MOST LIKELY PATH TO THE SHORTFALL RISK UNDER THE OPTIMAL HEDGING STRATEGY

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ABSTRACT

In this dissertation, we search for the most likely path to the shortfall risk in hedging a long-term supply commitment with short-term futures contracts, which leads to a class of optimization problems. Motivated by a simple model initially discussed in Culp and Miller [**3**], Mello and Parsons [**18**] and Glasserman [**10**], the optimal hedging strategy provided by Larcher and Leobacher [**12**], and the simple discussion about the most likely path by Glasserman [**10**], we studied the following optimization problem:

$$\min_{\phi \in \mathcal{A}_x} \frac{1}{2} \int_0^1 \left[\dot{\phi}(t)\right]^2 dt$$

where

$$\mathcal{A}_x = \left\{ \phi : \sigma \int_0^t [G(s) + s - t] dW_s \leq -x, \text{for some } t \in [0,1] \right\}.$$

We showed the general form of the most likely path under a hedging strategy. We obtained most likely paths under the optimal-fraction hedging strategy provided by Glasserman [**10**] and found that the most likely path is not unique. The main work in this dissertation is to search for the most likely path to the shortfall risk under the optimal hedging strategy provided by Larcher and Leobacher [**12**].

DEDICATION

This dissertation is dedicated to everyone who helped me and guided me through the trials and tribulations of creating this manuscript. In particular, my family and close friends who stood by me throughout the time taken to complete this work.

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It has been a long journey for me from the childhood days in a small town in China to the present day. Along the journey I have faced challenging and joyful moments. I have also met many people who have not only upheld faith in my abilities but also were able to inspire me to a greater goal. I dedicate this dissertation to my mother and father; without their support it would have been impossible for me to continue my studies.

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CHAPTER 1

INTRODUCTION

1.1. Background

The main work of this dissertation is to find the most likely path to the shortfall risk under the optimal hedging strategy in long-term hedging with short-term futures contracts. Our analysis is mainly motivated by the simple discussion about most likely path by Glasserman (2001) [**10**], which is based on the study on hedging strategies initiated by the debate regarding the widely publicized derivatives losses of Metallgesellschaft Refining and Marketing (MGRM).
Metallgesellschaft AG is a large industrial conglomerate engaged in a wide range of activities, from mining and engineering to trade and financial services. In December 1993, the firm reported huge derivatives-related losses which are estimated at over $1.4 billion, at its U.S. oil subsidiary MGRM.

In 1992, MGRM began implementing an aggressive marketing program which included several novel contracts and two of them are relevant to this study. (Mello and Parsons (1995a) [**16**] provided a detailed description of these contracts.) MGRM sought to offset the exposure resulting from its delivery commitments by buying a combination of short-dated oil swaps and futures contracts as part of a strategy known as a "stack-and-roll" hedge. However, the precipitous decline in oil prices in late 1993 caused funding problems. Upon learning of these circumstances, MG's board of supervisors instructed MG's new managers to begin liquidating MGRM's hedge and to enter into negotiations to cancel its long-term contracts with its customers. This action further complicated matters, however. (Culp and Miller, 1994a, [**1**])

The actions taken by MG's board of supervisors had spurred widespread debate and criticism. Culp and Miller (1994a, b, 1995a, b, c, d) [**1, 2, 3, 4, 5, 6**] and Culp and Hanke (1994) [**7**] were critical of MG's board of supervisors for terminating MGRM's marketing and hedging program. In contrast, Edwards and Canter (1995) [**9**] and Mello and Parsons (1995a, b) [**16, 17**] were more critical of MGRM's hedging strategy.

Because of the complexities of this case and the many aspects that remain undisclosed, Glasserman (2001) [**10**] focused instead on an admittedly simple model of a central aspect of MGRM's strategy: the use of a rolling stack of short-dated futures contract to hedge long-term supply commitments. In this strategy, futures contacts are rolled into the next maturity as they expire, but the number of contracts is decreased over time to reflect the decrease in the remaining commitment in the supply contracts.

A primary objective of such a hedging strategy is to protect the firm from the effects of large price fluctuations. It is therefore reasonable to examine how effectively the rolling stack accomplishes this. In the simple single-factor model Glasserman (2001) [**10**] studied, the rolling stack eliminates the effect of spot price fluctuations completely—but only at the end of the hedging horizon. Early in the life of the hedge, the use of short-dated contracts increases the risk of a cash shortfall; he quantified this effect. Glasserman argued that comparing spot risks understates the real shortfall risk resulting from the hedge. Indeed, one of his main conclusions, following from a result on Gaussian extremes, is that the unhedged variance should be compared with the running maximum of the hedged variance. He made several observations about risk in long-term hedging. These conclusions are elaborated in a model of spot prices that allows mean reversion. He also saw that the degree of mean reversion has a major impact on both the appropriate extent and the effectiveness of hedging with short-dated futures.

The search for an optimal strategy to reduce the running risk in hedging a long-term supply commitment with short-dated futures contracts leads to a class of intrinsic optimization problems. Larcher and Leobacher (2003) [**12**] gave an explicit analytic solution for this optimization problem if the market price of the commodity is based on a simple Gaussian model. Leobacher (2008) [**13**] generalized the work by Larcher and himself and gave solutions to more general models, i.e. a mean reverting model and geometric Brownian motion, and furthermore he allowed for interest rates greater than 0. Wu, Yu and Zheng (2011) [**21**] provided an explicit solution to the optimal deterministic strategy to reduce the running risk in hedging a long-term commitment with short-term futures contracts under the constraint of terminal risk.

In addition to comparing risks of a cash shortfall, Glasserman (2001) [**10**] identified the most likely path to a shortfall for each of the four basic cases: mean reverting or not, fully hedged or not, in a sense to be made precise. Since the most likely path gives information about how risky events occur and not just their probability of occurrence, it is important to identify the most likely path to a shortfall under the optimal hedging strategy. We found most likely paths by the theory of large deviations and calculus of variations.

1.2. Hedging Strategies Using Futures

Some good references for this section are John C. Hull's *Options, Futures, and Other Derivatives* [**11**] and Marek Capinski and Tomasz Zastawniak's *Mathematics for Finance-An Introduction to Financial Engineering* [**15**].

1.2.1. Futures. In finance, a futures contract is a standardized contract between two parties to exchange a specified asset of standardized quantity and quality for a price agreed today with delivery occurring at a specified future date, the delivery date. The party agreeing to buy the underlying asset in the future, the "buyer" of the contract, is said to be "long", and the party agreeing to sell the asset in the future, the "seller" of the contract, is said to be "short". In many cases, the underlying asset to a futures contract may not be traditional commodities at all, that is, for financial futures the underlying asset or item can be currencies, securities or financial instruments and intangible assets or referenced items such as stock indexes and interest rates.

1.2.2. Hedging with futures contracts.

1.2.2.1. *Basis risk.* Many of the participants in futures markets are hedgers. Their aim is to use futures markets to reduce a particular risk that they face. A perfect hedge is one that completely eliminates the risk. In practice, perfect hedges are rare and hedging is often not quite as straightforward. This problem gives rise to what is termed *basis risk*. The *basis* in a hedging situation is as follows:

$$\text{Basis} = \text{Spot price of asset to be hedged} - \text{Futures price of contract used.}$$

Prior to expiration, the basis may be positive or negative. When the spot price increases by more than the futures price, the basis increases, and vise versa.

To examine the nature of basis risk, we will use the following notation:

S_1: Spot price at time t_1,

S_2: Spot price at time t_2,

F_1: Futures price at time t_1,

F_2: Futures price at time t_2,

b_1: Basis at time t_1,

b_2: Basis at time t_2.

We will assume that a hedge is put in place at time t_1 and closed out at time t_2. From the definition of the basis, we have

$$b_1 = S_1 - F_1$$

and

$$b_2 = S_2 - F_2.$$

Consider first the situation of a hedger who knows that the asset will be sold at time t_2 and takes a short futures position at time t_1. The price realized for the asset is S_2 and the profit on the futures position is $F_1 - F_2$. The effective price that is obtained for the asset with hedging is therefore

$$S_2 + F_1 - F_2 = F_1 + b_2.$$

The value of F_1 is known at time t_1. If b_2 were also known at this time, a perfect hedge would result. The hedging risk is the uncertainty associated with b_2 and is known as *basis risk*. Consider next a situation where a company knows it will buy the asset at time t_2 and initiates a long hedge at time t_1. The price paid for the asset is S_2 and the loss on the hedge is $F_1 - F_2$. The effective price that is paid with hedging is therefore

$$S_2 + F_1 - F_2 = F_1 + b_2$$

which is the same expression as before.

1.2.2.2. *Rolling the hedge forward.* Sometimes the expiration date of the hedge is later than the delivery dates of all the futures contracts that can be used. The hedger must then roll the hedge forward by closing out one futures contract and taking the same position in a futures contract with a later delivery date. Hedges can be rolled forward many times. Consider a company that wishes to use a short hedge to reduce the risk associated with the price to be received for an asset at time T. If there are futures contracts $1, 2, 3, ..., n$ (not all necessarily in existence at the present time) with progressively later delivery dates, the company can use the following strategy:

Time t_1: Short futures contract 1.

Time t_2: Close out futures contract 1. Short futures contract 2.

Time t_3: Close out futures contract 2. Short futures contract 3.

$\vdots$

Time t_n: Close out futures contract $n-1$. Short futures contract n.

Time T: Close out futures contract n.

In this strategy there are n basis risks or sources of uncertainty. At time T there is uncertainty about the difference between the futures price for contract n and the spot price of the asset being hedged. In addition, on each of the $n-1$ occasions when the hedge is rolled forward, there is uncertainty about the difference between the futures price for the contract being closed out and the futures price for the new contract being entered into. In many situations the hedger has some flexibility on the exact time when a switch is made from one contract to the next. This can be used to reduce the rollover basis risk. For example, if the rollover basis is unattractive at the beginning of the period during which the rollover must be made, the hedger can delay the rollover in the hope that the rollover basis will improve. Sometimes rolling the hedge forward can lead to cash flow pressure. The problem was illustrated dramatically by the activities of MGRM which we described in previous section.

1.3. Stochastic Process and Brownian Motion

A good reference for this section is Bernt Øksendal's *Stochastic Differential Equations–An Introduction with Applications* [**20**].

1.3.1. Stochastic process. In probability theory, a stochastic process is a collection of random variables which is often used to represent the evolution of some random value, or system, over time. It is the probabilistic counterpart to a deterministic process (or deterministic system). Instead of describing a process which can only evolve in one way, in a stochastic or random process there is some indeterminacy: even if the initial condition (or starting point) is known, there are several (often infinitely many) directions in which the process may evolve.

DEFINITION 1.3.1. *A* stochastic process *is a parameterized collection of random variables*

$$\{X_t\}_{t\in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ *and assuming values in* $\mathbb{R}^n$.

Note that for each $t \in T$ fixed we have a random variable

$$\omega \to X_t(\omega); \quad \omega \in \Omega$$

on the other hand, fixing $\omega \in \Omega$, we can consider the function

$$t \to X_t(\omega); \quad t \in T$$

which is called a *path* of X_t.

Intuitively, it may be useful to think of t as "time" and each ω as an individual "particle"or "experiment". With this picture $X_t(\omega)$ would represent the position (or result) at time t of the particle ω. Sometimes it's convenient to write $X(t,\omega)$ instead of $X_t(\omega)$. Thus we may also regard the process as a function of two variables

$$(t,\omega) \to X(t,\omega)$$

from $T \times \Omega$ into $\mathbb{R}^n$. This is often a natural point of view in stochastic analysis.

In addition, we note that we may identify each ω with the function $t \to X_t(\omega)$ from T into $\mathbb{R}^n$. Thus we may regard Ω as a subset of the space $\tilde{\Omega} = (\mathbb{R}^n)^T$ of all functions from T into $\mathbb{R}^n$.

1.3.2. Brownian motion. In 1828 the Scottish botanist Robert Brown observed that pollen grains suspended in liquid performed an irregular motion. The motion was later explained by the random collisions with the molecules of the liquid. Mathematically, Brownian motion is described by a continuous-time stochastic process called Wiener process $\{W_t\}_{t\geq 0}$. The Wiener process is characterized by three facts:

i) $W_0 = 0$;

ii) the function $t \to W_t$ is almost surely continuous;

iii) $\{W_t\}_{t\geq 0}$ has stationary independent increments with $W_t - W_s$ which is normally distributed with mean 0 and variance $t - s$ (for $0 \leq s < t$).

1.3.3. Geometric Brownian motion. A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion.

DEFINITION 1.3.2. *A stochastic process S_t is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):*

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1.3.1}$$

where W_t is a standard Brownian motion or Wiener process and μ ('the percentage drift') and σ ('the percentage volatility') are constants.

For an arbitrary initial value S_0 the above SDE has the analytic solution (under Itô's interpretation):

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \tag{1.3.2}$$

which is (for any value of t) a log-normally distributed random variable with expected value and variance given by

$$E(S_t) = S_0 e^{\mu t} \tag{1.3.3}$$

and

$$Var(S_t) = S_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1). \tag{1.3.4}$$

1.4. Itô Integrals

DEFINITION 1.4.1. *Let* $\mathcal{V} = \mathcal{V}(S,T)$ *be the class of functions*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

i) $(t,\omega) \to f(t,\omega)$ *is* $\mathcal{B} \times \mathcal{F}$*-measurable, where* $\mathcal{B}$ *denotes the Borel* σ*-algebra on* $[0,\infty)$.

ii) $f(t,\omega)$ *is* $\mathcal{F}_t$*-adapted.*

iii) $E[\int_S^T f(t,\omega)^2 dt] < \infty$.

For functions $f \in \mathcal{V}$, we want to define the Itô integral

$$\mathcal{I}[f](\omega) = \int_S^T f(t,\omega) dB_t(\omega),$$

where $B_t(\omega)$ is 1-dimensional Brownian motion starting at the origin, for a wide class of functions $f : [0,\infty) \times \Omega \to \mathbb{R}$.

DEFINITION 1.4.2 (The Itô Integral). *Let* $f \in \mathcal{V}(S,T)$*. Then the Itô integral of* f *(from* S *to* T*) is defined by*

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dB_t(\omega) \qquad \text{(limit in } L^2(P)) \tag{1.4.1}$$

where $\{\phi_n\}$ *is a sequence of elementary functions such that*

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0 \qquad \text{as } n \to \infty. \tag{1.4.2}$$

The following is important for Itô integral.

Corollary 1.4.1 (The Itô Isometry).

$$E\left[\left(\int_S^T f(t,\omega)dB_t\right)^2\right] = E\left[\int_S^T f^2(t,\omega)dt\right] \text{ for all } f \in \mathcal{V}(S,T). \tag{1.4.3}$$

1.5. Large Deviations

A good reference for this section is Amir Dembo and Ofer Zeitouni's *Large Deviations Techniques and Applications* [**8**].

The area of large deviations is a set of asymptotic results on rare probabilities and a set of methods to derive such results. Large deviations theory is essentially a study of rates of convergence in probabilistic limit theorems. Let X be a real-valued random variable, and we need to compute or estimate the probability that X exceeds a particular level x, i.e.

$$P(X > x).$$

Law of large numbers (LLN), central limit theorem (CLT) and Legendre transform are important in the theory of large deviations. Consider an infinite sequence of i.i.d. integrable random variables $\{X_i\}$ with expected value $E(X_1) = E(X_2) = \cdots = \mu$ and $Var(X_1) = Var(X_2) = \cdots = \sigma^2$, and let $S_n = \sum_{i=1}^n X_i$, the LLN states that

$$\frac{1}{n}S_n \xrightarrow{a.s.} \mu.$$

The CLT provides an estimate of the probability

$$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} > x\right).$$

Thus, the CLT estimates the probability of $O(\sqrt{n})$ deviations from the mean of the sum of the random variables $\{X_i\}_{i=1}^n$. These deviations are small compared to the mean of S_n which is an $O(n)$ quantity. On the other hand, large deviations are of the order of the mean itself, i.e., $O(n)$ deviations.

Legendre transform is an operation that transforms one real-valued function of a real variable into another. Specifically, the Legendre transform of a convex function f is the function f^* defined by

$$f^*(p) = \sup_x (px - f(x)).$$

The Legendre transform produces a new function, in which the independent variable x is replaced by $p = \frac{df}{dx}$, which is the derivative of the original function with respect to x. The Legendre transform is its own inverse and is especially well behaved if $f(x)$ is a convex function.

1.5.1. Large deviations for i.i.d. sequences: Cramer's theorem.

THEOREM 1.5.1 (Cramer's Theorem). *Let $X_1, X_2, \cdots$ be i.i.d. and suppose that their common moment generating function $M_X(\theta) := E(e^{\theta x}) < \infty$ for all θ in some neighborhood B_0 of $\theta = 0$. Further suppose that the supremum in the following definition of the rate function $I(x)$ is obtained at some interior point in this neighborhood:*

$$I(x) = \sup_\theta \{\theta x - \Lambda(\theta)\},$$

where $\Lambda(\theta) := \ln M_X(\theta)$ is called the log moment generating function or the cumulant generating function. In other words, we assume that there exists $\theta^* \in Interior(B_0)$ such that

$$I(x) = \theta^* x - \Lambda(\theta^*).$$

Fix any $x > E(X_1)$. Then, for each $\epsilon > 0$, there exists N such that, for all $n \geq N$,

$$e^{-n(I(x)+\epsilon)} \leq P(\sum_{i=1}^{n} X_i \geq nx) \leq e^{-nI(x)}.$$

Since $\epsilon \to 0$ as $n \to \infty$, another way to state the result of the above theorem is

$$\lim_{n\to\infty} \frac{1}{n} \ln P(\sum_{i=1}^{n} X_i \geq nx) = -I(x).$$

1.5.2. Rate function. The existence of a rate function can be proved by Ruelle-Lanford approach. Once we established the existence of the rate function, we can apply Varadhan's Theorem to calculate it. The rate functions have the following properties:

i) $I(x)$ is a convex function.

ii) Let $I(x)$ be the rate function of a random variable X with mean μ, then

$$I(x) \geq I(\mu) = 0.$$

iii)

$$\Lambda(\theta) = \sup_x \{\theta x - I(x)\}.$$

1.5.3. Large deviations for Brownian motion. In many problems, the interest is in rare events that depend on random process and the corresponding asymptotic probabilities, usually called sample path large deviations, were developed by Freidlin-Wentzell and Donsker-Varadhan.

An important one is known as Schilder's theorem. Let W_t, $t \in [0, 1]$ denote a standard Brownian motion in $\mathbb{R}^d$. Consider the process

$$W_\epsilon(t) = \sqrt{\epsilon} W_t,$$

and let ν_ϵ be the probability measure induced by $W_\epsilon(\cdot)$ on $C_0([0, 1])$ equipped with the supremum norm. The process $W_\epsilon(\cdot)$ is a candidate for a large deviation principle (LDP). Indeed, $\|W_\epsilon\| \xrightarrow{\epsilon \to 0} 0$ in probability (actually, almost surely) and exponentially fast in $1/\epsilon$ as implied by the following useful lemma.

LEMMA 1.5.1. *For any integer* d *and any* $\tau, \epsilon, \delta > 0$,

$$\mathrm{P}\left(\sup_{0\le t\le\tau} |W_\epsilon(t)| \ge \delta\right) \le 4de^{-\delta^2/2d\tau\epsilon}.$$

The LDP for $W_\epsilon(\cdot)$ is stated in the Schilder's theorem as follows. Let $H_1 \triangleq \{\int_0^t f(s)ds : f \in L_2([0,1])\}$ denote the space of all absolutely continuous functions with square integrable derivative equipped with the norm $\|g\|_{H_1} = [\int_0^1 \dot{g}^2(t)dt]^{\frac{1}{2}}$.

THEOREM 1.5.2. (Schilder's Theorem) $\{\nu_\epsilon\}$ *satisfies, in* $C_0([0,1])$, *an LDP with good rate function*

$$I_W(\phi) = \begin{cases} \frac{1}{2}\int_0^1 \dot{\phi}^2(t)dt, & \phi \in H_1; \\ \infty, & \textit{otherwise.} \end{cases}$$

CHAPTER 2

A CLASS OF HEDGING STRATEGIES

Some good references for this chapter are Larcher and Leobacher's *An optimal strategy for hedging with short-term futures contracts* [**12**] and Glasserman's *Shortfall risk in long-term hedging with short-term futures contracts* [**10**].

2.1. Background of A Hedging Problem

A firm has a commitment to deliver a fixed quantity q of a commodity at a forward price a_n at dates $n = 1, \ldots, N$. By doing so the firm is taking the risk of the underlying commodities' future price movements. The firm will have to pay the market price for the underlying commodities to deliver the contract which might result in loss of cash if the spot price of the underlying commodity rises abruptly.
To reduce the risk of losses due to possible price fluctuations, the firm might enter into a sequence of short-dated futures contracts to protect itself from the effects of large price fluctuations. Assume that the market price of the underlying commodities is given by a simple stochastic differential equation, i.e.,

$$dS_t = \mu dt + \sigma dW_t, \tag{2.1.1}$$

where W_t is Wiener process on $[0, T]$, μ and σ are constants, $\sigma > 0$. We further assume that the interest rate $r = 0$. The firm commits to supplying at each time t in the interval $[0, T]$ a commodity at rate q and the deterministic price a_t. There is no loss in assuming $T = 1$ and $q = 1$. By setting

$$V_t = E[S_t] - S_t,$$

the unhedged cumulative cash flow from this contract is

$$C_t = \int_0^t (a_s - S_s)ds$$

with an exposure of

$$C_t - E[C_t] = \int_0^t (E[S_s] - S_s)ds = \int_0^t V_s ds,$$

where

$$dV_t = -\sigma dW_t, \quad V_0 = 0.$$

So the unhedged exposure is

$$\int_0^t V_s ds = -\sigma \int_0^t W_s ds = -\sigma \int_0^t (t-s)dW_s.$$

Consider now at time t a short term future with maturity $t+\delta$ and futures price $F_{t,t+\delta}$. The payoff of such a contract is

$$S_{t+\delta} - F_{t,t+\delta}.$$

If we write $F_{t,t+\delta} = S_t + b_{t,t+\delta}\delta$, then we get for the payoff

$$S_{t+\delta} - S_t - b_{t,t+\delta}\delta$$

where $b_{t,t+\delta}\delta$ is the *basis* of the future, i.e., the deviation from the "natural" price S_t. The payoff from a hedging strategy holding $G_{n\delta}$ futures at time $n\delta$ is therefore

$$H_{k\delta} := \sum_{n=0}^{k-1} G_{n\delta}(S_{t+\delta} - S_t) - \sum_{n=0}^{k-1} G_{n\delta} b_{t,t+\delta}\delta.$$

Letting $\delta \to 0$ gives the continuous form

$$H_t = \int_0^t G_s dS_s - \int_0^t G_s b_s ds$$

where we assume that $b_t := lim_{\delta \to 0} b_{t,t+\delta}$ exists and is regular enough to guarantee existence of the integral. Since $dS_t = \mu dt + \sigma dW_t$, then

$$H_t = \int_0^t G_s \sigma dW_s + \int_0^t G_s(\mu - b_s)ds.$$

Note that if $G_s(\mu - b_s)$ is deterministic for almost all s, the cash flow H_t from a hedging strategy G_t satisfies

$$H_t - E[H_t] = \int_0^t G_s \sigma dW_s.$$

The hedged actual cumulative cash flow from this contract is $D_t := C_t + H_t$. The deviation of the actual cash flow D_t from the expected cash flow $E[D_t]$ is

$$D_t - E[D_t] = \sigma \int_0^t (s - t + G_s) dW_s.$$

By Itô isometry we have

$$\begin{aligned} \sigma_t^2 := \text{Var}[D_t] &= E[(\sigma \int_0^t (s - t + G_s)dW_s)^2] \\ &= \sigma^2 \int_0^t E[(s - t + G_s)^2]ds \\ &= \sigma^2 \int_0^t E[(t - s - G_s)^2]ds. \end{aligned}$$

From Jensen's inequality it follows that

$$\int_0^t E[(t - s - G_s)^2]ds \geq \int_0^t (E[t - s - G_s])^2 ds$$

with equality if and only if G_s is the deterministic for almost all $s \in [0, 1]$. So we may restrict our search to deterministic strategies. If we write $g(s) := G_s + s$, then

$$\sigma_t^2 = \sigma^2 \int_0^t (g(s) - t)^2 ds. \tag{2.1.2}$$

2.2. Previous Work: Hedging Strategies and Most Likely Paths

Suppose that the firm hedges to try to prevent the actual cash balance from falling short of the expected cash balance by a large amount x. Write D_t for the actual cash balance at time t under an arbitrary hedging strategy G_t, and say that a shortfall occurs when $D_t \leq E[D_t] - x$ for large x.

The *spot risk* at time t is

$$P(D_t - E[D_t] < -x), \tag{2.2.1}$$

the probability of a shortfall at time t. If the cash balance is Gaussian, the spot variance σ_t^2 measures this risk perfectly. But a more relevant measure is

$$P(\min_{0 \leq s \leq t}(D_s - E[D_s]) < -x), \tag{2.2.2}$$

the probability of a shortfall any time up to t, which is called the *running risk* to t. The shortfall probability (hedged or not) can be written as

$$P(\min_{0 \leq s \leq t}(D_s - E[D_s]) < -x) = e^{-\gamma x^2 + o(x^2)}, \tag{2.2.3}$$

where

$$\gamma = -\lim_{x \to \infty} \frac{1}{x^2} log P(\min_{0 \leq s \leq t}(D_s - E[D_s]) < -x). \tag{2.2.4}$$

One philosophy is to reduce spot risk to zero at the terminal date $t = 1$, i.e.

$$\sigma_1^2 = 0, \tag{2.2.5}$$

that is to choose $G(t) = 1 - t$. This is called a *rolling stack* or *full* hedging strategy under which

$$\sigma_t^2 = \sigma^2 t(t-1)^2.$$

As pointed out in detail by Glasserman (2001) [**10**], this strategy, however, produces large spot variance for smaller t, in fact even larger spot variance in over large time periods than if no hedge were to be applied (i.e. $G(t) = 0$). Standard calculations give

$$\sigma_t^2 = \frac{1}{3}\sigma^2 t^3$$

for the variance of the unhedged exposure. We compared the spot variances of unhedged and fully hedged cash balance over the life of the exposure in figure 2.1:

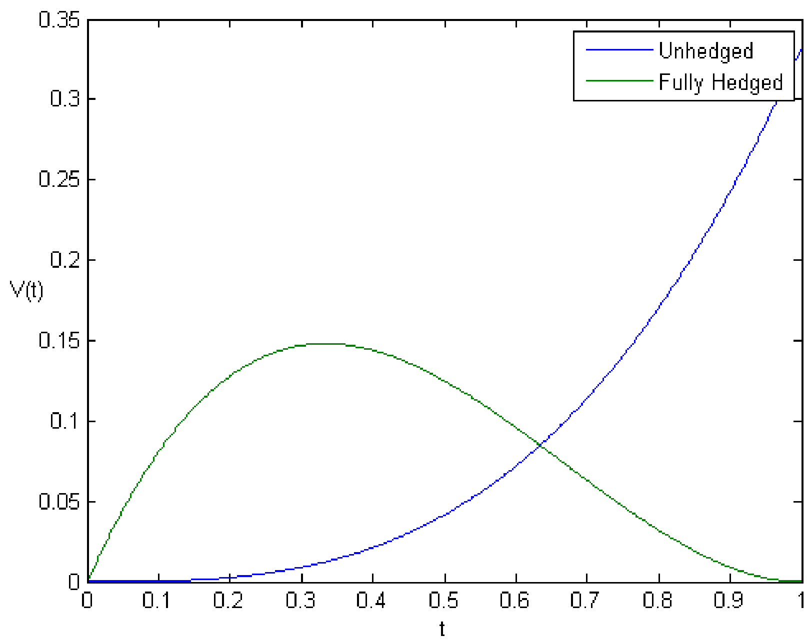

FIGURE 2.1. Spot variance of unhedged and fully hedged cash balance for $t \in [0, 1]$

Another philosophy is to keep the running risk as small as possible. There are two tools for evaluating γ in particular and the running cash flow risk in general. The first is a remarkable result of Marcus and Shepp (1972) [**14**] that, so long as D_t is Gaussian with sample paths that are bounded on bounded intervals (e.g., continuous)

$$\gamma = \frac{1}{2\nu_t^2}, \tag{2.2.6}$$

with

$$\nu_t = \sup_{0 \le s \le t} \sigma_t(G).$$

Thus, the running risk is measured by the running maximum standard deviation. If over some interval $[0, t]$, one hedging strategy has a larger maximum variance than another, then the shortfall probabilities are ordered the same way, for all sufficiently large x. (This is not true without the Gaussian assumption.) This means that we need to find a strategy for which the maximum of the spot risk attains its minimum, i.e.

$$\sup_{t \in [0,1]} \sigma_t^2 \to \min.$$

Glasserman (2001) [**10**] suggested two strategies to reduce the running risk: One is the optimal fixed-horizon hedging strategy in which he carries out a rolling stack strategy, but not for the whole time interval [0,1], but for [0,τ] with τ such that $\sup_{t \in [0,1]} \sigma_t^2$ becomes minimal, i.e.

$$G_\tau(t) = \begin{cases} \tau - t, & \text{if } t \in [0, \tau]; \\ 0, & \text{if } t \in (\tau, 1]. \end{cases}$$

The optimal fixed-horizon hedging strategy is the one that minimizes the running risk over the entire time interval $[0, 1]$. The maximal spot risk occurs either at $\tau/3$ (where the hedged portion is riskiest) or at 1 (where the unhedged portion is riskiest). The spot variances at these times are $4\sigma^2\tau^3/27$ and

$$\sigma^2 \int_0^\tau (1-\tau)^2 ds + \sigma^2 \int_\tau^1 (1-s)^2 ds = \sigma^2 (\frac{2}{3}\tau^3 - \tau^2 + \frac{1}{3}),$$

respectively. The optimal τ — the one that minimizes the running risk — makes the spot variances at these times equal. Numerically, $\tau \approx 0.733$ and $\sup_{t \in [0,1]} \sigma_t^2 \approx 0.0583$.

A further improvement gives the optimal-fraction hedge. The strategy is $G_\kappa(t) =$

$\kappa(1-t)$, which is rolling stack for a certain fraction κ carried out. κ is then chosen so that $\sup_{t\in[0,1]} \sigma_t^2$ is minimal. The spot variance under this strategy is

$$\sigma^2 \int_0^t (\kappa + (1-\kappa)s - t)^2 ds,$$

which achieves a local maximum at

$$t^* = \frac{\kappa(1+\kappa-\sqrt{\kappa})}{\kappa^2+\kappa+1}.$$

The other possible location of the maximal variance is the terminal time $t = 1$, where the spot variance is $(1-\kappa)^2\sigma^2$. The optimal κ sets the values of the spot variance at t^* and the terminal time $t = 1$ equal. Numerically, Glasserman found that the optimal $\kappa \approx 0.62996$, which appears to coincide with $(1/4)^{1/3}$, and $\sup_{t\in[0,1]} \sigma_t^2 \approx 0.0456$.

Based on Glasserman's work, Larcher and Leobacher (2003) [**12**] established the explicit optimal hedging strategy. The result is the following.

Theorem 2.2.1. *There is a unique continuous function $G_0 : [0,1] \to \mathbb{R}$ which satisfies the condition*

$$J := \max_{0\le t\le 1} \int_0^t (G_0(s) + s - t)^2 ds = \inf_{\substack{G:[0,1]\to\mathbb{R} \\ G \ \text{integrable}}} \sup_{0\le t\le 1} \int_0^t (G(s) + s - t)^2 ds.$$

G_0 is given by the formulae

$$G_0(t) = \begin{cases} 3t_0 - t, & \text{if } t \in [0, t_0); \\ e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}\eta}{2}) - t, & \text{if } t = \frac{1}{\sqrt{3}} e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}}{2}\eta + \frac{\pi}{6}) \text{ with } \eta \in [0, \frac{\pi}{3\sqrt{3}}]; \\ 1 - t, & \text{if } t \in (1/2, 1]; \end{cases}$$

where $t_0 = \frac{e^{-\frac{\pi}{6\sqrt{3}}}}{2\sqrt{3}}$. For the maximal risk $J = \frac{e^{-\frac{\pi}{2\sqrt{3}}}}{6\sqrt{3}} = 0.0388532\cdots$.

This optimal hedging strategy minimizes the maximum of the spot risk at the price of equally high risk at the terminal date. However, a relatively high terminal risk is certainly not what every customer prefers. Therefore, Wu, Yu and Zheng (2011) [**21**] provided an explicit solution to the optimal deterministic strategy to reduce the running risk in hedging a long-term commitment with short-term futures contracts under the constraint of terminal risk. The results are the following.

THEOREM 2.2.2. *For every* $x \in [0, \frac{e^{-\frac{\pi}{2\sqrt{3}}}}{6\sqrt{3}}]$, *there exists a unique non-decreasing function* g_x *such that* $F_{g_x}(1) = x$ *and*

$$\max_{t\in[0,1]} F_{g_x}(t) = F_x := \inf_{h\in R[0,1]} \sup_{t\in[0,1]} \{F_h(t) : F_h(1) \leq x\}.$$

For every $x \in [0, \frac{e^{-\frac{\pi}{2\sqrt{3}}}}{6\sqrt{3}}]$, *let* u_x *be the unique real solution of*

$$\frac{1}{2}(\frac{1}{3} + u^2)^{\frac{3}{2}} e^{-\sqrt{3}(\frac{\pi}{6} - tan^{-1}(\sqrt{3}u))} - \frac{1}{4}(1+u)^2 u = x.$$

Then the unique optimal function g_x *is given by*

$$g_x(t) := \begin{cases} 3\sigma_x, & \text{if } t \in [0, \sigma_x); \\ \beta(\gamma^{-1}(t)), & \text{if } t \in [\sigma_x, \tau_x]; \\ 1, & \text{if } t \in (\tau_x, 1]; \end{cases}$$

where the relations between σ_x, τ_x *and* u_x *are* $\sigma_x = \frac{1}{2\sqrt{3}}\sqrt{1+3u_x^2}e^{-\frac{1}{\sqrt{3}}(\frac{\pi}{6} - tan^{-1}(\sqrt{3}u_x))}$, $\tau_x = \frac{1-u_x}{2}$, *respectively, and the function* β *and* γ *are defined by*

$$\beta(\eta) = \sqrt{1+3u_x^2}e^{-\frac{\eta}{2}} cos\,(\frac{\sqrt{3}\eta}{2} + tan^{-1}(\sqrt{3}u_x)),$$

$$\gamma(\eta) = \frac{1}{\sqrt{3}}\sqrt{1+3u_x^2}e^{-\frac{\eta}{2}} cos\,(\frac{\sqrt{3}\eta}{2} + tan^{-1}(\sqrt{3}u_x) + \frac{\pi}{6}).$$

F_x *can be evaluated by* $F_x = \frac{1}{4}(1+u_x)^2 u_x + x$.

The second important tool for studying the running risk is the theory of large deviations, which is not restricted to the Gaussian case and gives more detailed information about when and how a shortfall is likely to occur. The "most likely path" identified by large deviations analysis illustrates the types of risks to which different strategies are exposed.

Glasserman (2001) [**10**] identified most likely paths to a shortfall over the time interval $[0,1]$ under no hedging and full hedging strategies:

- under no hedging strategy:

$$\phi^*(t) = -\frac{3}{2}t^2 + 3t, \quad t \in [0,1];$$

  with its graph as follows

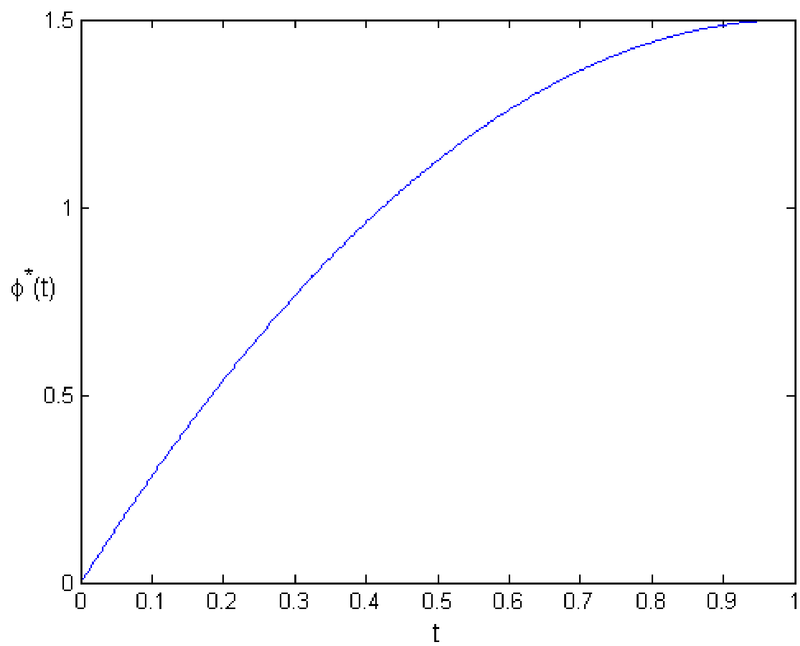

FIGURE 2.2. Most likely path under no hedging strategy

- under full hedging strategy:

$$\phi^*(t) = \begin{cases} -(9/2)t, & \text{if } t \in [0, \frac{1}{3}]; \\ -3/2, & \text{if } t \in (\frac{1}{3}, 1]; \end{cases}$$

with its graph as follows

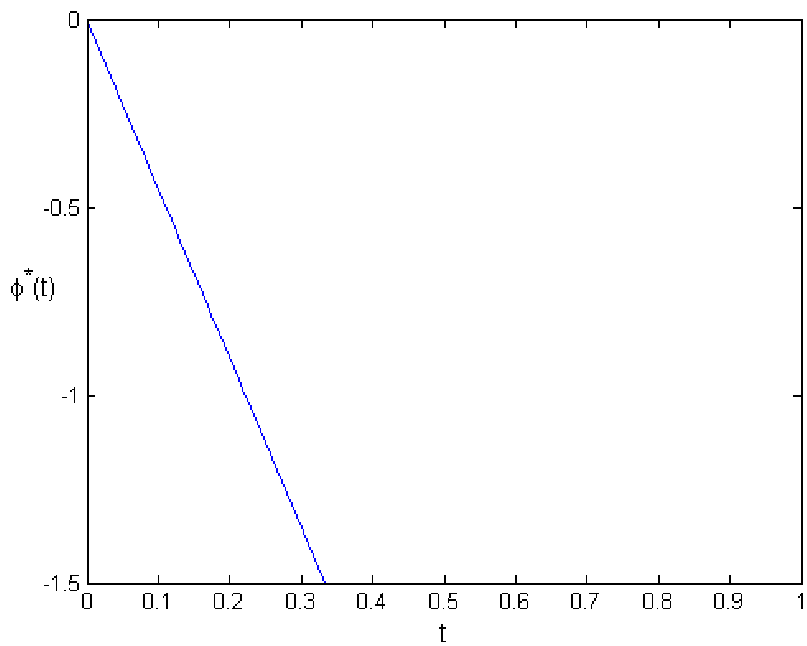

FIGURE 2.3. Most likely path under full hedging strategy

CHAPTER 3

MOST LIKELY PATH

In this chapter, based on Glasserman's work and Larcher and Leobacher's optimal hedging strategy, our main work is to find most likely paths under the optimal hedging strategy.

Based on the simple model (2.1.1), the exposure V_t is just a Wiener process. And event $\mathcal{A}_x$ like a shortfall of magnitude greater than x occurs in $[0, 1]$ is a set of sample paths of the Wiener process. The most likely path in a set like $\mathcal{A}_x$ is the path in the sense that when $\mathcal{A}_x$ occurs, it occurs with the Wiener process staying close to this path. This tendency to follow the most likely path becomes most pronounced as the event becomes rare, which corresponds to x becoming large in our setting. The most likely path $\phi^* \in \mathcal{A}_x$ has the following property: if we define a strip around ϕ^* of width ϵ, then the probability that the Wiener process leaves this strip conditional on $\mathcal{A}_x$ occurring vanishes exponentially as x increases.

The limit

$$\lim_{x\to\infty} \frac{1}{x^2} \log P(\mathcal{A}_x) = -\gamma$$

gives the exponential rate of decrease of $\mathcal{A}_x$ in x^2. The exponential rate of decrease of the shortfall probability is also the "energy" of the most likely path to a shortfall. For any absolutely continuous function ϕ on $[0, 1]$, denote by $\dot{\phi}$ its derivative with respect to time t. Finding the most likely path is a problem in the calculus of variations. The most likely path in $\mathcal{A}_x$ solves

$$\min_{\phi\in\mathcal{A}_x} \frac{1}{2} \int_0^1 [\dot{\phi}(t)]^2 dt \qquad (3.0.7)$$

by Schilder's theorem. Membership in $\mathcal{A}_x$ defines a constraint on ϕ.

If ϕ^* is the minimizing path, then

$$\gamma = \frac{1}{2}\int_0^1 [\dot{\phi}^*(t)]^2 dt, \tag{3.0.8}$$

with γ as defined in (2.2.4). Here, we let

$$E = \frac{1}{2}\int_0^1 [\dot{\phi}(t)]^2 dt$$

and E^* be the minimum energy $\frac{1}{2}\int_0^1 [\dot{\phi}^*(t)]^2 dt$.

3.1. Most Likely Path under Optimal-fraction Hedging Strategy

Before we search for the most likely path under Larcher and Leobacher's optimal hedging strategy, we study the most likely path under Glasserman's optimal-fraction hedging strategy

$$G_\kappa(t) = \kappa(1-t), \quad \text{where} \quad \kappa = \left(\frac{1}{4}\right)^{\frac{1}{3}}.$$

We found two most likely paths with $\tau = 2^{(\frac{1}{3})} - 1$ and $\tau = 1$, respectively, which implies that the most likely path is not unique. Our results are as follows.

If $\tau = 2^{(\frac{1}{3})} - 1$, the most likely path is

$$\phi^*(t) = \begin{cases} \frac{B}{2}(1-\kappa)t^2 + (B\kappa + C)t, & \text{if } t \in [0,\tau]; \\ -\frac{3}{2\kappa}, & \text{if } t \in (\tau, 1]; \end{cases}$$

with constraint function

$$F(t) = \begin{cases} \frac{B}{6}(1+\kappa-2\kappa^2)t^3 + \frac{1}{2}\left(2B\kappa^2 + C(1+\kappa)\right)t^2 - (B\kappa + C)\kappa t, & \text{if } t \in [0,\tau]; \\ -\frac{3}{2\kappa}t - \frac{B}{3}(1-\kappa)^2\tau^3 - \frac{1}{2}(2B\kappa + C)(1-\kappa)\tau^2 - (B\kappa + C)\kappa\tau, & \text{if } t \in (\tau, 1]; \end{cases}$$

where $B = -\frac{3(1+\kappa+\kappa^2)^2}{\kappa^3(1+\kappa\sqrt{\kappa})^2}$ and $C = \frac{3(1+\kappa+\kappa^2)}{\kappa^2(1+\kappa^2+(1+\kappa)\sqrt{\kappa})}$;

and the minimum energy

$$E^* = \frac{3(1+\kappa+\kappa^2)^2}{2\kappa^3(1+\kappa\sqrt{\kappa})^2} = 4 + 3\cdot 2^{(\frac{1}{3})} + 2\cdot 2^{(\frac{2}{3})} \approx 10.9546.$$

The graphs for this most likely path and the corresponding constraint function are showed in figure 3.1 and figure 3.2, respectively.

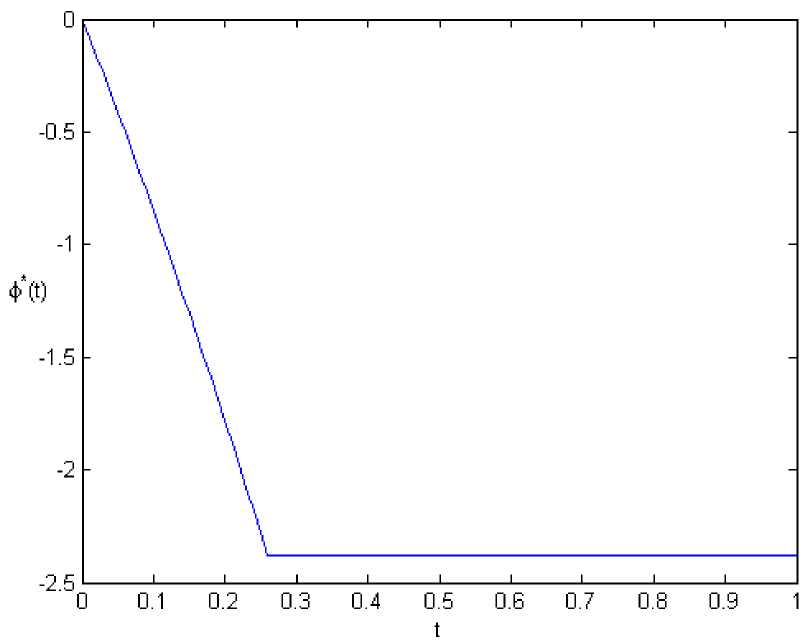

FIGURE 3.1. Most likely path under optimal-fraction hedging strategy with $\tau = 2^{(\frac{1}{3})} - 1$

If $\tau = 1$, the most likely path is

$$\phi^*(t) = -\frac{3}{2(1-\kappa)}t^2 + \frac{3}{1-\kappa}t, \quad t \in [0,1];$$

with constraint function

$$F(t) = \frac{-1-2\kappa}{2(1-\kappa)}t^3 + \frac{3(1+2\kappa)}{2(1-\kappa)}t^2 - \frac{3\kappa}{1-\kappa}t, \quad t \in [0,1],$$

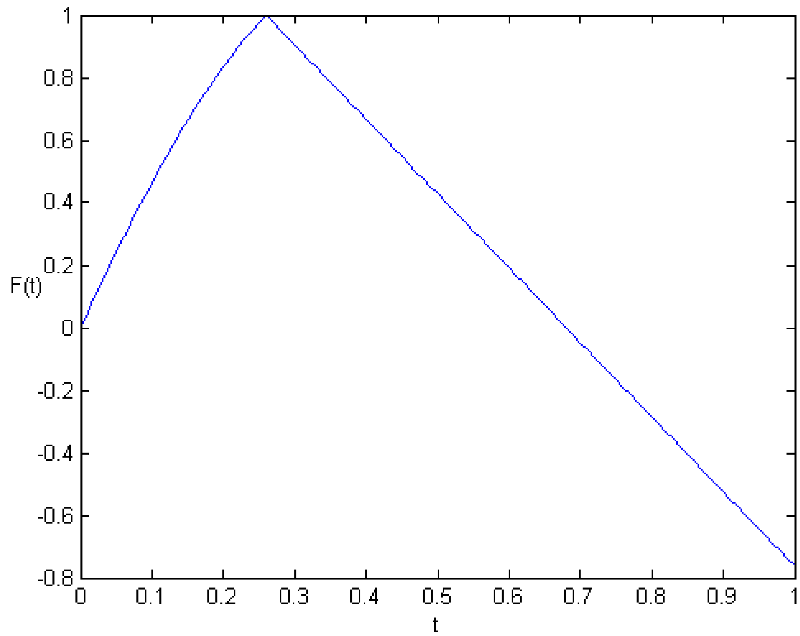

FIGURE 3.2. Constraint function for most likely path under optimal-fraction hedging strategy with $\tau = 2^{(\frac{1}{3})} - 1$

and the minimum energy

$$E^* = \frac{3}{2(1-\kappa)^2} \approx 10.9546.$$

The graphs for this most likely path and the corresponding constraint function are showed in figure 3.3 and figure 3.4, respectively.

We also compared the constraints for these two most likely paths and the spot variance under the optimal-fraction hedging strategy, and found that the times when the constraints attain the maximum are the times when the spot variance attains its maximum. The comparison is showed in figure 3.5.

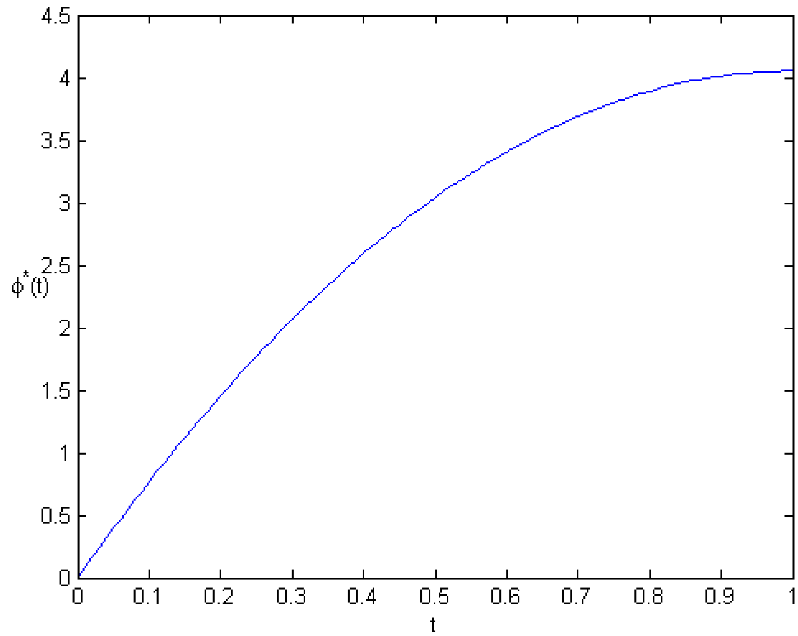

FIGURE 3.3. Most likely path under optimal-fraction hedging strategy with $\tau = 1$

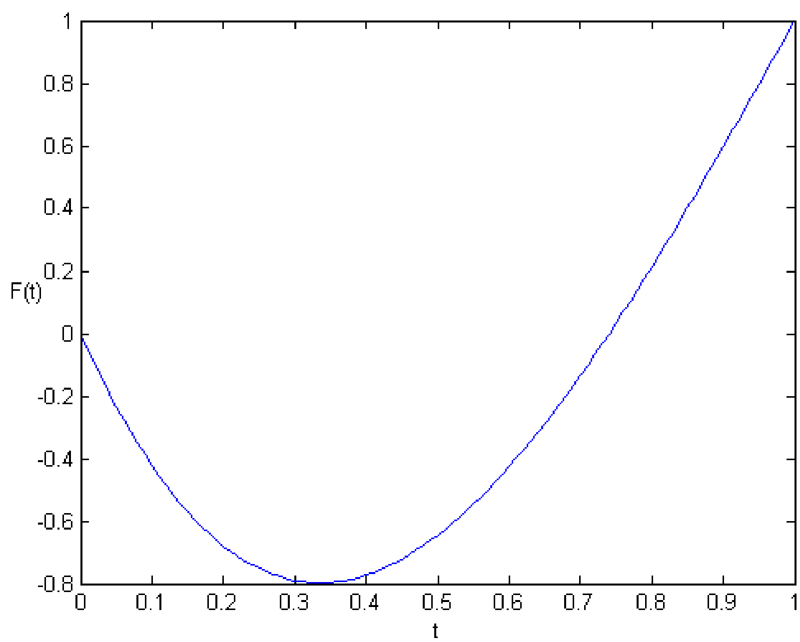

FIGURE 3.4. Constraint function for most likely path under optimal-fraction hedging strategy with $\tau = 1$

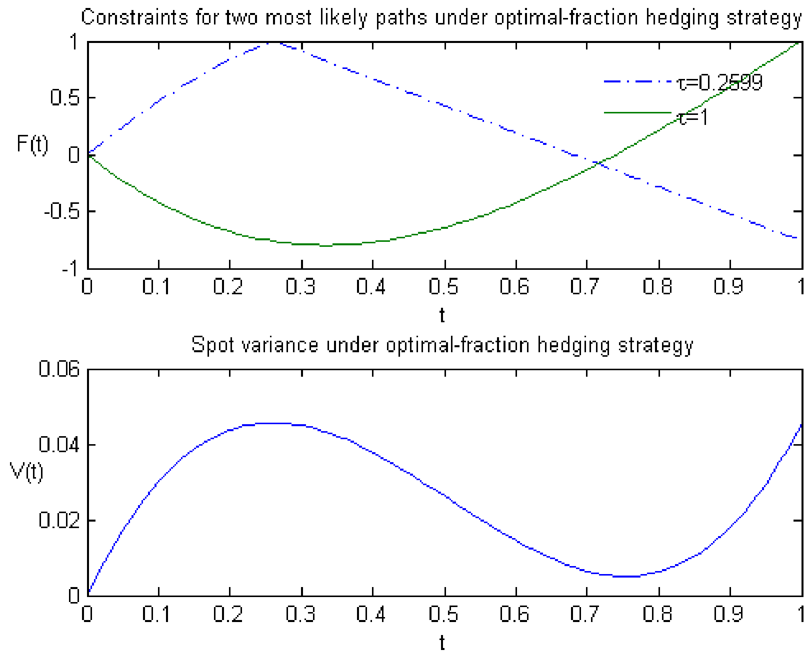

FIGURE 3.5. Comparison of constraint functions for most likely paths and spot variance under optimal-fraction hedging strategy

3.2. Most Likely Path under Optimal Hedging Strategy

Larcher and Leobacher's explicit optimal hedging strategy $G(t)$ which provides the minimal running risk on $[0, 1]$ is:

$$G(t) = \begin{cases} 3t_0 - t, & \text{if } t \in [0, t_0); \\ e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}\eta}{2}) - t, & \text{if } t = \frac{1}{\sqrt{3}} e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}}{2}\eta + \frac{\pi}{6}) \text{ with } \eta \in [0, \frac{\pi}{3\sqrt{3}}]; \\ 1 - t, & \text{if } t \in (\frac{1}{2}, 1]; \end{cases}$$

where $t_0 = \frac{1}{2\sqrt{3}} e^{-\frac{\pi}{6\sqrt{3}}}$.

We let $g(t) = G(t) + t$, then

$$g(t) = \begin{cases} 3t_0, & \text{for } t \in [0, t_0); \\ e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}\eta}{2}), & \text{for } t = \frac{1}{\sqrt{3}} e^{-\frac{\eta}{2}} \cos(\frac{\sqrt{3}}{2}\eta + \frac{\pi}{6}) \text{ with } \eta \in [0, \frac{\pi}{3\sqrt{3}}]; \\ 1, & \text{for } t \in (\frac{1}{2}, 1]. \end{cases}$$

From chapter 2, we know that the exposure is

$$\sigma \int_0^t [G(s) + s - t] dW_s \tag{3.2.1}$$

under a general hedging strategy $G(t)$. Then a shortfall occurs under the optimal hedging strategy $G(t)$ when

$$\sigma \int_0^t [G(s) + s - t] dW_s \leq -x.$$

So finding the most likely path is to solve

$$\min_{\phi \in \tilde{\mathcal{A}}_x} \frac{1}{2} \int_0^1 \left[\dot{\phi}(t)\right]^2 dt$$

where

$$\tilde{\mathcal{A}}_x = \left\{ \phi : \sigma \int_0^t [G(s) + s - t] d\phi(s) \leq -x, \text{for some } t \in [0, 1] \right\}.$$

LEMMA 3.2.1. *Assuming that the optimal solution $\phi^*(t)$ exists on $[0,1]$, the constraint $\tilde{\mathcal{A}}_x$ can be refined as*

$$\mathcal{A}_x = \left\{ \phi : \max_{0 \le t \le 1} \left\{ \sigma \left[\int_0^t \phi(s) dg(s) - G(t)\phi(t) \right] \right\} = x \right\}.$$

PROOF. Since $g(t) = G(t) + t$, we have

$$\begin{aligned}
-x &\ge \sigma \int_0^t [G(s) + s - t] d\phi(s) \\
&= \sigma \int_0^t (g(s) - t)\dot{\phi}(s) ds \\
&= \sigma \left[\int_0^t g(s)\dot{\phi}(s) ds - t\phi(t) \right] \\
&= \sigma \left[(g(t) - t)\phi(t) - \int_0^t \phi(s) dg(s) \right] \\
&= -\sigma \left[\int_0^t \phi(s) dg(s) - G(t)\phi(t) \right],
\end{aligned}$$

so the constraint can be refined as

$$\tilde{\mathcal{A}}_x = \left\{ \phi : \sigma \left[\int_0^t \phi(s) dg(s) - G(t)\phi(t) \right] \ge x, \text{for some } t \in [0,1] \right\}.$$

If

$$\sigma \left[\int_0^t \phi^*(s) dg(s) - G(t)\phi^*(t) \right] > x \quad \text{for some } t \in [0,1],$$

there exists $a > 1$ such that

$$\frac{1}{a}\sigma \left[\int_0^t \phi^*(s) dg(s) - G(t)\phi^*(t) \right] \ge x$$

and

$$\frac{1}{2} \int_0^1 [\frac{1}{a}\dot{\phi}^*(t)]^2 dt < \frac{1}{2} \int_0^1 [\dot{\phi}_*(t)]^2 dt,$$

which contradicts the fact that $\phi^*(t)$ is the optimal solution.

Thus, $\tilde{\mathcal{A}}_x$ can be further refined as

$$\mathcal{A}_x = \left\{ \phi : \max_{0\le t\le 1} \left\{ \sigma \left[\int_0^t \phi(s)dg(s) - G(t)\phi(t) \right] \right\} = x \right\}.$$

□

In our optimization problem, x merely serves to scale the solution: the solution for arbitrary x is just x times the solution for $x = 1$; hence, it suffices to give the solution for $x = 1$. The volatility parameter σ is also a scale parameter and may therefore be set to 1 as well. With these simplifications, we have

$$\mathcal{A} := \mathcal{A}_1 = \left\{ \phi : \max_{0\le t\le 1} \left\{ \int_0^t \phi(s)dg(s) - G(t)\phi(t) \right\} = 1 \right\}.$$

The optimization problem becomes

$$\min_{\phi\in\mathcal{A}} \frac{1}{2} \int_0^1 \left[\dot{\phi}(t) \right]^2 dt$$

where

$$\mathcal{A} = \left\{ \phi : \max_{0\le t\le 1} \left\{ \int_0^t \phi(s)dg(s) - G(t)\phi(t) \right\} = 1 \right\}.$$

We let ϕ^* be a most likely path and τ be the first time when

$$\int_0^t \phi^*(s)dg(s) - G(t)\phi^*(t)$$

attains the maximum value 1 on $[0, 1]$, then

$$\int_0^\tau \phi^*(s)dg(s) - G(\tau)\phi^*(\tau) = 1.$$

By integration by parts, we have

$$\tau\phi^*(\tau) - \int_0^\tau \dot{\phi}^*(t)g(t)dt = 1.$$

THEOREM 3.2.1. *The general form of the most likely path under the optimal hedging strategy is*

$$\phi^*(t) = \begin{cases} B\int_0^t g(s)ds + \int_0^t C(s)ds, & \text{if } t \in [0,\tau]; \\ B\int_0^\tau g(s)ds + \int_0^\tau C(s)ds, & \text{if } t \in (\tau,1]; \end{cases}$$

where (with $t_0 = \frac{1}{2\sqrt{3}}e^{-\frac{\pi}{6\sqrt{3}}}$*)*

$$C(t) = \begin{cases} C_1, & \text{for } t \in [0,t_0]; \\ C_2, & \text{for } t \in (t_0,\frac{1}{2}]; \\ C_3, & \text{for } t \in (\frac{1}{2},1]. \end{cases}$$

PROOF. Let $h \in C^1[0,1]$, *supp* $h \subset (0,\tau]$, and $\int_0^\tau h(t)dg(t) = 0$.

Considering a variation of ϕ^* by ϵh, we have

$$\int_0^\tau [\phi^*(t) + \epsilon h(t)]dg(t) - G(\tau)[\phi^*(\tau) + \epsilon h(\tau)] = 1,$$

which implies $\phi^* + \epsilon h \in \mathcal{A}$.

Since $\phi^*(t)$ is the optimal solution, then

$$\frac{1}{2}\int_0^1 [\dot{\phi}^*(t) + \epsilon\dot{h}(t)]^2 dt \geq \frac{1}{2}\int_0^1 [\dot{\phi}^*(t)]^2 dt,$$

or equivalently,

$$\epsilon\int_0^\tau \dot{\phi}^*(t)\dot{h}(t)dt + \frac{1}{2}\epsilon^2\int_0^\tau \dot{h}^2(t)dt \geq 0.$$

Since ϵ is arbitrary, we conclude that

$$\int_0^\tau \dot{\phi}^*(t)\dot{h}(t)dt = 0.$$

By integration by parts, we have

$$\dot{\phi}^*(t)h(t)|_{t=0}^{t=\tau} - \int_0^\tau \ddot{\phi}^*(t)h(t)dt = 0,$$

which is

$$\int_0^\tau \ddot{\phi}^*(t)h(t)dt = 0.$$

(1) If $\tau \in (0, t_0]$, since $dg(t) = 0$ on $[0, t_0]$, we have

$$\int_0^\tau h(t)dg(t) = 0 \quad \text{if } supp\ h \subset (0, \tau].$$

Therefore,

$$\ddot{\phi}^*(t) = 0 \quad \text{for } t \in [0, \tau].$$

Since $\dot{g}(t) = 0$ on $(0, t_0)$, we have

$$\ddot{\phi}^*(t) = B\dot{g}(t) \quad \text{for } t \in [0, \tau].$$

Since

$$\frac{1}{2}\int_0^1 [\dot{\phi}^*(t)]^2 = \frac{1}{2}\int_0^\tau [\dot{\phi}^*(t)]^2 + \frac{1}{2}\int_\tau^1 [\dot{\phi}^*(t)]^2$$

and we let

$$\frac{1}{2}\int_\tau^1 [\dot{\phi}^*(t)]^2 = 0,$$

we have

$$\dot{\phi}^*(t) = \begin{cases} Bg(t) + C_1, & \text{if } t \in [0, \tau]; \\ 0, & \text{if } t \in (\tau, 1]. \end{cases}$$

Since $\phi^*(0) = 0$ and $\phi^*(t)$ is continuous,

$$\phi^*(t) = \begin{cases} B\int_0^t g(s)ds + C_1 t, & \text{if } t \in [0, \tau]; \\ B\int_0^\tau g(s)ds + C_1\tau, & \text{if } t \in (\tau, 1]. \end{cases}$$

(2) If $\tau \in (t_0, \frac{1}{2}]$, since

$$\int_0^\tau h(t)dg(t) = 0 \quad \text{for } supp\ h \subset (0, t_0],$$

we have, as above,

$$\ddot{\phi}^*(t) = 0 \quad \text{for } t \in [0, t_0].$$

Therefore,

$$\int_{t_0}^{\tau} \ddot{\phi}^*(t)h(t)dt = \int_0^{\tau} \ddot{\phi}^*(t)h(t)dt = 0.$$

For any $H \in C^2[0,1]$, and *supp* $H \subset (t_0, \tau]$, we let

$$h(t) = \frac{\dot{H}(t)}{\dot{g}(t)},$$

where $\dot{g}(t_0) = \dot{g}(t_0+)$ and $\dot{g}(\tau) = \dot{g}(\tau-)$.

Then $h \in C^1[0,1]$, *supp* $h \subset (t_0, \tau]$, and

$$\int_0^{\tau} h(t)dg(t) = \int_{t_0}^{\tau} \frac{\dot{H}(t)}{\dot{g}(t)}\dot{g}(t)dt = H(\tau) - H(t_0) = 0.$$

So,

$$\int_{t_0}^{\tau} \ddot{\phi}^*(t)h(t)dt = \int_{t_0}^{\tau} \frac{\ddot{\phi}^*(t)}{\dot{g}(t)}\dot{H}(t)dt = \int_{t_0}^{\tau} \left[\frac{\ddot{\phi}^*(t)}{\dot{g}(t)}\right]' H(t)dt = 0.$$

Thus,

$$\left[\frac{\ddot{\phi}^*(t)}{\dot{g}(t)}\right]' = 0 \quad \text{for } t \in (t_0, \tau].$$

Therefore, we have

$$\begin{cases} \ddot{\phi}^*(t) = 0, & \text{if } t \in [0, t_0]; \\ \left[\frac{\ddot{\phi}^*(t)}{\dot{g}(t)}\right]' = 0, & \text{if } t \in (t_0, \tau]. \end{cases}$$

From $\left[\frac{\ddot{\phi}^*(t)}{\dot{g}(t)}\right]' = 0$, we get

$$\ddot{\phi}^*(t) = B\dot{g}(t).$$

Since

$$\dot{g}(t) = 0 \quad \text{for } t \in (0, t_0),$$

we have

$$\ddot{\phi}^*(t) = B\dot{g}(t) \quad \text{for } t \in [0, \tau] \setminus \{t_0\}.$$

Since

$$\frac{1}{2}\int_0^1[\dot{\phi}^*(t)]^2=\frac{1}{2}\int_0^\tau[\dot{\phi}^*(t)]^2+\frac{1}{2}\int_\tau^1[\dot{\phi}^*(t)]^2$$

and we let

$$\frac{1}{2}\int_\tau^1[\dot{\phi}^*(t)]^2=0,$$

we have

$$\dot{\phi}^*(t)=\begin{cases} Bg(t)+C(t), & \text{if } t\in[0,\tau];\\ 0, & \text{if } t\in(\tau,1];\end{cases}$$

where

$$C(t)=\begin{cases} C_1, & \text{if } t\in[0,t_0];\\ C_2, & \text{if } t\in(t_0,\frac{1}{2}].\end{cases}$$

Since $\phi^*(0)=0$ and $\phi^*(t)$ is continuous,

$$\phi^*(t)=\begin{cases} B\int_0^t g(s)ds+\int_0^t C(s)ds, & \text{if } t\in[0,\tau];\\ B\int_0^\tau g(s)ds+\int_0^\tau C(s)ds, & \text{if } t\in(\tau,1].\end{cases}$$

(3) If $\tau\in(\frac{1}{2},1]$, since

$$\int_0^\tau h(s)dg(s)=0 \quad \text{for } \mathit{supp}\ h\subset(\frac{1}{2},\tau],$$

we have

$$\int_{\frac{1}{2}}^\tau \ddot{\phi}^*(t)h(t)dt=\int_0^\tau \ddot{\phi}^*(t)h(t)dt=0.$$

Thus,

$$\ddot{\phi}^*(t)=0 \quad \text{for } t\in(\frac{1}{2},\tau].$$

For the same argument as above about $\phi^*(t)$ on $[0,\frac{1}{2}]$, we have

$$\begin{cases} \ddot{\phi}^*(t)=0, & \text{if } t\in[0,t_0];\\ \left[\frac{\ddot{\phi}^*(t)}{\dot{g}(t)}\right]'=0, & \text{if } t\in(t_0,\frac{1}{2}];\\ \ddot{\phi}^*(t)=0, & \text{if } t\in(\frac{1}{2},\tau].\end{cases}$$

From $\left[\frac{\ddot{\phi}^*(t)}{\dot{g}(t)}\right]' = 0$, we get

$$\ddot{\phi}^*(t) = B\dot{g}(t).$$

Since $\dot{g}(t) = 0$ on $(0, t_0)$ and $(\frac{1}{2}, 1)$, we have

$$\ddot{\phi}^*(t) = B\dot{g}(t) \quad \text{for } t \in [0, \tau] \diagdown \{t_0, \frac{1}{2}\}.$$

Since

$$\frac{1}{2}\int_0^1 [\dot{\phi}^*(t)]^2 = \frac{1}{2}\int_0^\tau [\dot{\phi}^*(t)]^2 + \frac{1}{2}\int_\tau^1 [\dot{\phi}^*(t)]^2$$

and we let

$$\frac{1}{2}\int_\tau^1 [\dot{\phi}^*(t)]^2 = 0,$$

we have

$$\dot{\phi}^*(t) = \begin{cases} Bg(t) + C(t), & \text{if } t \in [0, \tau]; \\ 0, & \text{if } t \in (\tau, 1]; \end{cases}$$

where

$$C(t) = \begin{cases} C_1, & \text{if } t \in [0, t_0]; \\ C_2, & \text{if } t \in (t_0, \frac{1}{2}]; \\ C_3, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Since $\phi^*(0) = 0$ and $\phi^*(t)$ is continuous,

$$\phi^*(t) = \begin{cases} B\int_0^t g(s)ds + \int_0^t C(s)ds, & \text{if } t \in [0, \tau]; \\ B\int_0^\tau g(s)ds + \int_0^\tau C(s)ds, & \text{if } t \in (\tau, 1]. \end{cases}$$

Therefore, the general form for the most likely path under the optimal hedging strategy is

$$\phi^*(t) = \begin{cases} B\int_0^t g(s)ds + \int_0^t C(s)ds, & \text{if } t \in [0, \tau];, \\ B\int_0^\tau g(s)ds + \int_0^\tau C(s)ds, & \text{if } t \in (\tau, 1]; \end{cases}$$

where

$$C(t) = \begin{cases} C_1, & \text{if } t \in [0, t_0]; \\ C_2, & \text{if } t \in (t_0, \frac{1}{2}]; \\ C_3, & \text{if } t \in (\frac{1}{2}, 1]; \end{cases}$$

and $t_0 = \frac{1}{2\sqrt{3}} e^{-\frac{\pi}{6\sqrt{3}}}$. □

THEOREM 3.2.2. *Under the optimal hedging strategy, there are three types of most likely paths $\phi_1^*(t)$, $\phi_2^*(t)$, and $\phi_3^*(t)$ corresponding to three cases: $\tau \in (0, t_0]$, $\tau \in (t_0, \frac{1}{2}]$, and $\tau \in (\frac{1}{2}, 1]$, respectively, with minimum energy $E^* = \frac{1}{8t_0^3}$.*

(1) If $\tau \in (0, t_0]$, then $\tau = t_0$ and

$$\phi_1^*(t) = \begin{cases} -\frac{1}{2t_0^2} t, & \text{if } t \in [0, t_0]; \\ -\frac{1}{2t_0}, & \text{if } t \in (t_0, 1]. \end{cases}$$

(2) If $\tau \in (t_0, \frac{1}{2}]$, then

$$\phi_2^*(t) = \begin{cases} \lambda(\tau - 3t_0)t, & \text{if } t \in [0, t_0]; \\ \lambda\left[\tau t - \zeta_1(t)\right], & \text{if } t \in (t_0, \tau]; \\ \lambda\left[\tau^2 - \zeta_1(\tau)\right], & \text{if } t \in (\tau, 1]. \end{cases}$$

(3) If $\tau \in (\frac{1}{2}, 1]$, then $\tau = 1$ and

$$\phi_3^*(t) = \begin{cases} \lambda(1 - 3t_0)t, & \text{if } t \in [0, t_0]; \\ \lambda\left[t - \zeta_1(t)\right], & \text{if } t \in (t_0, \frac{1}{2}]; \\ \frac{1}{8}\lambda, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Here $t_0 = \frac{1}{2\sqrt{3}} e^{-\frac{\pi}{6\sqrt{3}}}$, $\lambda = \frac{1}{4t_0^3}$, $\zeta_1(t) = \frac{1}{4\sqrt{3}} e^{-\eta(t)}[\cos(\sqrt{3}\eta(t) + \frac{\pi}{6}) + \sqrt{3}]$ and $\eta(t)$ satisfies $t = \frac{1}{\sqrt{3}} e^{-\frac{\eta(t)}{2}} \cos(\frac{\sqrt{3}\eta(t)}{2} + \frac{\pi}{6})$.

PROOF. By Theorem 3.2.1, we have

$$E^* = \frac{1}{2}\int_0^1 \left[\dot{\phi}^*(t)\right]^2 dt = \frac{1}{2}\int_0^\tau \left[Bg(t) + C(t)\right]^2 dt.$$

Since the constraint is

$$\tau\phi^*(\tau) - \int_0^\tau \dot{\phi}^*(t)g(t)dt - 1 = 0,$$

we have the constraint function

$$F(t) = t\phi^*(t) - \int_0^t \dot{\phi}^*(s)g(s)ds$$

and let

$$f = F(\tau) - 1.$$

(1) If $\tau \in (0, t_0]$:

$$\phi_1^*(t) = \begin{cases} (3Bt_0 + C_1)t, & \text{if } t \in [0, \tau]; \\ (3Bt_0 + C_1)\tau, & \text{if } t \in (\tau, 1]. \end{cases}$$

Then

$$\dot{\phi}_1^*(t) = \begin{cases} 3Bt_0 + C_1, & \text{if } t \in [0, \tau]; \\ 0, & \text{if } t \in (\tau, 1]; \end{cases}$$

$$E_1 = \frac{1}{2}\int_0^\tau (3Bt_0 + C_1)^2\, dt = \frac{1}{2}(3Bt_0 + C_1)^2\, \tau;$$

$$F_1(t) = t(3Bt_0 + C_1)(t - 3t_0) \quad \text{for} \quad t \in [0, t_0];$$

and

$$f_1 = F_1(\tau) - 1.$$

By the method of Lagrange Multipliers, we have

$$\begin{cases} \nabla E_1 = \lambda \nabla f_1, \\ f_1 = 0. \end{cases}$$

Since

$$\begin{cases} \frac{\partial E_1}{\partial B} = 3t_0\tau(3Bt_0 + C_1), \\ \frac{\partial E_1}{\partial C_1} = \tau(3Bt_0 + C_1), \\ \frac{\partial f_1}{\partial B} = 3t_0\tau(\tau - 3t_0), \\ \frac{\partial f_1}{\partial C_1} = \tau(\tau - 3t_0), \end{cases}$$

we have

$$\begin{cases} 3t_0\tau(3Bt_0 + C_1) = \lambda 3t_0\tau(\tau - 3t_0), \\ \tau(3Bt_0 + C_1) = \lambda\tau(\tau - 3t_0), \\ \tau(3Bt_0 + C_1)(\tau - 3t_0) - 1 = 0. \end{cases}$$

The first two equations are equivalent, so we get

$$3Bt_0 + C_1 = \lambda(\tau - 3t_0).$$

From the third equation, we have

$$\lambda\tau(\tau - 3t_0)^2 - 1 = 0.$$

Solving for λ, we have

$$\lambda = \frac{1}{\tau(\tau - 3t_0)^2}.$$

Then

$$E_1(\tau) = \frac{1}{2}\lambda^2(\tau - 3t_0)^2\tau = \frac{1}{2}\frac{1}{\tau^2(\tau - 3t_0)^4}(\tau - 3t_0)^2\tau = \frac{1}{2\tau(\tau - 3t_0)^2},$$

and

$$\dot{E}_1(\tau) = \frac{3t_0 - 3\tau}{2\tau^2(\tau - 3t_0)^3}.$$

By setting $\dot{E}_1(\tau) = 0$, we get

$$\tau = t_0,$$

so F_1 attains its maximum at $t = t_0$.

Then

$$\lambda = \frac{1}{t_0(t_0 - 3t_0)^2} = \frac{1}{4t_0^3}$$

and

$$3Bt_0 + C_1 = \lambda(t_0 - 3t_0) = -\frac{1}{2t_0^2}.$$

Thus the most likely path is

$$\phi_1^*(t) = \begin{cases} -\frac{1}{2t_0^2}t, & \text{if } t \in [0, t_0]; \\ -\frac{1}{2t_0}, & \text{if } t \in (t_0, 1]. \end{cases}$$

With this most likely path, the minimum energy is

$$E_1^* = \frac{1}{2t_0(t_0 - 3t_0)^2} = \frac{1}{8t_0^3},$$

and the constraint function is

$$F_1(t) = \begin{cases} -\frac{1}{2t_0^2}t^2 + \frac{3}{2t_0}t, & \text{if } t \in [0, t_0]; \\ -\frac{1}{2t_0}t + \frac{3}{2}, & \text{if } t \in (t_0, 1]; \end{cases}$$

with its maximum at $t = t_0$.

The graphs for this most likely path and the corresponding constraint function are showed in figure 3.6 and figure 3.7, respectively.

(2) If $\tau \in (t_0, \frac{1}{2}]$:

$$\phi_2^*(t) = \begin{cases} (3Bt_0 + C_1)t, & \text{if } t \in [0, t_0]; \\ (3Bt_0 + C_1)t_0 + \int_{t_0}^{t} [Bg(s) + C_2]\, ds, & \text{if } t \in (t_0, \tau]; \\ (3Bt_0 + C_1)t_0 + \int_{t_0}^{\tau} [Bg(s) + C_2]\, ds, & \text{if } t \in (\tau, 1]. \end{cases}$$

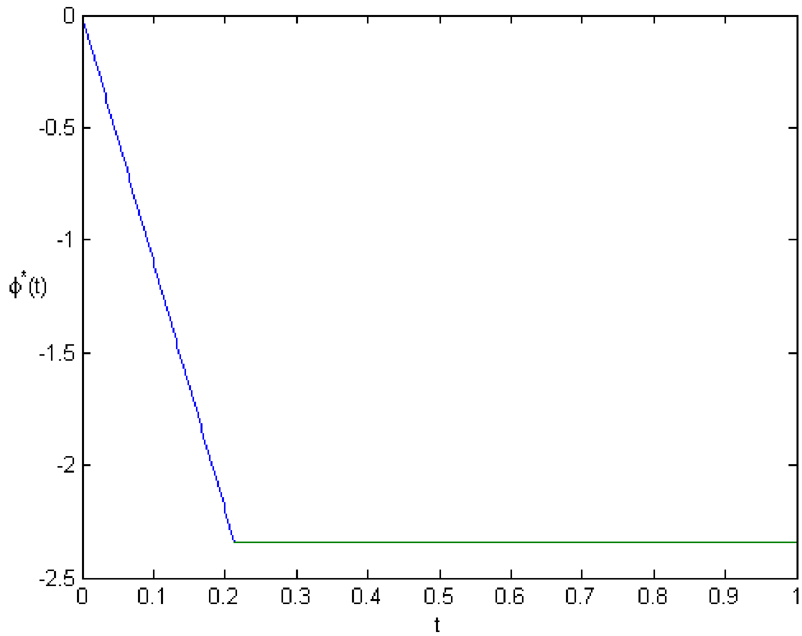

FIGURE 3.6. Most likely path under optimal hedging strategy if $\tau = t_0$

Then

$$\dot{\phi}_2^*(t) = \begin{cases} 3Bt_0 + C_1, & \text{if } t \in [0, t_0]; \\ Bg(t) + C_2, & \text{if } t \in (t_0, \tau]; \\ 0, & \text{if } t \in (\tau, 1]; \end{cases}$$

$$E_2 = \frac{1}{2}\left(3Bt_0 + C_1\right)^2 t_0 + \frac{1}{2}\int_{t_0}^{\tau}\left(Bg(t) + C_2\right)^2 dt;$$

$$F_2(t) = t_0(3Bt_0 + C_1)(t - 3t_0) + \int_{t_0}^{t}\left[Bg(s) + C_2\right]\left[t - g(s)\right] ds \quad \text{for} \quad t \in (t_0, \tau];$$

and

$$f_2 = F_2(\tau) - 1.$$

By the method of Lagrange Multipliers, we have

$$\begin{cases} \nabla E_2 = \lambda \nabla f_2, \\ f_2 = 0. \end{cases}$$

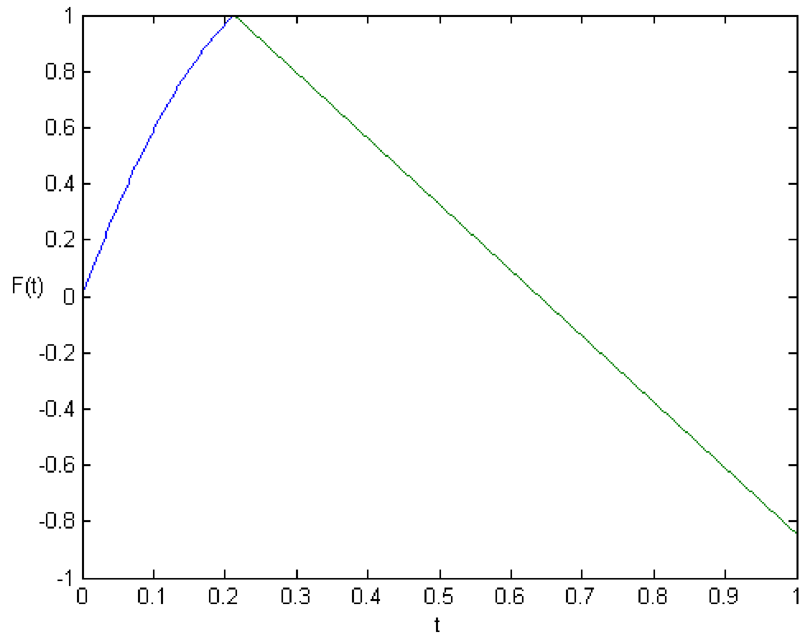

FIGURE 3.7. Constraint function under optimal hedging strategy if $\tau = t_0$

Since

$$\begin{cases} \frac{\partial E_2}{\partial B} = 3t_0^2(3Bt_0 + C_1) + \int_{t_0}^{\tau} \left[Bg(t) + C_2\right] g(t)dt, \\ \frac{\partial E_2}{\partial C_1} = t_0(3Bt_0 + C_1), \\ \frac{\partial E_2}{\partial C_2} = \int_{t_0}^{\tau} \left[Bg(t) + C_2\right] dt, \\ \frac{\partial f_2}{\partial B} = 3t_0^2(\tau - 3t_0) + \int_{t_0}^{\tau} \left[\tau - g(t)\right] g(t)dt, \\ \frac{\partial f_2}{\partial C_1} = t_0(\tau - 3t_0), \\ \frac{\partial f_2}{\partial C_2} = \int_{t_0}^{\tau} \left[\tau - g(t)\right] dt, \end{cases}$$

we have

$$\begin{cases} 3t_0^2(3Bt_0 + C_1) + \int_{t_0}^{\tau} \left[Bg(t) + C_2\right] g(t)dt = \lambda \left[3t_0^2(\tau - 3t_0) + \int_{t_0}^{\tau} \left[\tau - g(t)\right] g(t)dt\right], \\ t_0(3Bt_0 + C_1) = \lambda t_0(\tau - 3t_0), \\ \int_{t_0}^{\tau} \left[Bg(t) + C_2\right] dt = \lambda \int_{t_0}^{\tau} \left[\tau - g(t)\right] dt, \\ t_0(3Bt_0 + C_1)(\tau - 3t_0) + \int_{t_0}^{\tau} \left[Bg(t) + C_2\right] \left[\tau - g(t)\right] dt - 1 = 0. \end{cases}$$

From the first two equations, we get

$$\int_{t_0}^{\tau} \left[Bg(t) + C_2\right] g(t)dt = \lambda \int_{t_0}^{\tau} \left[\tau - g(t)\right] g(t)dt,$$

which is

$$(B + \lambda) \int_{t_0}^{\tau} g^2(t)dt = -(C_2 - \lambda\tau) \int_{t_0}^{\tau} g(t)dt.$$

By letting $G_1 = \int_{t_0}^{\tau} g(t)dt$ and $G_2 = \int_{t_0}^{\tau} g^2(t)dt$, we have

$$(B + \lambda)G_2 = -(C_2 - \lambda\tau)G_1. \tag{3.2.2}$$

From the third equation, we have

$$(B + \lambda)G_1 = -(C_2 - \lambda\tau)(\tau - t_0). \tag{3.2.3}$$

Combine equations (3.2.2) and (3.2.3), we get

$$(C_2 - \lambda\tau) \left[G_1^2 - (\tau - t_0)G_2\right] = 0. \tag{3.2.4}$$

By Holder's inequality, we have

$$G_1^2 < (\tau - t_0)G_2.$$

Thus,

$$C_2 = \lambda\tau.$$

Therefore,

$$B = -\lambda$$

and

$$C_1 = \lambda\tau.$$

Substitute B, C_1 and C_2 into $f_2 = 0$, we get

$$\lambda(\tau) = \frac{1}{t_0(\tau - 3t_0)^2 + \int_{t_0}^{\tau} \left[\tau - g(t)\right]^2 dt}.$$

Then

$$\phi_2^*(t) = \begin{cases} \lambda(\tau - 3t_0)t, & \text{if } t \in [0, t_0]; \\ \lambda \left[\tau t - 3t_0^2 - \int_{t_0}^{t} g(s)ds\right], & \text{if } t \in (t_0, \tau]; \\ \lambda \left[\tau^2 - 3t_0^2 - \int_{t_0}^{\tau} g(s)ds\right], & \text{if } t \in (\tau, 1]; \end{cases}$$

$$E_2 = \frac{1}{2}\lambda.$$

By Lemma 3.2.2, we have

$$\lambda = \frac{1}{4t_0^3},$$

then

$$E_2^* = \frac{1}{8t_0^3}.$$

Direct calculation yields

$$\begin{aligned} \int_{t_0}^{t} g(s)ds &= \frac{1}{4\sqrt{3}} e^{-\eta(t)} [\cos(\sqrt{3}\eta(t) + \frac{\pi}{6}) + \sqrt{3}] - \frac{1}{4} e^{-\frac{\pi}{3\sqrt{3}}} \\ &= \frac{1}{4\sqrt{3}} e^{-\eta(t)} [\cos(\sqrt{3}\eta(t) + \frac{\pi}{6}) + \sqrt{3}] - 3t_0^2 \end{aligned}$$

and

$$\int_{t_0}^{t} g^2(s)ds$$
$$= -\frac{1}{12\sqrt{3}}e^{-\frac{3\eta(t)}{2}}\left[\sin(\frac{3\sqrt{3}\eta(t)}{2}-\frac{\pi}{3})+\sqrt{3}\sin(\frac{\sqrt{3}\eta(t)}{2}-\frac{\pi}{6})-2\sqrt{3}\cos(\frac{\sqrt{3}\eta(t)}{2})\right]$$
$$-\frac{5}{24\sqrt{3}}e^{-\frac{\pi}{2\sqrt{3}}}$$
$$= -\frac{1}{12\sqrt{3}}e^{-\frac{3\eta(t)}{2}}\left[\sin(\frac{3\sqrt{3}\eta(t)}{2}-\frac{\pi}{3})+\sqrt{3}\sin(\frac{\sqrt{3}\eta(t)}{2}-\frac{\pi}{6})-2\sqrt{3}\cos(\frac{\sqrt{3}\eta(t)}{2})\right]$$
$$-5t_0^3.$$

We let

$$\zeta_1(t)=\frac{1}{4\sqrt{3}}e^{-\eta(t)}[\cos(\sqrt{3}\eta(t)+\frac{\pi}{6})+\sqrt{3}]$$

and

$$\zeta_2(t)=-\frac{1}{12\sqrt{3}}e^{-\frac{3\eta(t)}{2}}\left[\sin(\frac{3\sqrt{3}\eta(t)}{2}-\frac{\pi}{3})+\sqrt{3}\sin(\frac{\sqrt{3}\eta(t)}{2}-\frac{\pi}{6})-2\sqrt{3}\cos(\frac{\sqrt{3}\eta(t)}{2})\right].$$

Then the most likely path becomes

$$\phi_2^*(t)=\begin{cases}\lambda(\tau-3t_0)t, & \text{if } t\in[0,t_0];\\ \lambda\left[\tau t-\zeta_1(t)\right], & \text{if } t\in(t_0,\tau];\\ \lambda\left[\tau^2-\zeta_1(\tau)\right], & \text{if } t\in(\tau,1].\end{cases}$$

With this most likely path, the minimum energy is

$$E_2^*=\frac{1}{8t_0^3},$$

and the constraint function is

$$F_2(t) = \begin{cases} \lambda(\tau - 3t_0)(t^2 - 3t_0 t), & \text{if } t \in [0, t_0]; \\ \lambda\left[\tau t^2 + 4t_0^3 - (t+\tau)\zeta_1(t) + \zeta_2(t)\right], & \text{if } t \in (t_0, \tau]; \\ \lambda\left[\tau^2 t + 4t_0^3 - (t+\tau)\zeta_1(\tau) + \zeta_2(\tau)\right], & \text{if } t \in (\tau, 1]. \end{cases}$$

The graphs for this most likely path and the corresponding constraint function are showed in figure 3.8 and figure 3.9, respectively.

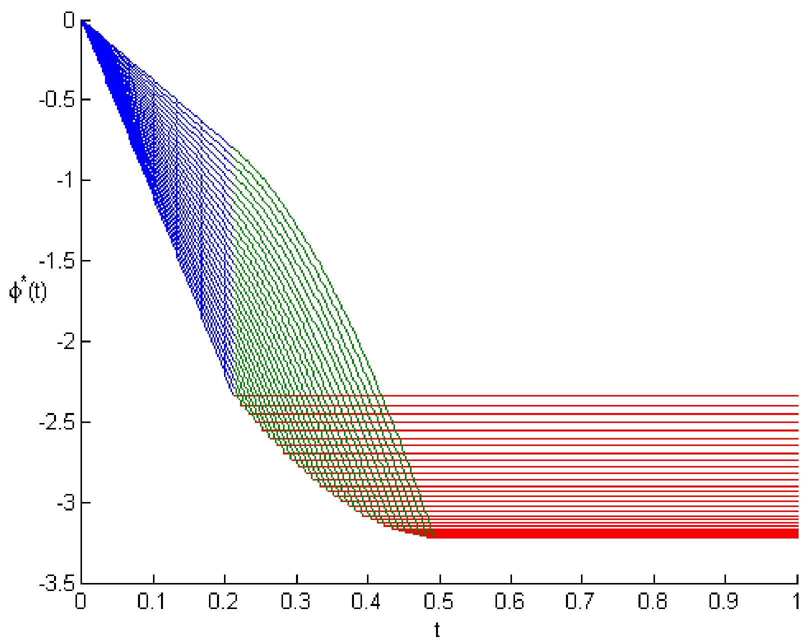

FIGURE 3.8. Most likely path under optimal hedging strategy if $\tau \in (t_0, \frac{1}{2}]$

(3) If $\tau \in (\frac{1}{2}, 1]$: let $t_1 = \frac{1}{2}$.

$$\phi_3^*(t) = \begin{cases} (3Bt_0 + C_1)t, & \text{if } t \in [0, t_0]; \\ (3Bt_0 + C_1)t_0 + \int_{t_0}^{t} [Bg(s) + C_2]\, ds, & \text{if } t \in (t_0, \frac{1}{2}]; \\ (3Bt_0 + C_1)t_0 + \int_{t_0}^{t_1} [Bg(s) + C_2]\, ds + (B + C_3)(t - t_1), & \text{if } t \in (\frac{1}{2}, \tau]; \\ (3Bt_0 + C_1)t_0 + \int_{t_0}^{t_1} [Bg(s) + C_2]\, ds + (B + C_3)(\tau - t_1), & \text{if } t \in (\tau, 1]. \end{cases}$$

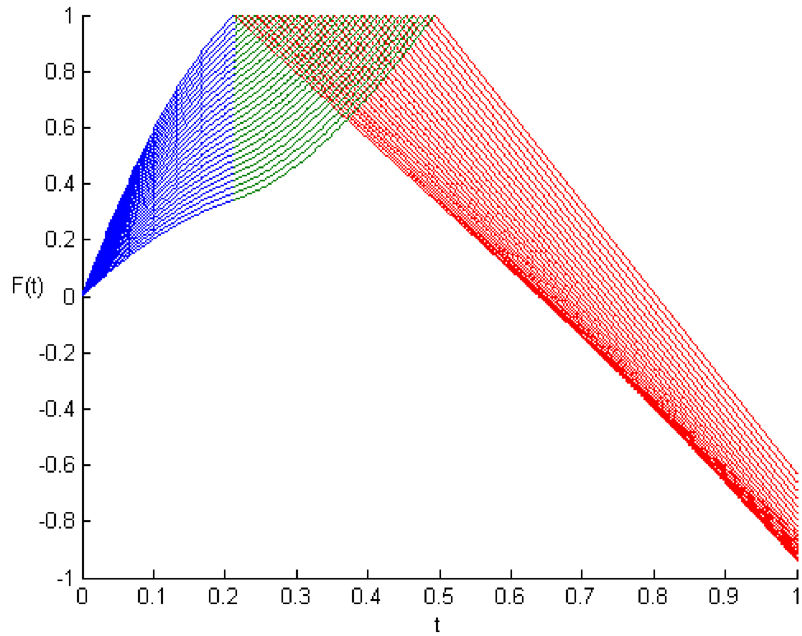

FIGURE 3.9. Constraint function under optimal hedging strategy if $\tau \in (t_0, \frac{1}{2}]$

Then

$$\dot{\phi}_3^*(t) = \begin{cases} 3Bt_0 + C_1, & \text{if } t \in [0, t_0]; \\ Bg(t) + C_2, & \text{if } t \in (t_0, \frac{1}{2}]; \\ B + C_3, & \text{if } t \in (\frac{1}{2}, \tau]; \\ 0, & \text{if } t \in (\tau, 1]; \end{cases}$$

$$E_3 = \frac{1}{2}\left\{[3Bt_0 + C_1]^2 t_0 + \int_{t_0}^{t_1} [Bg(t) + C_2]^2 \, dt + (B + C_3)^2(\tau - t_1)\right\};$$

$$F_3(t) = t_0(3Bt_0 + C_1)(t - 3t_0) + \int_{t_0}^{t_1} [Bg(s) + C_2]\,[t - g(s)]\, ds + (B + C_3)(t - 1)(t - t_1);$$

and

$$f_3 = F_3(\tau) - 1.$$

By the method of Lagrange Multipliers, we have

$$\begin{cases} \nabla E_3 = \lambda \nabla f_3, \\ f_3 = 0. \end{cases}$$

Since

$$\begin{cases} \frac{\partial E_3}{\partial B} = 3t_0^2(3Bt_0 + C_1) + \int_{t_0}^{t_1} \left[Bg(t) + C_2\right] g(t)dt + (B + C_3)(\tau - t_1), \\ \frac{\partial E_3}{\partial C_1} = t_0(3Bt_0 + C_1), \\ \frac{\partial E_3}{\partial C_2} = \int_{t_0}^{t_1} \left[Bg(t) + C_2\right] dt, \\ \frac{\partial E_3}{\partial C_3} = (B + C_3)(\tau - t_1), \\ \frac{\partial f_3}{\partial B} = 3t_0^2(\tau - 3t_0) + \int_{t_0}^{t_1} \left[\tau - g(t)\right] g(t)dt + (\tau - 1)(\tau - t_1), \\ \frac{\partial f_3}{\partial C_1} = t_0(\tau - 3t_0), \\ \frac{\partial f_3}{\partial C_2} = \int_{t_0}^{t_1} \left[\tau - g(t)\right] dt, \\ \frac{\partial f_3}{\partial C_3} = (\tau - 1)(\tau - t_1), \end{cases}$$

we have

$$\begin{cases} 3t_0^2(3Bt_0 + C_1) + \int_{t_0}^{t_1} \left[Bg(t) + C_2\right] g(t)dt + (B + C_3)(\tau - t_1) \\ = \lambda \left\{ 3t_0^2(\tau - 3t_0) + \int_{t_0}^{t_1} \left[\tau - g(t)\right] g(t)dt + (\tau - 1)(\tau - t_1) \right\}, \\ t_0(3Bt_0 + C_1) = \lambda t_0(\tau - 3t_0), \\ \int_{t_0}^{t_1} \left[Bg(t) + C_2\right] dt = \lambda \int_{t_0}^{t_1} \left[\tau - g(t)\right] dt, \\ (B + C_3)(\tau - t_1) = \lambda(\tau - 1)(\tau - t_1), \\ t_0(3Bt_0 + C_1)(\tau - 3t_0) + \int_{t_0}^{t_1} \left[Bg(t) + C_2\right] \left[\tau - g(t)\right] dt + (B + C_3)(\tau - 1)(\tau - t_1) - 1 = 0. \end{cases}$$

For the same argument in case (2), we have

$$B = -\lambda$$

and

$$C_1 = C_2 = C_3 = \lambda\tau.$$

Substitute B, C_1, C_2 and C_3 into $f_3 = 0$, we get

$$\begin{aligned}\lambda &= \frac{1}{t_0(\tau-3t_0)^2 + \int_{t_0}^{t_1}\left[\tau - g(t)\right]^2 dt + (\tau-1)^2(\tau-t_1)} \\ &= \frac{1}{4t_0^3 + \tau^3 - 2\tau^2 + \frac{5}{4}\tau - \frac{1}{4}},\end{aligned}$$

and

$$E_3 = \frac{1}{2}\lambda.$$

Let $M(\tau) = \tau^3 - 2\tau^2 + \frac{5}{4}\tau - \frac{1}{4}$, then

$$\dot{M}(\tau) = 3\tau^2 - 4\tau + \frac{5}{4}.$$

Setting $\dot{M}(\tau) = 0$, we get $\tau = \frac{1}{2}$ and $\tau = \frac{5}{6}$. Since $\ddot{M}(\tau) = 6\tau - 4$, $M(\tau)$ attains its maximum at $\tau = \frac{1}{2}$ and its minimum at $\tau = \frac{5}{6}$. But $\frac{1}{2} \notin (t_1, 1]$. And $M(\tau)$ attains its maximum at the endpoint $\tau = 1$ as well.

So

$$\lambda = \frac{1}{4t_0^3}.$$

Thus the most likely path is

$$\phi_3^*(t) = \begin{cases} \lambda(1-3t_0)t, & \text{if } t \in [0, t_0]; \\ \lambda\left[t - \zeta_1(t)\right], & \text{if } t \in (t_0, \frac{1}{2}]; \\ \frac{1}{8}\lambda, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

With this most likely path, the minimum energy is

$$E_3^* = \frac{1}{8t_0^3},$$

and the constraint function is

$$F_3(t) = \begin{cases} \lambda(1-3t_0)(t^2-3t_0t), & \text{if } t \in [0, t_0]; \\ \lambda\left[t^2+4t_0^3-(t+1)\zeta_1(t)+\zeta_2(t)\right], & \text{if } t \in (t_0, \frac{1}{2}]; \\ \lambda\left[\frac{1}{8}t+4t_0^3-\zeta_1(\frac{1}{2})+\zeta_2(\frac{1}{2})\right], & \text{if } t \in (\frac{1}{2}, 1]; \end{cases}$$

with its maximum at $t = 1$.

The graphs for this most likely path and the corresponding constraint function are showed in figure 3.10 and figure 3.11, respectively.

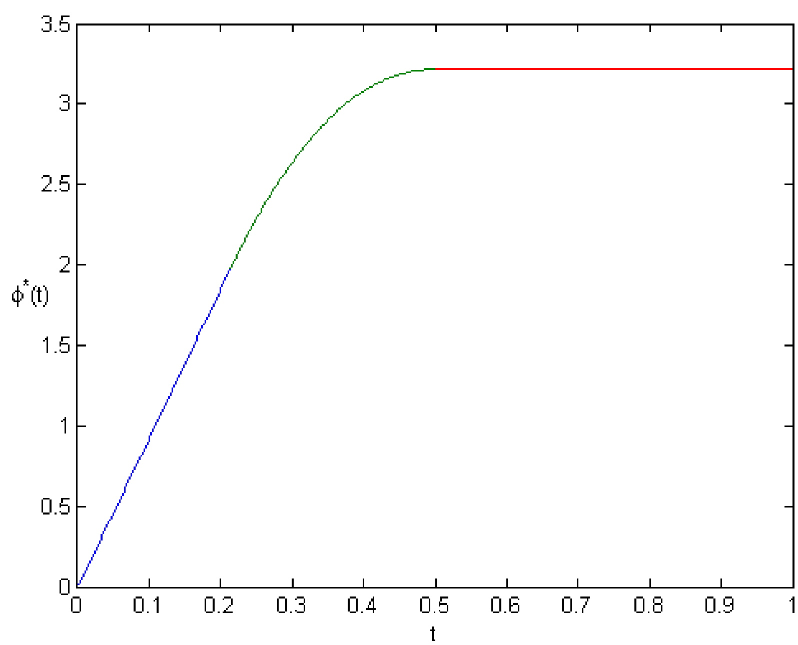

FIGURE 3.10. Most likely path under optimal hedging strategy if $\tau = 1$

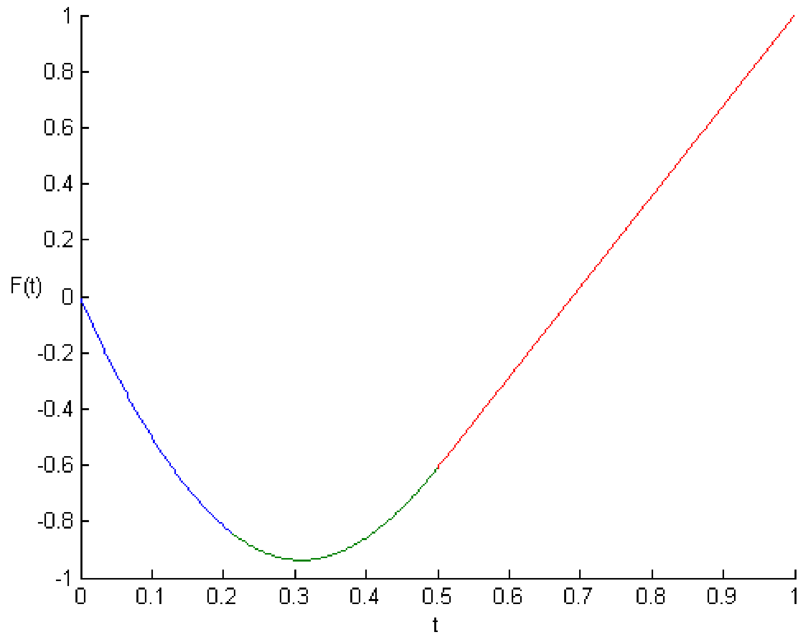

FIGURE 3.11. Constraint function under optimal hedging strategy if $\tau = 1$

Therefore, under the optimal hedging strategy, the minimum energy is

$$E^* = E_1^* = E_2^* = E_3^* = \frac{1}{8t_0^3}.$$

□

LEMMA 3.2.2.

$$\lambda = \frac{1}{t_0(\tau - 3t_0)^2 + \int_{t_0}^{\tau} [\tau - g(t)]^2 \, dt} = \frac{1}{4t_0^3}.$$

PROOF. Since

$$\lambda = \frac{1}{t_0(\tau - 3t_0)^2 + \int_{t_0}^{\tau} [\tau - g(t)]^2 \, dt},$$

then

$$\frac{1}{\lambda} = t_0(\tau - 3t_0)^2 + \int_{t_0}^{\tau} [\tau - g(t)]^2 \, dt$$
$$= \tau^3 - 6\tau t_0^2 + 9t_0^3 - 2\tau \int_{t_0}^{\tau} g(t)dt + \int_{t_0}^{\tau} g^2(t)dt.$$

Since

$$\tau = \frac{1}{\sqrt{3}} e^{-\frac{\eta(\tau)}{2}} \cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}),$$
$$\int_{t_0}^{\tau} g(t)dt = \frac{1}{4\sqrt{3}} e^{-\eta(\tau)}[\cos(\sqrt{3}\eta(\tau) + \frac{\pi}{6}) + \sqrt{3}] - 3t_0^2,$$

and

$$\int_{t_0}^{\tau} g^2(t)dt$$
$$= -\frac{1}{12\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \left[\sin(\frac{3\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{3}) + \sqrt{3}\sin(\frac{\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{6}) - 2\sqrt{3}\cos(\frac{\sqrt{3}\eta(\tau)}{2}) \right] - 5t_0^3,$$

then

$$\frac{1}{\lambda} = \tau^3 + 4t_0^3 - \frac{1}{6} e^{\frac{-3\eta(\tau)}{2}} \cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6})[\cos(\sqrt{3}\eta(\tau) + \frac{\pi}{6}) + \sqrt{3}]$$
$$- \frac{1}{12\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \left[\sin(\frac{3\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{3}) + \sqrt{3}\sin(\frac{\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{6}) - 2\sqrt{3}\cos(\frac{\sqrt{3}\eta(\tau)}{2}) \right].$$

Since

$$\tau^3 = \frac{1}{3\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \cos^3(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6})$$
$$= \frac{1}{6\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \cos(\sqrt{3}\eta(\tau) + \frac{\pi}{3}) + \frac{1}{6\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}),$$

then

$$
\begin{aligned}
\frac{1}{\lambda} &= 4t_0^3 + \frac{1}{6\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \cos(\sqrt{3}\eta(\tau) + \frac{\pi}{3}) + \frac{1}{6\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \\
&- \frac{1}{6} e^{\frac{-3\eta(\tau)}{2}} \cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6})[\cos(\sqrt{3}\eta(\tau) + \frac{\pi}{6}) + \sqrt{3}] \\
&- \frac{1}{12\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} \left[\sin(\frac{3\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{3}) + \sqrt{3}\sin(\frac{\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{6}) - 2\sqrt{3}\cos(\frac{\sqrt{3}\eta(\tau)}{2}) \right] \\
&= 4t_0^3 - \frac{1}{12\sqrt{3}} e^{-\frac{3\eta(\tau)}{2}} W
\end{aligned}
$$

where

$$
\begin{aligned}
W &= -2\cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \cos(\sqrt{3}\eta(\tau) + \frac{\pi}{3}) - 2\cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \\
&+ 2\sqrt{3}\cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \cos(\sqrt{3}\eta(\tau) + \frac{\pi}{6}) + 6\cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \\
&+ \sin(\frac{3\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{3}) + \sqrt{3}\sin(\frac{\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{6}) - 2\sqrt{3}\cos(\frac{\sqrt{3}\eta(\tau)}{2}) \\
&= \sin(\frac{3\sqrt{3}\eta(\tau)}{2}) - \sqrt{3}\cos(\frac{\sqrt{3}\eta(\tau)}{2}) \\
&+ \sqrt{3}\cos(\frac{3\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{3}) + 3\cos(\frac{\sqrt{3}\eta(\tau)}{2} + \frac{\pi}{6}) \\
&+ \sin(\frac{3\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{3}) + \sqrt{3}\sin(\frac{\sqrt{3}\eta(\tau)}{2} - \frac{\pi}{6}) = 0.
\end{aligned}
$$

Thus,

$$
\lambda = \frac{1}{4t_0^3}.
$$

□

Under the optimal hedging strategy, the spot variance is

$$
\sigma_t^2 = \int_0^t (G(s) + s - t)^2 ds,
$$

so

$$\sigma_t^2 = \begin{cases} (t-3t_0)^2 t, & \text{for } t \in [0, t_0]; \\ 4t_0^3, & \text{for } t \in (t_0, \frac{1}{2}]; \\ t^3 - 2t^2 + \frac{5}{4}t + 4t_0^3 - \frac{1}{4}, & \text{for } t \in (\frac{1}{2}, 1]. \end{cases}$$

We also compared the constraints for these most likely paths and the spot variance under the optimal hedging strategy, and found that the times when the constraints attain the maximum are the times when the spot variance attains its maximum. The comparison is showed in figure 3.12.

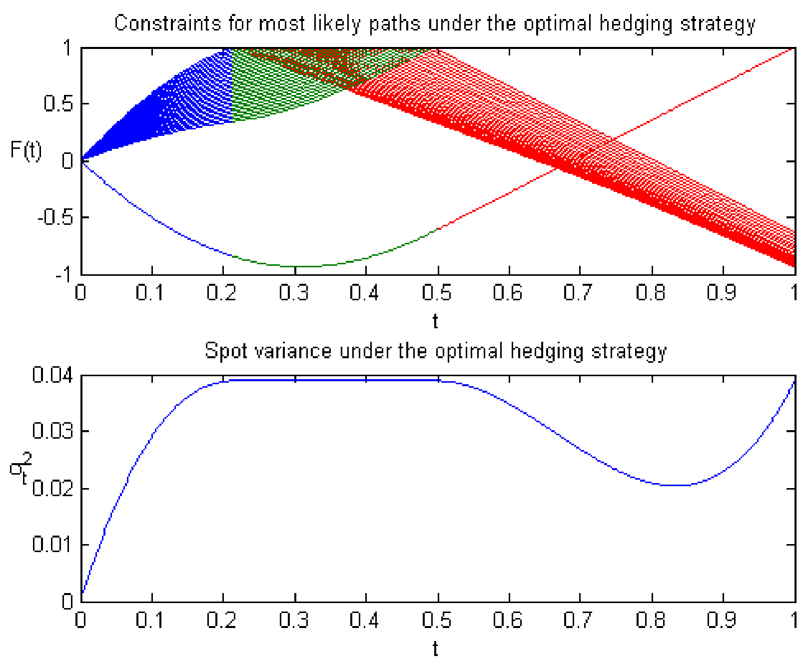

FIGURE 3.12. Comparison of constraint functions for most likely paths and spot variance under optimal hedging strategy

3.3. Future Research

Recall the comparisons of constraints for most likely paths and spot variance under both the optimal-fraction strategy and the optimal strategy in figure 3.13, we can see that the times at which the constraints attains the maximum 1 are the times,

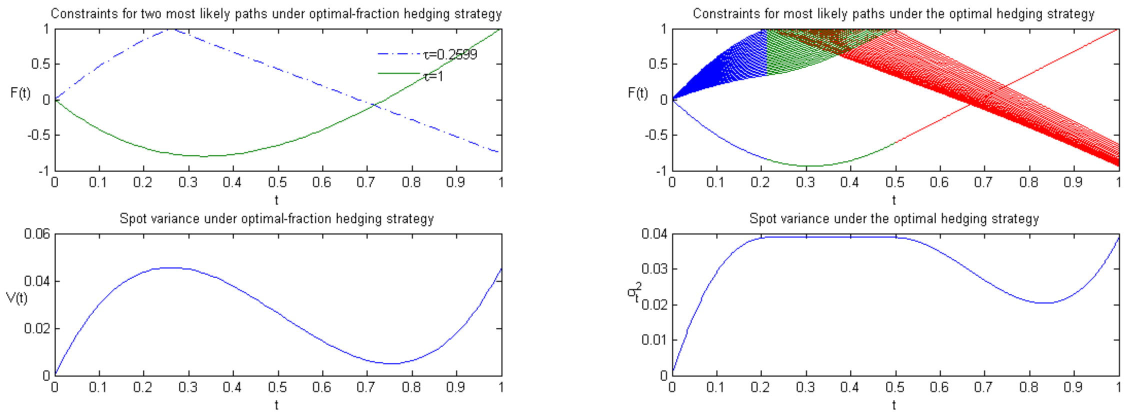

FIGURE 3.13. Comparisons of constraint functions for most likely paths and spot variance under optimal-fraction and optimal hedging strategies

noted by t_σ, at which the spot variance attains its maximum under both strategies, i.e.

$$\tau = t_\sigma.$$

This phenomena can be further studied.

In addition, it would be desirable to consider that the commodity price follows geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

and a general model with mean reversion

$$dS_t = -\alpha(S_t - c_t)dt + \sigma dW_t,$$

where $0 \leq \alpha < 1$ measures the speed of mean reversion, c_t is the level toward which the price reverts at time t.

Furthermore, to determine whether the most likely paths we found are relevant to the original setting, we can simulate the original discrete-time model, with and without mean reversion, under different hedging strategies. It would also be interesting to find the most likely path under Wu, Yu and Zheng (2011)'s optimal hedging strategy under the constraint of terminal risk. It will give us a dynamic solution for the most likely path.

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