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Analyticity of the charge-monopole scattering amplitude

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We study the analyticity in $\cos\theta$ of the exact quantum-mechanical electric-charge–magnetic-monopole scattering amplitude by ascribing meaning to its formally divergent partial-wave expansion as the boundary value of an analytic function. This permits us to find an integral representation for the amplitude which displays its analytic structure. On the physical sheet we find only a branch-point singularity in the forward direction, while on each of the infinitely many unphysical sheets we find a logarithmic branch-point singularity in the backward direction as well as the same forward structure.

I. INTRODUCTION

In investigating a new semiclassical approach to the scattering of electric charges by magnetic monopoles,[1] we became interested in the analyticity of the scattering amplitude in the cosine of the scattering angle θ. There are several reasons to expect nontrivial analytic structure for this amplitude. First there is a violation of crossing symmetry in charge-monopole scattering.[2] Second, this system exhibits infinitely many "rainbow" angles θ_n where the number of trajectories which contribute to the classical cross section at θ_n increases with n.[3] Third, at these rainbow angles the classical scattering cross section is divergent.[3] And fourth, the rainbow angles θ_n have a limit point at $\theta_n \to \pi$ as $n \to \infty$ (the backward "glory"). It is intriguing to ask what effects these classical features have on the quantum-mechanical amplitude.

To a large extent an answer to this question already exists. In their study of a semiclassical approximation to charge-monopole scattering, Ford and Wheeler[4] point out that apart from a forward "Rutherford" pole, the quantum-mechanical amplitude is continuous at the rainbow angles and even at the glory. The classical features are recovered in the limit that the Dirac quantization number $eg/4\pi$ becomes infinite. So it is already clear that the quantum-mechanical amplitude is much better behaved than the classical one and it only remains to ask whether there are any effects of the classical features on the *analytic* structure of the quantum amplitude.

Section II exhibits the exact quantum-mechanical scattering amplitude $f(\cos\theta)$ as a divergent series and defines the prescription for summing it. In Sec. III we find an integral representation for $f(\cos\theta)$. In Sec. IV we study the singularities of the integrand of this representation and determine their effect on the analytic structure of the integral.

II. THE SCATTERING AMPLITUDE

The Schrödinger equation for this system has been solved by Tamm[5] and the scattering amplitude given by Banderet.[6] The result is

$$f(x) = \frac{e^{-i\pi\mu}}{ik}\left(\frac{1-x}{2}\right)^{\mu} \times \sum_{l=0}^{\infty} (l+\mu+\tfrac{1}{2}) e^{-i\pi\Lambda(l+\mu+1/2)} P_l^{2\mu,0}(x) \qquad (1)$$

where $\hbar = c = 1$ and

$$\mu \equiv |eg|/4\pi, \quad \Lambda(\rho) \equiv (\rho^2-\mu^2)^{1/2} - \rho, \quad x \equiv \cos\theta, \qquad (2)$$

with $(\rho^2-\mu^2)^{1/2} > 0$ for $\rho > \mu$. Here θ is the scattering angle, e and g are electric and magnetic charges, μ is a positive integer or half-integer (the Dirac quantization number), and $P_l^{2\mu,0}(x)$ is the Jacobi polynomial.[7]

As Ford and Wheeler have pointed out,[4] (1) is a formally divergent series which can be summed by the methods of Euler.[8] They split off the leading behavior (in l) which they call the "Rutherford" sum[9]

$$f_1(x) \equiv -\frac{1}{ik}\left(\frac{1-x}{2}\right)^{\mu} \sum_{l=0}^{\infty}(l+\mu+\tfrac{1}{2})e^{-i\pi\mu}P_l^{2\mu,0}(x) . \quad (3)$$

This series they sum via Euler's method to

$$f_1(x) = e^{-i\pi\mu}\,\frac{i\mu}{2k}\left(\frac{1-x}{2}\right)^{-1} \quad (4)$$

With the "Rutherford" series f_1 removed, the termwise difference $f-f_1$ becomes conditionally convergent for $-1<x<1$, and Ford and Wheeler sum it numerically to arrive at their curves for the exact quantum-mechanical differential cross section.

A comparison of the series (3) with the generating function for the Jacobi polynomials[7] suggests a prescription for summing the series originally due to Abel.[8] We apply the method to the termwise difference $f-f_1$.

Consider a related series

$$F(x,w) \equiv \sum_{l=0}^{\infty}(l+\mu+\tfrac{1}{2})[e^{-i\pi\Lambda(l+\mu+1/2)}-1]P_l^{2\mu,0}(x)w^l \quad (5)$$

(for $-1<x<1$) which converges when $0<w<1$. Observe that formally

$$f(x)-f_1(x) = \frac{i}{k}\,e^{-i\pi\mu}\left(\frac{1-x}{2}\right)^{\mu}F(x,w)\bigg|_{w\to 1-} . \quad (6)$$

Abel's prescription as applied here consists in first summing the series $F(x,w)$ with $0<w<1$ and then taking the limit $w\to 1-$. This prescription applied to (3) yields (4). We propose to adopt (6) as the definition of $f-f_1$ and investigate the analytic structure in x of the right-hand side.

III. THE INTEGRAL REPRESENTATION

It is possible to find an integral representation for $F(x,w)$ by introducing a Laplace transform. We sketch the method.

Define

$$\psi(\rho) = \rho e^{-i\pi\Lambda(\rho)} - \rho - i\pi\mu^2/2 \quad (7)$$

with $\Lambda(\rho)$ defined above in (1). Note that $\psi(\rho)$ is certainly regular for $\mathrm{Re}\rho>\mu$ and $\psi(\rho)\to 0$ as $|\rho|\to\infty$. So $\psi(\rho)$ can be written as the Laplace transform of a function we will call χ:

$$\psi(\rho) = \int_0^{\infty} dy\, e^{-\rho y}\chi(y), \quad \mathrm{Re}\rho>\mu . \quad (8)$$

We may insert (7) into expression (5) for $F(x,w)$ to find

$$F(x,w) = +\frac{i\pi\mu^2}{2}\sum_{l=0}^{\infty}P_l^{2\mu,0}(x)w^l + \sum_{l=0}^{\infty}\psi(l+\mu+\tfrac{1}{2})P_l^{2\mu,0}(x)w^l . \quad (9)$$

The first series in (9) is just the generating function for the Jacobi polynomials[7] and can be summed to

$$F_0(x,w) \equiv \sum_{l=0}^{\infty}P_l^{2\mu,0}(x)w^l = 2^{+2\mu}R^{-1}(x,w)[1-w+R(x,w)]^{-2\mu}, \quad (10)$$

$$R(x,w) \equiv (1-2xw+w^2)^{1/2}.$$

Here $R(x,w)>0$ for $-1<x<1$ and $0<w<1$.

If we denote the second series in (9) by $F_1(x,w)$ and insert the Laplace integral (8) for ψ, we find upon interchanging summation and integration

$$F_1(x,w) = \int_0^{\infty}dy\,\chi(y)\sum_{l=0}^{\infty}e^{-(l+\mu+1/2)y}P_l^{2\mu,0}(x)w^l = \int_0^{\infty}dy\,\chi(y)e^{-(\mu+1/2)y}F_0(x,we^{-y}) .$$

The last step follows from the definition (10) of F_0. Change the integration variable to $t=e^{-y}$ and complete Abel's prescription by taking $w\to 1-$ to arrive at our final expression for $F(x,1)$:

$$F(x,1) = \frac{i\pi\mu^2}{4}\left(\frac{1-x}{2}\right)^{-\mu-1/2} + \int_0^1 dt\,\chi(-\ln t)t^{\mu-1/2}F_0(x,t) . \quad (11)$$

Finally, since we have defined χ by (8) as an inverse Laplace transform of ψ we may hope to find a closed-form expression for it. This is in fact possible, and the result is[10]

$$\chi(y) = i\pi\mu^2\,\frac{y+\pi i}{y^2+2\pi i}\,I_2(\mu(y^2+2\pi i y)^{1/2}) . \quad (12)$$

Thus our result for the integral representation of the scattering amplitude $f(x)$ is

$$f(x) = \frac{i\mu}{2k}e^{-i\pi\mu}\left(\frac{1-x}{2}\right)^{-1} - \frac{\pi\mu^2}{4k}e^{-i\pi\mu}\left(\frac{1-x}{2}\right)^{-1/2} + \frac{i}{k}e^{-i\pi\mu}(2-2x)^{\mu}\int_0^1 dt\,t^{\mu-1/2}\chi(-\ln t)R^{-1}(x,t)[1-t+R(x,t)]^{-2\mu} \quad (13)$$

with $R(x,t)$ defined in (10).

As is well known, the amplitude $f(x)$ has the kinematical singularities of a helicity amplitude.[11] The kinematical singularity-free amplitude is $[(1-x)/2]^{-\mu}f(x)$. The remainder of this paper is devoted to the study of the analytic structure of $F(x,1)$ which is related to $[(1-x)/2]^{-\mu}[f(x)-f_1(x)]$ through x-independent factors.

We may remark here that $[(1-x)/2]^{-\mu}f(x)$ does not have a simple pole at $x=1$. Its leading singularity as $x\to 1$ is given by the contribution from f_1 [cf. Eq. (4)]. It is a multiple pole or a branch point according to whether μ is integral or half-integral. (We assume of course that $\mu\neq 0$.) Compare this with the Coulomb amplitude which has a forward pole at $x=1$ as well as a forward branch point, but the latter appears only in an overall phase, so there is no effect of this singularity on the scattering cross section. We shall see in the next section that the remaining term $F(x,1)$ also has unusual analytic properties in x as compared to the Coulomb amplitude.

IV. ANALYTIC STRUCTURE OF $F(x,1)$

The leading singularity of $F(x,1)$ at $x=1$ is given by the first term of Eq. (11) [cf. Eq. (16) below]. It is a pole or a branch point depending on the value of μ. Next consider the integral in (11):

$$I(x)=\int_0^1 dt\, t^{\mu-1/2}\chi(-\ln t)R^{-1}(x,t)\times[1-t+R(x,t)]^{-2\mu}. \tag{14}$$

This integral is originally defined for $-1<x<1$. The integration in t is along the real axis. The appropriate branch of R is defined after (10).

We shall now study the analytic continuation of $I(x)$. In particular, we shall (a) show that $I(x)$ is holomorphic in the plane cut along the real axis from 1 to ∞, (b) determine the nature of the branch point at $x=1$, (c) analytically continue $I(x)$ to a second sheet and show that it has a branch point at $x=-1$ on this sheet, (d) determine the nature of this branch point.

The singularities of $I(x)$ will occur when there are end-point or pinch singularities in the integral.[12] In the absence of these singularities, the analytic continuation can be performed by contour deformation.[12] Thus we need to locate the singularities of the integrand and determine their motion in the complex t plane as functions of x.

An inspection of (14) reveals three singularities of the integrand. First there is a fixed (in x) branch point at $t=0$ due to the logarithm in the argument of χ and the presence of $t^{\mu-1/2}$. Second there is a pair of moving singularities located at

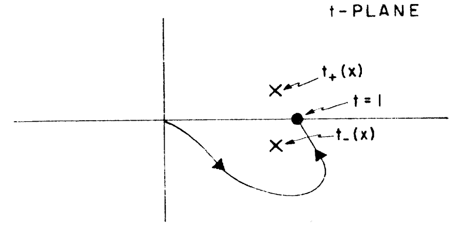

FIG. 1. Contour specifying $I(x)$ on an adjacent Riemann sheet.

$$t_\pm(x)=x\pm(x^2-1)^{1/2} \tag{15}$$

where R has zeros. Note that $t_+t_-=1$ and that the point $t=x$ bisects the line connecting $t_+(x)$ and $t_-(x)$. [There is also a potential singularity due to the vanishing of $1-t+R$ at $t=0$, but with the phase choice for R given after (10) this does not occur.]

By (15) it is apparent that there is an end-point singularity at $x=1$ since $t_\pm(1)=1$. In addition there is another coincidence of $t_\pm$ at $t=-1$. This can lead to a pinch singularity if the colliding branch points manage to catch the integration contour between them. We will now study when these possibilities actually occur and take up (a) to (d) above.

(a) The analytic continuation of $I(x)$ into the complex x plane is not single valued. For if x makes a small circuit of 2π about $x=1$, $t_\pm(x)$ rotate about $t=1$ by π and pull the contour around with them arriving at the situation of Fig. 1. This multivaluedness of $I(x)$ can be avoided by attaching a branch cut in the complex x plane to $x=1$ and extending it along the real axis to $x=\infty$; then $t_\pm(x)$ will always avoid the integration contour and $I(x)$ will be single valued in this cut x plane.

With $I(x)$ defined as above, it is also apparent that there can be no pinch singularity at $x=-1$ (on this Riemann sheet) since this would require one of $t_\pm(x)$ to cross the contour and drag it to $t=-1$. With the cut in x along $[1,\infty)$ this is plainly impossible. This proves (a).

(b) We can also determine the behavior of $I(x)$ when x is near the branch point at $x=1$. This only requires expanding $\chi(-\ln t)t^{\mu-1/2}$ in a power series about $t=1$ (where it is clearly analytic). Exchange the resulting sum and integration and evaluate the integrals term by term. The result is that in a neighborhood of $x=1$,

$$I(x)=f_1((1-x)/2)\ln(1-x)/2+[(1-x)/2]^{-\mu}f_2([(1-x)/2]^{1/2}), \tag{16}$$

where $f_1(z)$ and $f_2(z)$ are analytic at $z=0$.

(c) The integral representation (14) also allows us to determine the structure of $I(x)$ as an extended analytic function on its other Riemann sheets. Begin with x on the real axis near and to the left of x

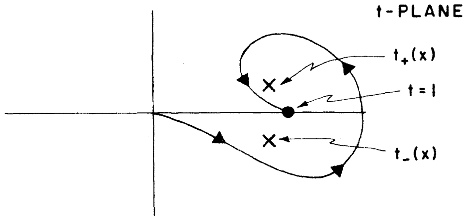

FIG. 2. Contour obtained from Fig. 3 after x executes a circuit of 4π about $x=1$.

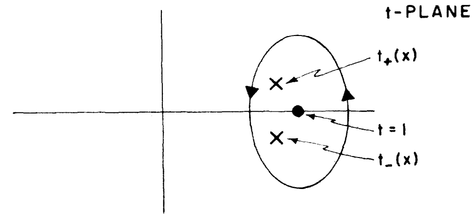

FIG. 4. Contour specifying the change in $I(x)$ after x makes a circuit of 4π about $x=1$. (The difference of integrals defined by Fig. 2 and Fig. 3.)

$=1$ and let x make a small counterclockwise circle about $x=1$ to an adjacent Riemann sheet. Then the singularities $t_\pm(x)$ execute counterclockwise semicircles about $t=x$ (with $t=1$ interior to this circle) and exchange positions—one passing through the contour which must be deformed to avoid it and the other passing around the right of $t=1$. The resulting configuration in the t plane is exhibited in Fig. 1.

Figure 1 defines $I(x)$ on an adjacent Riemann sheet. This one also has an end-point singularity at $x=1$. Furthermore, another rotation by π clockwise or counterclockwise yields different results so that $x=1$ is again a branch point and there is again a cut along the real axis, $1\leq x<\infty$. Another 2π counterclockwise rotation of x about $x=1$ (beginning from Fig. 1) brings us to Fig. 2. A comparison of Fig. 3 and Fig. 2 shows that the difference between the integrals they define is just another contour integral defined by Fig. 4. So a counterclockwise rotation of x from the first (physical) sheet by 4π about $x=1$ yields $I(x)$ again *plus* an additional function defined by Fig. 4. This is precisely the kind of behavior expected from (16).

$I(x)$ on the sheet defined by Fig. 1 (and indeed on all sheets but the first) has an additional singularity at $x=-1$. This time the possibility raised above of a pinch singularity at $x=-1$ is realized. To see this consider what happens to Fig. 1 as x moves along the real axis from $x=1$ toward $x=-1$. The two singularities in Fig. 1 move in opposite directions around the unit circle and collide at $t=-1$ as $x\to-1+$. The resulting pinch is illustrated in Fig. 5. So $x=-1$ is a singular point. If we continue to move x to real values to the left of $x=-1$, we need to pass either above or below the singularity, and the resulting integral depends on the choice. So $I(x)$ on this sheet (defined by Fig. 1) also has a singularity at $x=-1$ and requires a branch cut along x real and $x\leq-1$, say.

(d) The nature of this singularity is determined as before by making a small circuit of 2π about $x=-1$ and observing that $I(x)$ is recovered again with the addition of a contour integral defined by Fig. 6. So this singularity behaves as

$$I(x)=f_3((1+x)/2)\ln(1+x)/2+f_4((1+x)/2) \qquad (17)$$

in a neighborhood of $x=-1$. $f_3(z)$ and $f_4(z)$ are once more analytic at $z=0$. Equation (17) may also be verified by methods analogous to those which led to (16).

Thus a clear picture of the analytic structure of $I(x)$ emerges. On the first sheet (Fig. 3), there is a branch point at $x=1$ with a cut along the real axis $1\leq x<\infty$. The behavior of $I(x)$ near the branch point is described by (16). On each of the other sheets, in addition to the branch point and associated cut beginning at $x=1$, there is an additional branch point at $x=-1$ with a cut along real x, $-\infty<x\leq-1$. The behavior of $I(x)$ near $x=-1$ on these sheets is described by (17).

The change in $I(x)$ on any sheet after a small circuit of 4π about $x=1$ is given by a contour integral defined by Fig. 4, while the change after a small

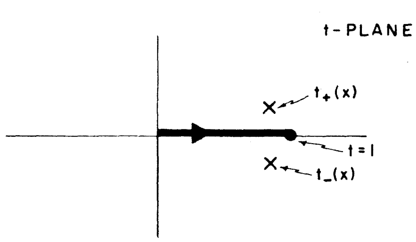

FIG. 3. Contour in the complex t plane defining the integral $I(x)$ on its physical sheet.

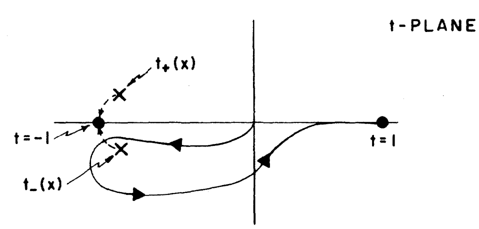

FIG. 5. Contour illustrating the pinch singularity of $I(x)$ at $x=-1$ on the Riemann sheet defined by Fig. 1. Arrows show the motion of $t_\pm(x)$ as $x\to-1+$.

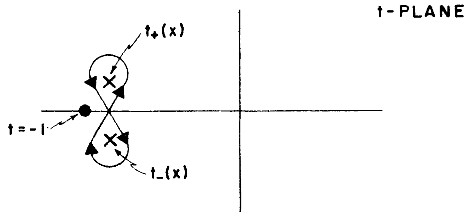

FIG. 6. Contour specifying the change in $I(x)$ on sheet of Fig. 1 after x executes a circuit of 2π about $x=-1$.

circuit about $x=-1$ by 2π is defined by Fig. 6.

One more remark is in order here. The analyticity we describe is that of $I(x)$, not $f(x)$. Recall the relation between these defined by (13). In particular the singularity $[(1-x)/2]^{-\mu}$ in $I(x)$ at $x=1$ does not appear in $f(x)$. Apart from this the discussion above for $I(x)$ also applies to $f(x)$.

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