

$$A = \sqrt{3}B$$

and

$$\phi + \frac{\pi}{6} = \psi$$

The solution follows. □

How do we determine B and ϕ ? Let $g(\eta = 0) = 1$ and $t(\eta = 0) = t_1$, then $A \cos(\phi) = 1$ and $B \cos(\psi) = t_1$. i.e.

$$\sqrt{3}B \cos(\phi) = 1$$

and

$$B \cos\left(\phi + \frac{\pi}{6}\right) = t_1$$

From this equation system, we can get

$$\frac{1}{2} - \frac{1}{2}B \sin \phi = t_1$$

and

$$B^2 = \frac{1}{3} + (1 - 2t_1)^2$$

Moreover, let $g(\eta_0) = 3t(\eta_0)$. We can get

$$\frac{\sqrt{3}}{2}\eta_0 + \phi = \frac{\pi}{6}$$

We summarize what we found here:

$$g(s) := \begin{cases} 3t_0 & \text{for } s \in [0, t_0], \\ \sqrt{3}Be^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}\eta}{2} + \phi) & \text{for } s = Be^{-\frac{1}{2}\eta} \cos(\frac{\sqrt{3}}{2}\eta + \phi + \frac{\pi}{6}) \text{ with } \eta \in [0, \eta_0], \\ 1 & \text{if } s \in [t_1, 1]. \end{cases}$$

Here, B , ϕ , η_0 and t_0 can be viewed as function of t_1 .

Let $u = 1 - 2t_1$, we have the following equations:

$$\frac{1}{4}(1 + u)^2 u = F - x, \tag{3.4.1}$$

$$B = \sqrt{\frac{1}{3} + u^2},$$

$$\phi = \tan^{-1}(\sqrt{3}u),$$

$$\frac{\sqrt{3}}{2}\eta_0 + \phi = \frac{\pi}{6},$$

and

$$t_0 = \frac{1}{2}Be^{-\frac{1}{2}\eta_0}.$$

Then

$$t_0 = \frac{1}{2}\sqrt{\frac{1}{3} + u^2}e^{-\frac{1}{\sqrt{3}}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u))}. \tag{3.4.2}$$

Recall the proof process of Lemma 3.2.4, we know that when $x \in (0, \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}})$, F is decreasing respect to x , and $F - x \in (0, \frac{4}{27})$. From equation (3.4.1), we know that $u \in (0, \frac{1}{3})$.

Since $F = 4t_0^3$, then

$$F = \frac{1}{2} \left(\frac{1}{3} + u^2 \right)^{\frac{3}{2}} e^{-\sqrt{3}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u))} \quad (3.4.3)$$

Define

$$f(u) = \frac{1}{2} \left(\frac{1}{3} + u^2 \right)^{\frac{3}{2}} e^{-\sqrt{3}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u))} - \frac{1}{4}(1+u)^2 u \quad (3.4.4)$$

So, we have $f(0) = \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}}$ and $f(\frac{1}{3}) = 0$.

When $x \in (0, \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}})$, we have $f(\frac{1}{3}) < x < f(0)$. Since $f(u)$ is continuous, it means that there exists at least one solution on $(0, \frac{1}{3})$ for the equation $f(u) = x$.

From the definition of $f(u)$, we obtain

$$f'(u) = \frac{1}{2}(1+3u)\sqrt{\frac{1}{3}+u^2}e^{-\sqrt{3}(\frac{\pi}{6}-\tan^{-1}(\sqrt{3}u))} - \frac{1}{4}(1+u)(1+3u)$$

and

$$f''(u) = (3u^2 + 2u + 1)\left(\frac{1}{3} + u^2\right)^{-\frac{1}{2}}e^{-\sqrt{3}(\frac{\pi}{6}-\tan^{-1}(\sqrt{3}u))} - \frac{3}{2}u - 1$$

We claim that $f'(u) < 0$ on $(0, \frac{1}{3})$. The proof is as following:

Since $(1+3u) > 0$, it is sufficient to show that

$$\frac{1}{2}\sqrt{\frac{1}{3}+u^2}e^{-\sqrt{3}(\frac{\pi}{6}-\tan^{-1}(\sqrt{3}u))} - \frac{1}{4}(1+u) < 0 \quad (3.4.5)$$

Let $\tan^{-1}(\sqrt{3}u) = \theta$, then $\theta \in (0, \frac{\pi}{6})$ and $u = \frac{1}{\sqrt{3}}\tan\theta$. Inequality (3.4.5) comes to

$$\frac{2}{\sqrt{3}}\sec\theta e^{-\sqrt{3}(\frac{\pi}{6}-\theta)} - \left(1 + \frac{1}{\sqrt{3}}\tan\theta\right) < 0 \quad (3.4.6)$$

Since $\frac{2}{\sqrt{3}} \sec \theta e^{-\sqrt{3}(\frac{\pi}{6}-\theta)} > 0$, divide the both sides of inequality (3.4.6) by $\frac{2}{\sqrt{3}} \sec \theta e^{-\sqrt{3}(\frac{\pi}{6}-\theta)}$, it is sufficient to show that

$$1 - \cos(\frac{\pi}{6} - \theta)e^{\sqrt{3}(\frac{\pi}{6}-\theta)} < 0 \quad (3.4.7)$$

Let $I(\theta) = 1 - \cos(\frac{\pi}{6} - \theta)e^{\sqrt{3}(\frac{\pi}{6}-\theta)}$. Then $I'(\theta) = -2 \cos(\frac{\pi}{6} - \theta)e^{\sqrt{3}(\frac{\pi}{6}-\theta)}$. Clearly, $I'(\theta) < 0$ on $(0, \frac{\pi}{6})$. So, $I(\theta)$ is decreasing on $(0, \frac{\pi}{6})$. With $I(\frac{\pi}{6}) = 0$, we know that the inequality (3.4.7) holds. Therefore, our claim $f'(u) < 0$ on $(0, \frac{1}{3})$ is correct. It implies that the solution for $f(u) = x$ on $(0, \frac{1}{3})$ is unique.

Since $f''(0) = \sqrt{3}e^{-\frac{\sqrt{3}\pi}{6}} - 1 < 0$ and $f''(\frac{1}{3}) = \frac{3}{2} > 0$, there exists at least one solution for $f''(u) = 0$, i.e. inflection point, on $(0, \frac{1}{3})$.

Note that

$$f^{(3)}(u) = (3u^3 + 3u^2 + 3u + \frac{5}{3})(\frac{1}{3} + u^2)^{-\frac{3}{2}}e^{-\sqrt{3}(\frac{\pi}{6}-\tan^{-1}(\sqrt{3}u))} - \frac{3}{2}$$

and

$$\begin{aligned} f^{(4)}(u) &= (1 - \frac{3}{2}u)(\frac{1}{3} + u^2)^{-\frac{5}{2}}(3u^3 + 3u^2 + 3u + \frac{5}{3})e^{-\sqrt{3}(\frac{\pi}{6}-\tan^{-1}(\sqrt{3}u))} \\ &+ (9u^2 + 6u + 3)(\frac{1}{3} + u^2)^{-\frac{3}{2}}e^{-\sqrt{3}(\frac{\pi}{6}-\tan^{-1}(\sqrt{3}u))} \end{aligned}$$

Clearly, when $u \in (0, \frac{1}{3})$, $f^{(4)}(u) > 0$. With $f^{(3)}(0) = 5\sqrt{3}e^{-\frac{\sqrt{3}}{6}\pi}$, we know that $f^{(3)}(u) > 0$ on $(0, \frac{1}{3})$. So, there is only one solution for $f''(u) = 0$ on $(0, \frac{1}{3})$.

Now, we are going to numerically solve for $u(x) \in (0, \frac{1}{3})$ as a solution of $f(u) = x$ with a given $x \in (0, \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}})$. Following is the algorithm of using Newton's method to solve it.

Let u_0 be the unique inflection point of $f(u)$, and

$$u_{n+1} = u_n - \frac{f(u_n) - x}{f'(u_n)}. \quad (3.4.8)$$

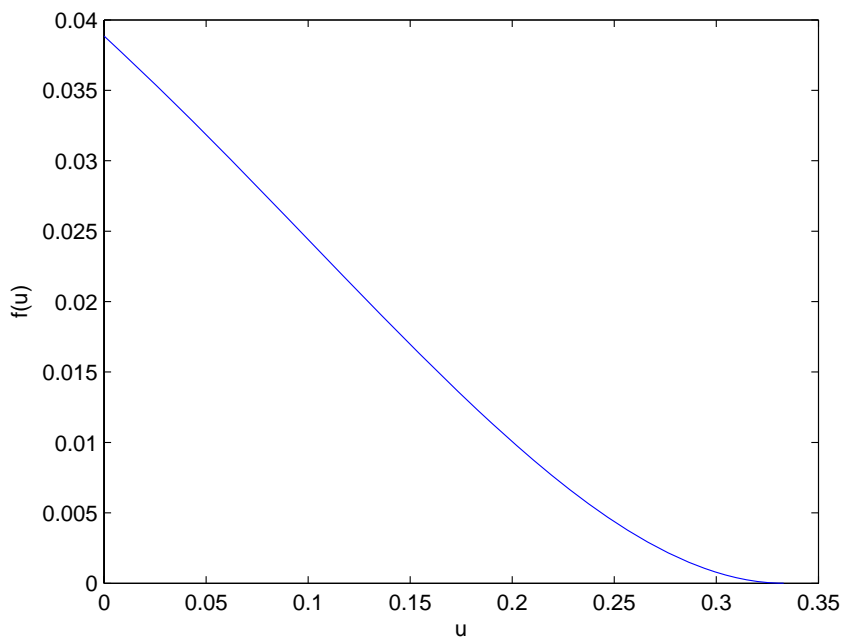


FIGURE 3.1. The Graph of $f(u)$

Suppose $u^* \in (0, \frac{1}{3})$ with $f(u^*) = x$. We will show that $\lim_{n \rightarrow \infty} u_n = u^*$. Let $\psi(u) = u - \frac{f(u)-x}{f'(u)}$. Then $u_{n+1} = \psi(u_n)$ and $\psi(u^*) = u^*$.

If $u^* < u_0$,

$$\begin{aligned} u_1 - u^* &= \psi(u_0) - \psi(u^*) \\ &= \psi'(\xi)(u_0 - u^*) \\ &= \frac{f''(\xi)f(\xi) - x}{[f'(\xi)]^2}(u_0 - u^*), \end{aligned}$$

where $u^* < \xi < u_0$. Since f is decreasing on $(0, \frac{1}{3})$ and f'' is increasing, we have $f(\xi) - x = f(\xi) - f(u^*) < 0$ and $f''(\xi) < f''(u^*) = 0$. So, with $u_0 > u^*$, we have $u_1 > u^*$. Inductively, we have $u_n > u^*$ for $n \geq 1$.

On the other hand, since $f'(u_n) < 0$ and $f(u_n) < x$, $u_{n+1} = u_n - \frac{f(u_n)-x}{f'(u_n)} < u_n$. Therefore, $\{u_n\}$ is a decreasing sequence with low bound and $\lim_{n \rightarrow \infty} u_n$ exists. Suppose

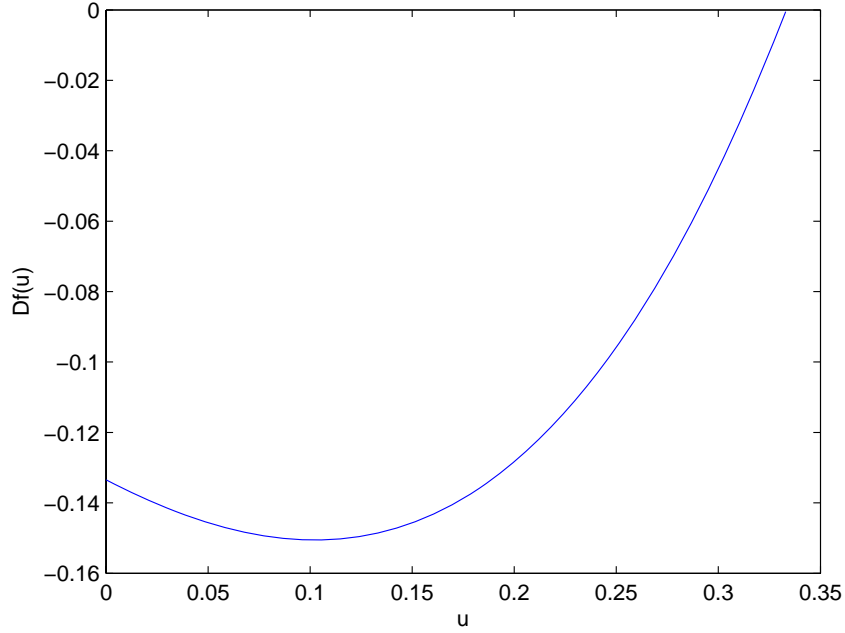


FIGURE 3.2. The Graph of $f'(u)$

$\lim_{n \rightarrow \infty} u_n = a$, from (4.1.8), we have $a = a - \frac{f(a)}{f'(a)}$. It implies that $f(a) = 0$, which means $a = u^*$.

If $u^* > u_0$, with similar argument, we can still get $\lim_{n \rightarrow \infty} u_n = u^*$.

Therefore, we proved the convergence of Newton's method for solving the equation $f(u) = x$.

Thus, for a given $x \in (0, \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}})$, we can numerically find $u(x)$ such that $f(u(x)) = 0$.

Once get the value of $u(x)$, we know $t_1(x) = \frac{1-u(x)}{2}$, $B(x) = \sqrt{\frac{1}{3} + u(x)^2}$, $\phi = \tan^{-1}(\sqrt{3}u(x))$, $\eta_0(x) = \frac{2}{\sqrt{3}}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u(x)))$ and $t_0(x) = \frac{1}{2}\sqrt{\frac{1}{3} + u(x)^2}e^{-\frac{1}{\sqrt{3}}(\frac{\pi}{6} - \tan^{-1}(\sqrt{3}u(x)))}$.

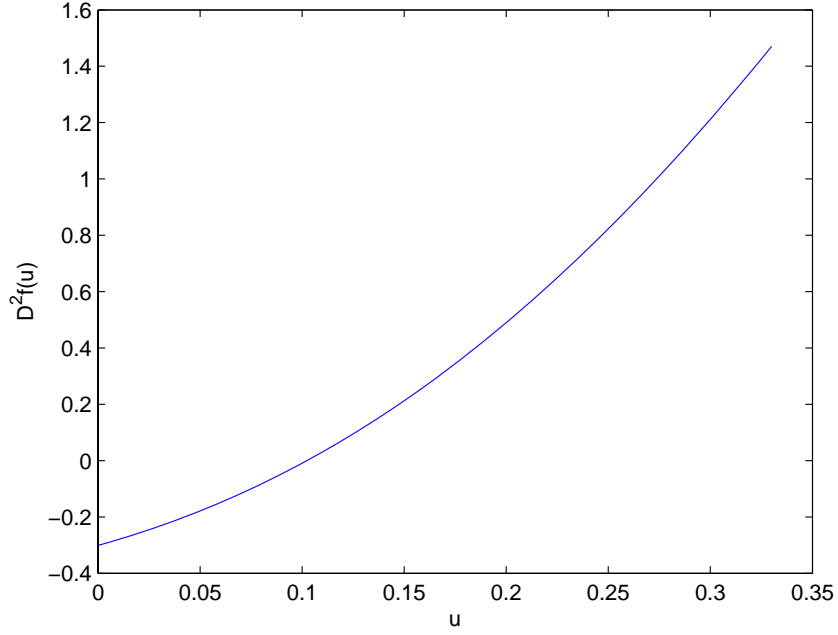


FIGURE 3.3. The Graph of $f''(u)$

Therefore, for a given $x \in (0, \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}})$, we can completely get the numerical solution of $g(s, x)$, which is

$$g(s, x) := \begin{cases} 3t_0(x) & \text{for } s \in [0, t_0(x)], \\ \sqrt{3}B(x)e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}\eta}{2} + \phi(x)) & \text{for } s = B(x)e^{-\frac{1}{2}\eta} \cos(\frac{\sqrt{3}}{2}\eta + \phi(x) + \frac{\pi}{6}), \\ & \text{with } \eta \in [0, \eta_0(x)]. \\ 1 & \text{if } s \in [t_1(x), 1]. \end{cases}$$

The corresponding maximum running risk is

$$F = \frac{1}{4}(1 + u(x))^2u(x) + x.$$

This is just the result of the Main Theorem II.

CHAPTER 4

A Dynamic solution for Best Hedging Strategies

In this chapter, by providing some examples to the solution of g and F , we will obtain a general dynamic solution for our hedging problem stated in Chapter 3. Specially, we will give some analytic solutions in certain cases. From these analytic solutions, we will see that the rolling stack hedge and optimal hedge by [5]G. Larcher and G. Leobacher are two special cases of our general solution. At last, we will discuss some open questions for future research.

4.1. Solutions for different x

We provide several examples with analytic solutions first and then generate a dynamic solution for our hedging problem.

Example 4.1. $x = 0$, i.e. $F = \max_{t \in [0,1]} F_g(t)$, $F_g(1) \leq 0$. We can get $u = t_0 = t_1 = \frac{1}{3}$ and $g(s) = 1$, which is *rolling stack hedging*.

Figure 4.1 displays the spot risk of Rolling stack hedging, just as we discussed in Chapter 2.

Example 4.2. $x = \frac{4}{(6+3\sqrt{3})^{\frac{3}{2}}} e^{-\frac{\sqrt{3}}{12}\pi} - \frac{\sqrt{3}-1}{3(3+\sqrt{3})}$, We can get $u = \frac{\sqrt{3}-1}{3+\sqrt{3}}$, $t_0 = \frac{\sqrt{2}}{3+\sqrt{3}} e^{-\frac{\sqrt{3}}{12}\pi}$, $t_1 = \frac{2}{3+\sqrt{3}}$, $B = \frac{2\sqrt{2}}{3+\sqrt{3}}$, $\phi = \frac{\pi}{12}$, and $\eta_0 = \frac{\pi}{6\sqrt{3}}$.

Therefore,

$$g(s) := \begin{cases} 3t_0 & \text{for } s \in [0, t_0], \\ \frac{2\sqrt{6}}{3+\sqrt{3}} e^{-\frac{\eta}{2}} \cos\left(\frac{\sqrt{3}\eta}{2} + \frac{\pi}{12}\right) & \text{for } s = \frac{2\sqrt{2}}{3+\sqrt{3}} e^{-\frac{1}{2}\eta} \cos\left(\frac{\sqrt{3}}{2}\eta + \frac{\pi}{4}\right) \text{ with } \eta \in [0, \frac{\pi}{6\sqrt{3}}], \\ 1 & \text{if } s \in [\frac{2}{3+\sqrt{3}}, 1]. \end{cases}$$

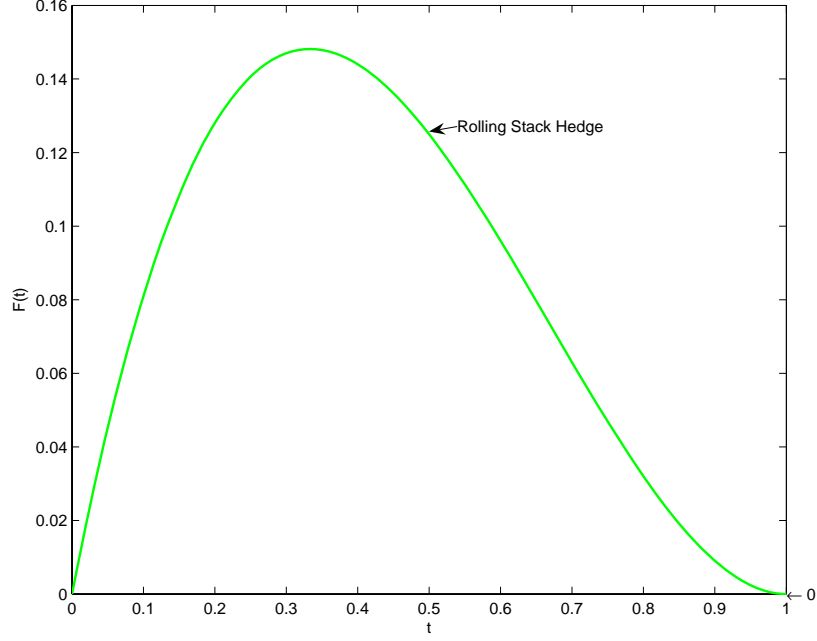


FIGURE 4.1. Spot risk of Rolling stack hedging, when $x = 0$.

where $t_0 = \frac{\sqrt{2}}{3+\sqrt{3}}e^{-\frac{\sqrt{3}}{12}\pi}$.

For the corresponding maximal risk we have $F = \frac{4}{(6+3\sqrt{3})^{\frac{3}{2}}}e^{-\frac{\sqrt{3}}{12}\pi} = 0.0678462\dots$

Figure 4.2 and 4.3 respectively display the graphs of optimal hedging strategy and the corresponding spot risk when $x = \frac{4}{(6+3\sqrt{3})^{\frac{3}{2}}}e^{-\frac{\sqrt{3}}{12}\pi} - \frac{\sqrt{3}-1}{3(3+\sqrt{3})}$.

Example 4.3. $x = \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}}$, then $u = 0$, and $t_1 = \frac{1}{2}$, $t_0 = \frac{1}{2\sqrt{3}}e^{-\frac{\pi}{6\sqrt{3}}}$, $\eta_0 = \frac{\pi}{3\sqrt{3}}$,
 $B = \frac{1}{\sqrt{3}}$, $\phi = 0$

The solution for $g(s)$ is:

$$g(s) := \begin{cases} 3t_0 & \text{for } s \in [0, \frac{1}{2\sqrt{3}}e^{-\frac{\pi}{6\sqrt{3}}}], \\ e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}\eta}{2}) & \text{for } s = \frac{1}{\sqrt{3}}e^{-\frac{1}{2}\eta} \cos(\frac{\sqrt{3}}{2}\eta + \frac{\pi}{6}) \text{ with } \eta \in [0, \frac{\pi}{3\sqrt{3}}], \\ 1 & \text{if } s \in [1/2, 1]. \end{cases}$$

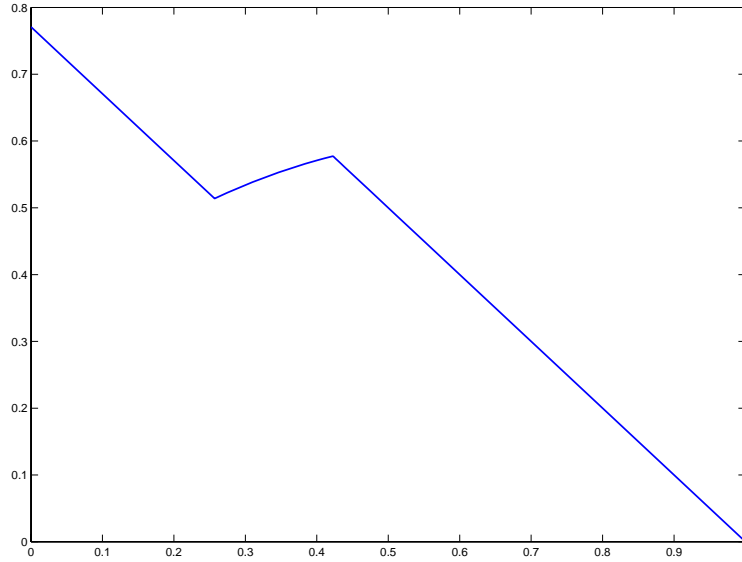


FIGURE 4.2. Strategy function $G(s) = g(s) - s$, when $x = \frac{4}{(6+3\sqrt{3})^{\frac{3}{2}}} e^{-\frac{\sqrt{3}}{12}\pi} - \frac{\sqrt{3}-1}{3(3+\sqrt{3})}$

For the corresponding maximal risk we have, $F = 4t_0^3 = \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}} = 0.0388532\dots$ This is just the result obtained by G.Larcher and G.Leobacher in [5]. Figure 4.4 and Figure 4.5 give the corresponding graphs, just as showed in chapter 2.

Actually, for given $x \in [0, \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}}]$, using the method provided in Chapter 3, we can numerically solve for g and $F_g(t)$. Graph them as a loop in Matlab, we will get the following graphs (Figure 4.6 to Figure 4.11):

Figure 4.6 and Figure 4.7 display the graphs of optimal strategies and corresponding spot risk function for different x .

Using Mesh plotter in Matlab, we also provide the 3 - D plots in Figure 4.8, Figure 4.9 and Figure 4.10, which can be viewed as graphs of a dynamic solution for the optimization problem stated in Chapter 3.

Figure 4.11 displays the graph of the maximal running risk F of the optimal hedge respect to the constraint value x . As we discussed in Section 4.1, F can be

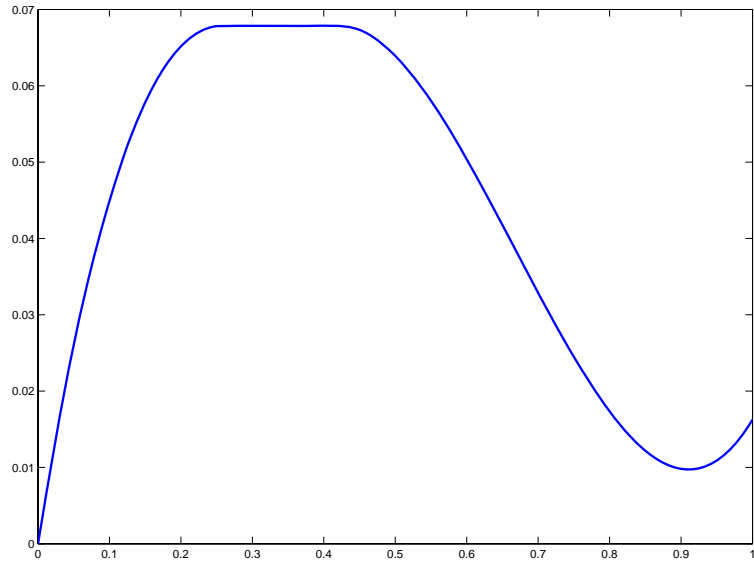


FIGURE 4.3. Spot risk function $F(t)$, when $x = \frac{4}{(6+3\sqrt{3})^{\frac{3}{2}}} e^{-\frac{\sqrt{3}}{12}\pi} - \frac{\sqrt{3}-1}{3(3+\sqrt{3})}$

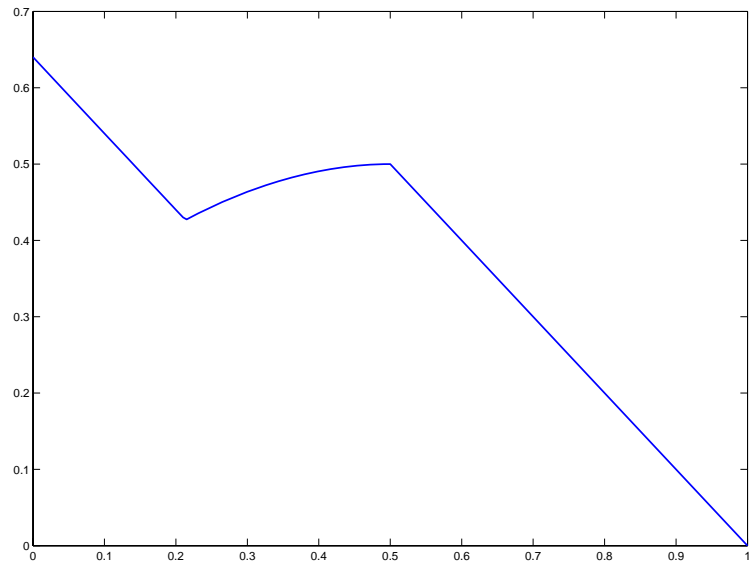


FIGURE 4.4. Optimal Strategy function $G(s) = g(s) - s$, when $x = \frac{1}{6\sqrt{3}} e^{-\frac{\pi}{2\sqrt{3}}}$.

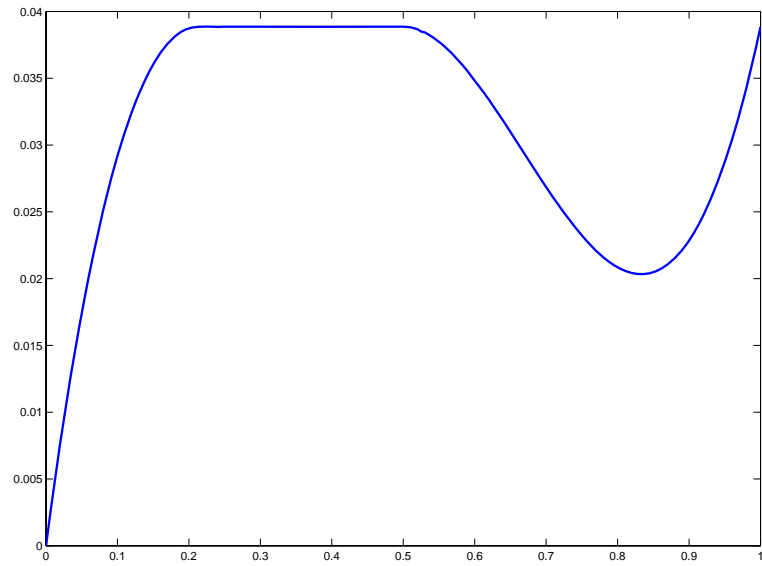


FIGURE 4.5. Spot risk function $F(t)$, when $x = \frac{1}{6\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}}$.

viewed as a decreasing function of x . From this graph, we can see that F decreasing sharply when x is close to 0.

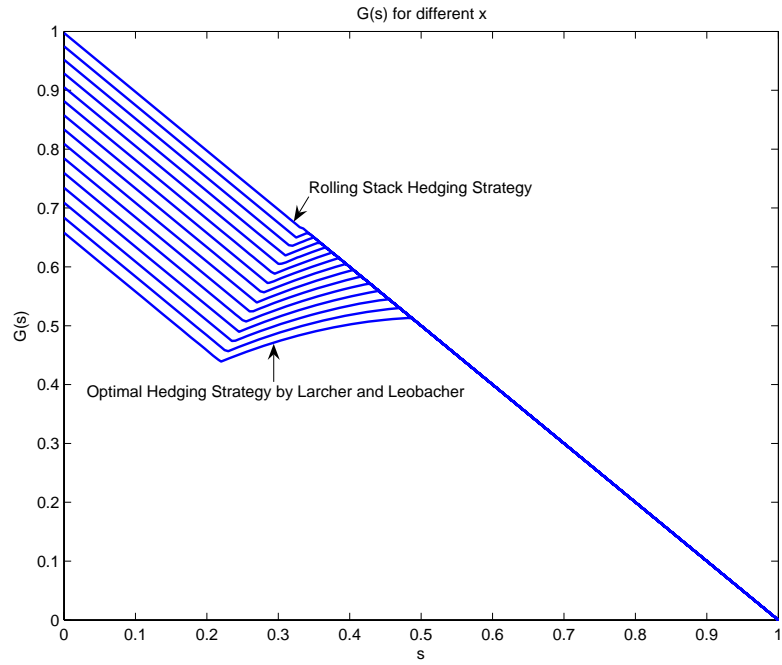


FIGURE 4.6. Graphs of Optimal Strategies for different x

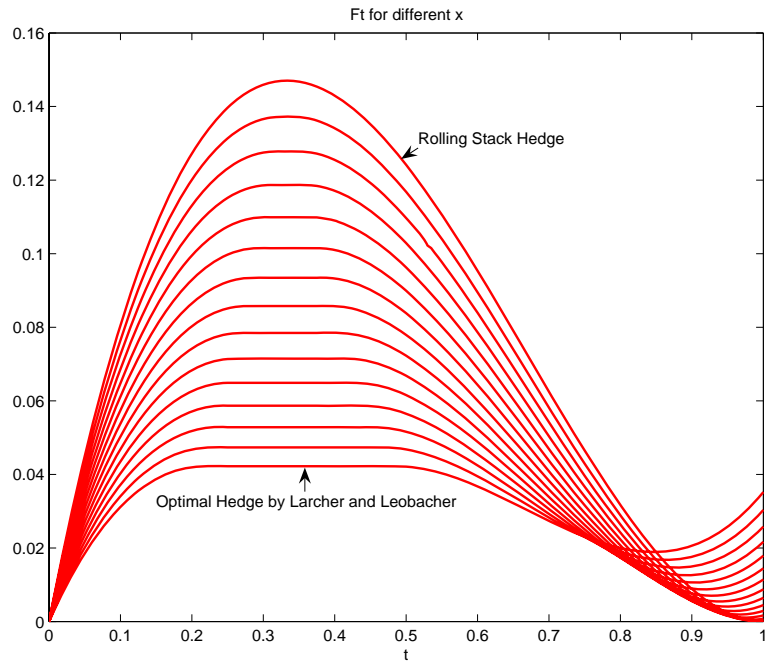


FIGURE 4.7. Graphs of spot risks corresponding to those Optimal Strategies

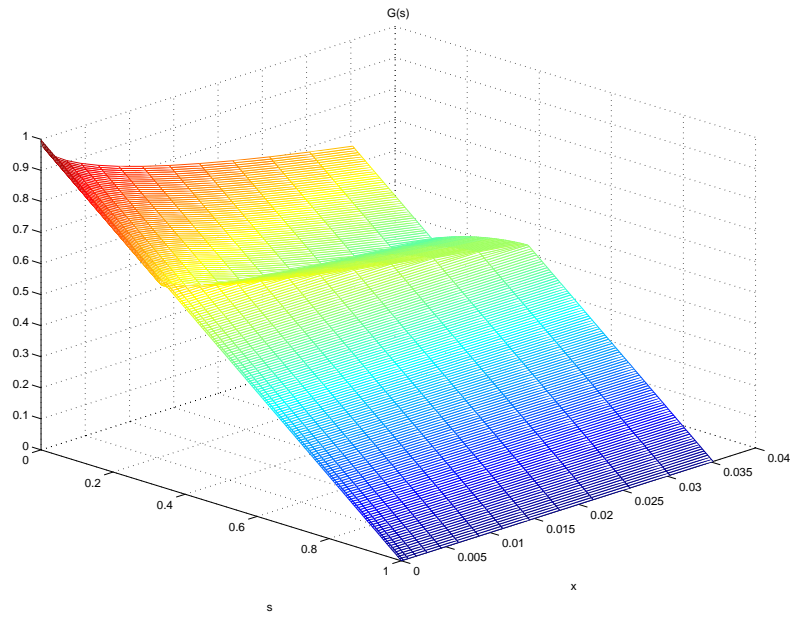


FIGURE 4.8. The 3-D Graph of $G(x, s)$

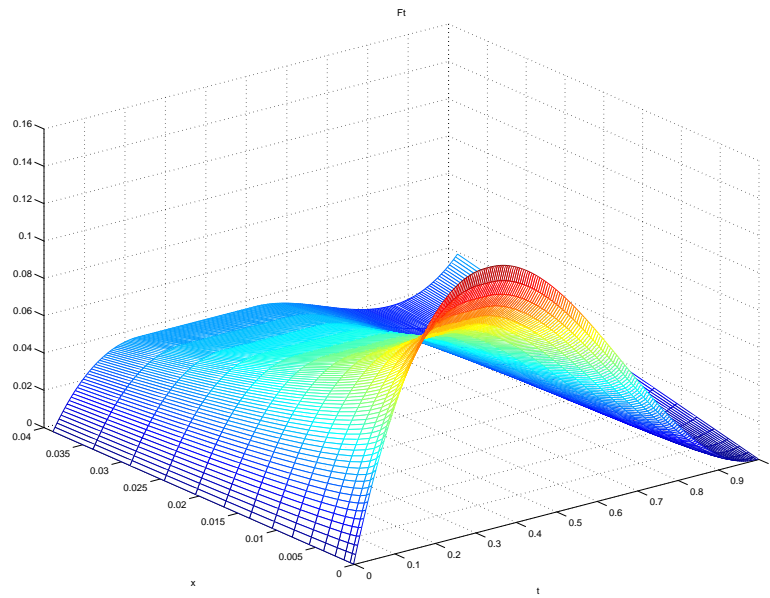


FIGURE 4.9. The 3-D Graph of $F(x, t)$

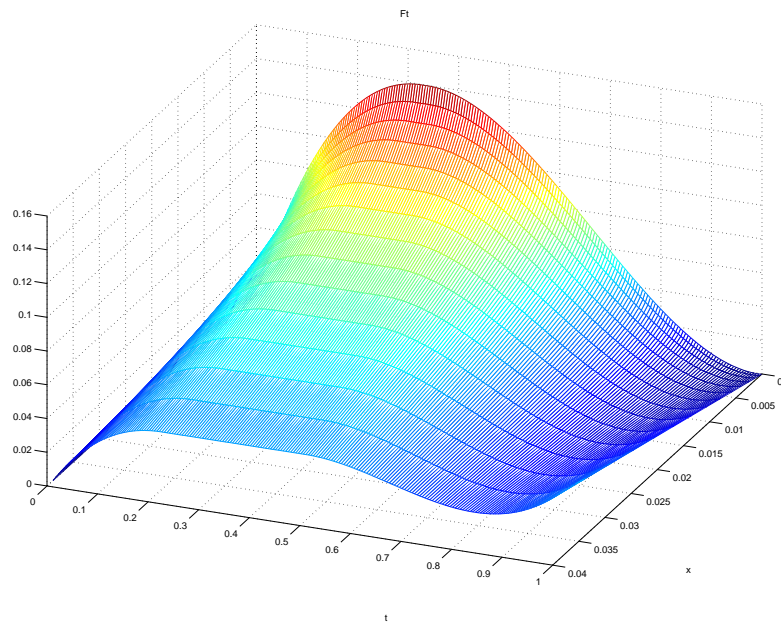


FIGURE 4.10. Another view of the 3-D Graph of $F(x, t)$

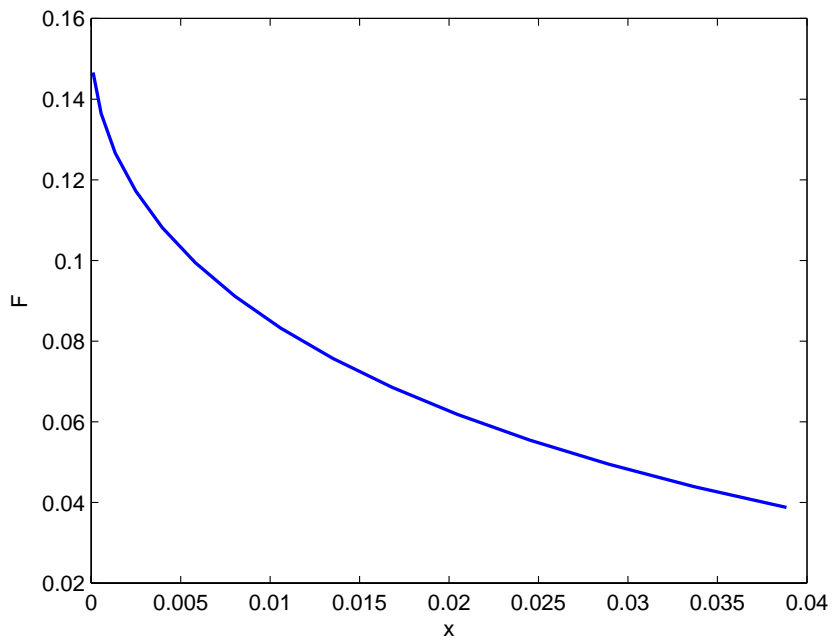


FIGURE 4.11. The Graph of maximal running risk F respect to x

4.2. Remarks and Future Research

It would be desirable to consider that the interest rate $r > 0$ and the commodity price follows a general model with mean reversion

$$dS_t = \alpha(c_t - S_t)dt + \sigma dB_t.$$

A similar calculation will get

$$V_t = \delta^2 \int_0^t (e^{-(\alpha+r)t} - g(s))^2 e^{2\alpha s} ds.$$

In 2008, G.Leobacher studied this kind of problems and got:

THEOREM 4. *Let $\varphi : [0, 1] \rightarrow R$ be an increasing or decreasing function and $\omega : [0, 1] \rightarrow R$ be measurable and positive. There exists a non-decreasing function $g : [0, 1] \rightarrow R$ such that*

$$\sup_{t \in [0, 1]} \int_0^t (\varphi(t) - g(s))^2 \omega(s) ds$$

attain its minimum. Moreover, we have $\varphi(s) \leq g(s) \leq \varphi(1)$ for all $s \in [0, 1]$.

If we add certain constraint on the terminal risk, under a similar work as we did in chapter 3, we can still prove the existence and uniqueness of the optimal solution.

However, since with the general ω and φ , we can not claim that $F_g(t)$ is still constant on $[t_0, t_1]$. It is still unknown for how to solve g on $[t_0, t_1]$. Our current thought is using step functions to approximate $\omega(s)$. We already proved existence and stability of this approximation.

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