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PEDAL CURVES

By

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INTRODUCTION

If from any given point P perpendiculars be drawn to the tangents to a given curve, the locus of their feet is the pedal curve of the original curve. The point P is called the pedal origin of the pedal curve. This pedal is commonly designated as the first positive pedal. The pedal curve of a first positive pedal is called the second positive pedal; the pedal curve of a second positive pedal, the third positive pedal; and so on. In this thesis attention is given generally to the first positive pedal, since everything applicable to the first positive pedal is equally applicable to any pedal.

The problem of finding the pedal curve of any curve in cartesian coordinates is obviously that of eliminating x , y , and y' from the equations

$$F: \quad y = f(x)$$

$$D: \quad y' = f'(x)$$

$$T: \quad y - y_0 = y'(x - x_0)$$

$$N: \quad y_1 - y_0 = -1/y'(x_1 - x_0),$$

where F is the function representing the given curve, D the derivative of the function with respect to x , T the tangent through a point (x,y) of the given curve and another point (x_0,y_0) , and N the normal to the tangent through the pedal origin (x_1,y_1) and the point (x_0,y_0) , a point on the desired locus. This elimination, however, becomes so cumbersome for even the simplest functions that we shall confine our efforts

to polar coordinates. Nevertheless, we shall make use of cartesian coordinates where we find them more expedient than polar coordinates. It is our aim to derive analogous equations for the solution of the pedal curve in polar coordinates.

If a given curve with equation, say

$$p = F(\psi),$$

represents the pedal of some curve, then the curve for which $F(\psi)$ is the pedal is the first negative pedal of the given curve. The negative pedal of a first negative pedal is the second negative pedal, and so on. However, we are concerned only with the first negative pedal, for all negative pedals are obtained in the same manner. We shall derive equations enabling us to write the negative pedal of any curve.

If from a point P lines be drawn to the tangents to the given curve intersecting the tangents in any constant angle, the locus of their intersection is a new curve, We shall designate this curve as the oblique pedal curve of the given curve. If the constant angle be a right angle, we have the orthogonal pedal just described. We shall derive a method enabling us, by use of the equations found for the first positive pedal, to write the oblique pedal equation. We shall prove that the oblique pedal and its corresponding orthogonal pedal are similar curves.

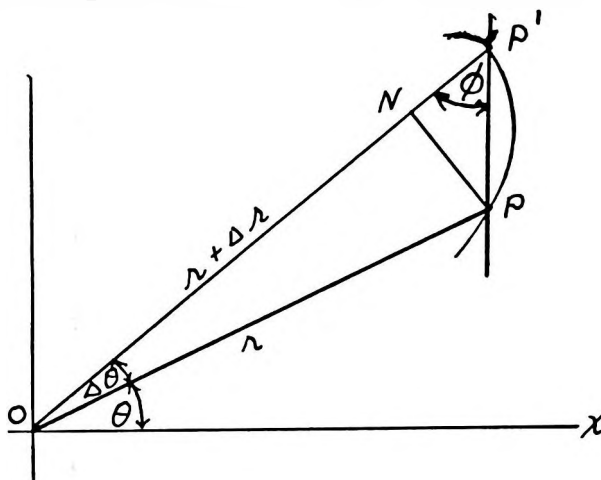
POSITIVE AND NEGATIVE PEDALS

1. The first positive pedal. Suppose that the given curve has the polar equation

$$(1) \quad r = f(\theta).$$

It is desired to find the first positive pedal of this curve with the polar origin being taken as pedal origin.

Let P and P' be two points on the curve with (r, θ) and $(r + \Delta r, \theta + \Delta \theta)$ as polar coordinates, respectively. If



we join PP' , and draw PN perpendicular to OP' , we have

$$PN = OP \sin \Delta \theta = r \sin \Delta \theta$$

$$\begin{aligned} P'N &= OP' - ON = r + \Delta r - r \cos \Delta \theta \\ &= \Delta r + r(1 - \cos \Delta \theta). \end{aligned}$$

From these relations it follows that

$$\begin{aligned} \cot PP'N &= \frac{P'N}{PN} = \frac{r + 2r \sin^2 \frac{1}{2} \Delta \theta}{r \sin \Delta \theta} \\ &= \frac{\frac{\Delta r}{\Delta \theta}}{r \frac{\sin \Delta \theta}{\Delta \theta}} + \frac{2 \sin^2 \frac{1}{2} \Delta \theta}{2 \sin \frac{1}{2} \Delta \theta \cos \frac{1}{2} \Delta \theta} \end{aligned}$$

$$\cot PP'N = \frac{1}{r} \frac{\Delta r}{\Delta \theta} \frac{1}{\frac{\sin \Delta \theta}{\Delta \theta}} + \tan \frac{1}{2} \Delta \theta$$

Hence, we have

$$(2) \quad \lim_{\Delta \theta \rightarrow 0} \cot PP'N = \cot \phi = \frac{1}{r} \frac{dr}{d\theta},$$

since, $\lim_{\Delta \theta \rightarrow 0} \frac{\sin \Delta \theta}{\Delta \theta} = 1$. Here, ϕ denotes the angle which

the tangent to the curve at P, drawn on the side of θ increasing, makes with the positive direction of the radius vector.¹

Draw a tangent to the curve (1) at a point P with polar coordinates (r, θ) .

Drop a perpendicular from

the origin O upon the tangent at Q with coordinates (p, ψ) .

The locus of Q is the pedal curve of the curve (1).

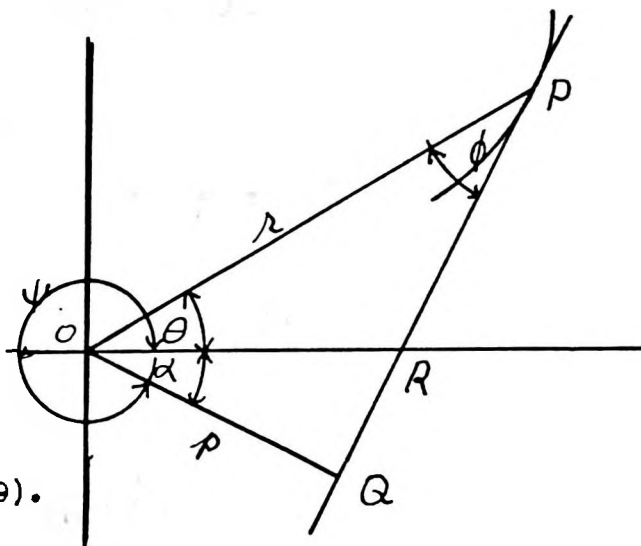
We may write

$$OQ = OP \sin \angle OPQ$$

or

$$(3) \quad p = r \sin \phi,$$

where ϕ is the angle which the tangent makes with the radius



1. Lamb, Infinitesimal Calculus, pp. 125-126.

vector of the curve and p is the radius vector of the pedal curve.

Denote the angle ROQ by α . Then, it follows that

$$\phi + \theta + \alpha = \frac{1}{2}\pi.$$

But from the figure we have

$$\alpha = 2\pi - \psi.$$

Hence, we may write

$$\theta + \phi + 2\pi - \psi = \frac{1}{2}\pi$$

or

$$(4) \quad \theta + \phi + 3/2\pi = \psi.$$

THEOREM I: The pedal curve

$$(5) \quad p = F(\psi)$$

of the curve (1) with the polar origin as pedal origin is obtained by eliminating r , θ , and ϕ from the relations (1), (2), (3), and (4).

As an example, let us take the parabola with the equation

$$r = \frac{a}{1 - \cos \theta} = \frac{a}{2 \sin^2 \frac{1}{2} \theta}.$$

If we take the logarithm of both sides of this equation and differentiate with respect to θ , we have, from (2),

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-\csc^2 \frac{1}{2} \theta \cot \frac{1}{2} \theta}{\csc^2 \frac{1}{2} \theta} = -\cot \frac{1}{2} \theta = \cot \phi.$$

This gives

$$-\frac{1}{2} \theta = \phi.$$

From (4) we obtain

$$\psi = \theta + \phi + 3/2 \pi = \frac{1}{2}(\theta + 3\pi).$$

This gives, substituting in (3),

$$\begin{aligned} p &= r \sin(-\frac{1}{2} \theta) = -r \sin \frac{1}{2} \theta = -r \cos \psi \\ &= -\frac{1}{2} a \csc^2(\psi - 3/2\pi) \cos \psi \\ &= -\frac{1}{2} a \sec^2 \psi \cos \psi \end{aligned}$$

or

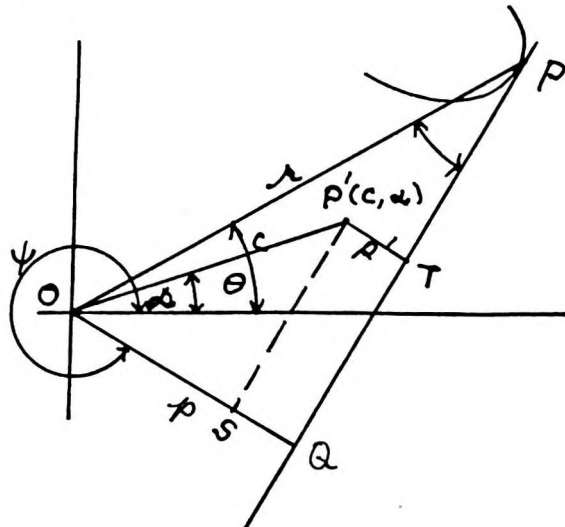
$$p = -\frac{1}{2} a \sec \psi.$$

This result enables us to state the following

THEOREM II: The pedal curve of a parabola with respect to the focus as pedal origin is a straight line tangent to the parabola at the vertex.

Suppose, now, that we are given $p = F(\psi)$ as the pedal of the given curve (1) with the pedal origin taken at the polar origin.

It is to be shown that



THEOREM III: The equation of the pedal curve with respect to any point (c, α) as pedal origin is given by

$$p' + c \cos(\psi - \alpha) = F(\psi).$$

From the figure it is seen that

$$P'OQ = 2\pi - \psi + \alpha,$$

and

$$OS = OP' \cos P'OQ$$

or

$$OS = c \cos(2\pi - \psi + \alpha) = c \cos(\psi - \alpha).$$

It is seen, too, that

$$QS + OS = OQ$$

from which we obtain

$$(6) \quad p' + c \cos(\psi - \alpha) = F(\psi).$$

Since $P'T$ and OQ are parallel, the polar angles of the two pedal curves are equal. The polar origin of the new curve is the new pedal origin (c, α) .

For an illustration we shall take the parabola of the previous example

$$r = \frac{a}{1 - \cos \theta}$$

for which the pedal with respect to the focus as pedal origin is the straight line

$$p = -\frac{1}{2} a \sec \psi.$$

It is desired to find the pedal with respect to the vertex as pedal origin.

We have, from the equation (6),

$$p' + \frac{1}{2} a \cos (\psi - 0) = -\frac{1}{2} a \sec \psi .$$

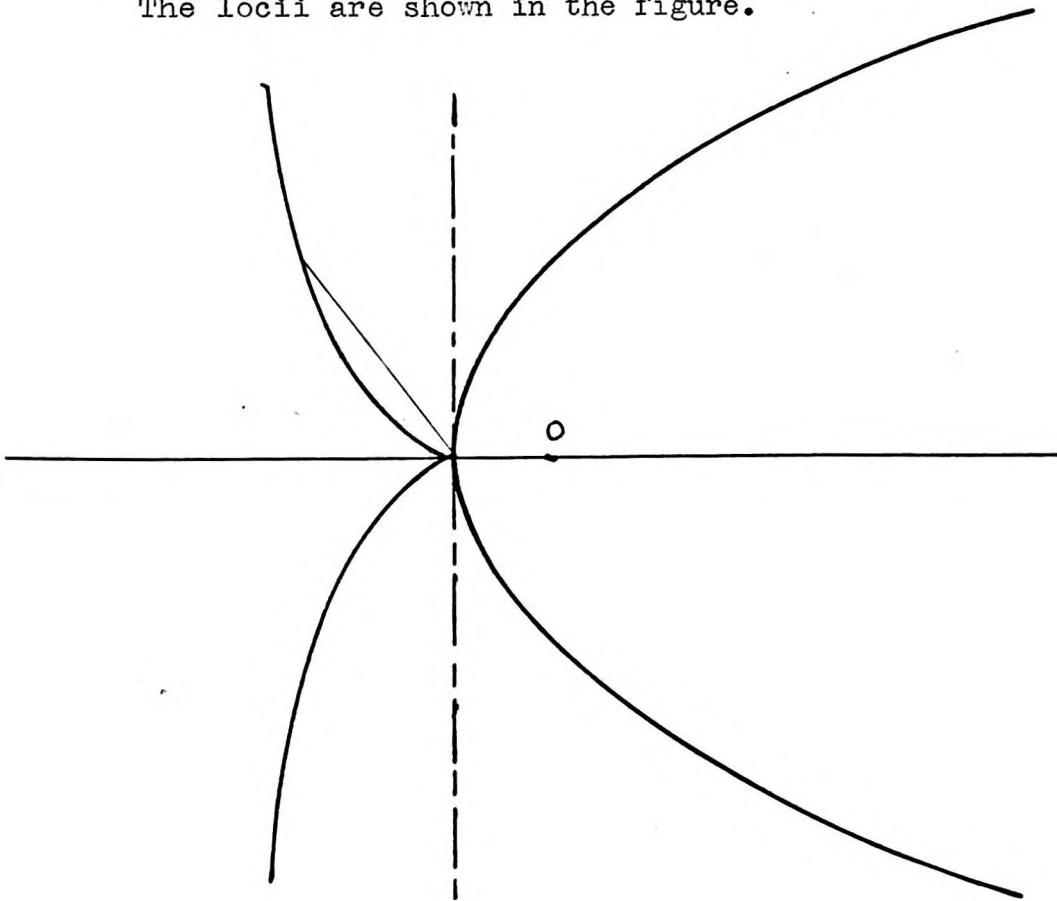
Transposing and solving for p' gives

$$\begin{aligned} p' &= -\frac{1}{2} a (\sec \psi - \cos \psi) = -\frac{1}{2} a \left(\frac{1}{\cos \psi} - \cos \psi \right) \\ &= -\frac{1}{2} a \left(\frac{1 - \cos^2 \psi}{\cos \psi} \right), \end{aligned}$$

or

$$p' = -\frac{1}{2} a \frac{\sin^2 \psi}{\cos \psi} = -\frac{1}{2} a \tan \psi \sin \psi .$$

The loci are shown in the figure.

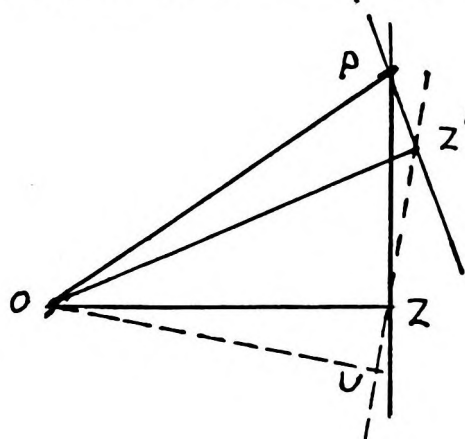


2. The first negative pedal. We are confronted with the problem of finding the curve for which a given curve is the pedal curve; that is, the negative pedal curve of a given curve.

In solving this problem, it is necessary to make use of the following

THEOREM IV: The angle which the tangent makes with the radius vector at corresponding points is the same for a curve and its pedal.¹

To prove this let OZ , OZ' be the perpendiculars from O on two consecutive tangents PZ and PZ' , and let OU be drawn perpendicular to ZZ' produced. The points Z and Z' lie on the circle described on OP as diameter. Hence, the exterior angle OZU of the quadrilateral $OZZ'P$ is equal to the interior and opposite angle OPZ' . In the limiting position these are the angles which OZ and OP make with the tangent to the pedal and with the tangent to the original curve, respectively. Hence, the theorem is proved.



Also, by similar triangles we have from the above figure

$$OU:OZ = OZ':OP.$$

Hence, if r be the radius vector of the original curve, p the

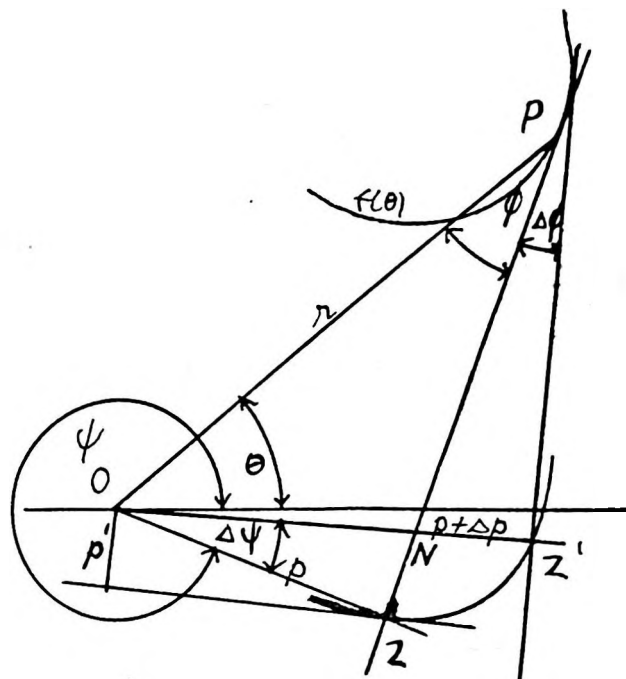
1. Lamb, Infinitesimal Calculus, pp. 383-384.

perpendicular from O on the tangent to the curve, and p' the perpendicular from O on the tangent to the pedal, we have, ultimately,

$$p':p = p:r$$

or

$$p' = p^2/r.$$



If OZ' meet PZ in N , we may write

$$OZ = p,$$

$$OZ' = p + \Delta p,$$

and

$$ZOZ' = ZPZ' = \Delta\psi.$$

Neglecting infinitesimals of second order, we have

$$p = NZ' = PZ' \tan \Delta\psi = PZ' \Delta\psi \frac{\tan \Delta\psi}{\Delta\psi}.$$

Now as Z' approaches Z along the pedal curve, $\Delta\psi$ approaches 0, and hence, $\frac{\tan \Delta\psi}{\Delta\psi}$ approaches 1. Then, we have

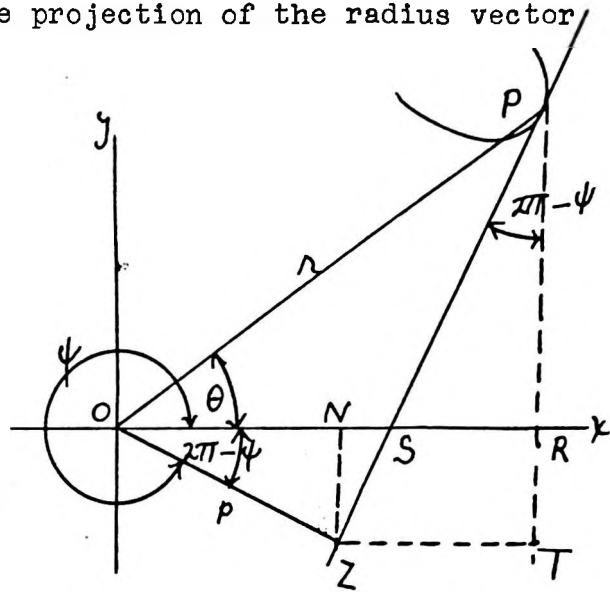
$$\lim_{\Delta\psi \rightarrow 0} \frac{\Delta p}{\Delta\psi} = \lim_{\Delta\psi \rightarrow 0} PZ'$$

or

$$PZ = \frac{dp}{d\psi},$$

which is the expression for the projection of the radius vector on the tangent of a curve.

This result enables us to easily solve the problem of the negative pedal. Taking O as origin, and the initial line of ψ as axis of x , the coordinates of the point of contact, P , are given in parametric form by



$$\begin{aligned} x &= ON + NR = ON + ZT \\ &= OZ \cos (2\pi - \psi) + PZ \sin (2\pi - \psi) \\ &= OZ \cos \psi - PZ \sin \psi \end{aligned}$$

or

$$(7) \quad x = p \cos \psi - \frac{dp}{d\psi} \sin \psi,$$

and

$$\begin{aligned} y &= RP = PT - TR = PT - ZN \\ &= PZ \cos (2\pi - \psi) - OZ \sin (2\pi - \psi) \\ &= PZ \cos \psi + OZ \sin \psi \end{aligned}$$

or

$$(8) \quad y = p \sin \psi + \frac{dp}{d\psi} \cos \psi.$$

For a simple illustration, let us find the negative pedal of the curve

$$p = a \sin \psi \cos \psi.$$

Substituting in equation (7), we find

$$x = a \sin \psi \cos^2 \psi - a \cos^2 \psi \sin \psi + a \sin^3 \psi$$

or

$$x = a \sin^3 \psi.$$

Substituting in (8), we find

$$y = a \sin^2 \psi \cos \psi - a \sin^2 \psi \cos \psi + a \cos^3 \psi$$

or

$$y = a \cos^3 \psi.$$

Solving these results for $\sin \psi$ and $\cos \psi$, respectively, and squaring and adding, we obtain

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

or

$$\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{a^{2/3}} = 1.$$

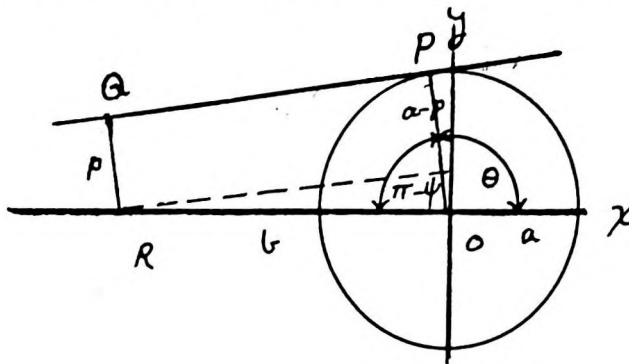
This result is the curve sometimes known as the astroid.

3. The circle. The problem of finding the pedal of the circle is best solved directly from geometry, since the radii are perpendicular to the tangents.

THEOREM V: The pedal of any circle is a limaçon.

We shall, without loss of generality, take the circle with the center at the origin and radius a. Take a point R at

a distance b from the origin on the initial line of the circle as pedal origin. Draw a line tangent to the circle at P . From R drop a perpendicular to the tangent at Q . Then QR is the radius vector of the pedal



curve and is designated as p . Here θ and ψ are the polar angles of the given circle and its pedal, respectively, and are equal, since both are perpendicular to the tangent QP . Hence, we may write

$$a - p = b \cos (\pi - \psi) = -b \cos \psi$$

or

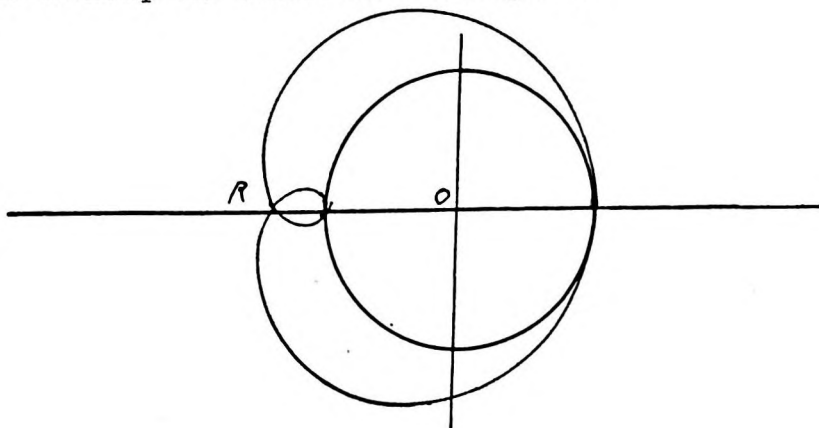
$$p = a + b \cos \psi,$$

which is the equation of a limaçon.

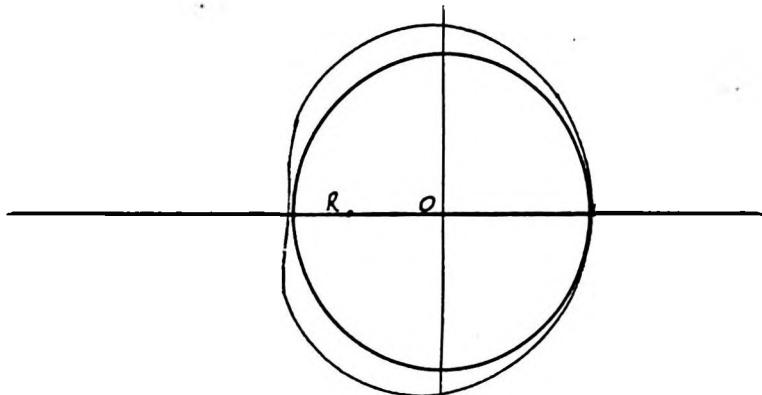
If $a < b$, the curve passes through the polar origin R when

$$\psi = \arccos (-a/b)$$

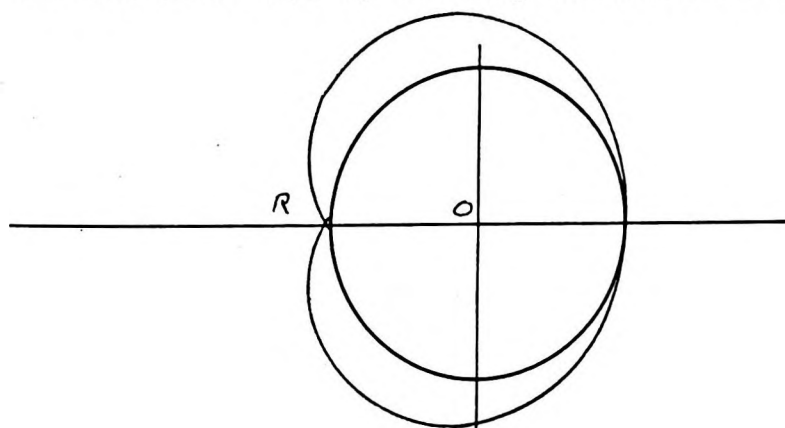
and forms a loop as shown in the figure.



If $a > b$, p cannot vanish and we have a curve as shown in the second figure.



In the case $a = b$, the loop shrinks into a cusp, and



the locus is a cardioid with the equation

$$p = a(1 + \cos \psi).$$

In the trivial case when $b = 0$, the equation becomes

$$p = a.$$

The locus is the original circle itself.

4. Some closure properties. It is our purpose to determine those curves whose first positive pedal and first negative pedal, with the polar origin as pedal origin, is a

curve of the same kind; that is, the curve for which $p = c r$ and $\psi = \theta + C$.

In order that the given curve (1) and its pedal curve (5) be similar, it is necessary that

$$\sin \phi = c,$$

so that we have, from (3),

$$p = c r.$$

This implies that,

$$\phi = \alpha,$$

where α is a constant, and hence we have from (4)

$$\psi = \theta + C.$$

If for $\cot \phi$, in (2), we put $\cot \alpha$ and solve the differential equation, we have

$$\log r = (\cot \alpha) \theta + a$$

or

$$r = e^{(\cot \alpha) \theta + a} = e^{(\cot \alpha) \theta} e^a = b e^{(\cot \alpha) \theta}.$$

This is the equation of the equiangular spiral.

We have, also, by differentiation that

$$dp = c dr \quad \text{and} \quad d\psi = d\theta.$$

Substituting these results in the differential equation

$$\frac{1}{r} \frac{dr}{d\theta} = \cot \alpha$$

and solving, we find

$$\log p = (\cot \alpha) \psi + b$$

or

$$p = c e^{(\cot \alpha) \psi}.$$

This, too, is the equation of an equiangular spiral.

We have proved the following

THEOREM VI: The only curve whose positive and negative pedals, with respect to the polar origin as pedal origin, are curves of the same kind is the equiangular spiral.

Let us consider the more special case where

$$p = r.$$

From (2) we see that $p = r$ is possible only if

$$\sin \phi = 1.$$

Now $\sin \phi = 1$ implies that $\phi = \frac{1}{2}\pi$. Therefore, replacing α by $\frac{1}{2}\pi$ in the equation of the equiangular spiral gives

$$p = r = a.$$

This is the equation of a circle about the polar origin as center. Also, since $\phi = \frac{1}{2}\pi$, we have from (4)

$$\psi = \theta + 2\pi = \theta.$$

Hence, it is true that the circle about the polar origin as center and its pedal curve with the polar origin as pedal origin are identical curves.

We may now state the following corollary to our theorem

COROLLARY I: The circle is the only curve whose positive and negative pedals, with polar origin as pedal origin, are identical with the original curve.

5. The general conic. We shall consider the general polar equation of the conic

$$(9) \quad r = \frac{a}{1 - e \cos \theta},$$

where a is one-half of the latus rectum and e is the eccentricity. The polar origin is the focus. If $e = 0$, the curve is a circle about the origin as center; and if $e = 1$, the curve is a parabola. When $e < 1$, the equation represents an ellipse. Correspondingly, when $e > 1$, the equation represents a hyperbola.

Our problem here is to determine the pedal curve of (9) about the origin or focus as pedal origin. Taking the logarithm of both sides of (9) we obtain

$$\log r = \log a - \log (1 - e \cos \theta).$$

Differentiating with respect to θ and making use of (2), we get

$$(10) \quad \frac{1}{r} \frac{dr}{d\theta} = \frac{-e \sin \theta}{1 - e \cos \theta} = \cot \phi = \frac{\cos \phi}{\sin \phi}.$$

From this we find

$$e (\cos \theta \cos \phi - \sin \theta \sin \phi) = \cos \phi$$

or

$$e \cos (\theta + \phi) = \cos \phi.$$

Equation (4) gives

$$\theta + \phi = \psi - 3/2\pi,$$

so that

$$\cos \phi = -e \sin \psi$$

and

$$\sin \phi = \pm (1 - e^2 \sin^2 \psi)^{\frac{1}{2}}.$$

From (10) we have, then,

$$\frac{-e(1 - \cos^2 \theta)^{\frac{1}{2}}}{1 - e \cos \theta} = \frac{-e \sin \psi}{(1 - e^2 \sin^2 \psi)^{\frac{1}{2}}}$$

Squaring both sides and solving for $\cos \theta$, gives

$$\cos \theta = e \sin^2 \psi \pm \cos (1 - e^2 \sin^2 \psi)^{\frac{1}{2}}.$$

Substituting this value of $\cos \theta$ in (3) gives

$$p = \frac{\pm a}{(1 - e^2 \sin^2 \psi)^{\frac{1}{2}} \pm e \cos \psi}$$

The determination of the plus or minus sign is difficult in polar coordinates; hence, we shall convert the equation to cartesian coordinates. We have, then, that

$$(x^2 + y^2)^{\frac{1}{2}} = \frac{\pm a}{\left(1 - e^2 \frac{y^2}{x^2 + y^2} \pm e \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}\right)}$$

This reduces to the simpler form

$$(1 - e^2)(x^2 + y^2) = a^2 \pm 2 e a x.$$

This equation always represents a circle. And since the pedal curve must, by definition, be tangent to the original curve in two points, we may state the following

THEOREM VII: The pedal with respect to the focus as pedal origin of any conic is always a circle about the major axis as diameter.

In the case of the parabola, the pedal is a circle with an infinite radius and hence a straight line tangent to the curve at the vertex.¹

We shall take the polar equation of the pedal of the conic in the following form

$$(11) \quad p = \frac{-a}{(1 - e^2 \sin^2 \psi)^{\frac{1}{2}} + e \cos \psi}.$$

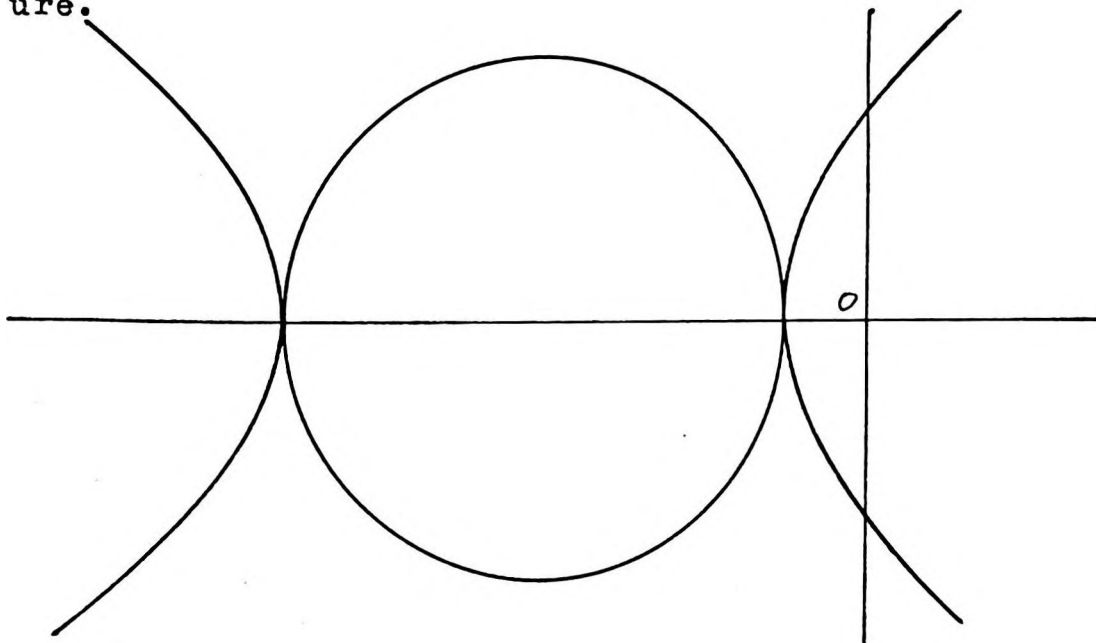
The signs are so chosen that the pedal curve will be tangent to both vertices of an ellipse or to both branches of a hyperbola. This is obviously a necessary requirement, as seen in the above theorem.

1. See THEOREM II, p. 6.

Let us take the hyperbola for which $e = 5/4$ as an illustrative example. This curve has the equation

$$p = \frac{-4a}{(16 - 25 \sin^2 \psi)^{\frac{1}{2}} + 5 \cos \psi}.$$

The locus of this equation is a circle as shown in the figure.



Let us also consider the negative pedal of the conic. Substitution in the parametric equations (7) and (8) for the negative pedal gives

$$x = \frac{a \cos \psi}{1 - e \cos \psi} + \frac{a e \sin^2 \psi}{(1 - e \cos \psi)^2}$$

and

$$y = \frac{a \sin \psi}{1 - e \cos \psi} - \frac{a e \sin \psi \cos \psi}{(1 - e \cos \psi)^2}.$$

These equations may be reduced to the simpler form

$$x = \frac{a \cos \psi - a e \cos 2\psi}{(1 - e \cos \psi)^2},$$

$$y = \frac{a \sin \psi - a e \sin 2\psi}{(1 - e \cos \psi)^2}.$$

For an example, take $e = 0$. The equations become

$$x = a \cos \psi,$$

$$y = a \sin \psi.$$

Eliminating ψ , we obtain

$$x^2 + y^2 = a^2,$$

a circle about the origin.

6. The curves $r^n = a^n \cos n \theta$. A rather general group of important curves are included in the family

$$r^n = a^n \cos n \theta,$$

where n is any rational real number. Thus if $n = \pm 1$, we have, respectively, the circle

$$r = a \cos \theta$$

and the straight line

$$r \cos \theta = a.$$

If $n = \pm 2$, we have, respectively, the lemniscate

$$r^2 = a^2 \cos 2\theta,$$

sometimes called the 'lemniscate of Bernoulli', and the rectangular hyperbola

$$r^2 \cos 2\theta = a^2.$$

If $n = \pm \frac{1}{2}$, we have the cardioid

$$r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta, \text{ or } r = \frac{1}{2} a (1 + \cos \theta)$$

and the parabola

$$r^{\frac{1}{2}} \cos \frac{1}{2}\theta = a^{\frac{1}{2}}, \text{ or } r = \frac{2a}{1 + \cos \theta}.$$

The pedals of this group of curves are very easily found by the simple manipulation of replacing n by $\frac{n}{n+1}$ in the equation of the curve.

To show this we take the four relations

$$(1) \quad r^n = a^n \cos n\theta$$

$$(2) \quad \frac{1}{r} \frac{dr}{d\theta} = \cot \phi$$

$$(3) \quad p = r \sin \phi$$

$$(4) \quad \psi = \theta + \phi + 3/2\pi.$$

We rewrite (1) in the form

$$r = a (\cos n\theta)^{1/n}$$

and take the logarithm of both sides, obtaining

$$\log r = \log a + \frac{1}{n} \log \cos n \theta.$$

Differentiating with respect to θ gives, by (2),

$$\frac{1}{r} \frac{dr}{d\theta} = - \frac{\sin n \theta}{\cos n \theta} = - \tan n \theta = \cot \phi.$$

This gives

$$n \theta = \phi - \frac{1}{2} \pi.$$

From (4) we have

$$\psi = \theta + n \theta + \frac{1}{2} \pi + \frac{3}{2} \pi = (n + 1) \theta$$

from which we find

$$n \theta = \frac{n}{n + 1} \psi.$$

Substituting in (3) we have, now,

$$p = a (\cos n \theta)^{1/n} \cos n \theta$$

or

$$p = a \left(\cos \frac{n}{n + 1} \psi \right)^{1/n} \cos \frac{n}{n + 1} \psi,$$

from which we have

$$p^n = a^n \cos^{n+1} \frac{n}{n + 1} \psi,$$

or

$$\frac{n}{p^{n+1}} = \frac{n}{a^{n+1}} \cos \frac{n}{n + 1} \psi.$$

This equation gives the pedal curve with respect to the origin or center as the pedal origin.

For illustration, let us take the rectangular hyperbola, for which $n = -2$. We have, then

$$r^{-2} = a^{-2} \cos(-n\theta).$$

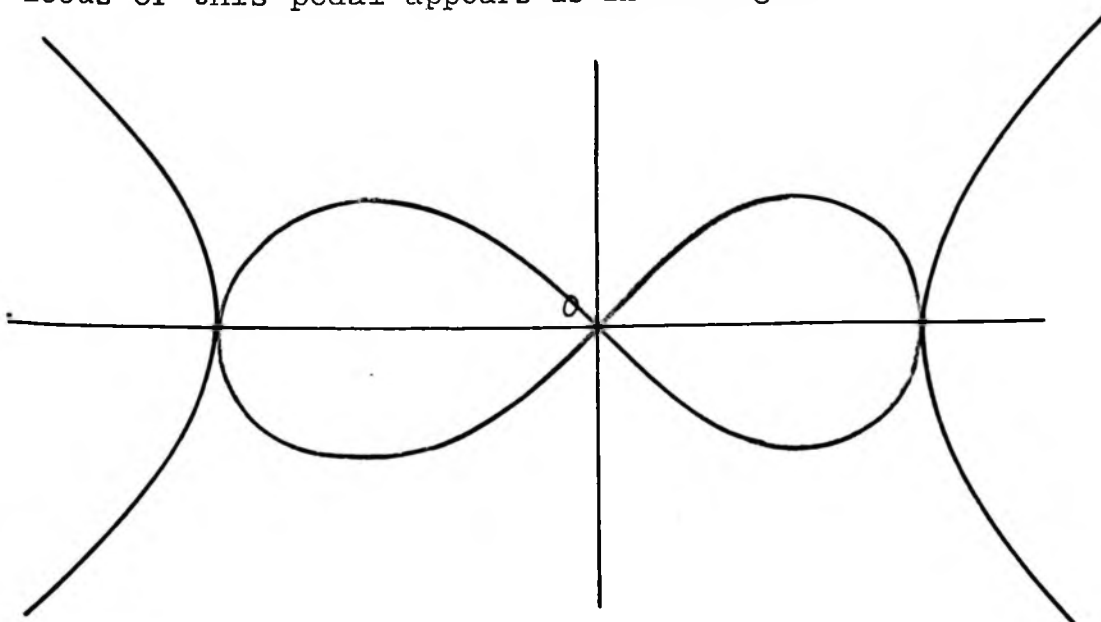
Replacing n by $\frac{n}{n+1}$, we find that

$$p^{\frac{-2}{-2+1}} = a^{\frac{-2}{-2+1}} \cos \frac{-2}{-2+1},$$

or

$$p^2 = a^2 \cos 2\psi.$$

We recognize this curve as the lemniscate of Bernoulli obtained for $n = 2$, above. There is, however, no significance attached to the similarity of the two curves in this case. The locus of this pedal appears as in the figure.



7. The oblique pedal. From a given point P let us drop a transversal to the tangent to a given curve, cutting the tangent at a constant angle not a right angle. The locus of their points of intersection is the oblique pedal curve, as previously defined. We wish to determine the nature of this oblique pedal relative to the orthogonal pedal.

Let us suppose that the orthogonal pedal or ordinary pedal curve is known or can be easily found with respect to the desired point as

pedal origin. From the figure, OQ is the perpendicular to the tangent and, hence, the radius vector, p , of the orthogonal pedal, with the vectorial angle ψ .

Through O, the desired point, drop a transversal upon PQ cutting PQ at Q' in a constant angle. Designate the complement of this angle, Q'OQ, as α . Let OQ', the radius vector of the desired locus, be ρ . We have, then,

$$\cos \alpha = p/\rho,$$

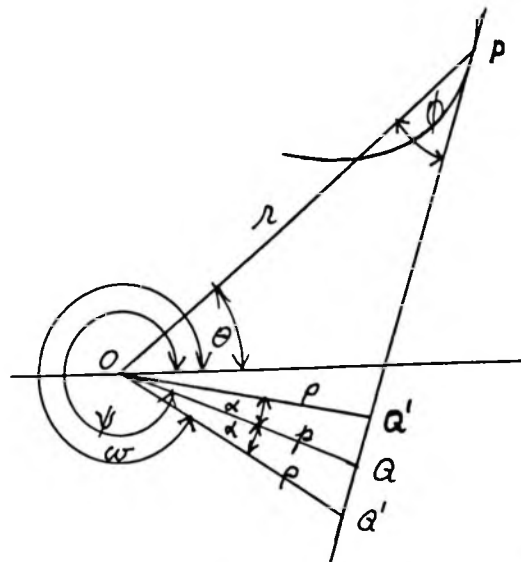
or

$$\rho = F(\psi) \sec \alpha.$$

But from the figure we find

$$\omega = \psi \pm \alpha$$

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or

$$\psi = \omega \pm d.$$

Hence, we have

$$(12) \quad \rho = \sec d F(\omega \pm d),$$

which is the equation of the desired locus.

Since we have presupposed that the orthogonal pedal curve could be found for any desired point as pedal origin, it is unnecessary to derive equations for the translation of the pedal origin of the oblique pedal.

As an example, we take the equation

$$r = 2a \cos \theta.$$

Since this is a member of the family $r^n = a^n \cos n \theta$, we have

$$p = (2a)^{\frac{1}{2}} \cos \frac{1}{2} \psi$$

or

$$p = a (1 + \cos \psi).$$

Translating the pedal origin to the point (a, π) gives

$$p' = a (1 + 2 \cos \psi).$$

Let us find the oblique pedal for $d = \pi/4$. We have

$$\begin{aligned} \rho &= \sqrt{2} a [1 + \sqrt{2} \cos (\omega \pm \pi/4)] \\ &= \sqrt{2} a [1 + \sqrt{2} \cos \omega \pm \sqrt{2} \sin \omega] \end{aligned}$$

or

$$\rho = \sqrt{2} a + 2 a (\cos \omega \pm \sin \omega).$$

Also, for $\alpha = \pi/6$ we have

$$\rho = \frac{2 a}{\sqrt{3}} (1 + \sqrt{3} \cos \omega \pm \sin \omega);$$

too, for $\alpha = \pi/3$, we have

$$\rho = 2 a (1 + \cos \omega \pm \sqrt{3} \sin \omega).$$

The accompanying figure shows these loci, with the orthogonal pedal. It is to be noted that each oblique pedal curve consists of two distinct curves.

We have seen illustrated in the previous example the following

THEOREM VIII: The oblique pedal curve is similar to the corresponding pedal curve.

In the general solution of the oblique pedal we found that

$$\rho = \sec \alpha$$

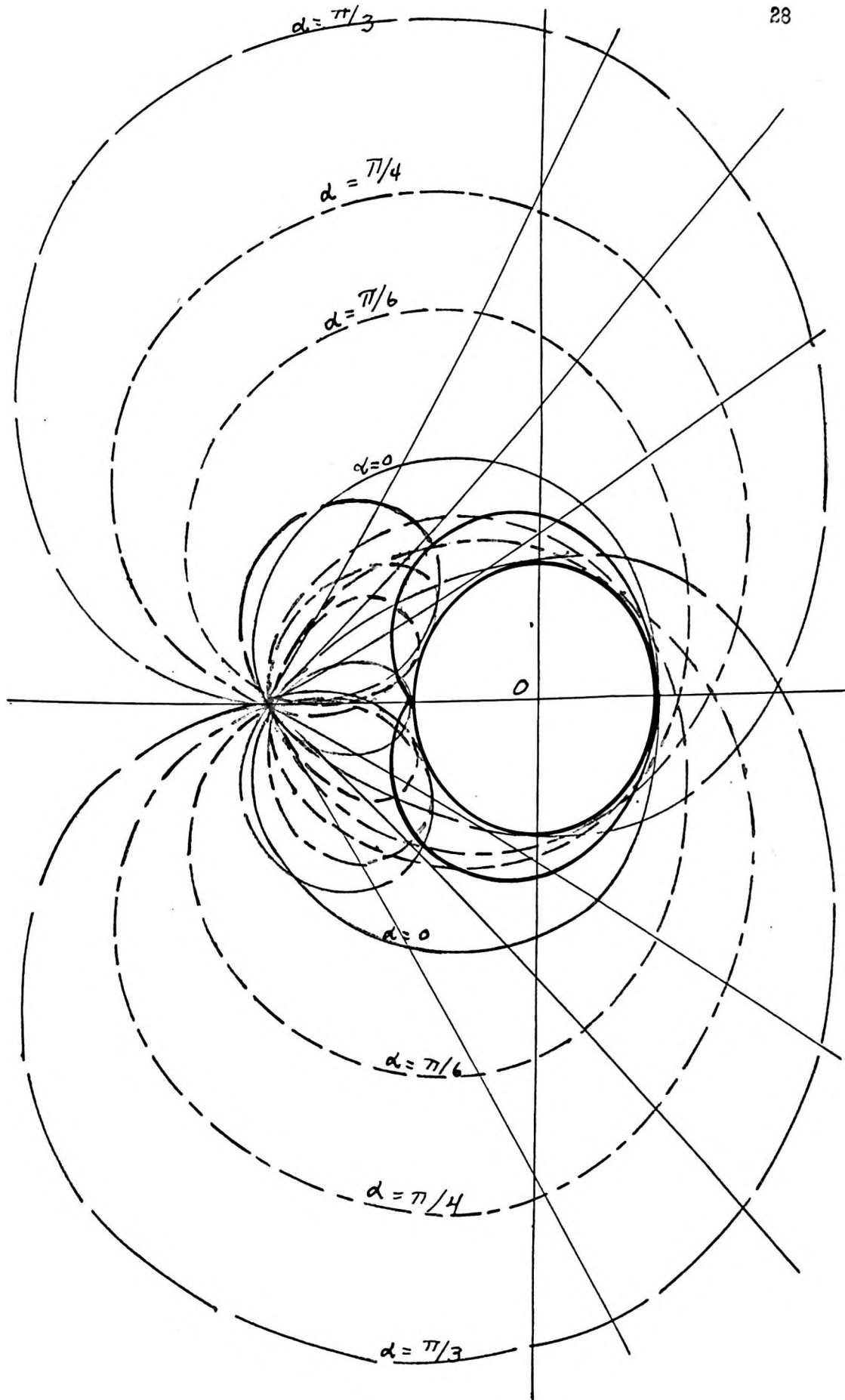
or

$$\rho = k p.$$

We found also that

$$\psi = \omega \pm \alpha.$$

Hence, the oblique pedal curve is similar to its orthogonal pedal. The oblique pedal is magnified by a constant factor, and the initial line of the oblique pedal is rotated through a constant angle.



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