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Hidden quantum group structure in Einstein's general relativity

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Abstract

A new formal scheme is presented in which Einstein's classical theory of General Relativity appears as the common, invariant sector of a one-parameter family of different theories. This is achieved by replacing the Poincaré group of the ordinary tetrad formalism with a q -deformed Poincaré group, the usual theory being recovered at $q = 1$. Although written in terms of *non-commuting* vierbein and spin-connection fields, each theory has the same metric sector leading to the ordinary Einstein–Hilbert action and to the corresponding equations of motion. The Christoffel symbols and the components of the Riemann tensor are ordinary *commuting* numbers and have the usual form in terms of a metric tensor built as an appropriate bilinear in the vierbeins. Furthermore, we exhibit a one-parameter family of Hamiltonian formalisms for general relativity, by showing that a canonical formalism à la Ashtekar can be built for any value of q . The constraints are still polynomial, but the Poisson brackets are *not* skewsymmetric for $q \neq 1$.

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1. Introduction

Currently, one of the most fascinating and challenging problems in gravity involves understanding the nature of space–time at distance scales where quantum mechanical

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effects enter. In the past decade, string theory has addressed this issue, but we will not be concerned with the string theory point of view here. Rather, we will be interested in approaching the problem in the general framework of field theory. In this context, a necessary step for going beyond the classical theory is to pick the “right set of variables” to construct a canonical formalism suitable for quantization. This task is both essential and non-trivial for a highly non-linear theory such as general relativity. Here one may recall the success obtained in $2 + 1$ Einstein’s gravity from showing its equivalence with a gauge theory described by a pure Chern–Simons action [1]. In fact, the essential progress allowed by the gauge theory formulation was to unravel the structure of the classical phase space opening the way to the canonical formalism and to quantization.

As is well known, it is not possible to formulate Einstein’s gravity in $3 + 1$ dimensions entirely as a Yang–Mills theory. Nevertheless, its description in terms of variables such as vierbeins and spin connections, from which one constructs the metric tensor and the Christoffel symbols, has been a fundamental step forward. In terms of these fields Einstein’s gravity appears as a gauge theory associated with the Poincaré group, although the action only exhibits an invariance under the local Lorentz subgroup (as well as under diffeomorphisms of the space–time manifold) [2,3]. Then gravitational interactions with matter are prescribed by a gauge principle in analogy with all other fundamental interactions as dictated by the Standard Model. Furthermore, following this approach, a much better understanding of the structure of the classical phase space and great simplicity in building a canonical formalism have been achieved. To be more specific, Ashtekar [4] has been able to construct a canonical formalism in which the pull-backs on the “space” manifold of the self-dual part of the spin connections play the rôle of dynamical variables. One can then identify the corresponding conjugate momenta and show that the constraints are polynomial in these variables. However, a reality condition on physical solutions must be further imposed.

There is still one more lesson that can be learned from $2 + 1$ -dimensional gravity. It has recently been shown [5] that the usual equivalence between Einstein’s theory and Chern–Simons theory with local Poincaré group invariance is only a specific case of a more general equivalence. Indeed, Einstein’s theory in $2 + 1$ dimensions (in the absence of a cosmological term) admits a one-parameter family of Chern–Simons formulations, corresponding to the q -deformed Poincaré gauge group [6]. Here, q is either a real dimensionless parameter or a phase and the ordinary Poincaré group, $ISO(2, 1)$ is recovered for $q = 1$. For $q \neq 1$, the system has a non-commutative structure, which we will elaborate on shortly. The question which then naturally arises is whether a similar non-commutative structure can also be present in Einstein’s gravity in $3 + 1$ dimensions. In this paper we give a positive answer to this question by showing that (torsionless) Einstein’s gravity may be formulated as a gauge theory associated with a q -deformed Poincaré group. The dynamics is determined from an action analogous to Palatini’s, and it has the usual local Lorentz invariance. We thus find that the usual description of Einstein’s gravity in terms of vierbeins and spin connections may be extended to a one-parameter family of gauge theories.

It should be stressed that such an equivalence not only holds for the pure gravity case,

but it also holds in the presence of matter, provided there are no sources for torsion. (In fact, it is only a non-zero torsion that distinguishes the different classical theories from one another, each one coupling to a different kind of “exotic” matter.) As a result, for zero torsion, our one-parameter family of systems has the metric sector of the theory in common. It is the latter which contains all the physically relevant information for *classical gravity*. On the other hand, concerning quantization, for each theory there exists a canonical formalism à la Ashtekar.

Before advancing further in this discussion, a note of warning should be made about the price paid for achieving these results. It concerns the non-commutative structure stated above. A quantum group G_q is defined as the *non-commutative* algebra of functions on the Lie-group G . As a consequence the gauge fields transforming locally under a quantum (or q -deformed) group are *not* ordinary functions, but exhibit non-trivial braiding relations among themselves [8,9]. (Let us stress that, in our approach, space-time is an ordinary manifold labeled by commuting variables.) Vierbeins and spin connections are fields, or more precisely differential one-forms, endowed with non-standard commutation relations. As in the ordinary undeformed theory, one can build a symmetric space-time metric as an appropriate bilinear in the vierbeins. We find that different metric components commute among themselves, but do *not* commute with vierbeins and spin connections. Nevertheless, the Christoffel symbols and the Riemann tensor are given by the usual expression in terms of the metric tensor and its inverse and, furthermore, they commute with *all* fields and therefore can be represented by ordinary numbers.² The entire metric sector is made out of objects which mutually commute and is identical to (torsionless) Einstein’s gravity. The non-commutative structure of the theory cannot then be probed with large scale gravitational experiments where quantum effects are not present. (For further discussion of the physical implications, see [7].)

The formal scheme which we propose, where classical Einstein’s gravity appears as the common, invariant part of a one-parameter family of gauge theories, may be quite interesting in itself, but more interesting are its physical consequences. As we already mentioned, for each of these theories, despite the non-commutative nature of the variables, a canonical formalism à la Ashtekar can be carried out. We find that the notion of self-duality is consistent with the fields braiding relations and the constraints are still polynomial, even if deformed with respect to the usual ones. For $q \neq 1$, the Poisson brackets, however, have new features, such as not being skewsymmetric due to the non-commuting nature of the conjugate variables. It may be too early to say what may be the consequences of having a family of canonical formalisms for general relativity, even if expressed in terms of exotic variables. However, it seems fair to say that the challenging problem of quantizing a deformed gauge theory may be physically relevant.

We begin in Section 2 by introducing the quantum Poincaré group. This quantum group contains the (undeformed) Lorentz group. In Section 2 we also give a heuristic description of a bicovariant differential calculus on the quantum group, which is nec-

² For the case when matter is present, the same result holds for the energy momentum tensor.

essary to formulate the associated gauge theory. The latter is done in Section 3. There we also write down an action principle for gravity which has local Lorentz invariance (as well as diffeomorphism symmetry) and it turns out to be equivalent to the one found by Castellani [9]. We next show how to recover the metric theory, for the case of pure gravity in Section 4, and the case of coupling to matter (with no torsion) in Section 5. We then apply Ashtekar’s procedure in Section 6, and give concluding remarks in Section 7.

2. Bicovariant differential calculus on the quantum Poincaré group

In this section we give a heuristic description of a quantum Poincaré group, which we denote by $ISO_q(3, 1)$, together with a bicovariant differential calculus on it. The procedure is the same as the one followed in Ref. [5] for the case of 2 + 1 dimensions. We refer the reader to it for more details. The mathematical structures discussed here are equivalent to those found in Ref. [6], where we refer the reader for rigorous proofs.

We begin with the 3 + 1 Poincaré group $ISO(3, 1)$. It is most easily described in terms of Lorentz matrices $\ell = [\ell_{ab}]$ and vectors $z = [z_a]$. Roman letters from the beginning of the alphabet, running from 1 to 4, denote Lorentz indices. We shall raise and lower them using the Lorentz metric tensor

$$\eta = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}. \tag{2.1}$$

Infinitesimal left and right transformations on the group are given by the variations

$$\delta_L \ell^{ba} = \tau_L^{bc} \ell_c^a, \quad \delta_L z^b = \tau_L^{bc} z_c + \rho_L^b, \tag{2.2}$$

and

$$\delta_R \ell_c^d = \ell_c^f \tau_R^{fd}, \quad \delta_R z^b = \ell^b_a \rho_R^a, \tag{2.3}$$

respectively, with $\tau_L^{ab} = -\tau_L^{ba}$ and $\tau_R^{ab} = -\tau_R^{ba}$ being infinitesimal Lorentz parameters and ρ_L^a and ρ_R^a being infinitesimal translations.

The quantum Poincaré group $ISO_q(3, 1) \equiv Fun_q(ISO(3, 1))$ can also be described in terms of matrix elements ℓ_{ab} and vector components z_a , but, unlike for $ISO(3, 1)$, they are not c -numbers. Instead they obey the following commutation relations:

$$z^a \ell_c^b = q^{A(b)} \ell_c^b z^a, \tag{2.4}$$

where

$$\Delta(1) = -1, \quad \Delta(2) = \Delta(3) = 0, \quad \Delta(4) = 1, \tag{2.5}$$

and all other commutation relations are trivial. Consistent hermiticity properties on the group parameters restrict q to be either real or a phase [7]. These commutation relations

are associated with an R -matrix satisfying the quantum Yang–Baxter equation [6]. They are consistent with the Lorentz constraint

$$\ell_{ab}\ell_c^b = \eta_{ac} \tag{2.6}$$

due to the identity

$$\eta_{ab} = q^{\Delta(a)+\Delta(b)}\eta_{ab}, \tag{2.7}$$

which follows from the metric (2.1) along with (2.5). As in Ref. [5], the commutation relations (2.4) are also consistent with the other constraints defining the $SO(3, 1)$ group, namely $\ell_{ba}\ell_c^b = \eta_{ac}$ and $\det(\ell) = 1$. After imposing all such constraints we therefore conclude that $ISO_q(3, 1)$ contains the ordinary Lorentz group.

Since ℓ_{ab} and z_a are not c -numbers, neither are the infinitesimal transformation parameters τ_L^{ab} , τ_R^{ab} , ρ_L^a and ρ_R^a . In fact, for the commutation relations (2.4) to be preserved, we need

$$\begin{aligned} \rho_L^a \ell_c^b &= q^{\Delta(b)} \ell_c^b \rho_L^a, \\ \rho_L^a z^b &= z^b \rho_L^a, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \rho_R^a \ell_c^b &= q^{\Delta(b)} \ell_c^b \rho_R^a, \\ \rho_R^a z^b &= q^{-\Delta(a)} z^b \rho_R^a, \\ \tau_R^{ab} z^c &= q^{-\Delta(a)-\Delta(b)} z^c \tau_R^{ab}. \end{aligned} \tag{2.9}$$

Also, the Lorentz transformation parameters τ_L commute with ℓ_c^b and z^a , while τ_R commute with ℓ_c^b .

In order to write a differential calculus on the space spanned by ℓ_{ab} and z_a , we must specify the commutation rules among l_{ab} , z_a and their exterior derivatives. A natural choice, consistent with (2.4), is

$$\begin{aligned} dz^a \ell_c^b &= q^{\Delta(b)} \ell_c^b dz^a, \\ z^a d\ell_c^b &= q^{\Delta(b)} d\ell_c^b z^a, \\ dz^a \wedge d\ell_c^b &= -q^{\Delta(b)} d\ell_c^b \wedge dz^a, \end{aligned} \tag{2.10}$$

while we assume the calculus on the space spanned by l_{ab} alone to be the usual one on $SO(3, 1)$

$$\begin{aligned} d\ell_c^b \ell_c^d &= \ell_c^d d\ell_c^b, \\ d\ell_c^b \wedge d\ell_c^d &= -d\ell_c^d \wedge d\ell_c^b, \end{aligned} \tag{2.11}$$

and the calculus on the space generated by z_a alone to be the usual one on \mathbf{R}^4

$$\begin{aligned} dz^b z^d &= z^d dz^b, \\ dz^b \wedge dz^d &= -dz^d \wedge dz^b. \end{aligned} \tag{2.12}$$

In addition to these commutation relations, we assume that the commutation relations (2.8) and (2.9) also hold when we replace ℓ and z by their exterior derivatives.

The bicovariant bimodule of one-forms on the group is spanned either by left or by right invariant one-forms. We choose to work with the left invariant basis and denote the one-forms by $\omega^{ab} = -\omega^{ba}$ and e^c . The following expressions are invariant under (global) left transformations:

$$\omega^{ab} = (\ell^{-1}d\ell)^{ab}, \quad e^c = (\ell^{-1}dz)^c. \tag{2.13}$$

On the other hand, under infinitesimal right transformations (2.3) they undergo the variations

$$\begin{aligned} \delta_R \omega^{ab} &= d\tau_R^{ab} + \omega^a{}_c \tau_R^{cb} - \omega^b{}_c \tau_R^{ca}, \\ \delta_R e^c &= d\rho_R^c + \omega^c{}_b \rho_R^b - \tau_R^{cb} e_b. \end{aligned} \tag{2.14}$$

From (2.9) we get the following commutation relations between gauge parameters and one-forms:

$$\begin{aligned} \rho_R^a \omega^{bc} &= q^{\Delta(b)+\Delta(c)} \omega^{bc} \rho_R^a, \\ \rho_R^a e^b &= q^{\Delta(b)-\Delta(a)} e^b \rho_R^a, \\ \tau_R^{ab} e^c &= q^{-\Delta(a)-\Delta(b)} e^c \tau_R^{ab}, \\ \tau_R^{ab} \omega^{cd} &= \omega^{cd} \tau_R^{ab}. \end{aligned} \tag{2.15}$$

The left invariant forms (2.13) satisfy the Maurer Cartan equations

$$\mathcal{R}^{ab} = 0, \tag{2.16}$$

$$T^a = 0, \tag{2.17}$$

where \mathcal{R}^{ab} and T^a have the usual expressions for the curvature and torsion

$$\begin{aligned} \mathcal{R}^{ab} &= d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \\ T^a &= de^a + \omega^a{}_b \wedge e^b, \end{aligned} \tag{2.18}$$

except that we no longer have the usual exterior product for the one-forms ω^{ab} and e^a . From (2.11), (2.12), (2.10) and (2.13) we obtain instead

$$\begin{aligned} \omega^{ab} \wedge \omega^{cd} &= -\omega^{cd} \wedge \omega^{ab}, \\ e^a \wedge \omega^{bc} &= -q^{\Delta(b)+\Delta(c)} \omega^{bc} \wedge e^a, \\ e^a \wedge e^b &= -q^{\Delta(b)-\Delta(a)} e^b \wedge e^a. \end{aligned} \tag{2.19}$$

3. One-parameter family of tetrad theories

Using the mathematical framework of the previous section, we shall construct a general q -Poincaré gauge theory and then write down an action principle for gravity having the usual symmetries, i.e. diffeomorphism invariance and local Lorentz invariance.

A general q -Poincaré gauge theory is obtained when we drop the assumption (2.13) that ω^{ab} and e^a be pure gauges, and hence that they satisfy the Maurer Cartan equations (2.16) and (2.17). Rather, we let them be arbitrary spin connections and vierbein

one-forms. In this case, the right transformations (2.14) are to be regarded as infinitesimal gauge transformations. Such transformations on the curvature and torsion are given by

$$\begin{aligned} \delta_R \mathcal{R}^{ab} &= \mathcal{R}^a{}_c \tau_R^{cb} - \mathcal{R}^b{}_c \tau_R^{ca}, \\ \delta_R T^c &= \mathcal{R}^c{}_b \rho_R^b - \tau_R^{cb} T_b. \end{aligned} \tag{3.1}$$

The Bianchi identities for this theory take the usual form

$$\begin{aligned} d\mathcal{R}^{ab} &= \mathcal{R}^a{}_c \wedge \omega^{cb} - \mathcal{R}^b{}_c \wedge \omega^{ca}, \\ dT^a &= \mathcal{R}^a{}_b \wedge e^b - \omega^a{}_b \wedge T^b, \end{aligned} \tag{3.2}$$

the ordering being crucial. We assume, consistent with (2.18) and (2.19), that

$$\begin{aligned} \omega^{ab} \wedge \mathcal{R}^{cd} &= \mathcal{R}^{cd} \wedge \omega^{ab}, \\ \omega^{bc} \wedge T^a &= q^{-\Delta(b)-\Delta(c)} T^a \wedge \omega^{bc}, \\ e^a \wedge \mathcal{R}^{bc} &= q^{\Delta(b)+\Delta(c)} \mathcal{R}^{bc} \wedge e^a, \\ e^a \wedge T^b &= q^{\Delta(b)-\Delta(a)} T^b \wedge e^a, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \mathcal{R}^{ab} \wedge \mathcal{R}^{cd} &= \mathcal{R}^{cd} \wedge \mathcal{R}^{ab}, \\ \mathcal{R}^{bc} \wedge T^a &= q^{-\Delta(b)-\Delta(c)} T^a \wedge \mathcal{R}^{bc}, \\ T^a \wedge T^b &= q^{\Delta(b)-\Delta(a)} T^b \wedge T^a. \end{aligned} \tag{3.4}$$

With a rescaling of the spin connection ω^{ab} and spin curvature \mathcal{R}^{ab} by a factor of $q^{-(1/2)(\Delta(a)+\Delta(b))}$, the above system becomes identical to that of Ref. [9].

The above gauge theory may be associated with a q -Lie algebra, defined by the algebraic relations

$$T_i T_j - A_{ij}^{kl} T_k T_l = C_{ij}^k T_k, \tag{3.5}$$

where T_i are the generators, A_{ij}^{kl} the braiding matrix elements and C_{ij}^k the q -structure constants. Following Ref. [8], from the commutation relations (2.15) we can identify the braiding matrix, while from the gauge transformations (2.14) we can identify the q -structure constants. For the former we get

$$A_{ab\ cd}^{cd\ ab} = 1, \quad A_{a\ bc}^{bc\ a} = q^{-\Delta(b)-\Delta(c)}, \quad A_{bc\ a}^{a\ bc} = q^{\Delta(b)+\Delta(c)}, \quad A_{a\ b}^{b\ a} = q^{\Delta(a)-\Delta(b)}, \tag{3.6}$$

with all other components vanishing. It is then verified that the square of the braiding matrix is the unit matrix and, furthermore, that A_{ij}^{kl} and C_{ij}^k satisfy all the necessary conditions (see Ref. [8]) of a minimally deformed gauge theory. The 10 generators T_i , which we denote by M_{ab} (Lorentz generators) and P_a (translations), are said to be dual to the one-forms ω^{ab} and e^a . In terms of them, (3.5) is expanded to

$$\begin{aligned} [M_{ab}, M_{cd}] &= \eta_{ac} M_{bd} - \eta_{bc} M_{ad} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc}, \\ [M_{ab}, P_c]_{q^{\Delta(a)+\Delta(b)}} &= -(\eta_{bc} P_a - \eta_{ac} P_b), \end{aligned}$$

$$[P_a, P_b]_{q^{\Delta(a)-\Delta(b)}} = 0, \tag{3.7}$$

where $[\alpha, \beta]_s \equiv \alpha\beta - s\beta\alpha$.

Next we write down a locally Lorentz invariant action

$$S = \frac{1}{4} \int_M \epsilon_{abcd} \mathcal{R}^{ab} \wedge \mathcal{E}^{cd}, \tag{3.8}$$

where \mathcal{E}^{cd} is the two-form

$$\mathcal{E}^{cd} = -\mathcal{E}^{dc} = q^{-\Delta(d)} e^c \wedge e^d, \tag{3.9}$$

M is a four manifold and ϵ_{abcd} is the ordinary, totally antisymmetric tensor with $\epsilon_{1234} = 1$. The expression (3.8) differs from that of the undeformed case by the $q^{-\Delta(d)}$ factor. Note that this factor can be written differently using the identity

$$q^{\Delta(a)+\Delta(b)+\Delta(c)+\Delta(d)} \epsilon_{abcd} = \epsilon_{abcd}. \tag{3.10}$$

It can be checked that the action (3.8) is identical to the one found in Ref. [9], up to an overall factor $q^{3/2}$. To check local Lorentz invariance we use the property

$$e^a \wedge \delta e^b = -q^{\Delta(b)-\Delta(a)} \delta e^b \wedge e^a. \tag{3.11}$$

Then

$$\delta S = \frac{1}{4} \int_M q^{-\Delta(d)} \epsilon_{abcd} (\delta \mathcal{R}^{ab} \wedge e^c + 2\mathcal{R}^{ab} \wedge \delta e^c) \wedge e^d. \tag{3.12}$$

After substituting in the variations (2.14) and (3.1) with $\rho_R^b = 0$ we get

$$\delta S = \frac{1}{2} \int_M \epsilon_{abcd} (\mathcal{R}^a{}_f \wedge \tau_R^{fb} \mathcal{E}^{cd} - \mathcal{R}^{ab} \wedge \tau_R^{cf} \mathcal{E}_f{}^d), \tag{3.13}$$

which vanishes due to the antisymmetry of τ_R^{fb} and \mathcal{E}^{cd} . Also, as in the undeformed case, the action is invariant under the full set of local Poincaré transformations (2.14), provided we impose the torsion to be zero upon making the variations.

The equations of motion obtained from varying the vierbeins have the usual form, i.e.

$$\epsilon_{abcd} \mathcal{R}^{ab} \wedge e^c = 0, \tag{3.14}$$

while varying ω^{ab} gives

$$d\tilde{\mathcal{E}}_{ab} = \omega_a{}^c \tilde{\mathcal{E}}_{bc} - \omega_b{}^c \tilde{\mathcal{E}}_{ac}, \quad \tilde{\mathcal{E}}_{ab} \equiv \epsilon_{abcd} \mathcal{E}^{cd}. \tag{3.15}$$

Due to the antisymmetry of \mathcal{E}^{cd} , we get the following expression in terms of the torsion from (3.15)

$$\epsilon_{abcd} T^c \wedge e^d q^{-\Delta(d)} = 0. \tag{3.16}$$

In the next section we show that this equation implies zero torsion (2.17), provided inverse vierbeins exist. This is necessary in order to recover Einstein’s gravity.

4. Recovering Einstein’s theory

In this section we prove that the metric formulation of the q -deformed Cartan theory of gravity discussed in the previous section is completely equivalent to the undeformed Einstein’s theory, for all values of q .

To make a connection with Einstein gravity, we need to introduce the space–time metric $g_{\mu\nu}$ on M . In order to do this we assume the following stronger form of the commutation relations (2.19):

$$\begin{aligned} \omega_{\mu}^{ab} \omega_{\nu}^{cd} &= \omega_{\nu}^{cd} \omega_{\mu}^{ab}, \\ e_{\mu}^a \omega_{\nu}^{bc} &= q^{\Delta(b)+\Delta(c)} \omega_{\nu}^{bc} e_{\mu}^a, \\ e_{\mu}^a e_{\nu}^b &= q^{\Delta(b)-\Delta(a)} e_{\nu}^b e_{\mu}^a, \end{aligned} \tag{4.1}$$

e_{μ}^a denoting the space–time components of the vierbein one-form e^a ; μ and ν being space–time indices. These commutation relations are consistent with the view that the space–time manifold be spanned by commuting coordinates. We now want to construct a bilinear from the vierbeins which is symmetric in the space–time indices and invariant under local Lorentz transformations. These requirements uniquely fix $g_{\mu\nu}$ to be

$$g_{\mu\nu} = q^{\Delta(a)} \eta_{ab} e_{\mu}^a e_{\nu}^b. \tag{4.2}$$

Using Eqs. (4.1) we see that $g_{\mu\nu}$ is symmetric, although the tensor elements are not c -numbers since

$$\begin{aligned} g_{\mu\nu} \omega_{\rho}^{ab} &= q^{2\Delta(a)+2\Delta(b)} \omega_{\rho}^{ab} g_{\mu\nu}, \\ g_{\mu\nu} e_{\rho}^a &= q^{2\Delta(a)} e_{\rho}^a g_{\mu\nu}. \end{aligned} \tag{4.3}$$

The components of $g_{\mu\nu}$ do however commute with themselves.

In order to go further, we need to be able to define the inverses e_{μ}^{μ} of the (co-)tetrads e_{μ}^a . This can be done if we enlarge our algebra by a new element e^{-1} such that

$$\begin{aligned} e^{-1} e_{\mu}^a &= q^{-4\Delta(a)} e_{\mu}^a e^{-1}, \\ e^{-1} \omega_{\mu}^{ab} &= q^{-4(\Delta(a)+\Delta(b))} \omega_{\mu}^{ab} e^{-1}, \end{aligned} \tag{4.4}$$

and such that

$$e^{-1} e = 1, \tag{4.5}$$

where e is the determinant

$$e = \epsilon^{\mu\nu\rho\sigma} e_{\mu}^1 e_{\nu}^2 e_{\rho}^3 e_{\sigma}^4. \tag{4.6}$$

Eq. (4.5) is consistent because its left-hand side commutes with everything, due to Eqs. (4.4). Moreover, one can check that $e^{-1} e = e e^{-1}$. The inverses of the vierbeins can now be written

$$e_{\mu}^{\mu} = \frac{1}{3!} \hat{\epsilon}_{abcd} \epsilon^{\mu\nu\rho\sigma} e_{\nu}^b e_{\rho}^c e_{\sigma}^d e^{-1}, \tag{4.7}$$

where the totally q -antisymmetric tensor $\hat{\epsilon}_{abcd}$ is defined such that

$$\hat{\epsilon}_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \quad \text{no sum on } a, b, c, d. \tag{4.8}$$

The solution to this equation can be expressed by

$$\hat{\epsilon}_{abcd} = q^{3\Delta(a)+2\Delta(b)+\Delta(c)+3} \epsilon_{abcd}. \tag{4.9}$$

Notice also the following useful identity satisfied by the q -antisymmetric tensor $\hat{\epsilon}^{abcd}$ obtained by raising the indices of $\hat{\epsilon}_{abcd}$ with the metric η^{ab}

$$q^{-6} \hat{\epsilon}^{abcd} e = -\epsilon^{\mu\nu\lambda\sigma} e_\mu^a e_\nu^b e_\lambda^c e_\sigma^d. \tag{4.10}$$

The explicit expression of $\hat{\epsilon}^{abcd}$ can be seen to be

$$\hat{\epsilon}^{abcd} = q^{-3\Delta(a)-2\Delta(b)-\Delta(c)+3} \epsilon^{abcd}, \tag{4.11}$$

where ϵ^{abcd} is the ordinary antisymmetric tensor obtained by raising the indices of ϵ_{abcd} with the metric η^{ab} . It is easy to prove that the inverses of the vierbeins (4.7) have the usual properties

$$\begin{aligned} e_\mu^a e_b^\mu &= e_b^\mu e_\mu^a = \delta_b^a, \\ e_\mu^a e_a^\nu &= e_a^\nu e_\mu^a = \delta_\mu^\nu. \end{aligned} \tag{4.12}$$

By using the inverses of the tetrads, we can now prove that Eq. (3.16) implies the vanishing of the torsion. To begin with, we introduce the components of the torsion two-form along the tetrads:

$$T_{bc}^a \equiv q^{\Delta(b)} T_{\mu\nu}^a e_b^\mu e_c^\nu; \tag{4.13}$$

the power of q ensures that they are antisymmetric in the lower indices, $T_{bc}^a = -T_{cb}^a$. Now we rewrite Eq. (3.16) as

$$\begin{aligned} 0 &= q^{-\Delta(d)} \epsilon_{abcd} \epsilon^{\lambda\mu\nu\rho} T_{\mu\nu}^c e_\rho^d = q^{-\Delta(d)-\Delta(h)} \epsilon_{abcd} \epsilon^{\lambda\mu\nu\rho} T_{gh}^c e_\mu^g e_\nu^h e_\rho^d \\ &= -q^{\Delta(d)-\Delta(f)-3} \epsilon_{abcd} \epsilon^{fghd} T_{gh}^c e_f^\lambda e \\ &= 2q^{-\Delta(a)-\Delta(b)-\Delta(c)-3} (q^{-\Delta(a)} T_{bc}^c e_a^\lambda + q^{-\Delta(b)} T_{ca}^c e_b^\lambda + q^{-\Delta(c)} T_{ab}^c e_c^\lambda) e, \end{aligned} \tag{4.14}$$

where we have used the identity

$$q^{-\Delta(d)-\Delta(h)} \epsilon^{\lambda\mu\nu\rho} e_\mu^g e_\nu^h e_\rho^d = -q^{\Delta(d)-\Delta(f)-3} \epsilon^{fghd} e_f^\lambda e, \tag{4.15}$$

which follows from Eqs. (4.10) and (4.11). Neglecting the overall factor of $q^{-\Delta(a)-\Delta(b)-3} e$ in (4.14) and multiplying it on the right by e_λ^d we finally get

$$q^{-\Delta(c)} T_{bc}^c \delta_a^d + q^{-\Delta(c)} T_{ca}^c \delta_b^d + q^{-\Delta(d)} T_{ab}^d = 0. \tag{4.16}$$

It is easy to verify that these equations imply the vanishing of all the T_{bc}^a and thus of the torsion.

We now define the Christoffel symbols $\Gamma_{\mu\nu}^\sigma$ by demanding that the covariant derivative of the vierbeins vanishes

$$D_\mu e_\nu^b = 0, \tag{4.17}$$

where

$$D_\mu e_\nu^b = \mathcal{D}_\mu e_\nu^b - \Gamma_{\mu\nu}^\sigma e_\sigma^b, \tag{4.18}$$

and

$$\mathcal{D}_\mu e_\nu^b = \partial_\mu e_\nu^b + \omega_\mu^{bc} e_{\nu c}. \tag{4.19}$$

The torsion being zero is consistent with the Christoffel symbols being symmetric in the lower two indices.

We can eliminate the spin connections from (4.17) if we multiply it on the left by $q^{\Delta(a)}\eta_{ab}e_\rho^a$, sum over the b index, and symmetrize with respect to the space–time indices ν and ρ . The result is

$$0 = q^{\Delta(a)}\eta_{ab}[e_\rho^a\partial_\mu e_\nu^b + e_\nu^a\partial_\mu e_\rho^b - e_\rho^a e_\sigma^b \Gamma_{\mu\nu}^\sigma - e_\nu^a e_\sigma^b \Gamma_{\mu\rho}^\sigma]. \tag{4.20}$$

Next we add to this the equation obtained by switching μ and ν , and subtract the equation obtained by replacing indices (μ, ν, ρ) by (ρ, μ, ν) . We can then isolate $\Gamma_{\mu\nu}^\sigma$ according to

$$2q^{\Delta(a)}\eta_{ab}e_\rho^a e_\sigma^b \Gamma_{\mu\nu}^\sigma = q^{\Delta(a)}\eta_{ab}[e_\rho^a(\partial_\mu e_\nu^b + \partial_\nu e_\mu^b) + e_\nu^a(\partial_\mu e_\rho^b - \partial_\rho e_\mu^b) + e_\mu^a(\partial_\nu e_\rho^b - \partial_\rho e_\nu^b)], \tag{4.21}$$

or

$$2g_{\rho\sigma}\Gamma_{\mu\nu}^\sigma = \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu}. \tag{4.22}$$

To solve this equation we need the inverse of the metric $g^{\mu\nu}$. The expression

$$g^{\mu\nu} = q^{\Delta(a)}\eta^{ab}e_a^\mu e_b^\nu \tag{4.23}$$

does the job as it can be checked that

$$g^{\mu\rho}g_{\rho\nu} = g_{\nu\rho}g^{\rho\mu} = \delta_\nu^\mu. \tag{4.24}$$

Notice that unlike in the usual Einstein Cartan theory

$$g^{\mu\nu}\eta_{ab}e_\nu^b = q^{\Delta(a)}e_a^\mu. \tag{4.25}$$

We are now able to solve Eq. (4.22). Upon multiplying it by $g^{\tau\rho}$ on any side, we get the usual expression for the Christoffel symbols in terms of the metric tensor and its inverse. It may be verified, using these expressions, that the Christoffel symbols commute with everything and, thus, even if written in terms of non-commuting quantities, they can be interpreted as being ordinary numbers.

The covariant derivative operator ∇_μ defined by the Christoffel symbols is compatible with the metric $g_{\mu\nu}$, i.e. $\nabla_\mu g_{\nu\rho} = 0$. This is clear because our Christoffel symbols have the standard expression in terms of the space–time metric $g_{\mu\nu}$, but it also follows from Eq. (4.17)

$$\nabla_\mu g_{\nu\rho} = D_\mu g_{\nu\rho} = D_\mu (q^{\Delta(a)} \eta_{ab} e_\nu^a e_\rho^b) = 0. \tag{4.26}$$

We now construct the Riemann tensor. It is defined as in the undeformed theory

$$R_{\mu\nu\rho}{}^\sigma v_\sigma = (D_\mu D_\nu - D_\nu D_\mu) v_\rho, \tag{4.27}$$

where v_μ is an ordinary co-vector. It follows from (4.17) that it has the standard expression in terms of the Christoffel symbols (and thus in terms of the space–time metric and its inverse) and therefore its components commute with everything. (This is also true for the Ricci tensor $R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma$, of course, but not for $R_{\mu\nu\rho\sigma}$ as the lowering of the upper index of the Riemann tensor implies contraction with $g_{\sigma\tau}$ which is not in the center of the algebra.) The relation among the Riemann tensor and the curvature of the spin connection follows from Eqs. (4.17) and (4.18)

$$e_\sigma^a R_{\mu\nu\rho}{}^\sigma v^\rho = e_\sigma^a (D_\nu D_\mu - D_\mu D_\nu) v^\sigma = (D_\nu D_\mu - D_\mu D_\nu) e_\sigma^a v^\sigma = -\mathcal{R}_{\mu\nu}^{ac} \eta_{bc} e_\sigma^b v^\sigma, \tag{4.28}$$

$\mathcal{R}_{\mu\nu}^{ab}$ being the space–time components of \mathcal{R}^{ab} . v^μ being an arbitrary ordinary vector, it follows from the above equation that

$$R_{\mu\nu\rho}{}^\tau = -\mathcal{R}_{\mu\nu}^{ac} \eta_{bc} e_\rho^b e_a^\tau. \tag{4.29}$$

Using this equation it can be checked directly that the components of the Riemann tensor commute with everything, as pointed out earlier. Our Riemann tensor has the usual symmetry properties

$$\begin{aligned} R_{\mu\nu\rho}{}^\sigma &= -R_{\nu\mu\rho}{}^\sigma, \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}, \\ R_{[\mu\nu\rho]}{}^\sigma &= 0. \end{aligned} \tag{4.30}$$

The first of these equations is obvious; the second can be proved starting from (4.29)

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho}{}^\tau g_{\tau\sigma} = -\mathcal{R}_{\mu\nu}^{ab} e_{\rho b} e_a^\tau g_{\tau\sigma} \\ &= -q^{\Delta(a)} \mathcal{R}_{\mu\nu}^{ab} e_{\rho b} e_{\sigma a} = -q^{\Delta(b)} \mathcal{R}_{\mu\nu}^{ab} e_{\sigma a} e_{\rho b} = -R_{\mu\nu\sigma\rho}, \end{aligned} \tag{4.31}$$

where we have made use of (4.25). The third of Eqs. (4.30) follows from the algebraic Bianchi identity, namely the second of Eqs. (3.2) with $\mathcal{T}^a = 0$, and from (4.29)

$$0 = -\epsilon^{\lambda\mu\nu\rho} \mathcal{R}_{\mu\nu}^{ac} \eta_{bc} e_\rho^b = \epsilon^{\lambda\mu\nu\rho} R_{\mu\nu\sigma}{}^\tau e_\tau^a e_b^\sigma e_\rho^b = 6\epsilon^{\lambda\mu\nu\rho} R_{[\mu\nu\rho]}{}^\tau e_\tau^a. \tag{4.32}$$

We now show that the action (3.8) becomes equal to the undeformed Einstein–Hilbert action, once the spin connection is eliminated using its equations of motion, namely the

zero torsion condition. For the purposes of the next sections we shall do it in two steps: first we rewrite (3.8) in a form analogous to Palatini’s action, which we shall use to develop the canonical formalism, and then show that the latter reduces to the *undeformed* Einstein–Hilbert action, once the spin connection is eliminated from it. Consider thus the following deformation of the Palatini action:

$$S = \frac{1}{2} \int_M d^4x q^{\Delta(a)-3} e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}^{ab}. \tag{4.33}$$

To see that it coincides with (3.8), we use the identity

$$q^{\Delta(a)-\Delta(b)-6} \hat{\epsilon}^{abcd} e_a^\mu e_b^\nu e = -\epsilon^{\mu\nu\lambda\sigma} e_\lambda^c e_\sigma^d. \tag{4.34}$$

The result (4.33) then follows after multiplying both sides of this equation on the left by $-1/8q^{-2\Delta(f)-\Delta(g)-3} \hat{\epsilon}_{f\sigma cd} \mathcal{R}_{\mu\nu}^{fg}$ and using the identity

$$\hat{\epsilon}_{f\sigma cd} \hat{\epsilon}^{abcd} = -2q^6 (\delta_f^a \delta_g^b - q^{\Delta(f)-\Delta(g)} \delta_g^a \delta_f^b),$$

along with (4.9).

We now show that Eq. (4.33) becomes in turn equal to the undeformed Einstein–Hilbert action upon eliminating the spin connection via its equation of motion. This amounts to expressing $\mathcal{R}_{\mu\nu}^{ab}$ in terms of the Riemann tensor by inverting Eq. (4.29) and then plugging the result in Eq. (4.33). We have

$$\begin{aligned} q^{\Delta(a)} e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}^{ab} &= -q^{\Delta(a)} R_{\mu\nu\rho}{}^\tau e_a^\mu e_b^\nu e_\tau^a e^{\rho b} \\ &= -q^{\Delta(b)} R_{\mu\nu\rho}{}^\mu e_b^\nu e^{\rho b} = R_{\nu\rho}{}^\mu{}_\mu g^{\nu\rho} = R, \end{aligned} \tag{4.35}$$

where we have made use of (4.12). Moreover, we have

$$\begin{aligned} g &\equiv \det || g_{\mu\nu} || = \frac{1}{4!} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4} g_{\mu_1\nu_1} g_{\mu_2\nu_2} g_{\mu_3\nu_3} g_{\mu_4\nu_4} \\ &= \frac{1}{4!} q^{\Delta(a_1)+\Delta(a_2)+\Delta(a_3)+\Delta(a_4)} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4} \eta_{a_1b_1} \eta_{a_2b_2} \eta_{a_3b_3} \eta_{a_4b_4} e_{\mu_1}^{a_1} e_{\nu_1}^{b_1} \dots e_{\mu_4}^{a_4} e_{\nu_4}^{b_4} \\ &= \frac{1}{4!} q^{[\Delta(a_2)+\Delta(b_2)]+2[\Delta(a_3)+\Delta(b_3)]+3[\Delta(a_4)+\Delta(b_4)]} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4} \\ &\quad \times \eta_{a_1b_1} \eta_{a_2b_2} \eta_{a_3b_3} \eta_{a_4b_4} (e_{\mu_1}^{a_1} \dots e_{\mu_4}^{a_4}) (e_{\nu_1}^{b_1} \dots e_{\nu_4}^{b_4}) \\ &= \frac{1}{4!} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4} \eta_{a_1b_1} \eta_{a_2b_2} \eta_{a_3b_3} \eta_{a_4b_4} (e_{\mu_1}^{a_1} \dots e_{\mu_4}^{a_4}) (e_{\nu_1}^{b_1} \dots e_{\nu_4}^{b_4}) \\ &= \frac{1}{4!} q^{-12} \hat{\epsilon}_{a_1a_2a_3a_4} \hat{\epsilon}^{a_1a_2a_3a_4} e^2 = -q^{-6} e^2, \end{aligned} \tag{4.36}$$

where we made use of (2.7), (3.10) and (4.10). Putting together (4.35) and (4.36) we see that the q -Palatini action (4.33) becomes equal to

$$S = \frac{1}{2} \int_M d^4x \sqrt{-g} R, \tag{4.37}$$

which is the *undeformed* Einstein–Hilbert action. Since the components of $g_{\mu\nu}$ and its inverse all commute among themselves, it is clear that the equations of motion of the metric theory will be equal to those of the undeformed Einstein’s theory in vacuum. One can obtain the same result starting directly from Eq. (3.14) and using (4.29).

Summarizing, the results of this section show that if we just consider the theory constructed in terms of the space–time metric $g_{\mu\nu}$, our theory is completely equivalent to Einstein’s theory. No trace of the q -structure existing in the tetrad formulation of the theory can be found at the metric level. In the complete theory, which includes the vierbeins and the spin connection, the space–time metric $g_{\mu\nu}$ can no longer be considered a c -number (at each point in space–time). Nevertheless, quantities such as the Christoffel symbols, the Riemann, Ricci and Einstein tensors, being in the center of the algebra of functions on the space–time manifold, can still be regarded as numbers. A complete discussion of the physical equivalence of this theory (in the absence of torsion) with the usual Einstein gravity is found in [7].

5. Inclusion of sources

Next let us introduce sources. We do this by replacing the free field equations (3.14) and (3.16) by

$$\begin{aligned} \kappa \epsilon_{abcd} \mathcal{R}^{ab} \wedge e^c &= \theta_d, \\ \kappa q^{-\Delta(b)} \epsilon_{abcd} T^a \wedge e^b &= \frac{1}{2} \Sigma_{cd}, \end{aligned} \tag{5.1}$$

where θ_d and Σ_{cd} are three-forms which are the analogues of the stress-energy and spin densities. κ is the gravitational coupling constant. The second equation differs from the undeformed case by factors of q . By substituting (5.1) into the Bianchi identities (3.2) we get a simple set of consistency conditions for the sources

$$\begin{aligned} d\theta_d &= \omega_{dc} \wedge \theta^c + \kappa \epsilon_{abcd} \mathcal{R}^{ab} \wedge T^c, \\ d\Sigma_{ab} &= q^{\Delta(a)} e_b \wedge \theta_a + \omega^c{}_b \wedge \Sigma_{ac} - (a \Rightarrow b). \end{aligned} \tag{5.2}$$

These expressions also differ from the undeformed case by factors of q . Moreover, we note that the three-forms θ_d and Σ_{cd} are not c -numbers. From the commutation relations (3.4), (3.3) and (4.1), the source terms should satisfy

$$\begin{aligned} \theta_a \wedge \theta_b &= -q^{\Delta(a)-\Delta(b)} \theta_b \wedge \theta_a, \\ \theta_a \wedge \Sigma_{bc} &= -q^{2\Delta(a)-\Delta(b)-\Delta(c)} \Sigma_{bc} \wedge \theta_a, \\ \Sigma_{ab} \wedge \Sigma_{cd} &= -q^{2\Delta(a)+2\Delta(b)-2\Delta(c)-2\Delta(d)} \Sigma_{cd} \wedge \Sigma_{ab}, \end{aligned} \tag{5.3}$$

where we used the identity (3.10). Of course, the above exterior products of three-forms vanish on a four-dimensional manifold. We shall interpret these equations as conditions on products of the components $\theta_d^{\mu\nu\rho}$ and $\Sigma_{cd}^{\mu\nu\rho}$ of the three-forms θ_d and Σ_{cd} . That is, the components satisfy

$$\theta_a^{\mu\nu\rho} \theta_b^{\sigma\lambda\eta} = q^{\Delta(a)-\Delta(b)} \theta_b^{\sigma\lambda\eta} \theta_a^{\mu\nu\rho},$$

$$\begin{aligned} \theta_a^{\mu\nu\rho} \Sigma_{bc}^{\sigma\lambda\eta} &= q^{2\Delta(a)-\Delta(b)-\Delta(c)} \Sigma_{bc}^{\sigma\lambda\eta} \theta_a^{\mu\nu\rho}, \\ \Sigma_{ab}^{\mu\nu\rho} \Sigma_{cd}^{\sigma\lambda\eta} &= q^{2\Delta(a)+2\Delta(b)-2\Delta(c)-2\Delta(d)} \Sigma_{cd}^{\sigma\lambda\eta} \Sigma_{ab}^{\mu\nu\rho}. \end{aligned} \tag{5.4}$$

We can then replace θ_d and Σ_{cd} in (5.3) by their corresponding dual one-forms. The commutation relations involving the source terms and the vierbeins or the spin connections may be derived from Eqs. (5.1), by consistency.

We next show how to recover the Einstein equations in the presence of matter. For this purpose we set the spin densities Σ_{cd} equal to zero. Then, assuming inverse vierbeins to exist, we get that the torsion vanishes. Next we rewrite the first of Eqs. (5.1). For this we start with

$$q^{\Delta(b)-\Delta(c)-4\Delta(e)} \epsilon^{bcde} e_b^\lambda e_c^\rho e_e^\tau = -q^6 \epsilon^{\lambda\rho\sigma\tau} e_\sigma^d e^{-1}, \tag{5.5}$$

which follows from (4.34), and multiply on the left with $\epsilon_{fgdh} \mathcal{R}_{\lambda\rho}^{fg}$. Upon substituting in the first of Eqs. (5.1), we then get

$$q^{\Delta(b)-\Delta(c)-4\Delta(e)} \epsilon^{bcde} \epsilon_{fgdh} \mathcal{R}_{\lambda\rho}^{fg} e_b^\lambda e_c^\rho e_e^\tau = -\frac{q^6}{3\kappa} \epsilon^{\lambda\rho\sigma\tau} \theta_{h\lambda\rho\sigma} e^{-1}, \tag{5.6}$$

$\theta_{h\lambda\rho\sigma}$ being the space–time components of the three-form θ_h . To evaluate the left-hand side we can use (3.10), (4.9) and the identity

$$\begin{aligned} -q^{-6} \epsilon^{bcde} \epsilon_{fgdh} &= q^{-2\Delta(b)-\Delta(c)+\Delta(e)-3} (\delta_f^b \delta_g^c \delta_h^e - \delta_f^c \delta_g^b \delta_h^e - \delta_f^b \delta_g^e \delta_h^c - \delta_h^b \delta_f^e \delta_g^c \\ &\quad + \delta_h^b \delta_g^e \delta_f^c + \delta_g^b \delta_f^e \delta_h^c). \end{aligned} \tag{5.7}$$

We then multiply on the right by $-1/4q^{2\Delta(h)+2\Delta(a)+3} e_\tau^a e_\mu^h$ to get the result

$$e_c^\lambda \mathcal{R}_{\mu\lambda}^{ac} - \frac{1}{2} e_d^\lambda e_\mu^a e_\rho^b \mathcal{R}_{\lambda\rho}^{de} = \frac{1}{12\kappa} \Theta_\mu^a, \tag{5.8}$$

where

$$\Theta_\mu^a = -q^{2\Delta(h)+2\Delta(a)+3} \epsilon^{\lambda\rho\sigma\tau} \theta_{h\lambda\rho\sigma} e^{-1} e_\tau^a e_\mu^h. \tag{5.9}$$

Upon multiplying (5.8) and (5.9) by $\eta_{af} e_\nu^f$ on the right and, using Eq. (4.29), we can see that the left-hand side becomes equal to the Einstein tensor

$$\begin{aligned} (e_c^\lambda \mathcal{R}_{\mu\lambda}^{ac} - \frac{1}{2} e_d^\lambda e_\mu^a e_\rho^b \mathcal{R}_{\lambda\rho}^{de}) \eta_{af} e_\nu^f &= \\ (-q^{\Delta(c)-\Delta(a)} R_{\mu\lambda\rho}{}^\sigma e_\sigma^a e_c^\lambda e^{\rho c} + \frac{1}{2} q^{\Delta(e)-\Delta(a)} e_d^\lambda e_\sigma^d e_\mu^a e_\rho^b e^{\tau e} R_{\lambda\rho\tau}{}^\sigma) \eta_{af} e_\nu^f & \\ = -q^{-\Delta(a)} (R_{\mu\lambda\rho}{}^\sigma e_\sigma^a \mathbf{g}^{\lambda\rho} - \frac{1}{2} R_{\lambda\rho\tau}{}^\lambda e_\mu^a \mathbf{g}^{\rho\tau}) \eta_{af} e_\nu^f & \\ = -R_{\mu\lambda\rho}{}^\sigma \mathbf{g}^{\lambda\rho} \mathbf{g}_{\sigma\nu} + \frac{1}{2} R_{\lambda\rho\tau}{}^\lambda \mathbf{g}^{\rho\tau} \mathbf{g}_{\mu\nu} & \\ = R_{\mu\lambda\nu\rho} \mathbf{g}^{\lambda\rho} - \frac{1}{2} R_{\rho\lambda\tau}{}^\lambda \mathbf{g}^{\rho\tau} \mathbf{g}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \mathbf{g}_{\mu\nu} R, \end{aligned} \tag{5.10}$$

where in the last line we made use of the second of Eqs. (4.30). Thus (5.8) implies the usual Einstein’s equation in the presence of matter

$$R_{\mu\nu} - \frac{1}{2} \mathbf{g}_{\mu\nu} R = \frac{1}{12\kappa} T_{\mu\nu}, \tag{5.11}$$

where $T_{\mu\nu}$ is the stress-energy tensor

$$\begin{aligned}
 T_{\mu\nu} &= \eta_{ab} \Theta_{\mu}^a e_{\nu}^b = -q^{2\Delta(h)+2\Delta(a)+3} \epsilon^{\lambda\rho\sigma\tau} \eta_{ab} \theta_{h\lambda\rho\sigma} e^{-1} e_{\tau}^a e_{\mu}^h e_{\nu}^b \\
 &= -q^{\Delta(h)+3} \epsilon^{\lambda\rho\sigma\tau} e_{\mu}^h \theta_{h\lambda\rho\sigma} \mathfrak{g}_{\tau\nu} e^{-1} = -q^{\Delta(h)} \epsilon^{\lambda\rho\sigma\tau} e_{\mu}^h \theta_{h\lambda\rho\sigma} \mathfrak{g}_{\tau\nu} \frac{1}{\sqrt{-g}}.
 \end{aligned}
 \tag{5.12}$$

We notice that the space–time components $T_{\mu\nu}$ of the matter stress-energy tensor, having the same commutation properties as the Einstein tensor, belong to the center of the algebra and thus can still be interpreted as ordinary numbers. Thus, in our formalism, the properties of matter, in the absence of sources for torsion, are left unaltered with respect to the undeformed theory. Even in the presence of matter, the metric version of our theory is then completely equivalent to Einstein’s theory, for all values of q .

6. Hamiltonian formulation

We derive here the Hamiltonian formulation of Einstein’s theory. We shall do it by introducing a set of deformed Ashtekar’s variables [4]. For simplicity we shall restrict ourselves to the source-free case.

The idea behind Ashtekar’s variables is to use, instead of the spin-connection components ω_{μ}^{ab} , its self-dual part A_{μ}^{ab} , defined as

$$A_{\mu}^{ab} = \frac{1}{2} \omega_{\mu}^{ab} - \frac{i}{4} \epsilon_{cd}^{ab} \omega_{\mu}^{cd}.
 \tag{6.1}$$

A_{μ}^{ab} satisfies the self-duality property

$$iA_{\mu}^{cd} - \frac{1}{2} \epsilon_{ab}^{cd} A_{\mu}^{ab} = 0.
 \tag{6.2}$$

Its curvature $F_{\mu\nu}^{ab}$ is self-dual as well

$$iF_{\mu\nu}^{cd} - \frac{1}{2} \epsilon_{ab}^{cd} F_{\mu\nu}^{ab} = 0,
 \tag{6.3}$$

a property which plays a crucial role in rendering polynomial the constraints of the theory. Now, our spin connections are non-commuting objects and thus Eq. (6.2) might be inconsistent. Fortunately, this is not the case: it can be checked that Eq. (6.1) only involves linear combinations of the spin connection with identical commutation properties and thus A_{μ}^{ab} have the same commutation properties as those of ω_{μ}^{ab}

$$\begin{aligned}
 A_{\mu}^{ab} A_{\nu}^{cd} &= A_{\nu}^{cd} A_{\mu}^{ab}, \\
 e_{\alpha}^{\mu} A_{\nu}^{bc} &= q^{-\Delta(b)-\Delta(c)} A_{\nu}^{bc} e_{\alpha}^{\mu}.
 \end{aligned}
 \tag{6.4}$$

Similarly, Eq. (6.2) involves components of A_{μ}^{ab} with identical commutation properties and thus is consistent as well.

Following what is done in the undeformed case [4], we replace the q -Palatini action (4.33) with

$$S = \int_M d^4x q^{\Delta(a)} e e_a^\mu e_b^\nu F_{\mu\nu}^{ab}. \tag{6.5}$$

The above action is equivalent to (4.33). To see this, we show that if we solve first the equations of motion for A_μ^{ab} implied by (6.5) (for a given arbitrary tetrad) and then plug the solution back in the action, we get back (4.33). Now, it is easy to verify that the equations of motion for A_μ^{ab} imply, as in the undeformed case, that $F_{\mu\nu}^{ab}$ be the self-dual part of the curvature of the spin connection for the torsion-free connection determined by the tetrad. Thus

$$F_{\mu\nu}^{ab} = \frac{1}{2} \mathcal{R}_{\mu\nu}^{ab} - \frac{i}{4} \epsilon_{cd}^{ab} \mathcal{R}_{\mu\nu}^{cd}. \tag{6.6}$$

As in the undeformed case, when we substitute the right-hand side of the above equation in (6.5) we see that it reduces to (4.33) (up to the irrelevant constant factor q^{-3}) since

$$\begin{aligned} -q^{\Delta(a)} \epsilon_{cd}^{ab} e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}^{cd} &= q^{\Delta(a)} \epsilon_{cd}^{ab} e_a^\mu e_b^\nu \mathbf{R}_{\mu\nu\rho}{}^\sigma e_\sigma^c e_\rho^d \eta^{de} \\ &= q^{2\Delta(a)+\Delta(b)} \epsilon^{cabd} e_a^\mu e_b^\nu e_d^\rho \mathbf{R}_{\mu\nu\rho}{}^\sigma e_{\sigma c} \\ &= q^{2\Delta(a)+\Delta(b)} \epsilon^{cabd} e_a^\mu e_b^\nu e_d^\rho \mathbf{R}_{[\mu\nu\rho]}{}^\sigma e_{\sigma c} \\ &= 0, \end{aligned} \tag{6.7}$$

where we used the Bianchi identity. Thus we have the result of the equivalence of the self-dual action with Palatini’s action.

We now sketch the canonical formalism for our q -deformed self-dual Palatini action. In the canonical formalism one needs to split first the manifold M into a 3-manifold Σ , playing the rôle of “space”, times a real line associated with “time”, and then view the field equations as giving the “time” evolution of fields living on Σ . We perform the split of M in the usual manner by foliating it by a collection of space-like surfaces coinciding with $t = \text{constant}$ level surfaces for some (real) function t . The “time” variable is then introduced by means of a time-like real vector field t^μ , whose integral curves intersect every leaf of the foliation at a unique point. In this way we can identify all the leaves with a standard space-like surface Σ , which is our “space”. t^μ is normalized such that

$$t^\mu \partial_\mu t = 1, \tag{6.8}$$

and the Lie derivative along t^μ will play the rôle of the “time derivative”.

Next, let n_μ be the unit co-vector normal to the leaves of the foliations. It must be proportional to the gradient of t

$$n_\mu = -N \partial_\mu t, \tag{6.9}$$

where N is the analogue of the lapse function. In our q -calculus N (and thus n_μ) is not an ordinary number. Rather it fulfills the commutation relations

$$\begin{aligned} N e_a^\mu &= q^{-\Delta(a)} e_a^\mu (N, N e) = e N, \\ N \omega_\mu^{ab} &= q^{\Delta(a)+\Delta(b)} \omega_\mu^{ab} N. \end{aligned} \tag{6.10}$$

The commutation properties of N^{-1} easily follow from the above formulae. N is chosen such that n_μ is a unit vector

$$n^\mu n_\mu \equiv g^{\mu\nu} n_\mu n_\nu = g^{\mu\nu} N^2 \partial_\mu t \partial_\nu t = -1. \tag{6.11}$$

The commutation properties of N render the above equation consistent, as its left-hand side commutes with everything. (Had N been an ordinary c -number this would not have been true.) We now decompose t^μ into normal and tangential components. It follows from (6.8) that

$$t^\mu = n^\mu N + N^\mu, \tag{6.12}$$

where N^μ (the shift) is an ordinary vector such that $N^\mu n_\mu = 0$. [This condition is consistent with (6.8) and (6.9).] Notice that even though neither N nor n^μ are commuting objects their product, being in the center of the algebra, can be considered to be so, which is the only thing we need in order to write (6.12).

In order to perform an analogous 3 + 1 split of the fields, we now introduce the projection operator q^μ_ν

$$q^\mu_\nu \equiv \delta^\mu_\nu + n_\nu n^\mu. \tag{6.13}$$

(Notice that the components of q^μ_ν commute with everything.) With its help we decompose the tetrads e^μ_a into components normal and tangential to Σ

$$e^\mu_a = E^\mu_a - n_a n^\mu, \tag{6.14}$$

where

$$E^\mu_a = q^\mu_\nu e^\nu_a, \tag{6.15}$$

$$n_a \equiv e^\mu_a n_\mu. \tag{6.16}$$

Even if it is written as a four-dimensional field, E^μ_a has to be thought of as a field living on Σ , because contracting it with any vector normal to Σ gives zero. In the following we shall stress the difference among three-dimensional fields like E^μ_a and four-dimensional fields by writing an arrow over the former.

Upon substituting Eq. (6.14) in (6.5) we get

$$\begin{aligned} \mathcal{S} &= \int_M d^4x q^{\Delta(a)} e e^\mu_a e^\nu_b F_{\mu\nu}^{ab} = \int_M d^4x (q^{\Delta(a)} e \mathbf{E}^\mu_a \mathbf{E}^\nu_b F_{\mu\nu}^{ab} - 2q^{\Delta(a)} e \mathbf{E}^\mu_a n_b n^\nu F_{\mu\nu}^{ab}) \\ &= \int_M d^4x (q^{\Delta(a)} e \mathbf{E}^\mu_a \mathbf{E}^\nu_b F_{\mu\nu}^{ab} + i q^{\Delta(a)} e \mathbf{E}^\mu_a n_b n^\nu \epsilon^{ab}_{cd} F_{\mu\nu}^{cd}), \end{aligned} \tag{6.17}$$

where we have made use of Eq. (6.3) in the last passage. Upon introducing the q -triad field $\tilde{E}^\mu_a = N^{-1} e \mathbf{E}^\mu_a$ and the quantity $\tilde{N} = N^2 e^{-1}$ (we shall write a tilde on field densities), we rewrite the last line of the above expression as

$$\begin{aligned}
 & \int_M d^4x [q^{-2\Delta(a)} \tilde{N} \tilde{E}_a^\mu \tilde{E}_b^\nu F_{\mu\nu}^{ab} + iq^{-\Delta(b)} \tilde{E}_a^\mu n_b (t^\nu - N^\nu) \epsilon^{ab}_{cd} F_{\mu\nu}^{cd}] \\
 &= \int_M d^4x [q^{-2\Delta(a)} \tilde{N} \tilde{E}_a^\mu \tilde{E}_b^\nu F_{\mu\nu}^{ab} - iq^{-\Delta(b)} \tilde{E}_a^\mu n_b \epsilon^{ab}_{cd} (\mathcal{L}_t A_\mu^{cd} - D_\mu (A_\nu^{cd} t^\nu) - N^\nu F_{\nu\mu}^{cd})] \\
 &= \int_M d^4x [-iq^{-\Delta(b)} \tilde{E}_a^\mu n_b \epsilon^{ab}_{cd} \mathcal{L}_t A_\mu^{cd} + iq^{-\Delta(b)} \tilde{E}_a^\mu n_b \epsilon^{ab}_{cd} N^\nu F_{\nu\mu}^{cd} + \\
 & \quad -iq^{-\Delta(b)} D_\mu (\tilde{E}_a^\mu n_b \epsilon^{ab}_{cd}) (A_\nu^{cd} t^\nu) + q^{-2\Delta(a)} \tilde{N} \tilde{E}_a^\mu \tilde{E}_b^\nu F_{\mu\nu}^{ab}]. \tag{6.18}
 \end{aligned}$$

In deriving the above equation we have used the identity $t^\nu (F_{\mu\nu}^{ab}) = -\mathcal{L}_t (A_\mu^{ab}) + D_\mu (A_\nu^{cd} t^\nu)$ in the second line, where \mathcal{L}_t is the Lie derivative along the vector field t^μ and D_μ is the covariant derivative relative to A_μ^{ab} . Notice also that in $D_\mu (\tilde{E}_a^\mu n_b \epsilon^{ab}_{cd})$, the field A_μ^{ab} has to be written on the right, while in $D_\mu (A_\nu^{cd} t^\nu)$, A_μ^{ab} has to be written on the left.

Now we observe that all fields in Eq. (6.18) are written in a 3+1 form. First consider the terms containing $F_{\mu\nu}^{ab}$. Since $F_{\mu\nu}^{ab}$ always appears contracted with vectors lying in Σ , we can replace it with the curvature of the pull-back A_μ^{ab} of A_μ^{ab} to Σ , which we denote by $\mathbf{F}_{\mu\nu}^{ab}$. ($\mathbf{F}_{\mu\nu}^{ab}$ is the analogue of the magnetic field.) Also, since $q_\nu^\rho \mathcal{L}_t q_\mu^\rho = 0$, we can replace $\mathcal{L}_t A_\mu^{cd}$ with $\mathcal{L}_t \dot{A}_\mu^{cd}$ in the first term of the result of (6.18). Finally, in the term containing $D_\mu (\tilde{E}_a^\mu n_b \epsilon^{ab}_{cd})$ we can replace D_μ by its projection \mathbf{D}_μ onto Σ , obtained by replacing in it A_μ^{ab} with A_μ^{ab} . As for $A_\nu^{cd} t^\nu$, it plays the rôle of the “time” component of the connection A_μ^{cd} . Thus we can rewrite Eq. (6.18) in the 3 + 1 form

$$\begin{aligned}
 S = \int_M d^4x [-iq^{-\Delta(b)} \tilde{E}_a^\mu n_b \epsilon^{ab}_{cd} \dot{A}_\mu^{cd} + iq^{-\Delta(b)} \tilde{E}_a^\mu n_b \epsilon^{ab}_{cd} N^\nu \mathbf{F}_{\nu\mu}^{cd} \\
 -iq^{-\Delta(b)} \mathbf{D}_\mu (\tilde{E}_a^\mu n_b \epsilon^{ab}_{cd}) (A_\nu^{cd} t^\nu) + q^{-2\Delta(a)} \tilde{N} \tilde{E}_a^\mu \tilde{E}_b^\nu \mathbf{F}_{\mu\nu}^{ab}], \tag{6.19}
 \end{aligned}$$

where we have adopted the notation \dot{A}_μ^{ab} for $\mathcal{L}_t A_\mu^{ab}$.

In Eq. (6.19) the action is written in the form $\int dt (P\dot{Q} - H)$ and we thus can read off the canonical coordinates. The rôle of Q is played by the pull-back to Σ , A_μ^{ab} , of the self-dual connection A_μ^{ab} , while its conjugate momentum $\tilde{\Pi}_{ab}^\mu$ is

$$\tilde{\Pi}_{cd}^\mu = q^{-\Delta(d)} \tilde{E}_c^\mu n_d - q^{-\Delta(c)} \tilde{E}_d^\mu n_c - \frac{i}{2} q^{-\Delta(b)} \tilde{E}_a^\mu n_b \epsilon^{ab}_{cd}. \tag{6.20}$$

Thus the non-vanishing Poisson brackets are

$$\begin{aligned}
 \{A_\mu^{ab}(x), \tilde{\Pi}_{cd}^\nu(y)\} &= -q^{-2(\Delta(a)+2\Delta(b))} \{\tilde{\Pi}_{cd}^\nu(y), A_\mu^{ab}(x)\} \\
 &= \frac{1}{2} \delta_\mu^\nu \left[\delta_{|c}^a \delta_{|d}^b - \frac{i}{2} \epsilon^{ab}_{cd} \right] \delta^3(x, y). \tag{6.21}
 \end{aligned}$$

Notice that our Poisson brackets are not skewsymmetric, as a consequence of the following non-trivial commutation properties of A_μ^{ab} and $\tilde{\Pi}_{ab}^\mu$:

$$\begin{aligned}
 \tilde{\Pi}_{ab}^\mu \tilde{\Pi}_{cd}^\nu &= q^{2(\Delta(a)+\Delta(b)-\Delta(c)-\Delta(d))} \tilde{\Pi}_{cd}^\nu \tilde{\Pi}_{ab}^\mu, \\
 \tilde{\Pi}_{ab}^\mu A_\nu^{cd} &= q^{2(\Delta(c)+\Delta(d))} A_\nu^{cd} \tilde{\Pi}_{ab}^\mu.
 \end{aligned}
 \tag{6.22}$$

We can now rewrite the action in terms of the canonical variables as

$$\begin{aligned}
 \mathcal{S} = \int_M d^4x \{ &\tilde{\Pi}_{ab}^\mu \dot{A}_\mu^{ab} + (D_\mu \tilde{\Pi}_{cd}^\mu)(A_\nu^{cd} t^\nu) \\
 &+ \tilde{N} q^{-2\Delta(a)+2\Delta(d)} \eta^{cd} \tilde{\Pi}_{ac}^\mu \tilde{\Pi}_{db}^\nu F_{\mu\nu}^{ab} - N^\mu \tilde{\Pi}_{ab}^\nu F_{\mu\nu}^{ab} \}.
 \end{aligned}
 \tag{6.23}$$

Since Eq. (6.23) contains no “time derivatives”, namely Lie derivatives with respect to t^μ , of the fields \tilde{N} , N^μ and $A_\nu^{cd} t^\nu$, we conclude that they are Lagrange multipliers for the constraints

$$\begin{aligned}
 q^{-2\Delta(a)+2\Delta(d)} \eta^{cd} \tilde{\Pi}_{ac}^\mu \tilde{\Pi}_{db}^\nu F_{\mu\nu}^{ab} &\approx 0, \\
 D_\mu \tilde{\Pi}_{cd}^\mu &\equiv \partial_\mu \tilde{\Pi}_{cd}^\mu + \tilde{\Pi}_{ed}^\mu A_{\mu c}^e \approx 0, \\
 \tilde{\Pi}_{ab}^\nu F_{\nu\mu}^{ab} &\approx 0.
 \end{aligned}
 \tag{6.24}$$

It is clear from the above formulae that our deformed canonical formalism reduces to the undeformed one, as presented, for example, in Ref. [4], for $q \rightarrow 1$.

7. Concluding remarks

In this paper we have shown how to build a theoretical scheme in which Einstein’s classical theory of general relativity enters as the invariant kernel common to a one-parameter family of theories. The main theoretical tool has been the construction of gauge theories based on a q -deformed rather than an ordinary Lie group [8]. This was made possible by extending, by means of purely algebraic techniques, the ordinary bi-covariant (right/left covariant) calculus on group manifolds to the deformed case [10]. This procedure has been successfully carried out for the Poincaré group [6,9].

We have seen in Section 6 how one can extend Ashtekar’s approach to build a canonical formalism for any value of q . Apart from a minor change in one of the constraints, the non-commutative nature of the conjugate variables is reflected in the structure of the Poisson brackets which are no longer skewsymmetric. We do not yet know the consequences of having an entire family of Hamiltonian formalisms for general relativity at our disposal. We recall that there exists an analogous occurrence of different Hamiltonian structures for two-dimensional integrable models (although there the fields are c -numbers) [12].

Apart from its formal aspects, the physical content of our theoretical scheme is bound to its quantization. One is then faced with the new technical problem of quantizing a canonical theory of non-commuting canonical variables. Clearly, before attempting to quantize the system presented here, initial efforts should be devoted to quantizing analogous q -deformed systems in classical mechanics, probably in terms of a path-integral formulation. New developments [11] in integration techniques on the quantum

plane seem to open the way in this direction. The first step toward quantum gravity could then be taken by tackling the problem in $2 + 1$ dimensions where a q -deformed Chern–Simons formulation of gravity is available [5]. As the topological nature of the theory is preserved by the deformation, one expects to be able to still characterize expectation values of physical quantities, related to non-contractible loops of the space-manifold, in terms of quantized knots invariants.

It is our hope that the existence of a one-parameter family of q -deformed formulations of general relativity, each endowed with a Hamiltonian structure, will shed some light on quantum gravity in physical space–time dimensions. One could imagine a scenario where q plays the rôle of a regularization parameter, with the advantage that the correct classical limit is obtained by taking the limit of vanishing Planck’s constant for any value of q .

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