SOME LINEAR ELASTOSTATIC BOUNDARY VALUE PROBLEMS FOR COMPOSITE MATERIALS

by

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A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Engineering Mechanics in the Graduate School of the University of Alabama

University, Alabama

1966

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ABSTRACT

Solutions are presented for three different linear elastostatic boundary value problems pertaining to infinite bodies which contain voids or inclusions. The first section of the thesis contains an analysis of the three dimensional axisymmetric mixed boundary value problem of an infinite body containing two identical partially bonded rigid spherical inclusions and subjected to an axisymmetric torsional stress field. In the second section a study is made regarding the problem of longitudinal or transverse shear stresses in a body which is uniformly stressed at infinity and contains two perfectly bonded circular cylindrical inclusions of arbitrary size, elastic properties, and spacing. The last section presents a solution for the three dimensional axisymmetric contact stress problem of an infinite body containing a smooth spherical elastic inclusion and subjected to an axisymmetric stress state at infinity. Numerical results are presented for each problem. Computer programs for the above problems are contained in appendices.

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professor Howard B Wilson, Jr., for his assistance and direction of this work and to Professor James L. Hill for many valuable suggestions.

The National Aeronautics and Space Administration, through a NASA Traineeship, made the completion of this dissertation possible.

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CHAPTER I

AXISYMMETRIC TORSIONAL STRESSES IN A SOLID CONTAINING TWO PARTIALLY BONDED RIGID SPHERICAL INCLUSIONS

A solution is presented in this chapter for the determination of the stresses and displacements in an infinite elastic body having two identical spherical cavities containing partially bonded rigid spherical inclusions. Denoting the line connecting the centers of the cavities as the z -axis, the bonded inclusions are then symmetric with respect to this axis as well as to a plane normal to and intersecting the z-axis midway between the two cavities as shown in Fig. 1. The problem of torsion of an infinite solid containing two stress free spherical cavities has been solved by Eubanks [1][1]. A solution for the analagous problem involving two perfectly bonded rigid inclusions was obtained by Hill [2]. The present analysis extends the work of the above references to include partially bonded inclusions as well as allowing for nonvanishing resultant torques acting on the inclusions.

The determination of the stresses and displacement reduces to the solution of a set of dual Legendre series relations generated from the boundary conditions on the spherical surfaces and the required stress and displacement conditions at infinity. Anticipating that the boundary stress will have a square root type singularity at the points of separation between the matrix and the inclusions, the boundary stresses are represented in terms of a singular stress field of known form and a regular stress field.

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[1]Numbers in square brackets refer to references at the end of the chapter.

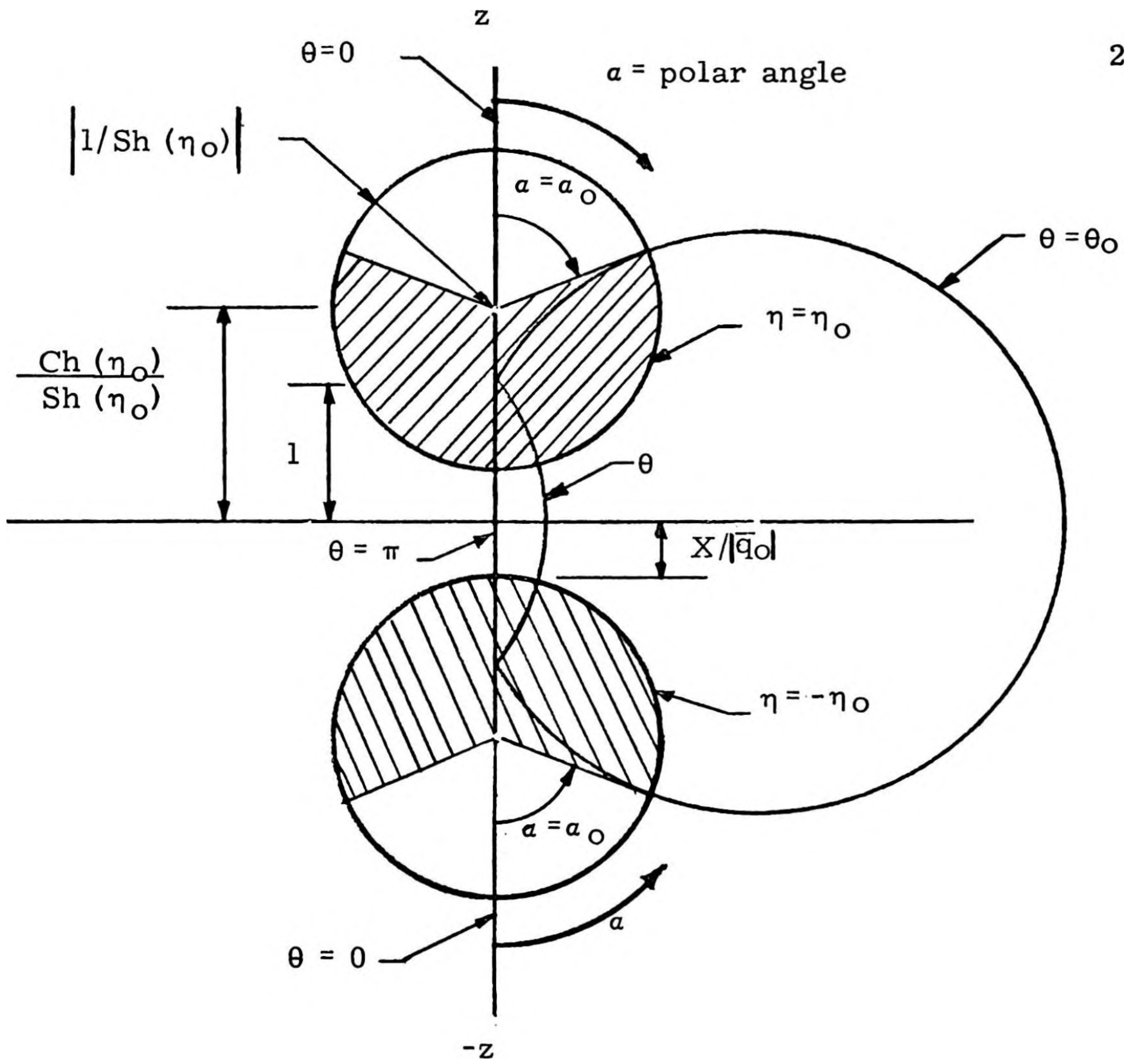

Fig.. 1 Partially Bonded Inclusions in an Infinite Space

An analysis is then made by reducing the dual series equations to a related system of linear simultaneous equations. The unknown components of stress and displacement are determined for various configurations of loading and geometry. The required boundary conditions are shown to be satisfied to a high degree of accuracy except for cases where the inclusions are closely spaced.

Geometrical Relations and Equilibrium Equations

It is convenient to formulate the problem in bispherical coordinates (η, θ, β) with the transformation being defined by [3]

$$x = \frac{\bar{p}}{Q^2} \cos(\beta), \quad y = \frac{\bar{p}}{Q^2} \sin(\beta), \quad z = \frac{\bar{q}}{Q^2}$$

where

$$p = \cos(\theta),\ \bar{p} = \sin(\theta),\ q = \cosh(\eta),\ \bar{q} = \sinh(\eta) \qquad (1)$$

and $Q = \sqrt{q - p}$.

The curvilinear coordinates (η , θ , β) take on all values within the ranges

$$-\infty < \eta < \infty,\ 0 \leq \theta \leq \pi, \text{ and } 0 \leq \beta \leq 2\pi . \qquad (2)$$

The spherical surfaces are defined by $\eta = \pm\eta_o$ with the center of the spheres located at $z = \coth(\pm\eta_o)$ and having a radius of $r = |1/\sinh(\eta_o)|$.

In the case of a torsional stress field symmetric about the z-axis, the only nonvanishing displacement component is $u(\eta, \theta) = u$ which acts in the β direction circumferential to the z-axis. It then follows from the equilibrium equations that ([4], page 325)

$$\nabla^2 \left(u \begin{matrix} \sin\beta \\ \cos\beta \end{matrix}\right) = 0. \qquad (3)$$

Appropriate particular solutions of Eq. 3 are [3]

$$u = Q\, C_n\, \bar{p}\, P_n'(p) \qquad \text{or} \qquad (4)$$

$$u = Q\, S_n\, \bar{p}\, P_n'(p) \qquad (5)$$

where $C_n = \cosh[(n+1/2)\eta]$, $S_n = \sinh[(n+1/2)\eta]$, $P_n(p)$ is the Legendre polynomial and $P_n'(p) = d\,P_n(p)/dp$. Eqs. 4 and 5 give displacements which are symmetric and antisymmetric respectively, about the plane of geometrical symmetry of the system.

In bispherical coordinates the nonvanishing stress components are [1]

$$\sigma_{\eta\beta} = \mu\, \bar{p}\, \frac{\partial}{\partial\eta}\left[u \frac{Q^2}{\bar{p}}\right] \text{ and} \qquad (6)$$

$$\sigma_{\theta\beta} = \mu\, \bar{p}\, \frac{\partial}{\partial\theta}\left[u \frac{Q^2}{\bar{p}}\right] \qquad (7)$$

where μ is the shear modulus.

Development of the Dual-Series Equations

It will be shown that the determination of a solution for the general case of partially bonded inclusions having nonzero torques appled to the inclusions as well as torsion of the elastic solid may be resolved into two fundamental problems.

The first of these is the stress boundary value problem corresponding to the infinite space containing two spherical cavities with tangential stresses acting on the surface of the cavities and stresses at infinity producing pure torsion about the z-axis. This was considered by Eubanks [1] for the particular condition of vanishing tangential stress on the spherical cavities. The second fundamental problem is the mixed boundary value problem arising when the cavities contain partially bonded inclusions which are rotated through angles ω_1 and ω_2 about the z-axis. The unbonded portions of the spherical surfaces are unloaded and the stresses and displacement vanish near infinity. It is always possible to resolve this problem into two separate cases; the first having equal symmetric rotations given by $\frac{\omega_1 + \omega_2}{2}$ and the second, equal asymmetric rotations given by $\frac{\omega_1 - \omega_2}{2}$. However, as can be seen from Eqs. 4 and 5, due to the particular form of the stress and displacement series it is only necessary to replace C_n by S_n to obtain a solution corresponding to symmetric rotations from one determined for asymmetric rotations. Therefore, only one characteristic analysis is required. As the shearing stresses are linear in the rotation angle ω, this problem could equally well be formulated in terms of applied torques; although the solution would reduce to determining the torque corresponding to a unit angle of twist and then multiplying the computed stresses and displacement by the ratio of the desired torque to the computed value. The same resolution as described above for arbitrary rotations ω_1 and ω_2 would also apply to arbitrary resultant torques T_1 and T_2.

For convenience then, in obtaining a solution to the above mixed boundary value problem, only the case associated with rotating the inclusions through opposite angles ω will be presented in detail. With reference to Fig. 1, it will also be assumed that the contact surface is specified by $\theta_o \le \theta \le \pi$. The related case of contact on the arc $0 \le \theta \le \theta_o$ will be shown to follow easily from this solution.

An appropriate form of u such that the displacement and stresses vanish near infinity and that $u(\eta, \theta) = -u(-\eta, \theta)$ is seen to be

$$u = Q\bar{p} \sum_{n=1}^{\infty} A_n S_n P_n'(p) \tag{8}$$

where the unknown constants A_n are to be determined from the boundary conditions on the cavity surface specified by $\eta = \eta_o$. The stresses are then given by Eqs. 6 and 7 as

$$\sigma_{\eta\beta} = \frac{\mu Q}{2} \bar{p} \sum_{n=1}^{\infty} A_n \left[3\bar{q} S_n + (2n+1) C_n Q^2 \right] P_n'(p) \tag{9}$$

and

$$\sigma_{\theta\beta} = \frac{\mu Q}{2} \sum_{n=1}^{\infty} A_n S_n \left[[3\bar{p}^2 - 4pQ^2] P_n'(p) + 2n(n+1)Q^2 P_n(p) \right]. \tag{10}$$

If the inclusions are rotated through an angle ω, the displacements on the contact surfaces will be

$$u = r\omega \sin(\alpha) \qquad \text{for } \eta = \eta_o \text{ and} \tag{11}$$

$$u = -r\omega \sin(\alpha), \qquad \eta = -\eta_o \tag{12}$$

where r is the radius of the sphere and α is the true polar angle as shown in Fig. 1.

As $\sin(\alpha) = \dfrac{y}{r \sin(\beta)}$ and $y = \dfrac{\bar{p}}{Q_o^2} \sin(\beta)$,

the displacement in terms of the curvilinear condinates is

$$u = \frac{\bar{p}\,\omega}{Q_o^2}, \qquad \theta_o \le \theta \le \pi \tag{13}$$

with $Q_o = \sqrt{q_o - p}$ and $q_o = \cosh(\eta_o)$.

It is also appropriate to define the following quantities for future reference:

$\bar{q}_o = \sinh(\eta_o)$, $S_n^o = \sinh[(n+1/2)\eta_o]$ and $C_n^o = \cosh[(n+1/2)\eta_o]$.

From the requirement that the unbonded portions of the spherical cavities have no applied loads and from Eq. 13 it then follows that the stress $\sigma_{\eta\beta}$ and displacement u must satisfy

$$\sigma_{\eta\beta} = 0 \qquad 0 \leq \theta < \theta_o \tag{14}$$

and

$$u = \frac{\bar{p}\ \omega}{Q_o^2} \qquad \theta_o \leq \theta \leq \pi \tag{15}$$

on the surface specified by $\eta = \eta_o$.

Substituting Eqs. 8 and 9 into Eqs. 14 and 15 there results

$$\sum_{n=1}^{\infty} A_n \left[3\,\bar{q}_o S_n^o + (2n+1)\, C_n^o\, Q_o^2 \right] P_n'(p) = 0, \quad 0 \leq \theta < \theta_o \tag{16}$$

and

$$\sum_{n=1}^{\infty} A_n S_n^o P_n'(p) = \frac{\omega}{Q_o^3} \qquad \theta_o \leq \theta \leq \pi \tag{17}$$

The right hand side of Eq. 17 can be expanded in a series of associated Legendre functions by noting that the generating function for the Legendre polynomials may be written as

$$\frac{1}{\sqrt{q_o - p}} = \sqrt{2} \sum_{n=0}^{\infty} P_n(p)\, e^{-(n+1/2)\eta_o} \tag{18}$$

If Eq. 18 is differentiated partially with respect to p the expansion

$$\frac{1}{Q_o^3} = 2\sqrt{2} \sum_{n=1}^{\infty} e^{-(n+1/2)\eta_o} P_n'(p) \tag{19}$$

is obtained. It then follows from Eqs. 9, 10, 16, 17, and 19 that the boundary conditions will be satisfied provided

$$\sum_{n=1}^{\infty} B_n \left[3\bar{q}_o + \frac{(2n+1)}{T_n^o} Q_o^2 \right] P_n'(p) = 0, \quad 0 \leq \theta < \theta_o \tag{20}$$

and

$$\sum_{n=1}^{\infty} B_n P_n'(p) = 2\sqrt{2}\,\omega \sum_{n=1}^{\infty} e^{-(n+1/2)\eta_o} P'_n(p), \qquad \theta_o \leq \theta \leq \pi \tag{21}$$

where

$B_n = A_n S_n^o$ and $T_n^o = \tanh[(n+1/2)\eta_o]$.

The discontinuous form of the boundary conditions at the end of the contact arc is accompanied by a stress field having a square root type singularity at the end of the contact arc. It is convenient to account for this stress by representing the resultant stress distribution on the cavity as being composed of a singular and a regular component. The resultant stress $\sigma_{\eta\beta}$ on the boundary is then written as

$$\sigma_{\eta\beta} = \sigma^R_{\eta\beta} + \sigma^P_{\eta\beta} \tag{22}$$

where $\sigma^R_{\eta\beta}$ represents a stress field which is regular on the entire cavity surface and $\sigma^P_{\eta\beta}$ is the singular component.

The singular component is selected such that it is zero for $0 \leq \theta < \theta_o$, becomes infinite at $\theta = \theta_o$, is readily expandable in a Legendre series, and vanishes at $\theta = \pi$. An appropriate form for the singular component $\sigma^P_{\eta\beta}$ is seen to be

$$\sigma^P_{\eta\beta} = \delta^{(1)} \frac{\mu\,\theta_o \bar{p}}{2\sqrt{p_o - p}} \qquad \begin{matrix} \theta_o \leq \theta \leq \pi \\ 0 \leq \theta < \theta_o \end{matrix} \tag{23}$$

where $\delta^{(1)}$ is a weighting coefficient to be determined such that $\sigma^P_{\eta\beta}$ represents the singular portion of the stress field. Thus $\delta^{(1)}$ specifies the order of the singularity in the mixed boundary value problem considered.

From Eq. 9 it is necessary that $\sigma^P_{\eta\beta}$ also admits the representation

$$\sigma^P_{\eta\beta} = \frac{\delta^{(1)} \mu\, Q_o}{2} \bar{p} \sum_{n=1}^{\infty} \sigma_n \left[3\bar{q}_o + \frac{(2n+1)}{T_n^o} Q_o^2 \right] P_n'(p), \tag{24}$$

$$0 \leq \theta \leq \pi .$$

In order to determine the constants σ_n it is necessary to first obtain the coefficients a_n in the expansion

$$\sigma^P_{\eta\beta} = \frac{\delta^{(1)} \mu Q_o}{2} \bar{p} \sum_{n=1}^{\infty} a_n P'_n(p) \qquad 0 \leq \theta \leq \pi . \quad (25)$$

Note that both Eq. 24 and 25 apply for all values of θ. Using the orthogonality of $P'_n(p)$ and substituting from Eq. 23 into 25 the a_n are

$$a_n = \frac{(2n+1)}{2n(n+1)} \int_{-1}^{p_o} \frac{\bar{p} P'_n(p)}{\sqrt{p_o - p}} dp . \quad (26)$$

By use of the Mehler-Direchlet integrals [5], the coefficients a_n are found to be

$$a_n = \sqrt{2} \left[\frac{\cos[(n-1/2)\theta_o]}{2n-1} - \frac{\cos[(n+3/2)\theta_o]}{2n+3} \right] . \quad (27)$$

By writing Eq. 24 as [1]

$$\sigma^P_{\eta\beta} = \frac{\delta^{(1)} \mu Q_o \bar{p}}{2} \sum_{n=1}^{\infty} \Big[(n+2) C^o_{n+1} [\sigma^*_n - \sigma^*_{n-1}] + (n-1) C^o_{n-1} [\sigma^*_n - \sigma^*_{n-1}] \Big] P'_n(p) \quad (28)$$

where $\sigma_o = 0$ and $\sigma_n = \sigma^*_n S^o_n$, the σ^*_n are then obtained in terms of a_n by equating coefficients of $P'_n(p)$ in Eqs. 25 and 28. This relation is

$$(n+2) C^o_{n+1} [\sigma^*_n - \sigma^*_{n-1}] + (n-1) C^o_{n-1} [\sigma^*_n - \sigma^*_{n-1}] = a_n . \quad (29)$$

If Eq. 29 is multiplied by $n(n+1) C^o_n$ and the result summed from $n=1$ to $n=k$, it follows that

$$k(k+1)(k+2) C^o_k C^o_{k+1} [\sigma^*_{k+1} - \sigma^*_k] = - \sum_{n=1}^{k} n(n+1) C^o_n a_n . \quad (30)$$

Now by summing on k, k=1, 2..., N, the equation

$$\sigma^*_{N+1} - \sigma^*_1 = - \sum_{k=1}^{N} \frac{1}{k(k+1)(k+2)C^o_k \; C^o_{k+1}} \sum_{n=1}^{k} n(n+1)C^o_n \; \alpha_n \qquad (31)$$

is obtained.

It is evident that convergence of the series representation for the displacement corresponding to the singular stress field

$$u = \delta^{(1)} Q_o \bar{p} \sum_{n=1}^{\infty} \sigma^*_n P'_n (p) \qquad (32)$$

demands that σ^*_N tends to zero with increasing N. Consequently Eq. 31 yields in the limit σ^*_1, from which it follows that

$$\sigma_r = S^o_r \sum_{k=r}^{\infty} \frac{1}{k(k+1)(k+2) \; C^o_k \; C^o_{k+1}} \sum_{n=1}^{k} n(n+1) \; C^o_n \; \alpha_n, \; r=1, 2, \ldots, \qquad (33)$$

Eqs. 20 and 21 may now be written in terms of the regular stress and displacement fields and the displacement corresponding to the singular stress field as

$$\sum_{n=1}^{\infty} D_n \left[3\bar{q}_o + \frac{(2n+1)}{T^o_n} Q^2_o \right] P'_n (p) = 0 \qquad 0 \leq \theta < \theta_o \qquad (34)$$

and

$$\sum_{n=1}^{\infty} (D_n + \delta^{(1)} \sigma_n) \; P'_n (p) = 2\sqrt{2}\omega \sum_{n=1}^{\infty} e^{-(n+1/2)\eta_o} P_n(p), \quad \theta_o \leq \theta \leq \pi \qquad (35)$$

where the D_n are the unknown coefficients of $\sigma^R_{\eta\beta}$. The coefficients B_n in Eqs. 20 and 21 are obviously related to D_n and $\delta^{(1)} \sigma_n$ by $B_n = D_n + \delta^{(1)} \sigma_n$.

Mention should be made of several errors in the above referenced paper by Eubanks. [1]. The statement by Eubanks that the coefficient A_1 which corresponds to σ_1 in the present analysis, is indeterminate and reflects a rigid body motion is incorrect. As has already been demonstrated, the coefficient σ_1 can be determined if it is only required that the displacement series converge. Further, the assumed form of the displacement series does not permit any rigid body motion. In addition, the final equation for D_n and Eqs. 34, 38, and 39 of [1] are in error, however the numerical results presented are correct. Eqs. 34 and 35 should have $A_k - A_s$ and the last term of Eq. 39 should be

$$+ \frac{(2n+1)(1-e^{-2\alpha_o})(1-e^{-4\alpha_o})}{[1+e^{-(2n+1)\alpha_o}][1+e^{-(2n+3)\alpha_o}][1+e^{-(2n+5)\alpha_o}]} \quad .$$

The equation for D_n should have n replaced by n+1 in the first two terms. Eq. 33, however, which follows directly from D_n is correct.

Determination of the Superposition Constants

A solution to Eqs. 34 and 35 can be obtained by replacing the dual-Legendre series by a related set of dual series in terms of circular functions. This related set of equations can then be easily reduced to an infinite system of simultaneous algebraic equations in the unknown coefficients D_n and $\delta^{(1)}$ which can be truncated and an approximate solution obtained.

Utilizing the following known relations involving the Legendre polynomials [6],

$$pP_n'(p) = \frac{1}{(2n+1)}\left[nP_{n+1}'(p) - (n+1)\,P_{n-1}'(p)\right] \quad \text{and}$$
$$p^{-2}P_n'(p) = -\frac{n(n+1)}{2n+1}\left[P_{n+1}(p) - P_{n-1}(p)\right] \tag{36}$$

and recalling that $Q_o^2 = q_o - p$, it is possible to express Eqs. 34 and 35 in terms of Legendre polynomials as

$$\sum_{n=1}^{\infty} D_n n(n+1)\left[\left[\frac{1}{T_n^o} + \frac{3\tanh(\eta_o)}{2n+1}\right]\left[P_{n+1}(p) - P_{n-1}(p)\right]\right.$$
$$\left. - \frac{1}{q_o T_n^o}\left[\frac{n+1}{2n+3}P_{n+2}(p) - \frac{n-1}{2n-1}P_{n-2}(p) - \frac{2n+3}{(2n-1)(2n+3)}P_n(p)\right]\right] = 0,$$
$$0 \le \theta < \theta_o \tag{37}$$

and
$$\sum_{n=1}^{\infty}(D_n + \delta^{(1)}\sigma_n)\frac{n(n+1)}{2n+1}\left[P_{n+1}(p) - P_{n-1}(p)\right] = 2\sqrt{2\omega}\sum_{n=1}^{\infty}\frac{n(n+1)}{2n+1}\,.$$
$$e^{-(n+1/2)\eta_o}\left[P_{n+1}(p) - P_{n-1}(p)\right], \quad \theta_o \le \theta \le \pi \quad . \tag{38}$$

As a direct consequence of the generating function for the Legendre polynomials, an integral transformation may easily be obtained which converts the above dual-Legendre series into dual-trigonometric series. This can be seen by writing the generating function as

$$\frac{1}{\sqrt{p-\xi}} \quad \sqrt{2} \sum_{n=1}^{\infty} \bar{e}^{i(n+1/2)\theta} P_n(\xi) \qquad \begin{array}{c} -1 \le p \le 1 \\ -1 \le \xi \le 1 \end{array} \tag{39}$$

and using the orthogonality of $P_n(p)$. The well known Mehler-Direchlet integrals [5] relating the Legendre polynomials to the circular functions are obtained as

$$\int_{-1}^{p} \frac{P_n(\xi)}{\sqrt{p-\xi}} d\xi = \sqrt{2} \frac{\cos[(C_n+1/2)\theta]}{n+1/2}, \tag{40}$$

$$\frac{d}{d\theta}\int_{-1}^{p} \frac{P_n(\xi)}{\sqrt{p-\xi}} d\xi = -\sqrt{2} \sin[(n+1/2)\theta], \tag{41}$$

$$\int_{p}^{1} \frac{P_n(\xi)}{\sqrt{\xi-p}} d\xi = \sqrt{2} \frac{\sin[(n+1/2)\theta]}{n+1/2}, \text{ and} \tag{42}$$

$$\frac{d}{d\theta}\int_{p}^{1} \frac{P_n(\xi)}{\sqrt{\xi-p}} d\xi = \sqrt{2} \cos[(n+1/2)\theta]. \tag{43}$$

Through the use of these integral transformations, the system of Eqs. 37 and 38 may be written as

$$\sum_{n=1}^{\infty} D_n n(n+1) \left[\left[\frac{1}{T_n^o} + \frac{3\tanh(\eta_o)}{2n+1}\right]\left[\frac{\sin[(n+3/2)\theta]}{2n+3} - \frac{\sin[(n-1/2)\theta]}{2n-1}\right]\right.$$

$$- \frac{1}{q_o T_n^o}\left[\frac{n+2}{(2n+3)(2n+5)} \sin[(n+5/2)\theta] - \frac{n-1}{(2n-1)(2n-3)} \sin[(n-3/2)\theta] -\right.$$

$$\left.\left.\frac{\sin[(n+1/2)\theta]}{(2n-1)(2n+3)}\right]\right] = 0, \qquad 0 \le \theta < \theta_o \tag{44}$$

and
$$\sum_{n=1}^{\infty} (D_n + \delta^{(1)} \sigma_n) \frac{n(n+1)}{2n+1}\left[\sin[(n+3/2)\theta] - \sin[(n-1/2)\theta]\right] =$$

$$2\sqrt{2}\, \omega \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} e^{-(n+1/2)\eta_o}\left[\sin[(n-3/2)\theta] - \sin[(n-1/2)\theta]\right],$$

$$\theta_o \leq \theta \leq \pi \quad . \tag{45}$$

An infinite system of linear algebraic equations in the unknowns D_n and $\delta^{(1)}$ may be obtained by multiplying each side of Eqs. 44 and 45 by $\sin[(k+1/2)\,\theta]$ and integrating between the appropriate limits. After some simplifications there results

$$\sum_{n=1}^{\infty} D_n\, n(n+1) \left[H_1\, E_3 - H_2 E_7 - \frac{1}{q_o T_n^o} [E_5 - E_9 - E_1] \right]$$

$$+ \delta^{(1)} \sum_{n=1}^{\infty} \sigma_n \frac{n(n+1)}{2n+1} [E_{11} - E_{13}] = 2\sqrt{2}\,\omega \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} e^{-(n+1/2)\eta_o}$$

$$[E_{11} - E_{13}] ; \qquad k = 1, 2, \ldots \infty \tag{46}$$

The constants in Eq. 46 are defined as

$$H_1 = \frac{1}{T_n^o} + \frac{3\tanh(\eta_o) - 2n - 3}{(2n+1)},$$

$$H_2 = \frac{1}{T_n^o} + \frac{3\tanh(\eta_o) - 2n + 1}{(2n+1)},$$

$$E_1 = \frac{2}{(2n-1)(2n+3)} \int_0^{\theta_o} \sin[(k+1/2)\,\theta]\, \sin[(n+1/2)\,\theta]\, d\theta,$$

$$E_3 = \frac{2}{2n+3} \int_0^{\theta} \sin[(k+1/2)\theta]\, \sin[(n+3/2)\theta]\, d\theta,$$

$$E_5 = \frac{2\,(n+2)}{(2n+3)(2n+5)} \int_0^{\theta_o} \sin[(k+1/2)\theta]\, \sin[(n+5/2)\theta]\, d\theta,$$

$$E_7 = \frac{2}{2n-1} \int_0^{\theta} \sin[(k+1/2)\theta]\, \sin[(n-1/2)\theta]\, d\theta,$$

$$E_9 = \frac{2(n-1)}{(2n-1)(2n-3)} \int_0^{\theta} \sin[(k+1/2)\theta]\, \sin[(n-3/2)\theta]\, d\theta,$$

$$E_{11} = 2 \int_{\theta_o}^{\pi} \sin[(k+1/2)\theta]\, \sin[(n+3/2)\theta]\, d\theta$$

$$E_{13} = 2 \int_{\theta_o}^{\pi} \sin[(k+1/2)\,\theta]\, \sin[(n-1/2)\theta]\, d\theta,$$

where for n = k-1

$$H_1 E_3 = \frac{H_1}{2n+3} EK + \frac{\pi}{2n+1}$$

and

n = k + 1

$$H_2 E_7 = \frac{H_2}{2n-1} EK + \frac{\pi}{2n+1}$$

with

$$EK = 2\int_0^{\theta_o} \sin^2[(k+1/2)\theta] \, d\theta .$$

An approximate solution to the system of equations may be obtained by truncating the above series and writing

$$\sum_{n=1}^{N-1} D_n A_{k,n} + \delta(1)\sum_{n=1}^{M} \sigma_n F_{k,n} = \sum_{n=1}^{J} G_{k,n}, \quad k=1,2\ldots,N \tag{47}$$

where

$$A_{k,n} = n(n+1)\left[H_1 E_3 - H_2 E_7 - \frac{1}{q_o T_n^o}[E_5 - E_9 - E_1]\right], \tag{48}$$

$$F_{k,n} = \frac{n(n+1)}{2n+1}[E_{11} - E_{13}] \quad \text{and}$$

$$G_{k,n} = 2\sqrt{2}\,\omega\frac{n(n+1)}{2n+1} e^{-(n+1/2)\eta_o}[E_{11} - E_{13}].$$

To obtain a solution in the instance where contact is specified by $0 \leq \theta \leq \theta_o$, it is necessary to interchange the ranges of θ in Eqs. 20, 21, and 23, along with writing $\sqrt{p - p_o}$ for $\sqrt{p_o - p}$ in Eq. 23. The results of these substitutions will be to interchange the ranges of θ in Eqs. 44 and 45 and replace the terms $\sin[(n+1/2)\theta]$ by $\cos[(n+1/2)\theta]$ throughout. The constants α_n will, of course, be altered due to the modification in Eq. 23. Eq. 46 remains unchanged in form with the obvious changes being made in the constants E_i and H_i.

The general solution for the case of torque free spheres with non-vanishing stresses at infinity is obtained from the above solution for antisymmetric rotation of the inclusions as described below.

The analysis presented above for antisymmetric torsion of the inclusions gives stresses and displacements such that

$$\sigma^{(1)}_{\eta\beta} = 0 \qquad , 0 \le \theta < \theta_o \tag{49}$$

and

$$u^{(1)} = r\sin(\alpha) \qquad , \theta_o \le \theta \le \pi \tag{50}$$

where the superscript (1) denotes that $\omega = 1$ in the solution. The displacement field corresponding to a uniform angle of twist Θ per unit length of the space is $u^s = \Theta Rz$ where R is the cylindrical radius with respect to the z-axis. In terms of the curvilinear coordinates, $u^s = \frac{\Theta \bar{p}\bar{q}}{Q^4}$ near infinity and therefore an appropriate solution of Eq. 3 for this problem is

$$u^s = \Theta \frac{\bar{p}\bar{q}}{Q^4} + \Theta \bar{p} Q \sum_{n=1}^{\infty} Y_n S_n P_n'(p) \tag{51}$$

from which it follows that

$$\sigma_{\eta\beta}^{s} = -\Theta \frac{\mu \bar{p}[qp - 1]}{Q^4} + \Theta \frac{\mu \bar{p} Q}{2} \sum_{n=1}^{\infty} Y_n \left[3\bar{q}_o S_n^o + (2n+1) C_n^o Q_o^2 \right]$$

$$P_n'(p) \tag{52}$$

where $Y_n = F_n^* + \delta^{(2)} \sigma_n^*$ with σ_n given by Eq. 33, page 9. It is necessary to seek a solution to the problem of having a known angle of twist Θ satisfying Eq. 51 along with the condition on the spherical surface that the resultant stress and displacement satisfy

$$\sigma_{\eta\beta} = 0 \quad \text{and} \qquad 0 \le \theta < \theta_o \tag{53}$$

$$u = r\omega * \sin(\alpha) \qquad , \theta_o \le \theta \le \pi \tag{54}$$

where $\sigma_{\eta\beta}$ and u are the total stress and displacement and $\omega*$ is the unknown total angle of twist of the inclusion due to the pure torsion of the space: Representing the stress and displacement as

$$\sigma_{\eta\beta} = \omega * \sigma^{(1)}_{\eta\beta} + \sigma^{s}_{\eta\beta} \quad \text{and} \tag{55}$$

$$u = \omega * u^{(1)} + u^s \tag{56}$$

Eqs. 51, 53, and 54 will obviously be satisfied if on the surface $\eta = \eta_o$

$$\sigma^{s}_{\eta\beta} = 0 \qquad \text{and} \qquad 0 \le 0 < \theta_o \tag{57}$$

$$u^s = 0, \qquad \theta_o \le \theta \le \pi \tag{58}$$

The equations specifying the unknowns F_n and $\delta^{(2)}$ are then

$$\frac{-2[q_o p-1]}{Q_o^5} + \sum_{n=1}^{\infty} F_n \left[3\bar{q}_o + \frac{(2n+1)}{T_n} Q_o^{(2)} \right] P_n'(p) = 0, \qquad 0\leq\theta<\theta_o \tag{59}$$

and

$$\frac{\bar{q}}{Q_o^5} + \sum_{n=1}^{\infty} \left[F_n + \delta^{(2)} \sigma_n \right] P_n'(p) = 0, \qquad \theta_o\leq\theta<\pi \tag{60}$$

where $F_n = F_n^* S_n^o$ and $\sigma_n = \sigma_n^* S_n^o$,

along with the condition that the net torques on the inclusions [2] vanish. Due to the symmetry it is only necessary to require vanishing of the torque on one sphere which gives

$$\int_s \sigma_{\eta\beta}\, r ds = 0, \qquad s = \text{surface area.} \tag{61}$$

The coefficient $\delta^{(2)}$ gives the order of the singularity for the boundary value problem corresponding to Eqs. 57 and 58. The order of the singularity for the desired problem of pure torsion of the space with torque free inclusions is then given by $\omega^* \delta^{(1)} + \delta^{(2)}$.

Eqs. 59 and 60 are solved in the same manner as Eqs. 34 and 35 for the N unknowns F_n and $\delta^{(2)}$. Once these are known, Eq. 61 gives ω^* in terms of Θ as

$$\omega^* = -\Theta \frac{\sum_{n=1}^{N-1} \frac{n(n+1)}{S_n^o} F_n + \delta^{(2)} \sum_{n=1}^{M} \frac{n(n+1)}{S_n^o} \sigma_n}{\sum_{n=1}^{N-1} \frac{n(n+1)}{S_n^o} D_n + \delta^{(1)} \sum_{n=1}^{M} \frac{n(n+1)}{S_n^o} \sigma_n}. \tag{62}$$

It should be noted that for total contact, $\theta_o = 0$, Eq. 57 is not present, and the problem is just that of reference [2]. For cavities $\theta_o = \pi$, Eqs. 58 and 61 are not present, and the problem is that of [1]. The case of contact specified by $0\leq\theta\leq\theta_o$ is handled in the same manner as before.

A computer program has been prepared for the above problem in Fortran IV language for the Univac 1107 computer. All of the above mentioned problems are included in the program, a complete description of which is given in Appendix I.

Numerical Results

The computer program will account for the problems of symmetric or asymmetric rotations of the inclusions with vanishing stresses and displacements at infinity or pure torsion producing a uniform angle of twist per unit length of the space with zero resultant torques acting on the inclusions. In each of the above cases contact may be specified as being from $\theta = \theta_o$ to $\theta = \pi$ or from $\theta = 0$ to $\theta = \theta_o$. The necessary input data consists of the polar contact angle α_o, the spacing between inclusions X normalized such that the diameter of each cavity is unity, and three control parameters. These parameters specify: (a) whether the displacements are asymmetric or symmetric, (b) whether the inclusions have an applied torque or are torque free, and (c) whether contact is from $\theta = \theta_o$ to $\theta = \pi$ or from $\theta = 0$ to $\theta = \theta_o$. It is also necessary to assign values to certain integers according to the number of terms to be taken in various series as described below.

In any of the above cases one must obtain solutions to systems of equations of the form

$$\sum_{n=1}^{N-1} X_n^{(i)} A_{k,n}^{(i)} + \delta^{(i)} \sum_{n=1}^{M} \sigma_n F_{k,n}^{(i)} = \sum_{n=1}^{J} G_{k,n}^{(i)} , \qquad (63)$$

$$i=1, 2; \quad k= 1, 2, \ldots, N$$

with σ_r obtained by truncating Eq. 33 as

$$\sigma_r = S_r^o \sum_{k=r}^{K+r} \gamma_k \sum_{n=1}^{k} \beta_n . \qquad (64)$$

For i = 1, Eq. 63 represents Eq. 47 and for i = 2 Eq. 63 represents the corresponding system obtained from Eqs. 59 and 60. The constant M is selected by the requirement that M terms of the series expansion, Eq. 24, for the singular stress field satisfy Eq. 23 to within a specified error everywhere except in the vicinity of $\theta = \theta_o$. The remaining constants are best chosen by studying their effects on the results. In the results presented in Figs. 2 - 4 the above parameters were taken as M=300, N=50, K=15, and $J=60/\eta_o$ but within the limits $50 \leq J \leq 300$. These values were found to be quite satisfactory for all but the closest spacings. The largest amount of computation time is required to obtain the coefficients σ_n which depend on M and K. For example, the above mentioned problems require approximately forty minutes each, over twenty minutes of which is necessary for the computation of the σ_n.

A selected group of problems has been worked to demonstrate the accuracy of the present formulation as well as to allow for a comparison with references [1] and [2]. All of the problems presented are for bonding from $\theta = \theta_o$ to $\theta = \pi$ and are for torque free inclusions. As the region of greatest interaction is near $\theta = \pi$, it was felt that the above examples would give a true indication of the validity of the method. A limited amount of computer time prevented more extensive results. The cases presented may also be directly compared with [1] and [2] which give the limiting cases of cavities and total contact respectively. The present analysis gives as a special case the results of [1] and [2].

The normalized stress $\sigma_{\eta\beta}^{(n)}$ is given in Figure 2 along with the results of [2] for comparison. In Fig. 3 the normalized stress $\sigma_{\theta\beta}^{(n)}$ is presented as well as the results of [1]. Both stresses are normalized as

$$\sigma_{\eta\beta}^{(n)} = \sigma_{\eta\beta} / \sigma^{(o)}$$

and

$$\sigma_{\theta\beta}^{(n)} = \sigma_{\theta\beta} / \sigma^{(o)}$$

where

$$\sigma^{(o)} = \mu\,\omega^* / \bar{q}_o$$

with $\sigma^{(o)}$ being the maximum stress that would occur on a spherical surface of radius $\left| 1/\bar{q}_o \right|$ in an elastic body without a cavity under pure torsion. Fig. 4 gives the normalized displacement $u^{(n)}$ around

the cavity. The normalized displacement is defined as $u^{(n)} = u\frac{\bar{q}_o}{\omega^*}$
As mentioned, all these results are obtained with N = 50, M = 300, K = 15, and $J = 60/\eta_o$ or $50 \leq J \leq 300$.

Table 1 illustrates the variation of $\delta^{(1)}$, $\delta^{(2)}$, and ω^* with M for the case of X = .1. Figs. 5 and 6 demonstrate the convergence of $\sigma^{(n)}_{\eta\beta}$ and $u^{(n)}$ respectively as a function of M with X = .1, N = 50, K = 15, and J = 300.

M	$\delta^{(1)}$	$\delta^{(2)}$	ω^*
100	4.6854	-14.0025	1.4973
200	4.5103	-13.4792	1.4889
300	4.4803	-13.3896	1.4857

Table 1 Variation of $\delta^{(1)}$, $\delta^{(2)}$ and ω^* with M for X = .1, N = 50, K = 15, J = 300

Conclusions

From Fig. 2 it is seen that the residual stress on the free surface increases as the spacing decreases with the maximum residual stress occurring near $\theta = \theta_o$. Expressing the residual in terms of percent of the maximum stress on the contact surface excluding the vicinity near $\theta = \theta_o$, it is found that the maximum residual stress error is 2.4, 5.3, and 8.1 percent for spacings of 5, .1, and .05 respectively. The displacement boundary condition is satisfied quite accurately with only a significant error occurring at the end of the contact surface. Assuming that the rotation angle of the inclusions is accurately determined, the displacement error is defined as the percent difference between the displacement computed from the truncated series solution and the displacement accompanying a rigid rotation of the spherical surfaces for $\theta_o < \theta \leq \pi$. For spacings of 5, .1, and .05 the error is .39, 4.9, and 6.18 percent of the correct value respectively and in all cases this error is less than one percent from 94 degrees on. The effects of the singular stress field then appears to be fairly localized, with the solution actually presented corresponding to a problem having a small amount of "relaxing" near the end of the contact surface.

It is seen from Fig. 3 that the stress $\sigma_{\theta\beta}$ has essentially the same value as in the case of cavities over about two-thirds of the free surface. There is considerable oscillation near $\theta = \theta_o$ and it is indicated that this stress component is more slowly converging than either $\sigma_{\eta\beta}$ or u. The stress $\sigma_{\theta\beta}$ should be identically zero on the open interval $\theta_o < \theta \leq \pi$ as

$$\sigma_{\theta\beta} = \mu \bar{p} \frac{\partial}{\partial \theta} [u \frac{Q_o^2}{\bar{p}}]$$

and (65)

$$u = \frac{\bar{p}\,\omega}{Q_o^2} \qquad \theta_o \leq \theta \leq \pi .$$

It is of interest to note that the coefficients $\delta^{(1)}$ $\delta^{(2)}$, and ω^* as given in Table 1 are relatively insensitive to the number of terms M. Therefore, it appears that the order of the singularity can be determined with reasonable accuracy even though not enough terms are taken to satisfy all the boundary conditions extremely well. This would be of considerable interest in investigating the separation of the inclusion from the matrix due to the propagation of a crack, originating at the end of the contact arc.

Figs. 5 and 6 indicate that an increasingly accurate solution can be obtained by taking successively more terms in the various series, with particular attention being given the series representing the singular solution.

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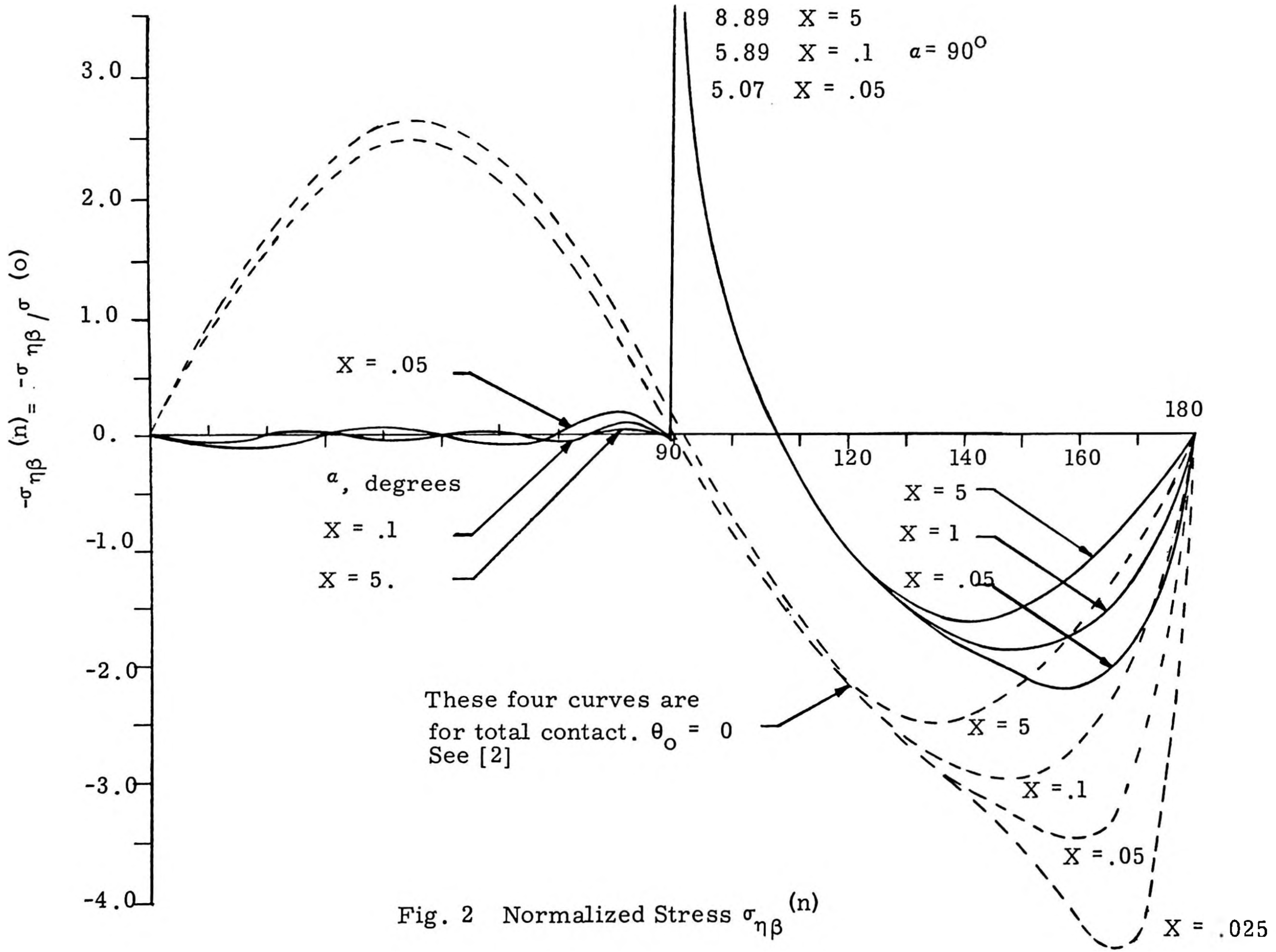

Fig. 2 Normalized Stress $\sigma_{\eta\beta}^{(n)}$

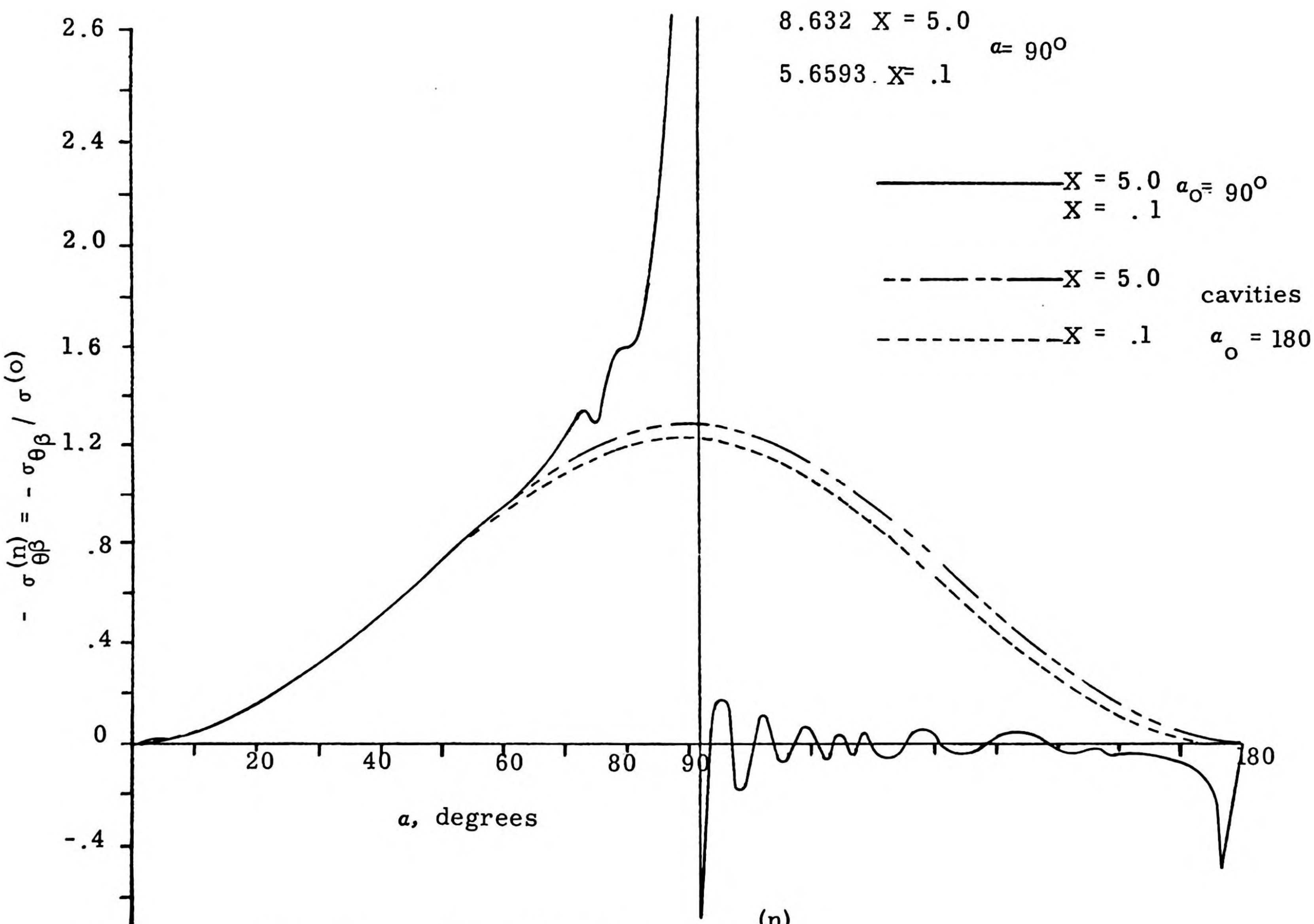

Fig. 3 Normalized Stress $\sigma^{(n)}_{\theta\beta}$

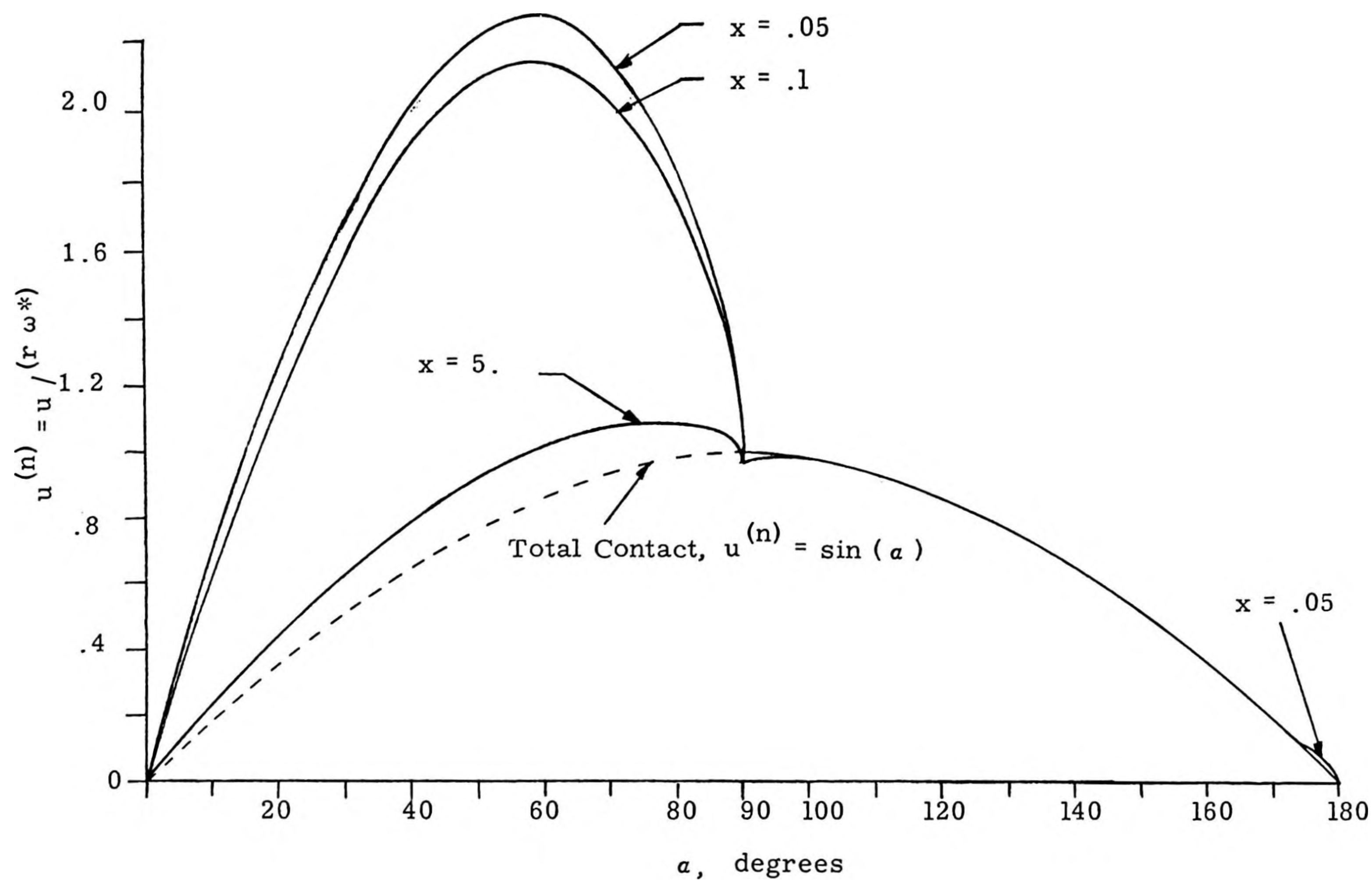

Fig. 4 Normalized Displacement on the Spherical Surface

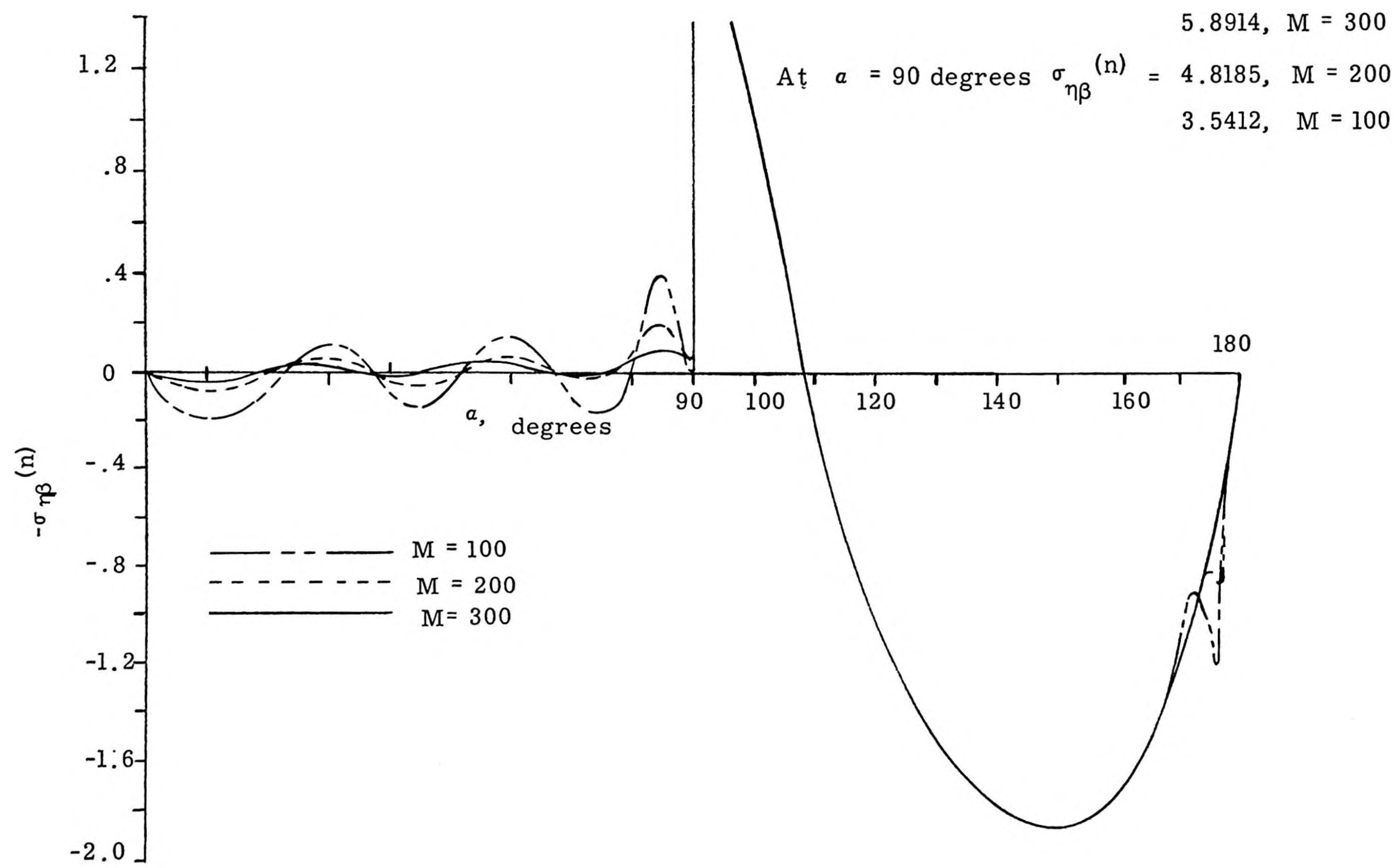

Fig. 5 Variation of $\sigma_{\eta\beta}^{(n)}$ with M

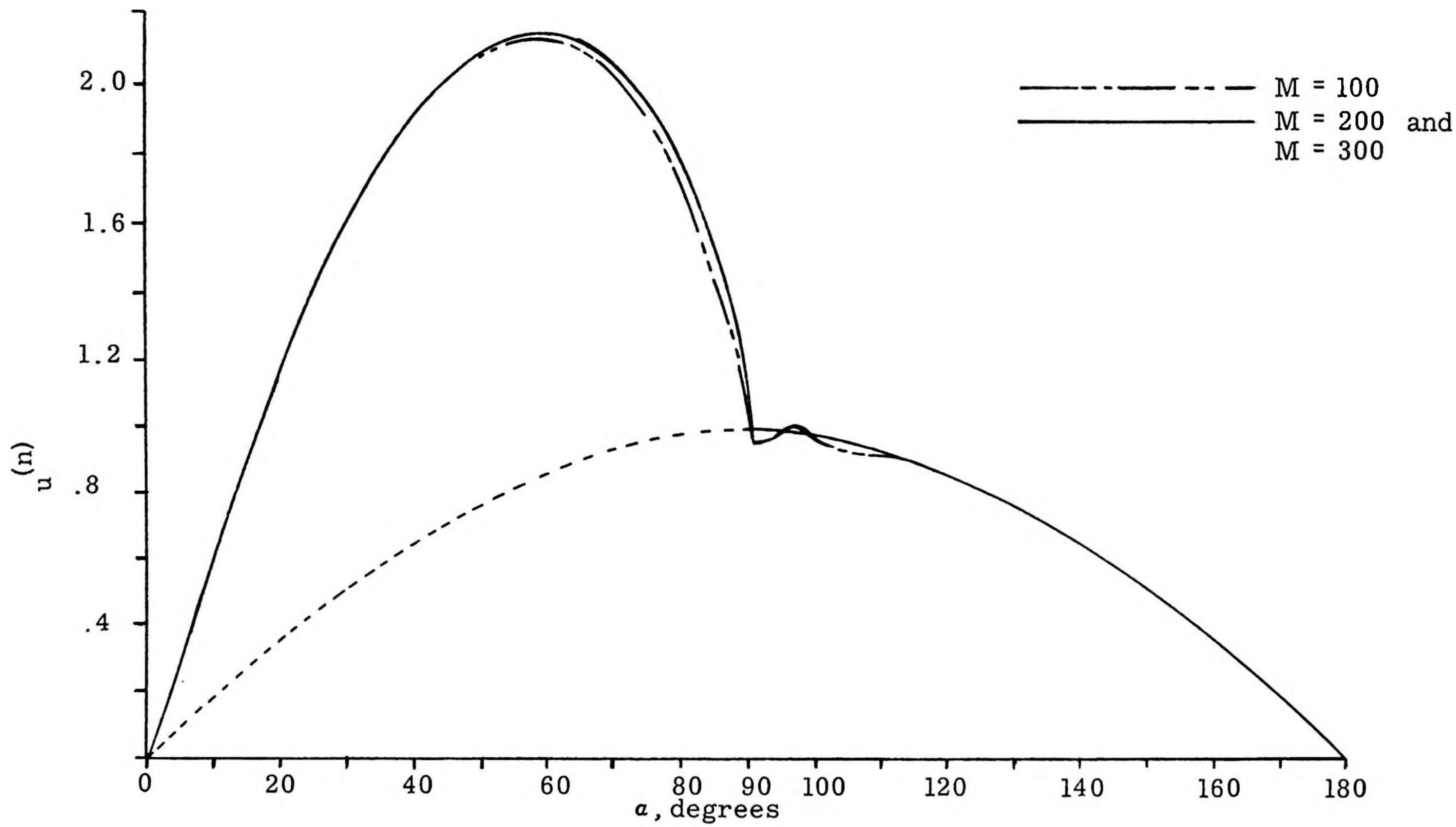

Fig. 6 Variation of $u^{(n)}$ with M

CHAPTER II

TRANSVERSE SHEAR LOADING IN AN ELASTIC MATRIX CONTAINING TWO ELASTIC CIRCULAR CYLINDRICAL INCLUSIONS

Introduction

Consider an infinite elastic solid containing several cylindrical inclusions with axes parallel to a direction p. The inclusions are perfectly bonded to the surrounding matrix material and have elastic properties different from those of the matrix. Let (x, y, p) define an orthogonal Cartesian coordinate system for which the related displacement components are u, v, and w. A state of transverse shear deformation in the system is defined by a displacement field of the form u = 0, v = 0, and w = w(x, y). In the corresponding stress field, the only non-zero stress components are σ_{xp} and σ_{yp}. When the matrix material is unbounded, a condition of uniform shear stress at infinity allows specification of two independent shear stress components.

In this chapter, a configuration consisting of an infinite medium containing two circular cylindrical inclusions is studied. The size, spacing, and elastic properties of the inclusions, as well as the shear stress components at infinity are arbitrary. Numerical results illustrating the effects of varying the inclusion spacing, elastic properties, and inclusion radius ratio are presented.

General Formulation

For the assumed displacement field, the two non-zero stress components are given by

$$\sigma_{xp} - i\sigma_{yp} = \mu \frac{dW}{dz} \quad , \; z = x + iy \tag{1}$$

where W(z) is a holomorphic function the real part of which is the displacement w(x, y) perpendicular to the (x, y) plane [1][1].

Let S^+ and S^- denote, respectively, the regions of the inclusions and the matrix which are separated by a contour $L = L_1 + \ldots + L_n$ where L_i, is the boundary of the i^{th} inclusion. The function μ in Eq. 1 is defined as

$$\mu = \mu_o \text{ in } S^- \; ; \; \mu = \mu_i \text{ in } S_i^+ \; , \; i = 1, \ldots, n \; , \tag{2}$$

with μ_i being the shear modulus for region S_i^+ bounded by L_i.

It is evident from Eq. 1 that a condition of uniform shear stress at infinity requires that for large z

$$W = Kz + O(1) \; , \quad K = (\sigma_{xp}^{\infty} - i\sigma_{yp}^{\infty}) / \mu_o \; . \tag{3}$$

Let C be a closed contour in any subregion of S^- or S^+ for which μ is constant. The one non-zero component of surface traction on an area element parallel to the p-axis is expressible as

$$T_p = \sigma_{xp} \frac{dy}{ds} - \sigma_{yp} \frac{dx}{ds} = \mu \, \text{Im} \, [W'(z) \frac{dz}{ds}] \tag{4}$$

where s is the arc length parameter along C. The requirement of zero resultant force on C shows, by virtue of the last equation, that

$$[\text{Im} \; W(z)]_C = [W(z)]_C = 0 \tag{5}$$

with use being made of the fact that the real part of W(z); which equals the displacement w, is evidently single valued. Consequently, W(z) will be a single valued function in S^+ and S^-. The conditions of displacement and stress continuity across the material interfaces can

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[1]Numbers in square brackets refer to references at the end of the chapter.

be expressed from Eq. 4 as

$$W^+(t) + \overline{W^+(t)} = W^-(t) + \overline{W^-(t)}, \quad t \in L_i \ , \tag{6}$$

and

$$\mu_o \ [F^+(t)t'(s) - \overline{F^+(t)t'(s)}] = \mu_i[F^-(t)\, t'(s) - \overline{F^-(t)\ t'(s)}],$$

$$t \in L_i \ , \ i = 1, \ldots, n \tag{7}$$

where F is the derivative of W with respect to z. The stress boundary condition can be formulated in terms of W(z) by integrating Eq. 7 with respect to arc length to yield.

$$\mu_o[W^+(t) - \overline{W^+(t)}] = \mu_i[W^-(t) - \overline{W^-(t)}] + K_i \quad ,$$

$$t \in L_i, \quad i = 1, \ldots, n \tag{8}$$

with the K_i being integration constants.

Analysis for Two Circular Inclusions

A solution is developed below for stresses in a composite material having two circular cylindrical inclusions and uniform stresses σ_{xp}^{∞} and σ_{yp}^{∞} acting at infinity. The pertinent geometry involves a distance h between the inclusion centers, and inclusion radii R_1 and R_2 as illustrated in Fig. 1. A corresponding region in the ζ-plane related to the z-plane by means of a bilinear transformation is also shown. Under this mapping, the region S_o of the matrix and regions S_1 and S_2 of the inclusions map into S_o', S_1' , and S_2'. Contours L_1 and L_2 defining the material interfaces map into concentric circles L_1' and L_2' of the ζ-plane.

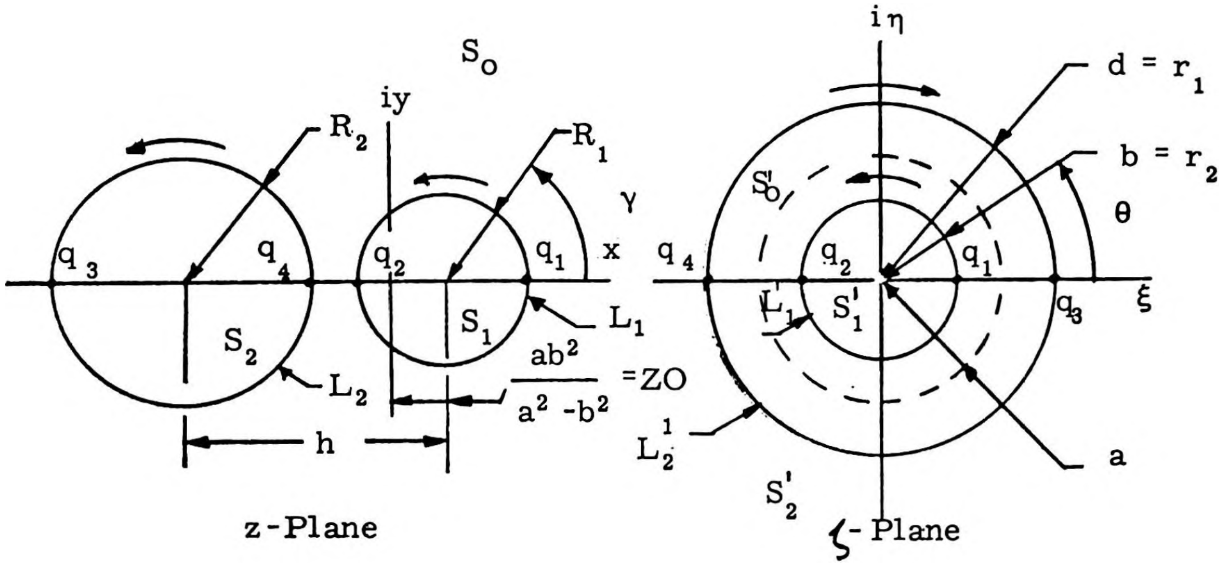

Fig. 1

The bilinear function for the desired mapping and its inverse can be written as

$$z = \omega(\zeta) = \frac{a\zeta}{a - \zeta} \quad , \qquad \zeta = \frac{az}{a + z} \ . \tag{9}$$

The parameter a and the radii $|\zeta| = b$ and $|\zeta| = d$ are uniquely determined by R_1 , R_2 , and h. For convenience, let parameters α and β be defined by

$$\alpha = \frac{h}{R_1 + R_2} > 1 \ , \quad \beta = \frac{R_1}{R_2} \leq 1 \ . \tag{10}$$

Some algebraic manipulation yields

$$\frac{a}{R_2} = \frac{\sqrt{(\beta - 1)^2 - 2\alpha^2(\beta^2 + 1) + \alpha^4(\beta + 1)^2}}{\alpha} \equiv E \ , \tag{11}$$

$$\frac{b}{R_2} = \frac{E\sqrt{E^2 + 4\beta^2} - E^2}{2\beta} \equiv G$$

and

$$\frac{d}{R_2} = \frac{E\sqrt{E^2+4} + E}{2} \equiv H . \tag{12}$$

Formulas similar to Eqs. 11 and 12 are derived in article 48 of Reference [2].

The function W, regarded as a function of ζ, will be sectionally holomorphic with a first order pole at ζ = a which corresponds to z = ∞. It follows from Eq. 3 that the following series expansions apply.

$$W = W_o(\zeta) = -a^2 K (\zeta - a)^{-1} + \sum_{n=-\infty}^{\infty} c_n \zeta^n , \quad b \le |\zeta| \le d, \quad K = (\sigma_{xp}^{\infty} - i\sigma_{yp}^{\infty})/\mu_o . \tag{13}$$

$$W = W_1(\zeta) = \sum_{n=o}^{\infty} a_n \zeta^n , \qquad |\zeta| \le b. \tag{14}$$

$$W = W_2(\zeta) = \sum_{n=o}^{\infty} b_n \zeta^{-n}, \qquad |\zeta| \ge d. \tag{15}$$

The unknown coefficients in these series can be determined by expanding $(\zeta - a)^{-1}$ in a geometric series for $|\zeta|$ = b and $|\zeta|$ = d and substituting for W_o, W_1, and W_2 in the boundary conditions expressed by Eqs. 6 and 8. Comparison of corresponding powers of $e^{i\theta}$, where θ = arg (ζ), yields a system of simultaneous equations for evaluation of a_n, b_n, and c_n. The algebra is straightforward and the solution of the system of equation gives for n>0

$$c_n = D_4 [\bar{K} D_1 a^{n+1} b^{-2n} + K D_3 a^{-n+1}]L , \tag{16}$$

$$c_{-n} = -D_3 [\bar{K} D_2 a^{-n+1} d^{2n} + K D_4 a^{n+1}] L , \tag{17}$$

$$a_n = K a^{-n+1} + [2 \bar{K} \mu_o D_4 a^{n+1} b^{-2n} + K D_3 D_4 a^{-n+1}$$
$$-K D_2 D_3 a^{-n+1} d^{2n} b^{-2n}]L, \tag{18}$$

$$b_n = -K a^{n+1} + [2 \bar{K} \mu_o D_3 a^{-n+1} d^{2n} + K D_3 D_4 a^{n+1}$$
$$-K D_1 D_4 a^{n+1} d^{2n} b^{-2n}]L, \tag{19}$$

which depend on constants L and D_i defined by

$$L = [D_1 D_2 d^{2n} b^{-2n} - D_3 D_4]^{-1} , \tag{20}$$

$$D_1 = \mu_1 + \mu_o , \quad D_2 = \mu_2 + \mu_o, \quad D_3 = \mu_1 - \mu_o , \text{ and}$$

$$D_4 = \mu_2 - \mu_o . \tag{21}$$

The constants a_o, b_o, and c_o need not be evaluated since they do not influence the stress distribution.

The stresses relative to the curvilinear coordinate system defined by the mapping can readily be expressed in terms of ζ . It follows from Eq. 1 that

$$\sigma_{rp} - i\sigma_{\theta p} = (\sigma_{xp} - i\sigma_{yp}) \frac{\omega'(\zeta)\zeta}{|\omega'(\zeta)\zeta|} = \frac{\mu\zeta}{|\omega'(\zeta)\zeta|} \frac{dW}{d\zeta} \tag{22}$$

where σ_{rp} and $\sigma_{\theta p}$ are the shear stress components directed along the curves corresponding to constant values of arg (ζ) and $|\zeta|$, respectively. Only the stresses in the matrix material are discussed here. Computations for the interior of the inclusions are similar. When the series for W_o is substituted into the last equation, it is found that for $b < |\zeta| < d$

$$\sigma_{rp} - i\sigma_{\theta p} = \mu_o \frac{|\zeta(a-\zeta)^2|}{a^2} \left[\frac{Ka^2}{(\zeta - a)^2} + \sum_{n=1}^{\infty} n[c_n \zeta^{n-1} - c_{-n}\zeta^{-n-1}] \right] . \tag{23}$$

Numerical Results

A digital computer program in Fortran language was written to evaluate stresses in the matrix by use of Eq. 23. The computations require evaluation of series of the form

$$S = \sum_{n=o}^{\infty} \frac{n\zeta^n}{c-f^n} ,$$

where $f<c$, $0<f<1$, and ζ is a complex number such that $|\zeta| <1$. The last relation can be written as

$$S = \sum_{n=0}^{m-1} \frac{n\zeta^n}{c-f^n} + R_m \tag{24}$$

where

$$\left|R_m\right| = \left|\sum_{n=m}^{\infty} \frac{n\zeta^n}{c-f^n}\right| \leq \sum_{n=m}^{\infty} \frac{n|\zeta|^n}{|c-f^n|} . \tag{25}$$

Since $f^n \leq f^m$ for $n \geq m$, it follows that

$$\left|R_m\right| \leq \frac{1}{c-f^m} \sum_{n=m}^{\infty} n|\zeta|^n = \frac{m|\zeta|^m - (m-1)|\zeta|^{m+1}}{(c-f^m)(1-|\zeta|)^2} . \tag{26}$$

This error bound was used to evaluate the stresses within an accuracy of three significant figures.

The computer program allows for specification of the shear moduli of the matrix and each inclusion, as well as the stress components σ_{xp}^{∞} and σ_{yp}^{∞}. The basic geometrical parameters are the inclusion radius ratio R_1 / R_2 and the spacing ratio X/R_1. The computed output quantities are shearing stresses along curvilinear coordinate lines in the interior and on the boundary of the matrix. Although provision was not made for computation of stresses in the inclusions, this option could readily be incorporated.

Shear stress distributions around the holes in the matrix are shown in Fig. 2 for various inclusion spacings when the inclusion radii are equal ($R_1 = R_2$). The combinations of elastic properties considered are for rigid inclusions ($\mu_1/\mu_o = \mu_2/\mu_o = \infty$) and for empty cavities ($\mu_1/\mu_o = \mu_2/\mu_o = 0$). The stresses shown in the figure are the radial component σ_{Rp} and the circumferential component $\sigma_{\gamma p}$ plotted as functions of an angle γ measured counterclockwise from the center of the right inclusion in Fig. 1 with $\gamma = 0$ corresponding to the positive x-axis. It can be shown from (23) that $\sigma_{Rp}=0$ or $\sigma_{\gamma p}=0$ for empty cavities or rigid inclusions, respectively. Moreover, the non-zero components of stress for these two cases are identical.

when the stress components at infinity are interchanged, i.e., with regard to the stresses at the hole σ_{Rp} in one case equals $\sigma_{\gamma p}$ for the other case. A considerable reduction therefore occurs in the number of curves required to describe the stress distributions around the holes for these particular cases.

It can be seen from Fig. 2 that unbounded stresses apparently occur at $\gamma = 180^{o}$ provided the circles touch and $\sigma_{xp}^{\infty} \neq 0$ for rigid inclusions or $\sigma_{yp}^{\infty} \neq 0$ for empty cavities. Because the series in (23) could not be summedin closed form, the author was unable to demonstrate conclusively that infinite stresses can actually occur when the inclusions touch. However, the possibility seems physically reasonable and is borne out by the numerical results. The computations also indicate that infinite stresses can occur only when the inclusions touch and loading the conditions and material properties correspond to either of the two cases cited.

In Fig. 3, the variation of stresses around the right inclusion of Fig. 1 are shown as a function of R_1/R_2 for a spacing distance of $X/R_1 = 0.1$. The combinations of elastic properties considered in these results are for either rigid inclusions or empty cavities.

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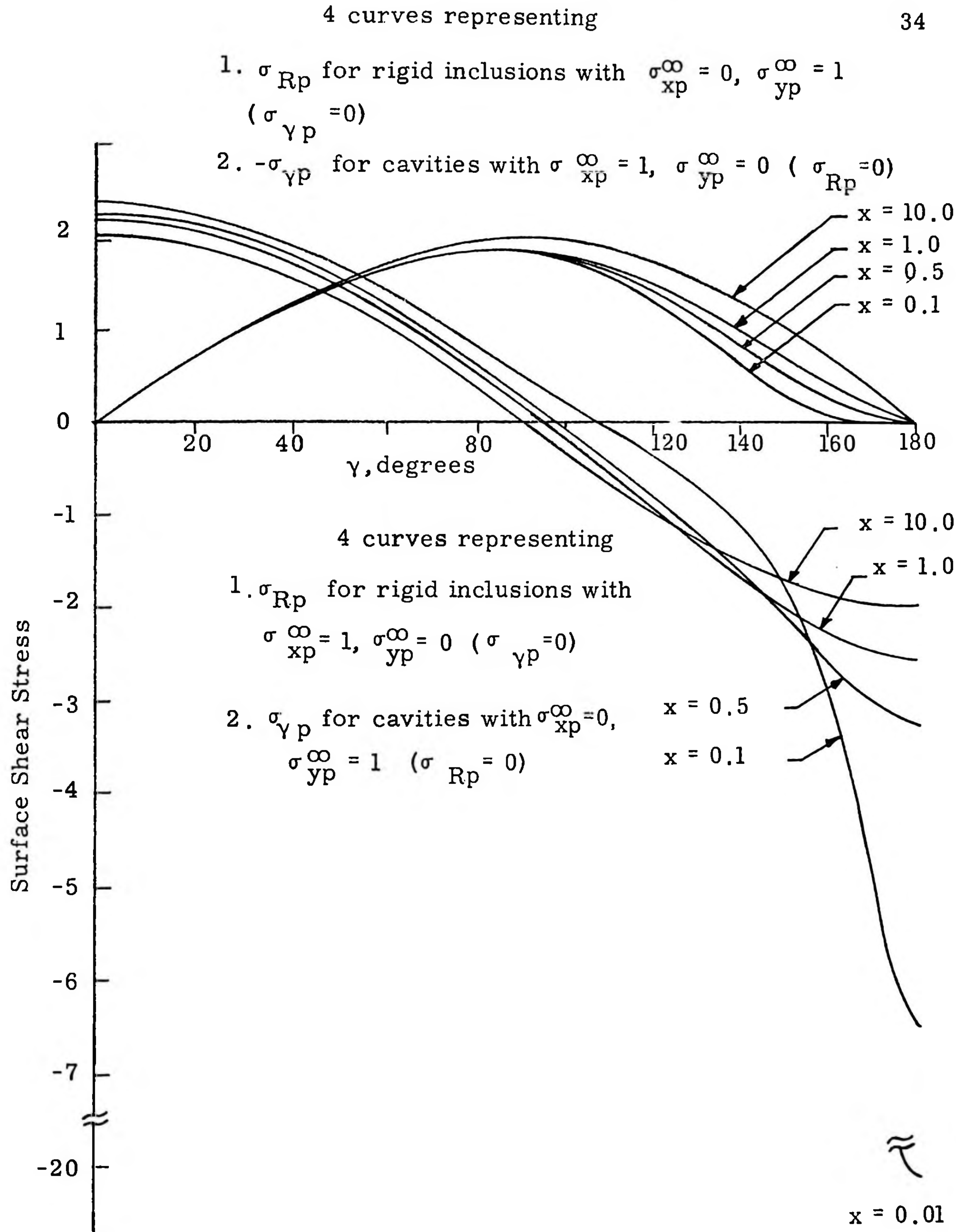

Fig. 2 Effects of Spacing on the Shearing Stresses on the Surface of Inclusion (1) for Inclusion of Equal Size

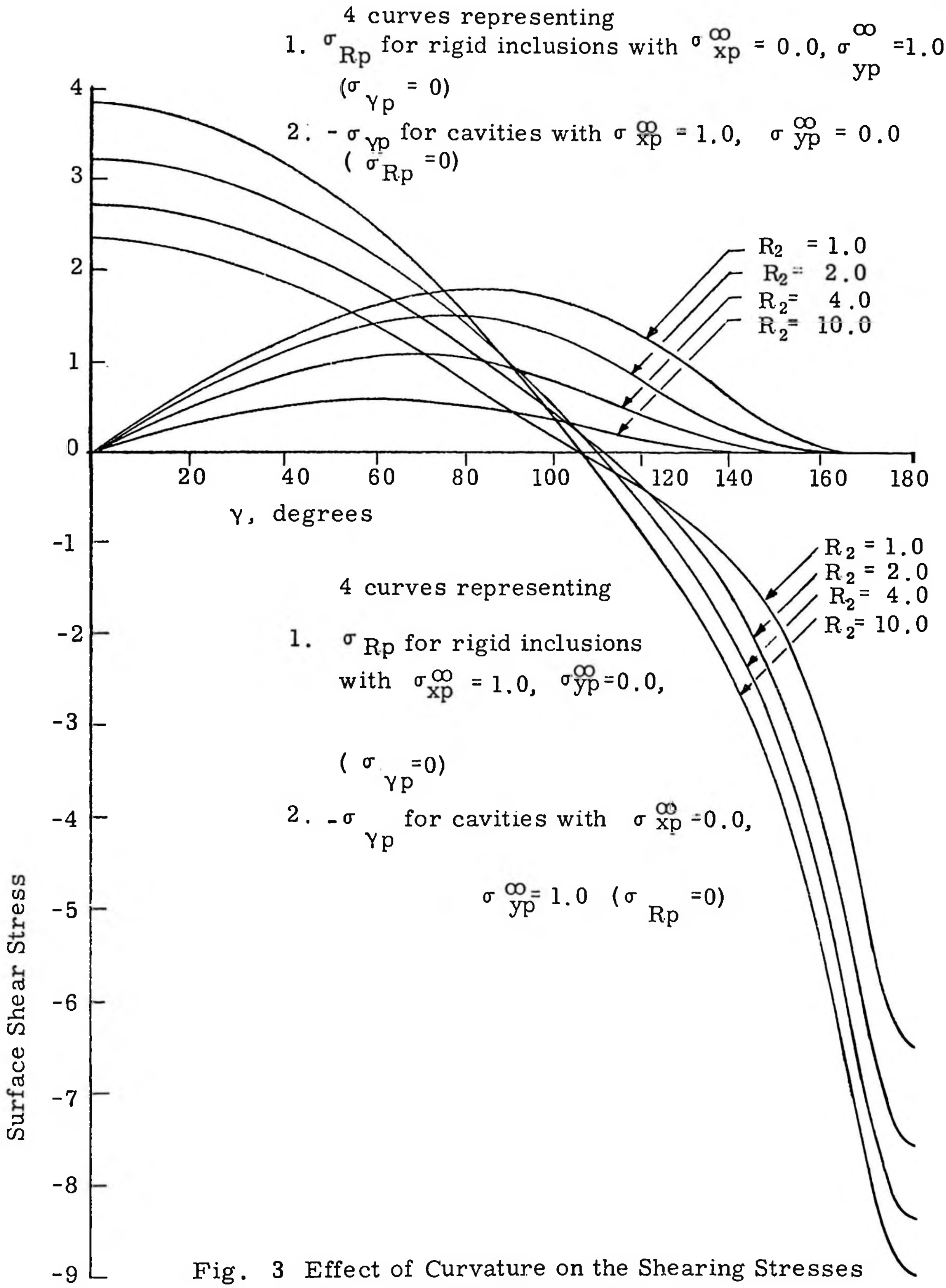

Fig. 3 Effect of Curvature on the Shearing Stresses on the Surface of Inclusion (1) with X = 0.1.

CHAPTER III

AXISYMMETRIC CONTACT STRESSES ABOUT A SMOOTH ELASTIC SPHERE IN AN INFINITE SOLID STRESSED UNIFORMLY AT INFINITY

A method of evaluating non-Hertzian contact stresses in an infinite elastic body which has a spherical cavity containing a smooth elastic sphere is developed in this chapter. The applied loading is specified by two independent axisymmetric stress components at infinity. For arbitrary combinations of elastic properties and stress components, partial separation generally occurs between the sphere and the surrounding material to yield a contact surface consisting of either a spherical band or two spherical caps. The boundary conditions correspond to vanishing shear stress on the entire boundary, zero normal stress on the surface of separation, and continuous normal displacement across the contact surface. Determination of the contact zone and the accompanying normal and circumferential stresses on the cavity are sought.

The analysis involves a mixed boundary value problem leading to dual series relations containing coefficients in the Legendre polynomial expansion of the unknown normal stress on the contact surface. Similar series relations have been studied in [1][1] through [4]. In all these references except for [4], the dual series are reduced to equivalent Fredholm integral equations or infinite systems of simultaneous equations amenable to approximate numerical solution. A different procedure which consists of representing the unknown normal stress

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[1] Numbers in square brackets refer to references at the end of the chapter.

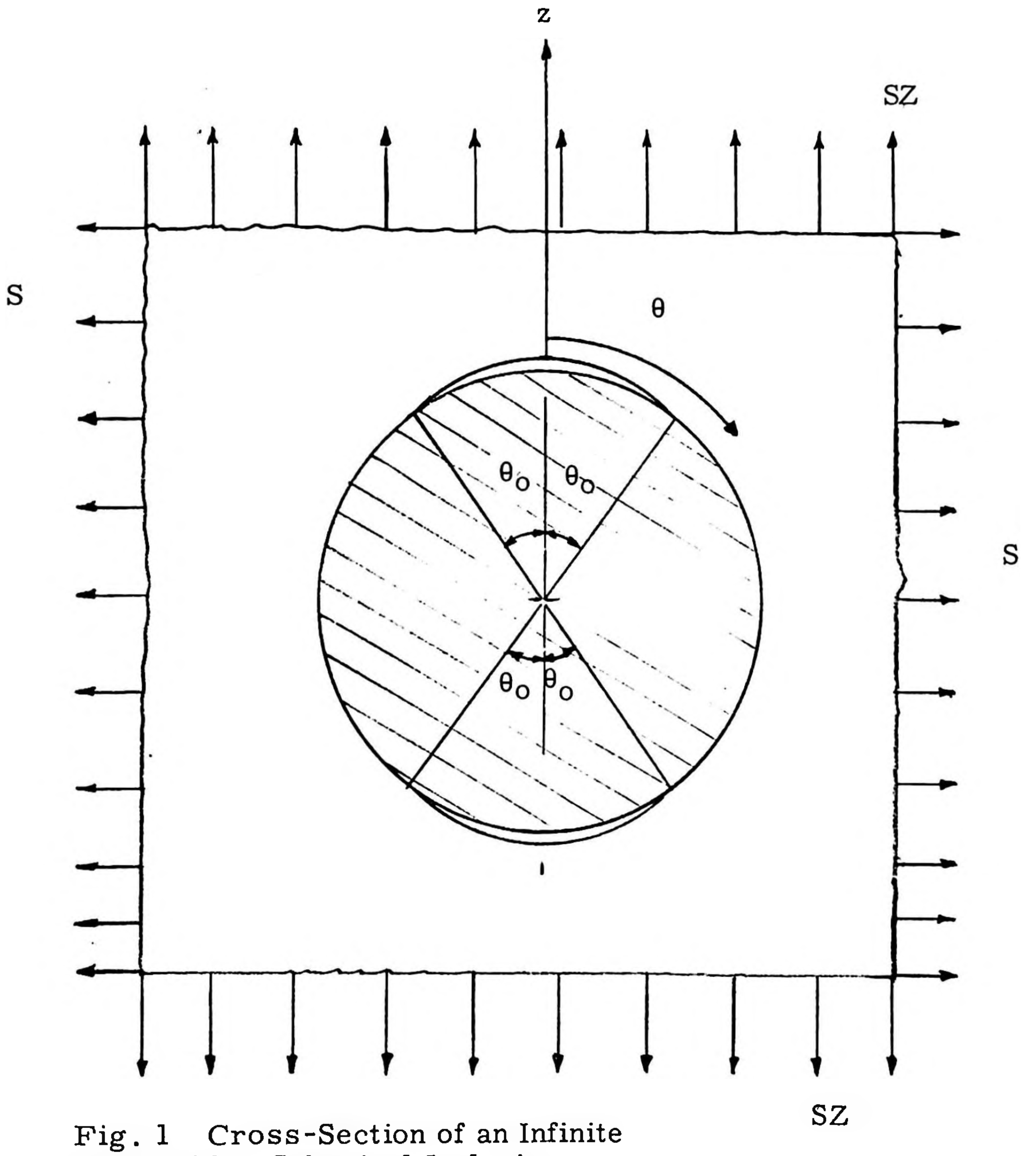

Fig. 1 Cross-Section of an Infinite Body with a Spherical Inclusion

by a series of properly chosen functions which satisfy one of the boundary conditions identically with the other boundary condition being satisfied by point matching was used for non-Hertzian plane contact stress problems in [6] and [7] and for axisymmetric contact problems in [4]. Point matching is also utilized in this paper. The solution obtained satisfies the boundary conditions to a high degree of accuracy for general loading conditions. Furthermore, it is shown that the solution differs negligibly from the exact solution for cases of total contact corresponding to appropriate combinations of the stress components at infinity.

A special case of the problem investigated here was studied previously by Wang [1]. That author treated a problem in which the spherical inclusion was rigid and only one stress component acted at infinity. Unfortunately, Wang's solution contains an error in the theoretical formulation and inaccuracies in the numerical results obtained by truncating an infinite system of linear simultaneous equations without retaining a finite system of sufficiently high order.

It is necessary to make some critical remarks concerning equations derived in [1] for specification of identical normal displacement on the sphere and the cavity in the contact zone. A general series solution has been given in [5] for stresses and displacements due to axisymmetric loading of a sphere or an infinite solid having a loaded spherical cavity and zero stresses at infinity. The solution involves the coefficients in the Legendre polynomial expansions of the boundary values of the normal and shearing stresses. When the results of [5] are used to calculate displacements in an infinite body having non-zero stresses at infinity, the displacements can conveniently be resolved into the following three components, (a) displacements for the given surface tractions and no stress at infinity (b) displacements in a body which has no cavity and is uniformly stressed (c) displacements in a body unstressed at infinity and having surface tractions on the cavity equal to minus the tractions given by (b). Wang [1] failed to include the component (b). This leads to an incorrect equation for normal displacement on the contact surface, and his subsequently computed values of boundary stresses are wrong.

Aside from the fact that dual series in [1] are invalid, the calculations made for the solution of those series should be discussed. The analysis entails solving an infinite system of simultaneous equations having individual matrix coefficients defined by infinite series. The approximate normal stress curve which Wang obtained by truncation of the infinite system exhibited pronounced oscillation near the line of separation between the sphere and the surrounding material. Thus, it was difficult to accurately evaluate the contact angle which is determined by the requirement that the normal stress should vanish at the edge of the contact surface. The oscillation of the approximate normal stress curve can apparently be reduced by increasing the order of the truncated infinite system. It seems, however, that a truncated system having approximately one hundred unknowns would probably be necessary to reduce the residual stress oscillation to a reasonable percentage of the maximum normal stress on the contact arc. Since formulation of the system would require accurate evaluation of ten thousand matrix coefficients each of which are infinite series, the computation time would be significant. The solution developed below is comparatively efficient and less than five minutes on a Univac 1107 is needed for evaluation of the contact angle as well as the normal and circumferential stresses on the cavity.

<u>Formulation of the Problem</u>

It is convenient to formulate the problem in spherical coordinates (r, θ, ϕ) related to the Cartesian system according to

$$x = r\sin(\theta)\cos(\phi), \quad y = r\sin(\theta)\sin(\phi), \quad z = r\cos(\theta) \tag{1}$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $r \geq 0$ and the origin is assumed to be located at the center of the cavity which is assumed, for convenience, to be of unit radius. For given axisymmetric tractions defined by $\sigma_r = N(\theta)$ and $\sigma_{r\theta} = T(\theta)$ on the surface of a sphere or an infinite solid with a spherical cavity, the corresponding stresses and displacements can be computed by elementary manipulation of the formulas in [5]. The boundary stresses pertinent in the problem studied here are representable as

$$T(\theta) = 0$$

$$N(\theta) = \sum_{n=o}^{\infty} \sigma_n P_{2n}(p) \tag{2}$$

where $p = \cos(\theta)$.

Only even ordered Legendre polynomials $P_{2n}(p)$ occur because of the symmetry with respect to the xy-plane. The accompanying displacement component normal to the surface of a sphere of unit radius is readily found to be [5]

$$u_r^o = \frac{1}{2\mu_o} \sum_{n=o}^{\infty} \frac{(4n^2+4n-1+2\nu_o)n-(4n^2-1)(n-1+2\nu_o)}{(2n+1)[4n^2+2n+1+(4n+1)\nu_o]} \sigma_n P_{2n}(p) \tag{3}$$

where a superscript or subscript zero indicates a quantity pertinent to the sphere.

It can be shown similarly that when an infinite solid is uniformly stressed at infinity and has stresses applied on the cavity according to Eq. 2, the related normal displacement on the cavity is expressible as

$$u_r^1 = \frac{1}{2\mu_1} \sum_{n=o}^{\infty} \frac{(2n+1)(2n^2-1+\nu_1)-(2n^2+2n)(2n+3-4\nu_1)}{(2n+2)[4n^2+2n+1-(4n+1)\nu_1]} \sigma_n P_{2n}(p)$$

$$+ \frac{(1 - \upsilon_1)(2S + S_z)}{4\mu_1 \ (1 + \upsilon_1)} P_o(p) + \frac{5(1 - \upsilon_1)(S - S_z)}{\mu_1 \ (7 - 5\upsilon_1)} P_2(p) \tag{4}$$

where the last two terms give the displacement for a solid having an unloaded cavity and uniform stresses $\sigma_z = S_z$ and $\sigma_x = \sigma_y = S$ at infinity. The superscript or subscript $_1$ on u_r, μ and υ defines a quantity pertinent to the infinite solid.

It follows from the boundary conditions that the coefficients σ_n defining the normal stress distribution between the sphere and the cavity are determined from dual Legendre series relations of the form

$$\sum_{n=o}^{\infty} \sigma_n P_{2n}(p) = o \ , \qquad \text{on the free surface} \tag{5}$$

and

$$\sum_{n=o}^{\infty} \left[\frac{[8n^2(1 - \upsilon_o) + 2\upsilon_o n - (1-2\upsilon_o)]\delta}{(2n - 1)[4n^2 + 2n + 1 + (4n+1)\upsilon_o]} + \frac{(1+8n^2)(1-\upsilon_1) + 2n(4-5\upsilon_1)}{2(n+1)[4n^2 + 2n + 1 - (4n+1)\upsilon_1]} \right] \sigma_n P_{2n}(p)$$

$$= \frac{(1-\upsilon_1)(2S+S_z)}{2(1+\upsilon_1)} P_o(p) - \frac{10(1-\upsilon_1)(S - S_z)}{(7 - 5\upsilon_1)} P_2(p) \ , \tag{6}$$

on the contact surface

where $\delta = \mu_1/\mu_o$. Eq. 6 is a generalization of Eq. 25 of Reference [1] which applies only for a rigid inclusion and an axial stress component at infinity. However, Eq. 25 of [1] is incorrect because of the omission of certain displacement terms related to a uniform stress field in an infinite solid.

For the special case in which the sphere and cavity do not separate, Eq. 6 is valid on the entire surface. Consequently,

$$\left[\frac{1-2\upsilon_0}{1+\upsilon_0}\delta + 1/2\right]\sigma_0 = \frac{(1-\upsilon_1)(2S+S_z)}{2(1+\upsilon_1)} \quad (7)$$

and

$$\left[\frac{7-4\upsilon_1}{7+5\upsilon_1}\delta + \frac{17-19\upsilon_1}{4(7-5\upsilon_1)}\right]\sigma_1 = -\frac{10(1-\upsilon_1)(S-S_z)}{(7-5\upsilon_1)}. \quad (8)$$

The remaining σ_n are zero and the normal stress equation becomes

$$\sigma_r = \sigma_0 \; P_0(p) + \sigma_1 P_2(p) = (\sigma_0 + \frac{\sigma_1}{4}) + 3/4\,\sigma_1 \cos(2\theta) \quad .(9)$$

The last equation can be used to ascertain the form of the contact configuration for arbitrary combinations of S and S_z. If S and S_z are related so that the coefficients σ_0 and σ_1 determined from Eqs. 7 and 8 give values of σ_r from Eq. 9 which are negative or zero for all values of θ, then contact occurs over the entire surface. However, separation between the sphere and solid occurs if Eq. 9 indicates tension on part of the boundary. Eq. 9 is then invalid. When tension is predicted for $\theta = 0$ and compression is predicted for $\theta = \pi/2$, the contact surface then consists of a spherical band, and Eqs. 5 and 6 apply for $0 \leq \theta \leq \theta_0$ and $\theta_0 \leq \theta \leq \pi/2$ respectively. The angle θ_0 must be determined by the requirement that $\sigma_r = 0$ for $\theta = \theta_0$. Alternately, the contact surface consists of two spherical caps when Eq. 9 indicates compression and tension for $\theta = 0$ and $\theta = \pi/2$, respectively. Then Eq. 5 applies for $\theta_0 \leq \theta \leq \pi/2$ and Eq. 6 applies for $0 \leq \theta \leq \theta_0$.

Solution of the Dual Series Equations

An accurate approximate solution of the dual series equations can be obtained by representing the normal stress as a finite series of judiciously chosen functions similar to those utilized in [4]. The case of polar contact will be discussed first. Approximating functions were selected which attain maximum amplitude and zero slope at $\theta=0$, are readily expandable in terms of Legendre polynomials, and vanish for $\theta_o < \theta \leq \pi/2$. The solution was also required to account exactly for a loading condition such that total contact exists with maximum normal stress at $\theta = 0$ and zero normal stress at $\theta = \pi/2$. It follows from Eq. 9 that in this case

$$\sigma_1 = 2\,\sigma_o \quad \text{and} \quad \sigma_r = 3\,\sigma_o \cos^2(\theta)\,, \qquad 0 \leq \theta \leq \pi/2\,. \quad (10)$$

Following the criteria just cited, N(θ) can be conveniently approximated as

$$N(\theta) = \sum_{k=o}^{K} A_k \left[\frac{p - p_o}{1 - p_o} \right]^{k/2}, \qquad 0 \leq \theta \leq \theta_o \quad (11)$$

and

$$N(\theta) = 0\,. \qquad \theta_o < \theta \leq \pi/2 \quad (12)$$

Note that when $\theta_o = \pi/2$, the term for k=4 in Eq. 11 accounts for the form required by Eq. 10. Except for $k = 0$, all terms in Eq. 11 vanish for $p = p_o$ corresponding to the edge of the contact surface. The term for k=0 gives a step function normal stress to be utilized in evaluating the contact angle θ_o which depends on the requirement that the normal stress should vanish at $\theta = \theta_o$. In the computations presented here it was found that the boundary conditions could be satisfied quite accurately by taking K=5 and determining $A_o, \ldots A_5$ by the stipulation that for a chosen value of θ_o, Eq. 6 should be satisfied at six equally spaced values of θ in the range $0 \leq \theta \leq \theta_o$. The contact angle was then adjusted to reduce the A_o component until the computed normal stress at $\theta = \theta_o$ vanished. It is shown in the numerical results discussed later in the article that the solution so obtained satisfies the boundary condition accurately at all points on the contact surface.

It follows from Eqs. 2, 12, and 13 and the orthogonality properties of the Legendre polynomials that

$$N(\theta) = \sum_{k=o}^{K} A_k \sum_{n=o}^{\infty} \sigma_n^k \; P_{2n}(p) \tag{13}$$

where

$$\sigma_n^k = (4n+1) \int_{p_o}^{1} \left[\frac{p - p_o}{1 - p_o}\right]^{k/2} P_{2n}(p)dp \tag{14}$$

and the integer N is taken sufficiently large to give a negligible error in satisfaction of Eq. 12.

The integrals in the last equation can be evaluated in closed form by integrating by parts and using the Mehler-Dirichlet integrals. [8]. Eq.6 can then be approximated as

$$\sum_{k=o}^{K} A_k \sum_{n=o}^{N} \left[\frac{[8n^2 \;(1-\upsilon_o) + 2\upsilon_o n - (1 - 2\upsilon_o)]\delta}{(2n-1)[4n^2 + 2n+1+(4n+1)\upsilon_o]} + \frac{(1+8n^2)(1-\upsilon_1)+2n(4-5\upsilon_1)}{2(n+1)\;[4n^2 + 2n+1-(4n+1)\upsilon_1]}\right]$$

$$\sigma_n P_{2n}(p) = \frac{(1-\upsilon_1)(2S+S_z)}{2(1+\upsilon_1)} P_o(p) - \frac{10(1-\upsilon_1)(S-S_z)}{(7-5\upsilon_1)} P_2(p) \tag{15}$$

where the coefficients A_k are determined by point matching as outlined earlier. The required value of θ_o was found by choosing an initial estimate known to be too small and computing a corresponding tensile value of normal stress at the edge of the contact surface. The estimated value of θ_o was increased incrementally until a sign change in σ_r occurred. The value of θ_o to make σ_r zero was then obtained by linear interpolation. When the required value of θ_o was found, the final values of normal stress σ_r and circumferential stresses σ_θ and σ_ϕ on the cavity were evaluated.

In the instance where the stress components at infinity are such that contact occurs along a spherical band about the equator, a suitable approximating series for the normal stress curve was found to be

$$N(\theta) = \sum_{k=o}^{K} A_k \left[\frac{p_o^2 - p^2}{p_o^2} \right]^{k/2} \qquad \theta_o \leq \theta \leq \pi/2$$

and (16)

$$N(\theta) = 0 \qquad 0 \leq \theta < \theta_o$$

where the p^2 terms are used so that $N'(\theta)$ vanishes at $\theta = \pi/2$ as required by the loading symmetry. The truncated series form of the first of Eq. 16 is

$$N(\theta) = \sum_{k=o}^{K} A_k \sum_{n=o}^{N} \sigma_n^k \, P_{2n}(p) \qquad 0 \leq p \leq p_o \qquad (17)$$

where

$$\sigma_n^k = (4n+1) \int_0^{p_o} \left[\frac{p_o^2 - p^2}{p_o^2} \right]^{k/2} P_{2n}(p) \qquad (18)$$

and the integer N is once again chosen sufficiently large to yield a negligible normal stress residual on the surface of separation between the sphere and the infinite solid. Although the integrals in the last equation were not found to be easily computed in closed form, an accurate numerical evaluation was effected by breaking the interval of integration into subintervals and approximating the function which multiplies $P_{2n}(p)$ in Eq. 18 by a straight line coinciding with the original function at the ends of the pertinent subintervals. When the known exact integral for a linear function times $P_{2n}(p)$ was used and the contributions from the various subintervals were added, a satisfactory approximate form of Eq. 18 was obtained. This manipulation amounts to replacing the original approximating functions by new ones defined by a series of chords between closely spaced points. The new functions still have the desired shape properties and, moreover, their truncated series forms are used in Eq. 17. Consequently no difficulty was caused by this step.

It was found that the boundary conditions could be satisfied accurately by taking $K = 3$ in Eq. 16 with the coefficients being determined by point matching in the interval $\theta_0 \leq \theta \leq \pi/2$. The value of contact angle θ_0 was computed by starting with an initial estimate which was too large and decreasing θ_0 to give $\sigma_r = 0$ at θ_0 in the manner described earlier.

With the value of θ_0 and the A_k determined, the circumferential stresses are given by

$$\sigma_\theta = \sum_{k=o}^{K} A_k \sum_{n=o}^{N} \Bigg[\left[8n^2 - 5n - 1 + (4n+1)\upsilon_1\right] P_{2n}(p) - \frac{2n^2 - 1 + \upsilon}{2(n+1)} P'_{2n+1}(p)$$

$$+ (n-2+2\upsilon_1)\, P'_{2n-1}(p)\Bigg] \frac{\sigma_n^k}{[4n^2 + 2n + 1 - (4n+1)\upsilon_1]}$$

$$+ \frac{3}{2(7 - 5\upsilon_1)} \left[S_z[9 - 5\upsilon_1 - 10p^2] - 2S[1 - 5p^2] \right], \qquad (19)$$

and

$$\sigma_\phi = \sum_{k=o}^{K} A_k \sum_{n=o}^{N} \Bigg[-n(4n+1)(1-2\upsilon_1)\, P_{2n}(p) + \frac{2n^2 - 1 + \upsilon_1}{2(n+1)} P'_{2n+1}(p)$$

$$- (n-2+2\upsilon_1)\, P'_{2n-1}(p)\Bigg] \frac{\sigma_n^k}{[4n^2 + 2n \pm 1 - (4n+1)\upsilon_1]}$$

$$- \frac{3}{2(7-5\upsilon_1)} \left[S_z [1 - 5\upsilon_1 + 10p^2 \upsilon_1] - 2S[4 - 5\upsilon_1 + 5\upsilon_1 p^2] \right] \qquad (20)$$

where the terms involving S_z and S in Eqs. 19 and 20 give the stresses on the surface of an unloaded cavity due to applied stresses S_z and S at infinity..

Numerical Results

The approximate solution developed above was coded as a FORTRAN IV program for the Univac 1107 digital computer. Provision was made for computation of the contact angle and the normal and circumferential stress components on the cavity corresponding to arbitrary stress components at infinity and arbitrary values of Poisson's ratio and modulus of elasticity for the sphere and the surrounding medium.

The accuracy of the solution was checked for two special cases of loading at infinity such that total contact occurred with separation being incipient at $\theta = 0$ in one case and at $\theta = \pi/2$ in the other case. The approximate solution gives boundary stress values correct within a maximum error of 0.05% in these cases. In the instances where separation occurred between the sphere and the solid, the accuracy of the solution was assessed by evaluating the residual normal stress given by Eqs. 13 or 17 for points on the arc of separation. It was found that with $N = 125$ for cap or band contact, the maximum stress residual on the arc of separation was less than .5 percent of the maximum normal stress on the contact arc for all cases discussed below. This residual could have easily been reduced further by increasing the value of N. The error in satisfaction of the displacement condition on the contact surface was measured by computing the difference between the radial displacements on the sphere and the cavity as a percentage of the largest displacement difference on the surface of separation. The maximum displacement error in any example considered was 0.89%. The typical displacement error was generally much smaller.

Computations were made for various combinations of elastic properties and stress components at infinity as shown in the following table which also lists the accompanying extremal values of normal and circumferential stresses on the cavity.

case	S	S	E /E_o	υ_1	υ_o	contact	θ_o	$(-N)_{max}$	$(\sigma_\theta)_{max}$	$(-\sigma_\theta)_{min}$
1	1	0	0	.3	-	band	71.19	.939	2.124	.588
2	1	0	0	.5	-	band	70.11	1.138	2.249	.946
3	1	0	1	.3	.3	band	70.49	.457	1.920	.634
4	0	-1	0	.3	-	band	44.68	2.370	.415	1.551
5	0	-1	0	.5	-	band	53.96	2.079	.661	1.975
6	0	-1	1	.3	.3	band	35.61	1.217	.570	1.784
7	-1	0	0	.3	-	cap	57.29	3.101	.491	1.325
8	-1	0	0	.5	-	cap	50.66	3.010	.135	1.458
9	-1	0	1	.3	.3	cap	63.65	1.433	.640	1.656
10	0	1	0	.3	-	cap	31.18	1.854	2.146	.400
11	0	1	0	.5	-	cap	34.26	2.289	2.289	.408
12	0	1	1	.3	.3	cap	30.74	.853	1.928	.477

Table 1 Combinations of Applied Stress, Elastic Parameters, Contact Angle and Maximum Boundary Stresses

The related curves for N, σ_θ, and σ_ϕ appear in Figs. 2 - 7. The results are quantitatively similar to those reported in [6] for the analagous plane problem. It should be noted that the correct maximum values of normal stress and circumferential stress σ_θ for case 1 equal approximately 2.49 and 1.05 times, respectively, the values obtained by Wang [1].

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Case	Sz	S	E_1/E_0	υ_1	υ_0
1	1	0	0	.3	-
2	1	0	0	.5	-
3	1	0	1	.3	.3
4	0	-1	0	.3	-
5	0	-1	0	.5	-
6	0	-1	1	.3	.3

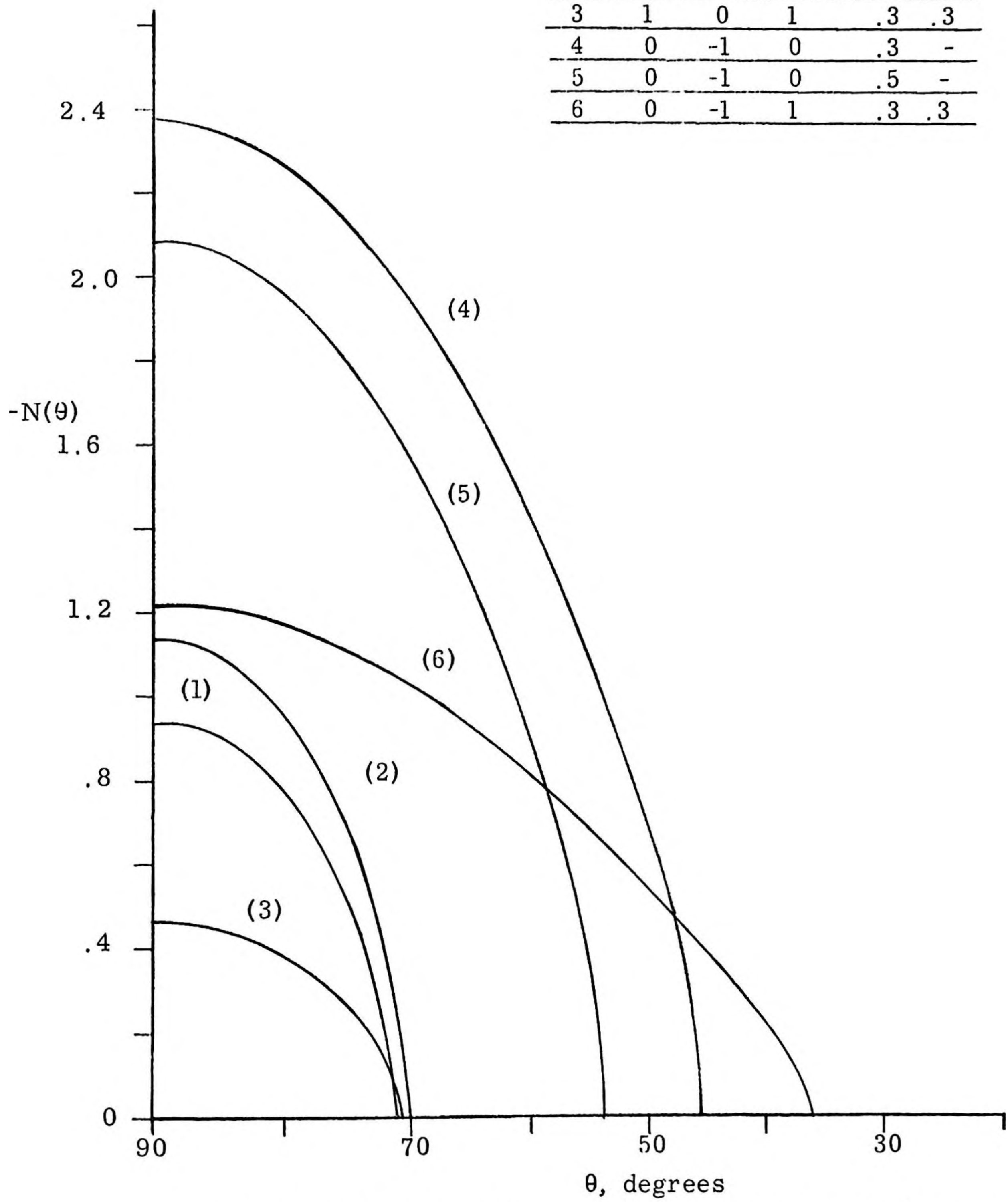
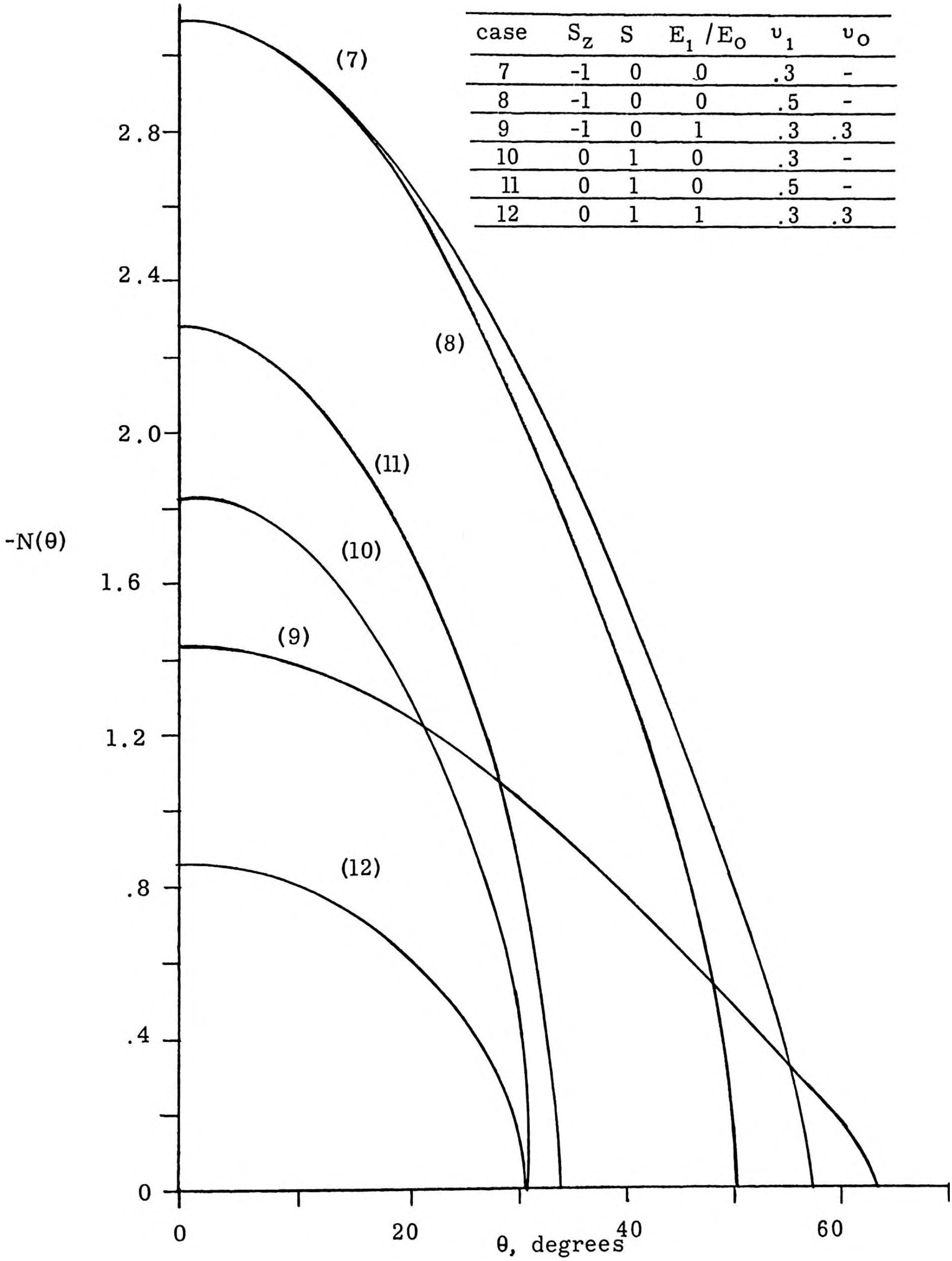
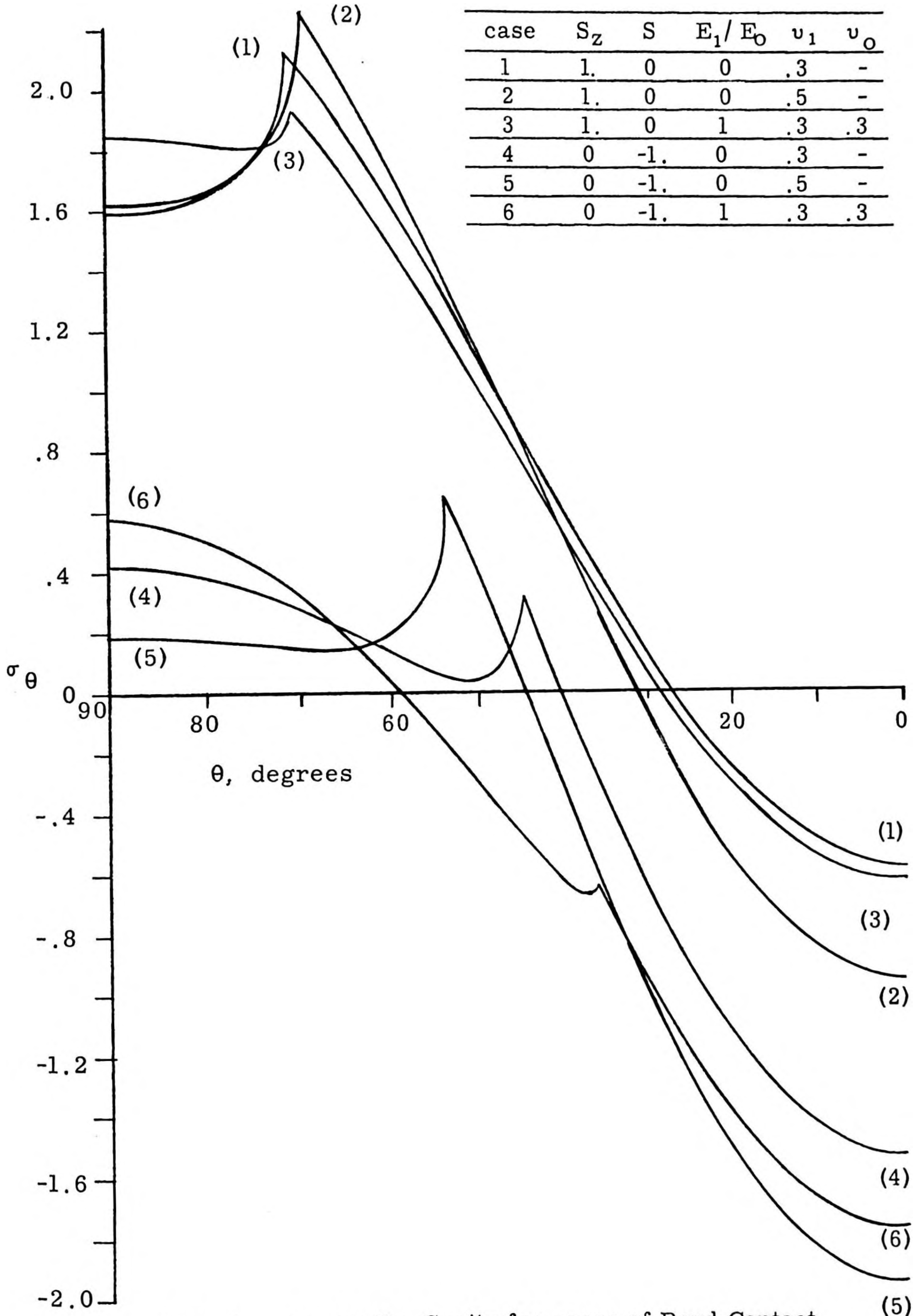


Fig. 2 Normal Stress on the Contact Surface for Cases of Band Contact



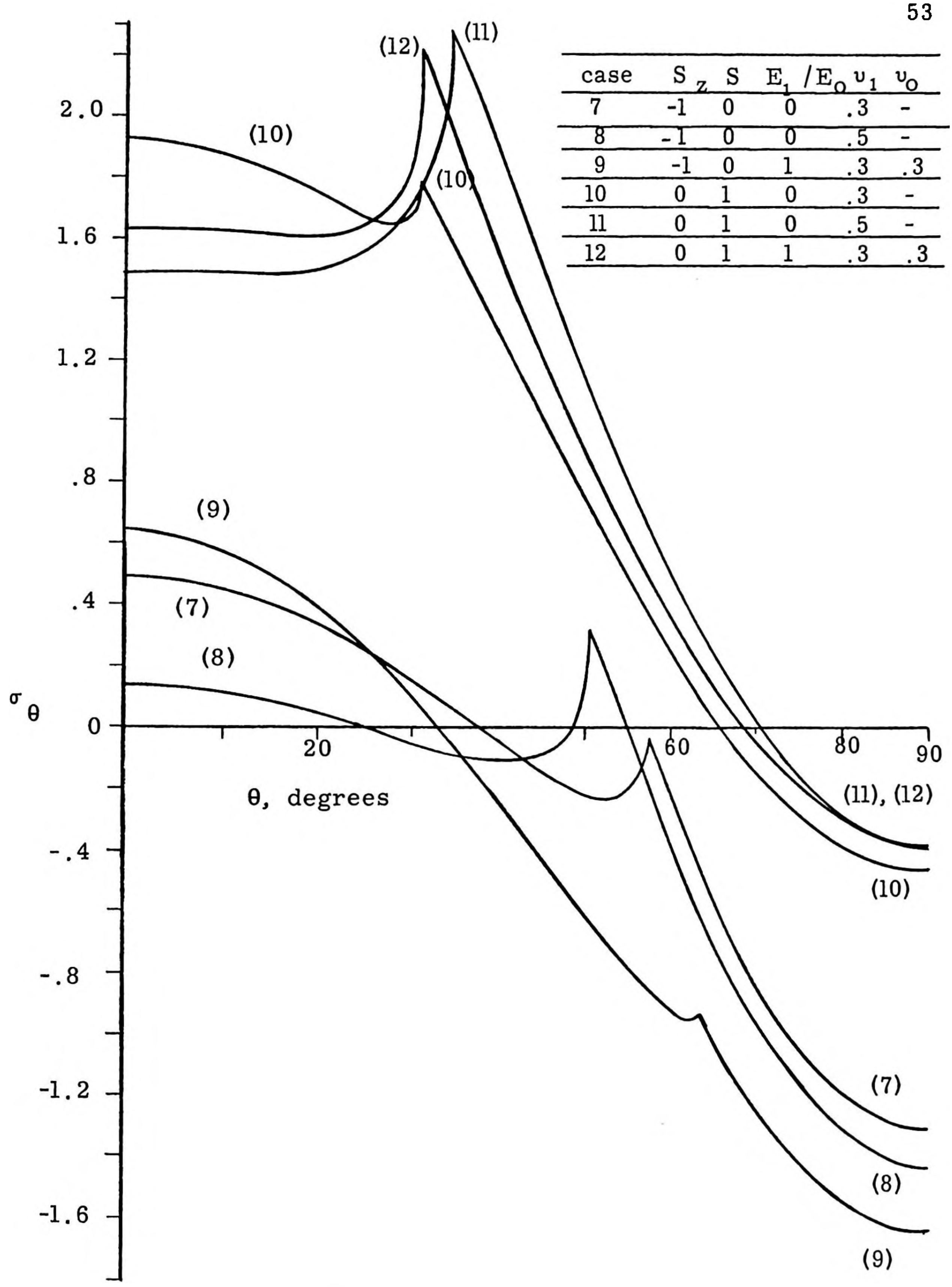
case	S_z	S	E_1/E_o	υ_1	υ_o
7	-1	0	0	.3	-
8	-1	0	0	.5	-
9	-1	0	1	.3	.3
10	0	1	0	.3	-
11	0	1	0	.5	-
12	0	1	1	.3	.3

Fig. 3. Normal Stress on the Contact Surface for Cases of Cap Contact



case	S_z	S	E_1/E_0	υ_1	υ_0
1	1.	0	0	.3	-
2	1.	0	0	.5	-
3	1.	0	1	.3	.3
4	0	-1.	0	.3	-
5	0	-1.	0	.5	-
6	0	-1.	1	.3	.3

Fig. 4. σ_θ on the Cavity for cases of Band Contact.



case	S_z	S	E_1/E_0	υ_1	υ_0
7	-1	0	0	.3	-
8	-1	0	0	.5	-
9	-1	0	1	.3	.3
10	0	1	0	.3	-
11	0	1	0	.5	-
12	0	1	1	.3	.3

Fig. 5 σ_θ on the Cavity for Cases of Cap Contact

case	S_z	S	E_1/E_o	ν_1	ν_o
1	1	0	0	.3	-
2	1	0	0	.5	-
3	1	0	1	.3	.3
4	0	-1	0	.3	-
5	0	-1	0	.5	-
6	0	-1	1	.3	.3

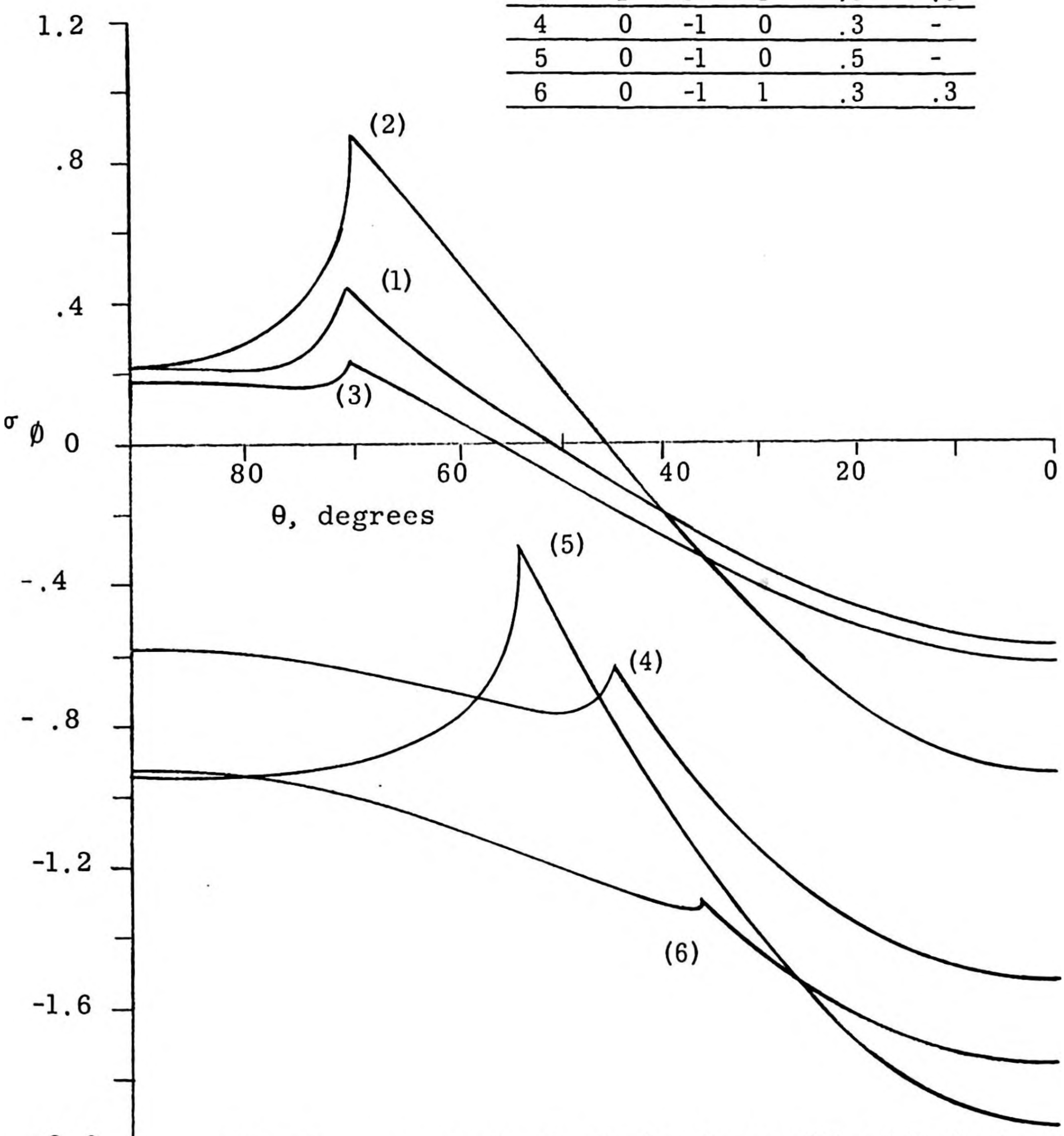
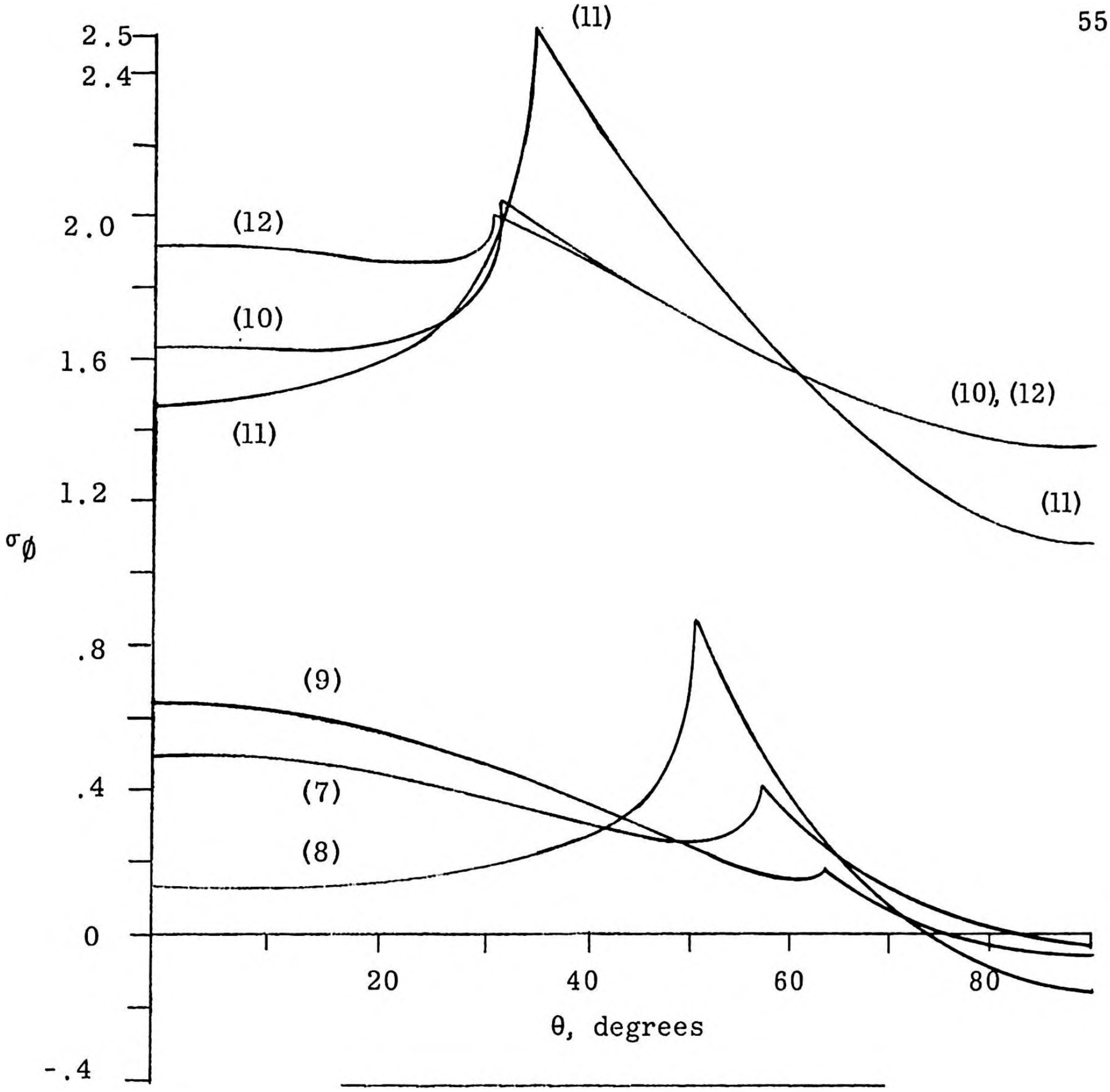


Fig. 5 σ_ϕ on the Cavity for Cases of Band Contact



case	Sz	S	E_1/E_0	ν_1	ν_0
7	-1	0	0	.3	-
8	-1	0	0	.5	-
9	-1	0	1	.3	.3
10	0	1	0	.3	-
11	0	1	0	.5	-
12	0	1	1	.3	.3

Fig. 7 σ_ϕ on the Cavity for Cases of Cap Contact

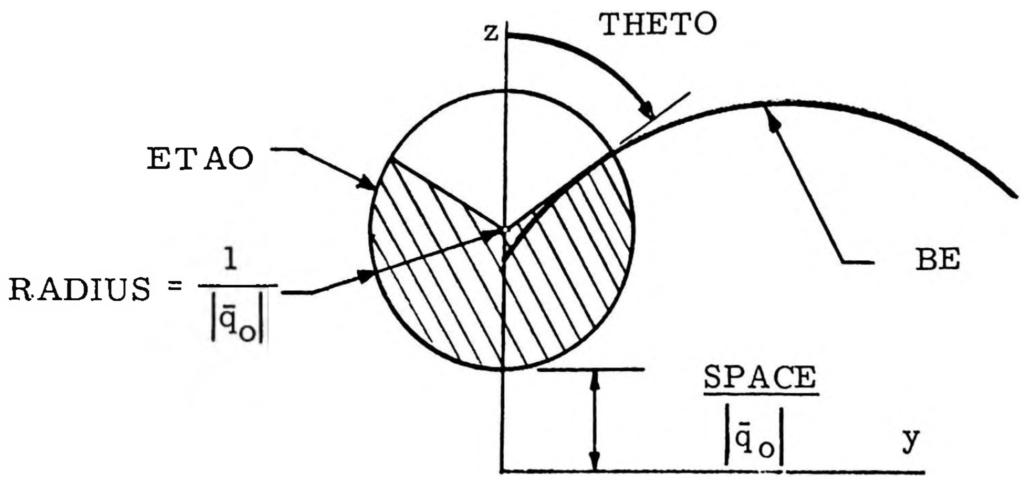
APPENDIX I

DESCRIPTION OF FORTRAN PROGRAM

Torsional Stresses in an Infinite Body Containing Two Partially Bonded Rigid Inclusions

I. Purpose

This program evaluates the boundary stresses and displacements for an infinite body containing two rigid partially bonded spherical inclusions. The body may be subjected to pure torsion about the z-axis as well as having torques applied to the inclusions. It is intended that the reader be familiar with the formulation of the problem as presented in the first chapter of this thesis before using this program. The following figure gives a comparison between the parameters used in the computer program and those defined in chapter one.



II. Input Parameter Definition

Parameter	Definition
NN	NN specifies the number of sets of data the program will execute. The data deck consist of one card.
LA, LB, LC	LA, LB, and LC are the month,day, and year respectively.
N	N is the number of terms taken in the residual stress and displacement series.
KK	KK specifies the number of terms taken in the series determining the coefficients for the singular stress field, i.e. $\sigma_r = S_r^o \sum_{k=r}^{r+KK} \gamma_k \sum_{n=1}^{k} \beta_n$.
MM	MM is the number of coefficients σ_n taken in the singular solution .
I1	I1 specifies whether the displacements are asymmetric or symmetric with respect to the η coordinate. I1 must be either $\pm$ 1 with + 1 corresponding to anti-symmetric torsion and -1 corresponding to symmetric torsion.
I2	I2 specifies the contact surface. I2 must be $\pm$1 with +1 corresponding to contact on the arc $\theta_o \leq \theta \leq \pi$ and -1 corresponding to contact on the arc $0 \leq \theta \leq \theta_o$.
I3	I3 specifies the resulting torque conditions on the inclusions. I3 must be $\pm$1 with + 1 corresponding to non-vanishing torques applied to the inclusions.

Parameter	Definition
	and -1 corresponding to zero resultant torque.
DTHET	DTHET is the increment in the true polar angle at which the stresses and displacement will be computed.
THETO	THETO is the true polar contact angle.
SPACE	SPACE is the distance between inclusions as shown in the figure on page 56. SPACE = X
W	W is the angle, in radians, through which the inclusions are rotated by means of an applied torque acting on the inclusions. If 13 = -1 W will be set equal to one.
ANGLE	ANGLE is the angle of twist per unit length in radians through which the body is rotated. ANGLE = Θ

III. Input Card Listing

Card No.	Parameter	Data Field	Format
1	NN	1-5	I5
	LA	6-7	I2
	LB	8-9	I2
	LC	10-11	I2
2	N	1-5	I5
	KK	6-10	I5
	MM	11-15	I5
	I1	16-20	I5
	I2	21-25	I5
	13	26-30	I5

Card No.	Parameter	Data Field	Format
2	DTHET	31-38	F8.4
	THETO	39-46	F8.4
	SPACE	47-54	F8.4
	W	55-62	F8.4
	ANGLE	63-70	F8.4

IV. Output of Program

A. Repeated Input Data

B. A statement, depending on the values of I1, I2, and I3, specifying the problem considered.

C. DEL1 = $\delta^{(1)}$ and DEL2 = $\delta^{(2)}$

D. W1 = ω^*, where ω^* is the computed angle of twist in radians necessary for zero resultant torque on the inclusions.

E. Tabulation of the boundary stresses, displacement, percent error in satisfy ing the displacement boundary condition, and a two line grouping giving on the first line the value of the truncated series representation for the singular stress and on the second line the true value of the singular stress. At $\theta = \theta_o$ the approximate value is printed on both lines.

TNSI = $\sigma_{\eta\beta}$

USI = u

TTSI = $\sigma_{\theta\beta}$

V. Program Listing

A listing of the program is given on pages 60 to 68 and a sample problem is given on page 69 . Only partial results, sufficient for checking program operations, are given for the sample problem.

```
C       TORSIONAL STRESSES IN AN INFINITE BODY CONTAINING
C             TWO PARTIALLY BONDED RIGID INCLUSIONS
      DIMENSION A(150,150),X(150),B(150),P(400),AS(400),
     1 P1(400),Z(150)
      COMMON A,B,X,N
      TH(G,B)=(1.-E*EXP(-2.*(G+.5)*B))/(1.+E*EXP(-2.*(G+.5)*B))
      SH(G)=0.5*(EXP(G)-1./EXP(G))
      CH(G)=0.5*(EXP(G)+1./EXP(G))
      EN1(G,B)=1./TH(G,B)+(3.*TANH(B)-2.*G-3.)/(2.*G+1.)
      EN2(G,B)=1./TH(G,B)+(3.*TANH(B)-2.*G+1.)/(2.*G+1.)
      EK(G,B)=3.14159265*(1.-F)/2.+F*B-SIN((2.*G+1.)*B)/(2.*G+1.)
      E1(G,H,B)=(F*SIN((G-H)*B)/(G-H)-SIN((G+H+1.)*B)/(G+H+1.))/
     1 ((2.*H-1.)*(2.*H+3.))
      E3(G,H,B)=(F*SIN((G-H-1.)*B)/(G-H-1.)-SIN((G+H+2.)*B)/(G+H+2.))/
     1 (2.*H+3.)
      E5(G,H,B)=(F*SIN((G-H-2.)*B)/(G-H-2.)-SIN((G+H+3.)*B)/(G+H+3.))
     1 *(H+2.)/((2.*H+3.)*(2.*H+5.))
      E7(G,H,B)=(F*SIN((G-H+1.)*B)/(G-H+1.)-SIN((G+H)*B)/(G+H))
     1 /(2.*H-1.)
      E9(G,H,B)=(F*SIN((G-H+2.)*B)/(G-H+2.)-SIN((G+H-1.)*B)/(G+H-1.))
     1 *(H-1.)/((2.*H-1.)*(2.*H-3.))
      E4(G,H,B)=F*SIN((G-H-1.)*B)/(G-H-1.)-SIN((G+H+2.)*B)/(G+H+2.)
      E6(G,H,B)=F*SIN((G-H+1.)*B)/(G-H+1.)-SIN((G+H)*B)/(G+H)
      E10(E,G,B)=1.+E*EXP(-G*B)
      ALPH(F,G,B)=SQRT(2.)*((1.+F)/2.*(COS((G-.5)*B)/(2.*G-1.)-COS((G+
     1 1.5)*B)/(2.*G+3.))+(1.-F)/2.*(SIN((G-.5)*B)/(2.*G-1.)-SIN((G
     2 +1.5)*B)/(2.*G+3.0)))
      ASIN(G)=ATAN(G/SQRT(1.-G**2))
      ACOS(G)=ATAN(SQRT(1.-G**2)/G)
      READ(5,20)NN,LA,LB,LC
      DO 710 NE=1,NN
      READ(5,30)N,  KK,MM,I1,I2,I3,DTHET,THETO,SPACE,W,ANGLE
      E0=ALOG(1.+SPACE+SQRT(SPACE*(2.+SPACE)))
      RADIUS=1./SINH(E0)
      DIST=2./TANH(E0)-2.*RADIUS
      Y0=COS(THETO*.01745329)
      Y8=ABS(COSH(E0)*Y0+1.)
      IF(COSH(E0)*Y0+1.) 2,3,3
 2    BE=3.14159265-ACOS(Y8/(COSH(E0)+Y0))
      GO TO 4
 3    BE=ACOS(Y8/(COSH(E0)+Y0))
 4    JJ1=MM+KK
      JJ=300
      F7=JJ
      YS=COS(BE)
      JJ2=MM-1
      N1=N-1
      N2=N-2
      N3=N-3
      F4=N
      F5=MM
      E=I1
      F=I2
      F1=I3
```

```
      F6=KK
      SS=10.0
      WRITE(6,40)LA,LB,LC,THETO,EO  ,BE,W,ANGLE,RADIUS,DIST,SPACE,
     1 E,F,F1,F4,F5,F6,F7
20    FORMAT(I5,3I2)
30    FORMAT(6I5,5F8.4)
40 FORMAT(////7X49HTORSIONAL STRESSES IN AN INFINITE BODY CONTAINING/
    113X37HTWO PARTIALLY BONDED RIGID INCLUSIONS//23XI2,1H/I2,1H/I2//
    2 7X6HTHETO=F8.4/8X5HETAO=F8.4/10X3HBE=F8.4/11X2HW=F8.4/
    3 7X6HANGLE=F8.4/6X7HRADIUS=F8.4/8X5HDIST=F8.4/5X8HSPACING=F8.4/
    4 10X3HI1=F8.4/10X3HI2=F8.4/10X3HI3=F8.4/11X2HN=F8.4/10X3HMM=F8.4/
    5 10X3HKK=F8.4/10X3HJJ=F8.4//)
      IF(E) 11,10,10
11    WRITE(6,21)
      GO TO 12
10    WRITE(6,22)
12    IF(F) 13,13,14
13    WRITE(6,23)
      GO TO 31
14    WRITE(6,24)
31    IF(F1) 32,32,33
32    W=1.0
      WRITE(6,34)
      GO TO 873
33    WRITE(6,35)
34    FORMAT(7X57HAND HAVING ZERO RESULTANT TORQUE ACTING ON THE INCLUS
     1IONS///)
35    FORMAT(7X45HAND HAVING A TORQUE APPLIED TO THE INCLUSIONS///)
21    FORMAT(7X56HTHE PROBLEM CONSIDERED IS THAT OF SYMMETRIC TORSION W
     1ITH)
22    FORMAT(7X61HTHE PROBLEM CONSIDERED IS THAT OF ANTI-SYMMETRIC TORS
     1ION WITH)
23    FORMAT(7X64HTHE FREE SURFACE BEING FROM THETA EQUAL THETO TO THET
     1A EQUAL 180)
24    FORMAT(7X65HTHE FREE SURFACE BEING FROM THETA EQUAL ZERO TO THETA
     1 EQUAL THETO)
42    FORMAT(////16X44HSTRESSES AND DISPLACEMENTS ON THE INCLUSIONS//
     1 10X5HTHETA,16X4HTNSI,17X3HUSI,17X4HTTSI,13X10HDISP ERROR,
     2 7X15HSINGULAR STRESS//)
380   FORMAT(//10X3HW1=F8.4//)
705   FORMAT(6E20.8/100XE20.8)
709   FORMAT(8X5HDEL1=E20.8)
711   FORMAT(8X5HDEL2=E20.8)
C     COMPUTE THE COEFFICIENTS OF THE SINGULAR SOLUTION
873   IF(THETO) 39,37,39
39    IF(THETO-180.0) 29,37,29
37    DO 38 K=1,MM
38    AS(K)=0.0
      GO TO 48
29    DO 851 NP=1,MM
      SN14=0.0
      I=NP
      NNP=KK+NP
      DO 850 K=NP,NNP
```

```
      SN12=0.0
      DO 825 L=1,K
      R=K
      S=L
825   SN12=SN12+ALPH(F,S,BE)*S*(S+1.)*EXP(-(2.*R-S-T+1.)*EO)*E10(E,(
     1 2.*S+1.),EO)/(R*(R+1.)*(R+2.)*E10(E,(2.*R+1.),EO)*E10(E,(2.*R
     2 +3.),EO))*E10(-E,(2.*T+1.),EO)
850   SN14=SN14+SN12
851   AS(NP)=SN14
C     CONSTRUCT THE MATRIX FOR THE CASE OF TORQUE APPLIED TO THE SPHERE
48    DO 120 K=1,N
      DO 101 L=1,N
      R=K
      S=L
      IF(L-K) 50,60,70
50    IF(L-K+1) 51,55,53
53    R=K
51    IF(L-K+2) 76,52,53
52    A(K,L)=E3(R,S,BE)*EN1(S,EO)-E7(R,S,BE)*EN2(S,EO)-
     1 (1./(TH(S,EO)*CH(EO)))  *(R*EK(R,BE)/((2.*R-1.)*(2.*R+1.))
     2 -E9(R,S,BE)-E1(R,S,BE))
      GO TO 101
55    A(K,L)=EK(R,BE)*EN1(S,EO)/(2.*R+1.)+3.14159265/(2.*R-1.)
     1 -E7(R,S,BE)*EN2(S,EO)-
     2 (1./(TH(S,EO)*CH(EO)))  *(E5(R,S,BE)-E9(R,S,BE)-E1(R,S,BE))
      GO TO 101
60    A(K,L)=E3(R,S,BE)*EN1(S,EO)-E7(R,S,BE)*EN2(S,EO)-
     1 (1./(TH(S,EO)*CH(EO)))  *(E5(R,S,BE)-E9(R,S,BE)-EK(R,BE)/((
     2 2.*R-1.)*(2.*R+3.)))
      GO TO 101
70    IF(L-K-1) 53,72,73
72    A(K,L)=E3(R,S,BE)*EN1(S,EO)-EK(R,BE)*EN2(S,EO)/(2.*R+1.)
     1 -3.14159265/(2.*R+3.)-
     2 (1./(TH(S,EO)*CH(EO)))  *(E5(R,S,BE)-E9(R,S,BE)-E1(R,S,BE))
      GO TO 101
73    IF(L-K-2) 53,75,76
75    A(K,L)=E3(R,S,BE)*EN1(S,EO)-E7(R,S,BE)*EN2(S,EO)-
     1 (1./(TH(S,EO)*CH(EO)))  *(E5(R,S,BE)-EK(R,BE)*(R+1.)/
     2((2.*R+3.)*(2.*R+1.))-E1(R,S,BE))
      GO TO 101
76    A(K,L)=E3(R,S,BE)*EN1(S,EO)-E7(R,S,BE)*EN2(S,EO)-
     1 (1./(TH(S,EO)*CH(EO)))  *(E5(R,S,BE)-E9(R,S,BE)-E1(R,S,BE))
101   CONTINUE
      SN1=0.0
      GS1=0.
      DO 108 J=1,JJ
      IF(SS) 115,105,105
115   V=J
C     COMPUTE R.H.S. FOR TORGUE FREE SPHERES
      IF(K-J) 112,117,113
112   IF(K-J+1) 117,116,117
113   IF(K-J-1) 117,114,117
114   GS2=GS1+2.*SQRT(2.)*V*(V+1.)/3.*(((V-1.)*EXP(-(V-.5)*EO)-(V+2.)*
     1 EXP(-(V+1.5)*EO))*2./(CH(EO)*(2.*V+1.))*(EK(R,BE)/(2.*V+3.)
```

```
    2 -E8(R,V,BE)
    3 /(2.*V-1.))+EXP(-(V+.5)*EO)*(EK(R,BE)-3.14159265-E8(R,V,BE)))
      GO TO 108
116   GS2=GS1+2.*SQRT(2.)*V*(V+1.)/3.*(((V-1.)*EXP(-(V-.5)*EO)-(V+2.)*
    1 EXP(-(V+1.5)*EO))*2./(CH(EO)*(2.*V+1.))*(E4(R,V,BE)/(2.*V+3.)
    2 -EK(R,BE)
    3 /(2.*V-1.))+EXP(-(V+.5)*EO)*(E4(R,V,BE)-EK(R,BE)+3.14159265))
      GO TO 108
117   GS2=GS1+2.*SQRT(2.)*V*(V+1.)/3.*(((V-1.)*EXP(-(V-.5)*EO)-(V+2.)*
    1 EXP(-(V+1.5)*EO))*2./(CH(EO)*(2.*V+1.))*(E4(R,V,BE)/(2.*V+3.)
    2 -E8(R,V,BE)
    3 /(2.*V-1.))+EXP(-(V+.5)*EO)*(E4(R,V,BE)-E8(R,V,BE)))
      GO TO 108
105   V=J
      IF(K-J) 102,107,103
102   IF(K-J+1) 107,106,107
103   IF(K-J-1) 107,104,107
104   GS2=GS1+(2.*SQRT(2.)*W*EXP(-(V+.5)*EO))*V*(V+1.)/(2.*V+1.)*
    1 (E8(R,V,BE)-EK(R,BE)+3.14159265)
      GO TO 108
106   GS2=GS1+(2.*SQRT(2.)*W*EXP(-(V+.5)*EO))*V*(V+1.)/(2.*V+1.)*
    1 (EK(R,BE)-3.14159265-E4(R,V,BE))
      GO TO 108
107   GS2=GS1+(2.*SQRT(2.)*W*EXP(-(V+.5)*EO))*V*(V+1.)/(2.*V+1.)*
    1 (E8(R,V,BE)-E4(R,V,BE))
108   GS1=GS2
109   IF(THETO) 41,120,41
41    IF(THETO-180.) 111,120,111
111   DO 208 J=1,MM
      V=J
      IF(K-J) 202,207,203
202   IF(K-J+1) 207,206,207
203   IF(K-J-1) 207,204,207
204   SN1=SN1+AS(J)*V*(V+1.)/(2.*V+1.)*(E8(R,V,BE)-EK(R,BE)+3.14159265)
      GO TO 208
206   SN1=SN1+AS(J)*V*(V+1.)/(2.*V+1.)*(EK(R,BE)-3.14159265-E4(R,V,BE))
      GO TO 208
207   SN1=SN1+AS(J)*V*(V+1.)/(2.*V+1.)*(E8(R,V,BE)-E4(R,V,BE))
208   CONTINUE
      A(K,N)=SN1
120   B(K)=GS2
      CALL SOLVE(A,X,B,N,150)
      IF(SS) 1120,1089,1089
1089  DO 1092 K=1,N
1092  Z(K)=X(K)
      WRITE(6,709) Z(N)
      IF(F1) 1095,1110,1110
1095  SS=-10.0
      GO TO 48
1110  W1=1.0
      THETA=0.0
      GO TO 699
1120  IF(THETO-180.0) 1122,1121,1122
1121  W1=0.0
```

```
      GO TO 1331
 1122 W5=0.0
      WRITE(6,711) X(N)
      W6=0.0
      W7=0.0
      DO 1310 K=1,N1
      R=K
 1310 W5=W5+X(K)*EXP(-(R+.5)*EO)/E10(-1.,(2.*R+1.),EO)
      DO 1320 K=1,N1
      R=K
 1320 W6=W6+Z(K)*EXP(-(R+.5)*EO)/E10(-1.,(2.*R+1.),EO)
      DO 1330 K=1,MM
      R=K
 1330 W7=W7+AS(K)*R*(R+1.)*EXP(-(R+.5)*EO)/E10(-1.,(2.*R+1.),EO)
      W1=-ANGLE*(W5+X(N)*W7)/(W6+Z(N)*W7)
      WRITE(6,380) W1
 1331 THETA=0.0
  699 WRITE(6,42)
  700 Y3=COS(THETA*.01745329)
      Y7=ABS(COSH(EO)*Y3+1.)
      IF(COSH(EO)*Y3+1.) 900,950,950
  900 PHI=3.14159265-ACOS(Y7/(COSH(EO)+Y3))
      GO TO 975
  950 PHI=ACOS(Y7/(COSH(EO)+Y3))
  975 Y1=COS(PHI)
      Z1=SIN(PHI)
      Q=SQRT(CH(EO)-Y1)
      Y0=Z1/(CH(EO)-Y1)
      Z0=SH(EO)/(CH(EO)-Y1)
C     COMPUTE THE LEGENDRE POLYNOMIALS
      P(1)=Y1
      P(2)=1.5*Y1*Y1-.5
      P1(1)=1.
      P1(2)=3.*Y1
      DO 300 K=3,MM
      R=K
      P(K)=((2.*R-1.)*Y1*P(K-1)-(R-1.)*P(K-2))/R
  300 P1(K)=((2.*R-1.)/(R-1.))*Y1*P1(K-1)-(R/(R-1.))*P1(K-2)
      IF(F1) 1410,409,409
  409 SN20=0.0
      SN7=0.0
      SN9=0.0
      GO TO 410
 1410 SN20=0.0
      SN7=0.0
      SN9=0.0
      DO 1420 K=1,N1
      W3=ANGLE
      R=K
     SN20=SN20+  Q*Z1*X(K)*(1.5*SH(EO)/(R*(R+1.))+Q*Q*(R+.5)
     1 /(TH(R,EO)*R*(R+1.)))*P1(K)*W3
      SN7=SN7+G*X(K)*(((1.5*Z1*Z1-2.*Y1*Q*G)*P1(K)+R*(R+1.)*G*Q*P(K))
     1 *W3/(R*(R+1.))
 1420 SN9=SN9+G*Z1*X(K)*P1(K)/(R*(R+1.))*W3
```

```
      SN20=SN20-W3*Z1*(COSH(EO)*Y1-1.)/(Q**4)
      SN7=SN7-W3*Z1*Z1*SINH(EO)/(Q**4)
      SN9=SN9+W3*SINH(EO)*Z1/(Q**4)
410   SN2=0.0
      SN3=0.0
      SN4=0.0
      UA=RADIUS*W1*SIN(THETA*.01745329)
      DO 400 K=1,N1
      R=K
       SN2=SN2+  Q*Z1*Z(K)*(1.5*SH(EO)/(R*(R+1.))+Q*Q*(R+.5)
    1 /(TH(R,EO)*R*(R+1.)))*P1(K)*W1
      SN3=SN3+Q*Z(K)*((1.5*Z1*Z1-2.*Y1*Q*Q)*P1(K)+R*(R+1.)*Q*Q*P(K))*W1
    1 /(R*(R+1.))
400   SN4=SN4+Q*Z1*Z(K)*P1(K)/(R*(R+1.))*W1
      SN5=0.0
      SN6=0.0
      SN8=0.0
      SN10=0.0
      SN11=0.0
      SN12=0.0
      SN15=0.0
      SN16=0.0
      SN18=0.0
      DO 600 K=1,MM
      R=K
     SN6=SN6+Q*Z1*W1*AS(K)*Z(N)*P1(K)
      SN16=SN16+Q*Z1*AS(K)*X(N)*P1(K)
     SN8=SN8+Q*Z1*W1*AS(K)*Z(N)
    1 *(1.5*SH(EO)+Q*Q*(R+.5)/(TH(R,EO)))*P1(K)
      SN18=SN18+Q*Z1*AS(K)*X(N)
    1 *(1.5*SH(EO)+Q*Q*(R+.5)/(TH(R,EO)))*P1(K)
      SN15=SN15+Q*AS(K)*X(N)*((1.5*
    1Z1*Z1-2.*Y1*Q*Q)*P1(K)+R*(R+1.)*Q*Q*P(K))
600  SN5=SN5+Q*W1*AS(K)*Z(N)*((1.5*
    1Z1*Z1-2.*Y1*Q*Q)*P1(K)+R*(R+1.)*Q*Q*P(K))
      SN30=SN4+SN6
      SN31=SN9+SN16
      USI=SN30+SN31
      SN40=(SN3+SN5)*SINH(EO)/ANGLE
      SN41=(SN7+SN15)*SINH(EO)/ANGLE
      TTSI=SN40+SN41
      TNP=SN8*SINH(EO)/ANGLE
      IF(THETO) 629,613,629
629   IF(THETO-180.0) 619,721,619
619   IF(F) 621,625,625
621   IF(THETA-THETO) 622,612,721
622   SN10=.5*Q*Z1*W1          /(ANGLE*SQRT(Y1-Y5))*Z(N)
      SN12=.5*Q*Z1/(SQRT(Y1-Y5))*X(N)
      SN11=SN10*SINH(EO)/ANGLE
      GO TO 613
625   IF(THETA-THETO) 721,612,627
627   SN10=.5*Q*Z1*W1          /(ANGLE*SQRT(Y5-Y1))*Z(N)
      SN12=.5*Q*Z1/(SQRT(Y5-Y1))*X(N)
      SN11=SN10*SINH(EO)/ANGLE
```

```
      GO TO 613
612   SN10=SN8
      SN12=SN18
      SN11=SN10*SINH(EO)/ANGLE
613   IF(THETA) 722,721,722
722   IF(THETA-180.) 723,721,723
723   ERROR=(UA-USI)/USI*100.0
      GO TO 610
721   ERROR=0.0
610   SN50=(SN2+SN10)*SINH(EO)/ANGLE
      SN51=(SN20+SN12)*SINH(EO)/ANGLE
      TNSI=SN50+SN51
      WRITE(6,705) THETA,TNSI,USI,TTSI,ERROR,TNP,SN11
      THETA=THETA+DTHET
      IF(THETA-180.0) 700,700,710
710   CONTINUE
      STOP
      END
```

```
      SUBROUTINE SOLVE(A,X,B,N,M)
      DIMENSION A(M,M),X(M),B(M),K1(210),Y(210)
      DOUBLE PRECISION Y,D1
 1071 DO 61 I=1,N
   61 K1(I)=I
      NI=N-1
      DO 8 L=1,N1
C     SEARCH FOR LARGEST ELEMENT
      D=0.
      DO 17 L1=L,N
      DO 1 L2=L,N
      IF(ABS(A(L1,L2))-ABS(D)) 1,1,15
   15 D=A(L1,L2)
      ID=L1
      JD=L2
    1 CONTINUE
   17 CONTINUE
      IF(D) 2,99,2
C     INTERCHANGE ROWS AND COLUMNS TO PUT LARGEST ELEMENT ON DIAGONAL
    2 TEMP=K1(L)
      K1(L)=K1(JD)
      K1(JD)=TEMP
      DO 3 J=L,N
      TEMP=A(L,J)
      A(L,J)=A(ID,J)
    3 A(ID,J)=TEMP
      TEMP=B(L)
      B(L)=B(ID)
      B(ID)=TEMP
      B(L)=B(L)/D
      DO 4 I=1,N
      TEMP=A(I,L)
      A(I,L)=A(I,JD)
    4 A(I,JD)=TEMP
C     ELIMINATE ELEMENTS IN COLUMN UNDER LARGEST ELEMENT
      L1=L+1
      DO 5 J=L1,N
    5 A(L,J)=A(L,J)/D
      DO 7 I=L1,N
      IF(A(I,L)) 16,7,16
   16 D1=A(I,L)
      DO 6 J=L1,N
    6 A(I,J)=A(I,J)-D1*A(L,J)
      B(I)=B(I)-D1*B(L)
    7 CONTINUE
    8 CONTINUE
      IF(A(N,N)) 9,99,9
C     BACK SUBSTITUTE TO SOLVE
    9 Y(N)=B(N)/A(N,N)
      DO 63 L=1,N
      LL=N-L+1
      D1=B(LL-1)
      DO 62 J=LL,N
   62 D1=D1-A(LL-1,J)*Y(J)
```

```
   63  Y(LL-1)=D1
C      REORDER ANSWER
       DO 64 I=1,N
       J=K1(I)
   64  X(J)=Y(I)
       GO TO 65
   99  WRITE(6,100) L
  100 FORMAT(39HOMATRIX IS SINGULAR  NO SOLUTION GIVEN    I5)
       CALL EXIT
   65  RETURN
       END
```

TORSIONAL STRESSES IN AN INFINITE BODY CONTAINING
TWO PARTIALLY BONDED RIGID INCLUSIONS

6/16/66

```
  THETO= 90.0000
   ETAO=   .4436
     BE=   .4297
      W=  1.0000
  ANGLE=  1.0000
 RADIUS=  2.1822
   DIST=   .4364
SPACING=   .1000
     I1=  1.0000
     I2=  1.0000
     I3= -1.0000
      N= 50.0000
     MM=100.0000
     KK= 15.0000
     JJ=300.0000
```

THE PROBLEM CONSIDERED IS THAT OF ANTI-SYMMETRIC TORSION WITH
THE FREE SURFACE BEING FROM THETA EQUAL ZERO TO THETA EQUAL THETO
AND HAVING ZERO RESULTANT TORQUE ACTING ON THE INCLUSIONS

```
DEL1=        .46053986+01
DEL2=       -.14002492+02
```

W1= 1.4973

STRESSES AND DISPLACEMENTS ON THE INCLUSIONS

THETA	TNSI	USI	TTSI	DISP ERROR	SINGULAR STRESS
.00000000	.00000000	.00000000	.30473958-05	.00000000	.00000000 .00000000
.20000000+01	.59025032-01	.43494985-00	-.27551424-01	.00000000	-.11542268+00 .00000000
.40000000+01	.11243347+00	.85623606-00	-.97295047-01	.00000000	-.18106596-00 .00000000
.60000000+01	.15523092-00	.12556342+01	-.17729925-00	.00000000	-.17069957-00 .00000000
.80000000+01	.18344951-00	.16322812+01	-.23261468-00	.00000000	-.93526343-01 .00000000
.10000000+02	.19457681-00	.19918653+01	-.24297348-00	.00000000	.10762280-01 .00000000
.12000000+02	.18784630-00	.23429285+01	-.21302408-00	.00000000	.91803860-01 .00000000
.14000000+02	.16432615-00	.26918896+01	-.16954068-00	.00000000	.11339224+00 .00000000
.16000000+02	.12683900-00	.30394913+01	-.14686049-00	.00000000	.70970504-01 .00000000
.18000000+02	.79700816-01	.33806310+01	-.16783533-00	.00000000	-.70435832-02 .00000000
.20000000+02	.28288838-01	.37074332+01	-.23042735-00	.00000000	-.75035115-01 .00000000
.22000000+02	-.21518843-01	.40139226+01	-.30762196-00	.00000000	-.95029571-01 .00000000
.23999999+02	-.63982859-01	.42997387+01	-.36157464-00	.00000000	-.57487994-01 .00000000
.26000000+02	-.94169095-01	.45706780+01	-.36508884-00	.00000000	.13472947-01 .00000000
.27999999+02	-.10856056+00	.48355445+01	-.31873032-00	.00000000	.74002857-01 .00000000
.30000000+02	-.10555111+00	.51009255+01	-.25322217-00	.00000000	.86489186-01 .00000000
.32000000+02	-.85745686-01	.53668274+01	-.21423090-00	.00000000	.42902489-01 .00000000
.34000000+02	-.52010387-01	.56258107+01	-.23689912-00	.00000000	-.28947517-01 .00000000
.36000000+02	-.92411209-02	.58664700+01	-.32462211-00	.00000000	-.81605908-01 .00000000
.38000000+02	.36161283-01	.60793872+01	-.44586825-00	.00000000	-.78383247-01 .00000000
.40000000+02	.77081729-01	.62619852+01	-.55314145-00	.00000000	-.19329564-01 .00000000
.41999999+02	.10667224+00	.64189689+01	-.61402240-00	.00000000	.55600121-01 .00000000
.44000000+02	.11942735+00	.65578274+01	-.63427581-00	.00000000	.91631325-01 .00000000
.45999999+02	.11219960+00	.66821830+01	-.65503418-00	.00000000	.58552329-01 .00000000
.48000000+02	.84994735-01	.67879041+01	-.72187462-00	.00000000	-.22902424-01 .00000000
.50000000+02	.41372439-01	.68655515+01	-.84485628-00	.00000000	-.90665201-01 .00000000
.52000000+02	-.11691060-01	.69083126+01	-.98044485-00	.00000000	-.86402865-01 .00000000
.54000000+02	-.64581801-01	.69197128+01	-.10561104+01	.00000000	-.59938282-02 .00000000
.55999999+02	-.10646811+00	.69142080+01	-.10273815+01	.00000000	.87220585-01 .00000000
.58000000+02	-.12725063-00	.69081693+01	-.92506960-00	.00000000	.10774451+00 .00000000
.59999999+02	-.11980087+00	.69069510+01	-.84604491-00	.00000000	.25069697-01

.62000000+02	-.82092085-01	.68989865+01	-.87963676-00	.00000000	.00000000 -.93607806-01
.64000000+02	-.18741970-01	.68642866+01	-.10232519+01	.00000000	.00000000 -.12939188-00
.66000000+02	.58559800-01	.67922180+01	-.11716023+01	.00000000	.00000000 -.26921660-01
.68000000+02	.13218055-00	.66919900+01	-.12158767+01	.00000000	.00000000 .12485102+00
.70000000+02	.18145370-00	.65818171+01	-.11774359+01	.00000000	.00000000 .15377167-00
.72000000+02	.18721408-00	.64631670+01	-.12247582+01	.00000000	.00000000 -.88847557-02
.73999999+02	.13730145-00	.63082586+01	-.14981774+01	.00000000	.00000000 -.20026953-00
.76000000+02	.31852201-01	.60845076+01	-.18861789+01	.00000000	.00000000 -.15732444-00
.77999999+02	-.11322852+00	.57999633+01	-.20489454+01	.00000000	.00000000 .14362035-00
.80000000+02	-.26630285-00	.55134282+01	-.18042331+01	.00000000	.00000000 .32782258-00
.82000000+02	-.38479232-00	.52717269+01	-.15131294+01	.00000000	.00000000 .70889019-02
.84000000+02	-.42485896-00	.50261292+01	-.18554663+01	.00000000	.00000000 -.54175080-00
.86000000+02	-.35436326-00	.46482449+01	-.29404194+01	.00000000	.00000000 -.30652040-00
.88000000+02	-.16554771-00	.40063543+01	-.37445281+01	.00000000	.00000000 .13896581+01
.90000000+02	-.35412019+01	.35033324+01	-.29891926+01	-.67345292+01	.00000000 .36736920+01
.91999999+02	-.34227206+01	.31727149+01	-.86425924-00	.29216039+01	.36736920+01 .46269663+01
.94000000+02	-.20800793+01	.31793387+01	.75897223-00	.25194405+01	.38804262+01 .35457909+01
.96000000+02	-.14190698+01	.33042165+01	.58293168-00	-.16559499+01	.28323894+01 .20339682+01
.98000000+02	-.10495383+01	.33070004+01	-.32521212-00	-.21590251+01	.23873289+01 .18128723+01
.10000000+03	-.83945093-00	.32197341+01	-.31675286-00	-.61298812-01	.21343910+01 .22661176+01
.10200000+03	-.68301457-00	.31907324+01	.17814749-00	.16506135-00	.19709899+01 .19766515+01
.10400000+03	-.49491847-00	.31895284+01	.45028133-02	-.60148482-00	.18578093+01 .15066624+01
.10600000+03	-.24202034-00	.31370743+01	-.16990974-00	.11959501+00	.17761665+01 .17826113+01
.10800000+03	.41124254-01	.31059466+01	.21249803-00	.49430870-01	.17159251+01 .18341851+01
.11000000+03	.28192231-00	.31042597+01	.78066667-01	-.10923144+01	.16710597+01 .14447431+01
.11200000+03	.43576556-00	.30426455+01	-.21909316-00	-.43268746-00	.16377450+01 .16741302+01
.11400000+03	.53518510-00	.29860879+01	.98362464-01	-.39167043-01	.16134240+01 .16833256+01
.11600000+03	.66265398-00	.29505787+01	-.45005618-01	-.46970863-00	.15963173+01 .14256174+01
.11800000+03	.85514131-00	.28840167+01	.31146360-02	.32117912-01	.15851440+01 .17328146+01
.12000000+03	.10471370+01	.28512856+01	.12976923-00	-.75875007-00	.15789543+01 .14679164+01
.12200000+03	.11531634+01	.27809437+01	-.17216519-00	-.36071185-00	.15770241+01 .16404759+01
.12400000+03	.12066379+01	.27166997+01	.11389490+00	-.29087517-00	.15787858+01 .15528274+01
					.15837803+01

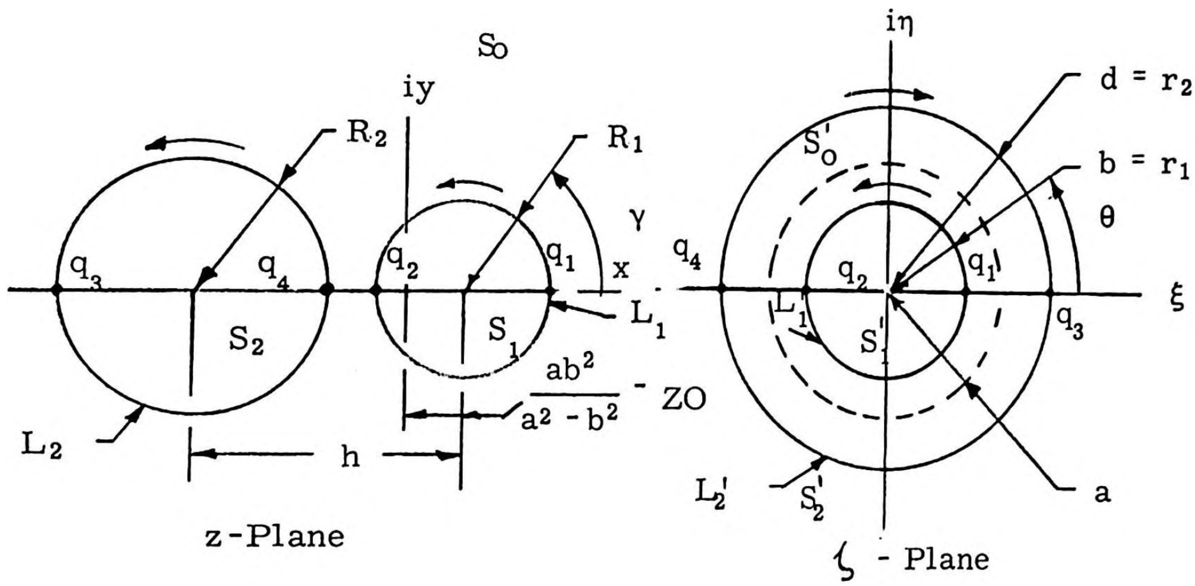
APPENDIX II

DESCRIPTION OF FORTRAN PROGRAM

Evaluation of Stresses Caused by the Transverse Shear Loading of a Solid Containing Two Circular Cylindrical Elastic Inclusions

Let an infinite elastic matrix contain two circular cylindrical inclusions of infinite length with the cylinder axes oriented normal to the x, y plane and the nonzero stress components at point remotely distant from the inclusions reducing to σ_{xp}= constant and σ_{yp} = constant. For this configuration the only nonvanishing displacement component is normal to the x, y plane and is a function of x and y only. Equilibrium conditions lead to the requirement that the displacement w be harmonic. For the chosen geometry it is appropriate to map the inclusions surfaces onto two concentric circles in the ζ -plane by use of the bilinear transformation. The matrix region then corresponds to the interior of an annulus while the two inclusions correspond to the regions exterior to the annulus as indicated in the figure below. The displacement w is representable by power series in each region with the coefficients in these series being evaluated by the required state of stress at $\sqrt{x^2 + y^2} = \infty$ and by the conditions of stress and displacement continuity between the inclusions and the matrix.

A solution of the problem was obtained by complex variable methods and a computer code prepared to account for two independent shear stress components at infinity, arbitrary values of inclusion spacing, ratio of inclusion radii, and different shear moduli of the matrix and each inclusion. The program yields as output the stresses



at the material interfaces and stresses in the matrix along a straight line drawn between the inclusion centers. Stresses can also be computed for a curvilinear grid in the x, y plane as defined by the mapping of curves corresponding to $|\zeta|$ = constant and $\arg(\zeta)$ = constant. Stresses are output as Cartesian components, curvilinear components, and resultant stress = $\sqrt{\sigma_{xp}^2 + \sigma_{yp}^2}$. No provision was made to compute stresses inside the inclusions. However, this generalization could readily be incorporated into the code if necessary.

I. Input Parameter Definition

Parameter	Definition
NN	NN specifies the number of sets of data the program will execute. The data deck consists of two cards.
L1, L2, L3	L1, L2, and L3, represent the month, day and year respectively.
DPHI	DPHI is the increment in angle in the ZETA-Plane. (The angle in the ZETA Plane is PHI and after one calculation this becomes PHI+DPHI etc.)
DR	DR is the increment in radius in the ZETA-Plane.
QZ	QZ is the radius of the inclusion on the left side of the y-axis in the z-plane
X	X is the spacing between inclusions as shown in the above figure.

Parameter	Definition
SOX, SOY	SOX and SOY are shear stresses applied at infinity.
C	C is the maximum error in summing the series for the stresses as defined by the equation $\sigma_{xp} - i\sigma_{yp} = G(\zeta) + \sum_{n=0}^{N} f_n(\zeta) + R_N$
where	$\lvert R_N \rvert \leq C \left\lvert \sum_{n=0}^{N} f_n(\zeta) \right\rvert$.
UO, Ul, U2	UO, Ul, and U2 are the shear moduli of the three corresponding regions.
T	T is a control quantity such that if T = zero the program will only compute stresses along the surface of the inclusions and on the negative x-axis between the inclusions. If T = positive the program will compute the above and then compute stresses in the matrix by incrementing R and PHI.

II. Input Card Listing

Card No.	Parameter	Data Field	Format
1	NN	1-5	I5
	L1	6-7	I2
	L2	8-9	I2
	L3	10-11	I2
2	DPHI	1-7	F7.3
	DR	8-14	F7.3
	QZ	15-21	F7.3
	X	22-28	F7.3
	SOX	29-35	F7.3
	SOY	36-42	F7.3
3	C	1-6	F6.5
	U 0	7-15	F9.0
	Ul	16-24	F9.0

Card No.	Parameter	Data Field	Format
	U2	25-33	F9.0
	T	34-42	F9.0

III. Output of Program

A. Repeated Input Data

B. The output of the stresses is in a two line grouping such that the first line of the heading corresponds to the first line of the output and the second line of the heading corresponds to the second line of the output. The location of the points in the z-plane at which the stresses are computed are given by the curvilinear coordinates RADIUS and ALPHA with the origin of this curvilinear coordinate system being on the real axis a distance ZO from the origin of the z-plane.

Stresses are computed first on the surface of inclusion 1, then on the surface of inclusion 2 and finally along the line connecting the centers of the inclusions. If the control parameter T is positive, stresses will then be computed at points on a curvilinear grid in the matrix.

C. The output quantities are interpreted as follows.

RADIUS (z-plane) = R as shown on the figure.

RADIUS (zeta-plane) = r as shown on the figure.

GAMA = Curvilinear coordinate angle in the z-plane.

PHI = Curvilinear coordinate angle in the ζ -plane.

SX = σ_{xp}

SY = σ_{yp}

SR = σ_{Rp}

ST = $\sigma_{\gamma p}$

RESULTANT = $(\sigma_{xp}^2 + \sigma_{yp}^2)^{\frac{1}{2}}$

PSI = $\text{Tan}^{-1}(\sigma_{xp}/\sigma_{yp})$

FN = N = The number of terms required in the series such

that $|R_N| \leq C \sum_{n=o}^{N} |f_n(\zeta)|$

ZO = The location of the center of the curvilinear coordinates

with respect to the origin of the z-plane.

IV. Program Listing

A listing of the program is given on pages 76 to 78 and a sample problem is given on page 79. Only partial results, sufficient for checking operation, are given for the sample problem.

```
C       EVALUATION OF STRESSES CAUSED BY THE TRANSVERSE SHEAR LOADING
C       OF A SOLID CONTAINING TWO CIRCULAR CYLINDRICAL ELASTIC INCLUSIONS
        COMPLEXSXY,SXYC,SRT,SRTC,SO,SOC,Z,ZETA,SN2,SN1,RN
        ASIN(X)=ATAN(X/SQRT(1.-X**2))
        READ(5,2)NN,L1,L2,L3
        DO 50 NE=1,NN
         READ(5,3)DPHI,DR,QZ,X,SOX,SOY
        READ(5,7)C,U0,U1,U2,T
        WRITE(6,4)L1,L2,L3,QZ,X,SOX,SOY,C,U0,U1,U2
  2     FORMAT(I5,3I2)
  3     FORMAT(6F7.3)
  7     FORMAT(F6.5,4F9.0)
  4     FORMAT(//////7X47HEVALUATION OF STRESSES CAUSED BY THE TRANSVERSE
       1/11X39HSHEAR LOADING OF A SOLID CONTAINING TWO/11X39HCIRCULAR CYLI
       2NDRICAL ELASTIC INCLUSIONS//
       2                    23X12,1H/I2,1H/I2//7X4H QZ=F7.3/7X4H  X=F7.3/
       37X4HSOX=F7.3/7X4HSOY=F7.3/7X4H  C=F6.5/7X4H U0=F9.0/7X4H U1=F9.0/
       37X4H U2=F9.0////
       340X53HSTRESSES AT THE INTERFACE OF THE INCLUSIONS AND ALONG/
       445X42HTHE NEGATIVE X AXIS BETWEEN THE INCLUSIONS//2X16HRADIUS (Z P
       5LANE),11X5H GAMA,16X2HSX,18X2HSY,15X9HRESULTANT,14X2HFN/1X19HRADIU
       6S (ZETA PLANE),10X3HPHI,17X2HSR,18X2HST,17X3HPSI,19X2HZO////)
 132    FORMAT(6E20.8/6E20.8//)
  48    FORMAT(///40X22HSTRESSES IN THE MATRIX//2X16HRADIUS'(Z'PLANE),11X
       15HALPHA,16X2HSX,18X2HSY,15X9HRESULTANT,14X2HFN/1X19HRADIUS'(ZETA'P
       2LANE),10X3HPHI,17X2HSR,18X2HST,17X3HPSI,19X2HZO//////)
        EL=1.+QZ+X
        AL=EL/(1.+QZ)
        BE=1./QZ
        A=SQRT((1.-QZ**2)**2-2.*(EL**2)*(1.+QZ**2)+EL**4)/EL
         P=A*(SQRT(A*A+4.)-A)/2.
        Q=A*(SQRT(A*A+4.*QZ*QZ)+A)/(2.*QZ)
        F=SQRT((BE-1.)**2-2.*(AL**2)*(BE**2+1.)+(AL**4)+(BE+1.)**2)/AL
         G=F*(SQRT(F**2+4.*BE**2)-F)/(2.*BE)
        H=F*(SQRT(F**2+4.)+F)/2.
        SOC=CMPLX(SOX,SOY)
        SO=CONJG(SOC)
        AB=SQRT(SOX**2+SOY**2)
        D1=U1/U0+1.
        D2=U2/U0+1.
        D3=U1/U0-1.
        D4=U2/U0-1.
        DA=ABS(D3)
        DB=ABS(D4)
        V=0.
        R=P
  14    W=0.
  15    PHI=0.
  16    SN1=CMPLX(0.,0.)
        N=1
        X1=COS(PHI*.01745329)
        Y1=SIN(PHI*.01745329)
        ZETA=CMPLX(R*X1,R*Y1)
  18    FN=N
```

```
      SN2=SN1+(FN/(D1*D2-D3*D4*(G/H)**(2*N)))*(SOC*D1*D4*(F*F)/(H*H)*((
     1ZETA*H)/(QZ*H*H))**(N-1)+SO*D3*D4*(G*G)/(H*H)*((ZETA*G*G)/(QZ*F*H
     2*H))**(N-1)+SOC*D2*D3*(((QZ*G)/ZETA)**2)*((QZ*G*G)/(ZETA*F))**(N-1
     3)+SO*D3*D4*((QZ*G*F)/(ZETA*H))**2*((QZ*G*G*F)/(ZETA*H*H))**(N-1))
      W1=AB *D1*D4*((F/H)**2)*(FN*((R    *F)/(QZ*H*H))**(N-1)-(FN-1.)*((
     1R    *F)/(QZ*H*H))**N)/((1.-(R    *F)/(QZ*H*H))**2)
      W2=AB*DA*DB*((G/H)**2)*(FN*((R    *G*G)/(QZ*F*H*H))**(N-1)-(FN-1.)
     1*((R    *G*G)/(QZ*F*H*H))**N)/((1.-(R    *G*G)/(QZ*F*H*H))**2)
      W3=AB *D2*DA* (((QZ*G)/(R    ))**2)*(FN*((QZ*G*G)/(R    *F))**(N-1)
     1-(FN-1.)*((QZ*G*G)/(R    *F))**N)/((1.-(QZ*G*G)/(R    *F))**2)
      W4=AB*DA*DB*(((QZ*F*G)/(R    *H))**2)*(FN*((QZ*F*G*G)/(R    *H*H))*
     1*(N-1)-(FN-1.)*((QZ*F*G*G)/(R    *H*H))**N)/((1.-(QZ*F*G*G)/(R    *H
     2*H))**2)
      RN=(1./(D1*D2-DA*DB*(G/H)**(2*N)))*(W1+W2+W3+W4)
   22 IF(REAL(CABS(RN))-REAL(CABS(C*SN2))) 26,26,24
   24 SN1=SN2
      N=N+1
      GO TO 18
C     COMPUTATION OF STRESSES
   26 SXYC=SO*(((ZETA-A)**2)/(A*A))*SN2
      SRTC=ZETA*(CABS((ZETA-A)**2))*SXYC/(R*((ZETA-A)**2))
      SXY=CONJG(SXYC)
      SRT=CONJG(SRTC)
      SX=REAL(SXY)
      SY=AIMAG(SXY)
      SR=REAL(SRT)
      ST=AIMAG(SRT)
       RES=REAL(CABS(SXY))
      IF(SX) 106,103,100
  100 PSI=(ATAN(SY/SX))*57.2957795
      GO TO 109
  103 IF(SY) 105,104,104
  104 PSI=90.0
      GO TO 109
  105 PSI=-90.0
      GO TO 109
  106 PSI=180.0+(ATAN(SY/SX))*57.2957795
  109 IF(R-A) 110,110,120
  110 IF(X1-R/A) 116,113,111
  111 THETA=(ATAN(Y1/(X1-R/A)))*57.2957795
      GO TO 121
  113 IF(Y1) 115,114,114
  114 THETA=90.0
      GO TO 121
  115 THETA=-90.0
      GO TO 121
  116 THETA=180.0+(ATAN(Y1/(X1-R/A)))*57.2957795
      GO TO 121
  120 THETA=180.0+(ATAN(Y1/(X1-R/A)))*57.2957795
  121 X2=COS(THETA*.01745329)
      Y2=SIN(THETA*.01745329)
      ZO=A*R*R/(A*A-R*R)
      RZ=A*R/(SQRT(A*A+R*R-2.*A*R*X1))
      IF(R-A) 122,122,126
```

```
122  R1=R/(1.-(R*R)/(A*A))
     IF(RZ*X2-ZO) 124,123,123
123  ALPHA=(ASIN(RZ*Y2/R1))*57.2957795
     GO TO 130
124  ALPHA=180.0-(ASIN(RZ*Y2/R1))*57.2957795
     GO TO 130
126  R1=R/((R*R)/(A*A)-1.)
     IF(RZ*X2-ZO) 128,128,127
127  ALPHA=(ASIN(RZ*Y2/R1))*57.2957795
     GO TO 130
128  ALPHA=180.0-(ASIN(RZ*Y2/R1))*57.2957795
130  WRITE(6,132)R1,ALPHA,SX,SY,RES,FN,R,PHI,SR,ST,PSI,ZO
     IF(W) 28,28,35
 28  PHI=PHI+DPHI
     IF(PHI-180.0) 16,16,29
 29  IF(V) 30,30,41
 30  IF(R-Q) 31,32,32
 31  R=Q
     GO TO 15
 32  PHI=180.0
     W=10
     R=P
     GO TO 16
 35  R=R+DR
     IF(Q-R) 39,16,16
 39  IF(T) 50,50,40
 40  V=10
     WRITE(6,48)
      R=P
41    R=R+DR
     IF(R-A) 42,41,42
 42  IF(R-Q) 14,50,50
 50  CONTINUE
     STOP
     END
```

EVALUATION OF STRESSES CAUSED BY THE TRANSVERSE
SHEAR LOADING OF A SOLID CONTAINING TWO
CIRCULAR CYLINDRICAL ELASTIC INCLUSIONS

6/16/66

```
 QZ=  1.000
  X=   .100
SOX=  1.000
SOY=   .000
  C=.00100
 U0=       10.
 U1=10000000.
 U2=10000000.
```

STRESSES AT THE INTERFACE OF THE INCLUSIONS AND ALONG
THE NEGATIVE X AXIS BETWEEN THE INCLUSIONS

RADIUS (Z PLANE)	GAMA	SX	SY	RESULTANT	FN
RADIUS (ZETA PLANE)	PHI	SR	ST	PSI	ZO
.99999993-00	.00000000	.23281347+01	-.00000000	.23281347+01	.29000000+02
.46732801-00	.00000000	.23281347+01	-.00000000	.00000000	.72984374-00
.99999993-00	.31230404+02	.17594026+01	.10669859+01	.20576580+01	.30000000+02
.46732801-00	.50000000+01	.20576578+01	-.13594149-03	.31234625+02	.72984374-00
.99999993-00	.58515132+02	.75132206-00	.12273823+01	.14390803+01	.30000000+02
.46732801-00	.10000000+02	.14390802+01	.31744064-03	.58527794+02	.72984374-00
.99999993-00	.80260841+02	.13536637-00	.78816269-00	.79970274-00	.32000000+02
.46732801-00	.15000000+02	.79970273-00	-.88559056-04	.80254570+02	.72984374-00
.99999993-00	.96937080+02	-.34355789-01	.28245331-00	.28453504-00	.33000000+02
.46732801-00	.20000000+02	.28453503-00	-.99158962-05	.96935022+02	.72984374-00
.99999993-00	.10967412+03	.32580554-01	-.91482047-01	.97110541-01	.34000000+02
.46732801-00	.25000000+02	-.97110463-01	.12062998-03	-.70397074+02	.72984374-00
.99999993-00	.11952860+03	.19199689-00	-.33849712-00	.38915692-00	.34999999+02
.46732801-00	.30000000+02	-.38915685-00	-.22746553-03	-.60437923+02	.72984374-00
.99999993-00	.12729974+03	.38143146-00	-.50112269-00	.62977289-00	.36000000+02
.46732801-00	.34999999+02	-.62977283-00	.25296320-03	-.52723283+02	.72984374-00
.99999993-00	.13355321+03	.58457398-00	-.61462156-00	.84822544-00	.37000000+02
.46732801-00	.40000000+02	-.84822539-00	-.16969126-03	-.46435326+02	.72984374-00
.99999993-00	.13868342+03	.79765090-00	-.70114525-00	.10620036+01	.38000000+02
.46732801-00	.45000000+02	-.10620036+01	-.13087474-04	-.41315878+02	.72984374-00
.99999993-00	.14296698+03	.10213883+01	-.77075621-00	.12795699+01	.38999999+02
.46732801-00	.50000000+02	-.12795699+01	.12901650-03	-.37038798+02	.72984374-00
.99999993-00	.14660077+03	.12575853+01	-.82901265-00	.15062479+01	.38999999+02
.46732801-00	.55000000+02	-.15062478+01	-.15681523-03	-.33393264+02	.72984374-00
.99999993-00	.14972735+03	.15055109+01	-.87896588-00	.17433141+01	.38999999+02
.46732801-00	.59999999+02	-.17433140+01	.15565050-03	-.30277761+02	.72984374-00
.99999993-00	.15245186+03	.17649288+01	-.92048228-00	.19905429+01	.38999999+02
.46732801-00	.65000000+02	-.19905429+01	-.14815113-03	-.27543876+02	.72984374-00
.99999993-00	.15485317+03	.20342048+01	-.95518019-00	.22473002+01	.38000000+02
.46732801-00	.70000000+02	-.22473001+01	.23627602-03	-.25152846+02	.72984374-00
.99999993-00	.15699148+03	.23122302+01	-.98159865-00	.25119603+01	.38000000+02
.46732801-00	.75000000+02	-.25119601+01	-.26717360-03	-.23002416+02	.72984374-00
.99999993-00	.15891341+03	.25961949+01	-.10013792+01	.27826226+01	.38000000+02
.46732801-00	.80000000+02	-.27826224+01	.27004462-03	-.21092141+02	.72984374-00
.99999993-00	.16065560+03	.28846788+01	-.10124241+01	.30571840+01	.38000000+02
.46732801-00	.84999999+02	-.30571838+01	-.27014674-03	-.19339337+02	.72984374-00
.99999993-00	.16224720+03	.31754229+01	-.10168834+01	.33342709+01	.38000000+02
.46732801-00	.90000000+02	-.33342709+01	.23977415-03	-.17756914+02	.72984374-00
.99999993-00	.16371174+03	.34657522+01	-.10124704+01	.36106141+01	.38000000+02
.46732801-00	.95000000+02	-.36106140+01	-.20657361-03	-.16284975+02	.72984374-00

.99999993-00	.16506838+03	.37545543+01	-.10013700+01	.38857996+01	.38000000+02
.46732801-00	.10000000+03	-.38857996+01	.14476730-03	-.14933743+02	.72984374-00
.99999993-00	.16633293+03	.40379820+01	-.98180856-00	.41556283+01	.38000000+02
.46732801-00	.10500000+03	-.41556282+01	-.64058194-04	-.13665903+02	.72984374-00
.99999993-00	.16751053+03	.43160262+01	-.95537634-00	.44205005+01	.38000000+02
.46732801-00	.11000000+03	-.44205005+01	.14746519-05	-.12481478+02	.72984374-00
.99999993-00	.16863625+03	.45840503+01	-.92136247-00	.46757273+01	.38000000+02
.46732801-00	.11500000+03	-.46757272+01	.73502923-04	-.11364639+02	.72984374-00
.99999993-00	.16969548+03	.48427651+01	-.88031054-00	.49221256+01	.38000000+02
.46732801-00	.12000000+03	-.49221255+01	-.16040641-03	-.10302643+02	.72984374-00
.99999993-00	.17070426+03	.50077741+01	-.83225358-00	.51553943+01	.37000000+02
.46732801-00	.12500000+03	-.51553942+01	-.50529522-03	-.92901138+01	.72984374-00
.99999993-00	.17166957+03	.53185038+01	-.77927801-00	.53752913+01	.37000000+02
.46732801-00	.13000000+03	-.53752912+01	.50175004-03	-.83357791+01	.72984374-00
.99999993-00	.17259746+03	.55328193+01	-.71833542-00	.55792557+01	.37000000+02
.46732801-00	.13500000+03	-.55792555+01	-.49728004-03	-.73974300+01	.72984374-00
.99999993-00	.17349331+03	.57246164+01	-.65395975-00	.57668160+01	.37000000+02
.46732801-00	.13999999+03	-.57668158+01	.47219788-03	-.65113761+01	.72984374-00
.99999993-00	.17436189+03	.59062319+01	-.58262977-00	.59348994+01	.37000000+02
.46732801-00	.14500000+03	-.59348994+01	-.44451945-03	-.56338073+01	.72984374-00
.99999993-00	.17520751+03	.60631134+01	-.50873321-00	.60844189+01	.37000000+02
.46732801-00	.15000000+03	-.60844188+01	.39942604-03	-.47962404+01	.72984374-00
.99999993-00	.17603408+03	.61966806+01	-.42925698-00	.62115305+01	.37000000+02
.46732801-00	.15499999+03	-.62115304+01	-.35161345-03	-.39626676+01	.72984374-00
.99999993-00	.17684520+03	.63088498+01	-.34801573-00	.63184412+01	.37000000+02
.46732801-00	.16000000+03	-.63184412+01	.28936400-03	-.31574129+01	.72984374-00
.99999993-00	.17764424+03	.63953511+01	-.26287168-00	.64007513+01	.37000000+02
.46732801-00	.16500000+03	-.64007512+01	-.22518686-03	-.23537353+01	.72984374-00
.99999993-00	.17843437+03	.64593478+01	-.17669936-00	.64617641+01	.37000000+02
.46732801-00	.16999999+03	-.64617641+01	.15212861-03	-.15669699+01	.72984374-00
.99999993-00	.17921863+03	.64962165+01	-.08518575-01	.64968196+01	.37000000+02
.46732801-00	.17500000+03	-.64968193+01	-.77460862-04	-.78067393-00	.72984374-00

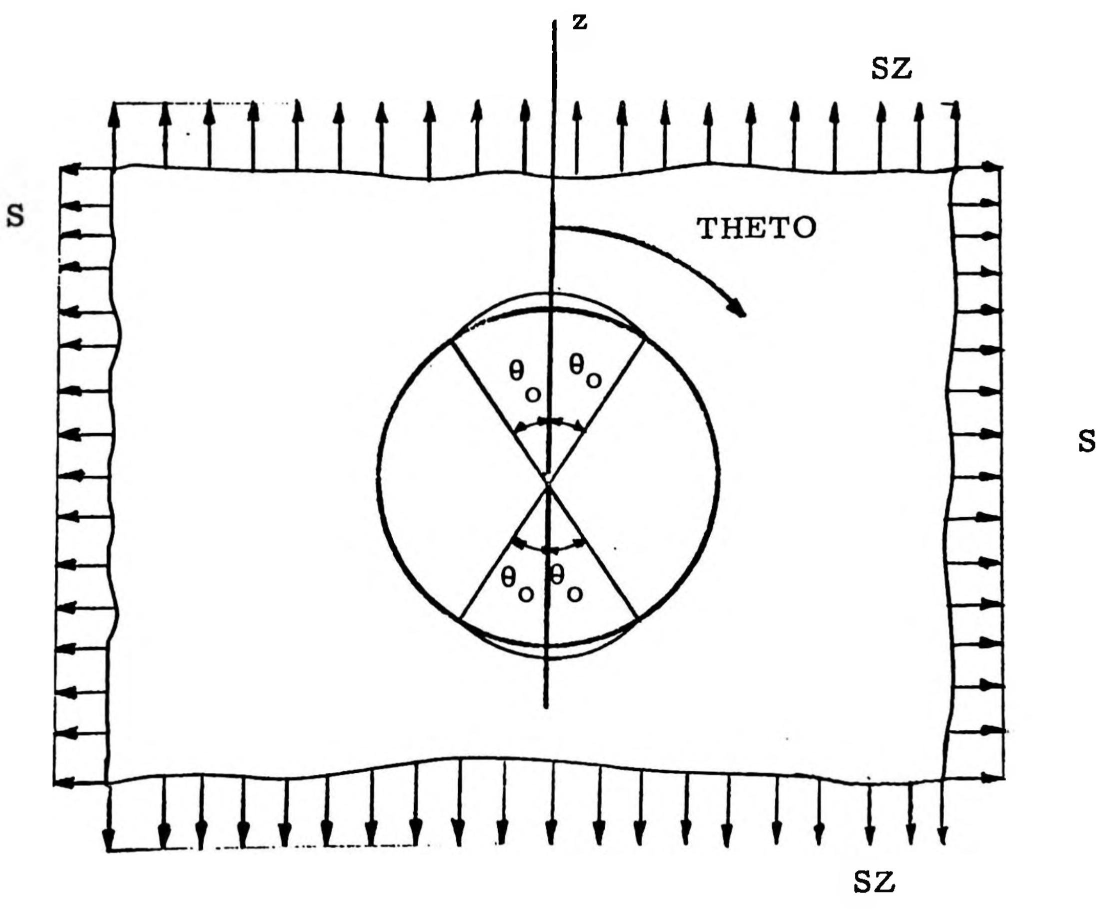
APPENDIX III

DESCRIPTION OF FORTRAN PROGRAM

Axisymmetric Contact Stresses Around a Smooth Elastic Sphere in an Infinite Solid, Uniformly Stressed at Infinity

I. Purpose

This program determines the contact angle θ_o, the normal stresses σ_r on the contact arc, and the circumferential stresses σ_θ and σ_ϕ around the spherical cavity for the axisymmetric configuration shown below.



II. Input Parameter Definition

Parameter	Definition
NN	NN specifies the number of sets of data the program will execute. The data deck consists of one card.
LA, LB, LC	LA, LB, and LC are the month, day, and year respectively.
JJ	JJ specifies the number of terms taken in the approximating functions. i.e. $N^k(\theta) = A_k \sum_{n=1}^{JJ} \sigma_n^k P_{2n}(p)$
MM	MM is a parameter such that if MM = + 1 the program will iterate to determine a contact angle, and if MM=-1 the superposition constants will be determined based on the given value of contact angle and the corresponding stresses computed.
I1	I1 specifies whether the problem considered is cap contact or band contact. I1 must be ± 1, with +1 and -1 corresponding to polar and equatorial contact respectively.
MS	MS specifies the number of intervals to be taken in the numerical integration used in evaluating the A_k for the case of equatorial contact.
S, SZ	S and SZ are the applied stress components at infinity.
VO, V1	VO and V1 are Poisson's ratio for the inclusion and the solid respectively.
BEO	BEO is the initial estimate for the contact angle and should be smaller than the correct value in the case of cap contact and larger than the correct value in the case of band contact.

DTHET — DTHET is the increment in the polar angle at which the stresses and displacement are computed.

DE — DE = μ_1 / μ_o where μ_1 and μ_o are the shear moduli of the space and inclusion respectively.

III. Input Card Listing

Card No.	Parameter	Data Field	Format
1	NN	1–5	I5
	LA	6-7	I2
	LB	8-9	I2
	LC	10-11	I2
2	JJ	1-5	I5
	MM	6-10	I5
	I1	11-15	I5
	MS	16-20	I5
	S	21-27	F7.4
	SZ	28-34	F7.4
	VO	35-41	F7.4
	V1	42-48	F7.4
	BEO	49-55	F7.4
	DTHET	56-62	F7.4
	DE	63-71	F9.4

IV. Output of Program

A. Repeated Input Data

B. A statement, depending on the value of I1, specifying the problem considered.

C. A(K) = A_k , where the A_k are the superposition constants. There are six constants in the case of cap contact and four in the case of band contact.

D. THETO, where THETO is the computed contact angle if MM = +1 and is the input value BEO if MM = -1.

E. Tabulation of the boundary stresses and displacement error on the cavity surface.

TR = N (θ)

TTHET = σ_θ

TGAMA = σ_ϕ

V. Program Listing

A listing of the program is given on pages 85 to 94 and a sample problem is given on page 95 .

```
C       AXISYMMETRIC CONTACT STRESSES AROUND A SMOOTH ELASTIC SPHERE
C            IN AN INFINITE SOLID, UNIFORMLY STRESSED AT INFINITY
        DIMENSION P(400),PO(400),A(51,51),B(51),X(51),SN(51),PS(400,75),
      1 YS(400),GAMA1(400),GAMA3(400),P1(400)
        COMMON A,B,X,N,RS1
        GO(G)=SQRT(2.)*( SIN((G-.5)*BI)/(2.*G-1.)- SIN((G+1.5)*BI)/(2.*G
      1 +3.))
        GN(G)=(8.*G*G*(1.-VO)+2.*VO*G-(1.-2.*VO))*DE/((2.*G-1.)*
      1 (4.*G*G+2.*G
      2   +1.+VO*(4.*G+1.)))+((1.+8.*G*G)*(1.-V1)+2.*G*(4.-5.*V1))/
      3 (2.*(G+1.)*(4.*G*G+2.*G+1.-V1*(4.*G+1.)))
        GN1(G)=(8.*G*G-5.*G+4.*G*V1-(1.-V1))
        GN2(G)=(4.*G*G+2.*G+1.-(4.*G+1.)*V1)
        READ(5,20) NN,LA,LB,LC
        DO 710 NE=1,NN
        READ(5,30)      JJ,MM,I1,MS,S,SZ,VO,V1,BEO,DTHET,DE
        N=6
        M=51
        F1=N
        F2=JJ
        F3=MS
        MS1=MS+1
        WRITE(6,40) LA,LB,LC,S,SZ,VO,V1,DE,F2,F3
   20   FORMAT(I5,3I2)
   30   FORMAT(4I3,6F7.4,F9.4)
   40   FORMAT(/////7X60HAXISYMMETRIC CONTACT STRESSES AROUND A SMOOTH EL
      1ASTIC SPHERE/11X52HIN AN INFINITE SOLID, UNIFORMLY STRESSED AT INF
      2INITY//
      3 23XI2,1H/I2,1H/I2//11X2HS=F8.4/10X3HSZ=F8.4/10X3HVO=F8.4/
      4 10X3HV1=F8.4/10X3HDE=F8.4/             10X3HJJ=F8.4/10X3HMS=F8.4/)
  704   FORMAT(7X6HTHETO=F8.4//)
  701 FORMAT(////16X37HSTRESSES AND DISP ERROR ON THE CAVITY//
      1 10X5HTHETA,18X2HTR,14X10HDISP ERROR,11X5HTTHET,15X5HTGAMA//)
  700   FORMAT(5E20.8/)
   49   FORMAT(7X2HA(,I1,2H)=F8.4//)
  652   FORMAT(24X22HTR DID NOT CHANGE SIGN)
   45   FORMAT(7X36HCONTACT EXISTS IN THE FORM OF A BAND//)
   46   FORMAT(7X44HCONTACT EXISTS IN THE FORM OF SPHERICAL CAPS//)
  607   FORMAT(7X3HBE=E20.8//)
        E=I1
        IF(E) 43,43,42
   43   N=4
        WRITE(6,45)
        GO TO 47
   42   WRITE(6,46)
   47   DBE=2.0
        J3=JJ+JJ+6
        BE=BEO
        SS1=5.
        SS2=10.0
        SS3=10.0
        SS4=10.0
        SS5=3.0
        RS1=10.0
        T=MM
```

```
      R1=N-1
      J1=JJ+1
      N1=N+1
      N2=N+2
   41 C2=0.0
      SM2=0.0
  250 BI=DE*.01745329
      Y1= COS(BI)
      B3=Y1/F3
      Y5=SQRT(1.-Y1)
      PO(1)=1.5*Y1*Y1-.5
      PO(2)=Y1
      PO(3)=1.
      PO(4)=1.
      PO(5)=Y1
      DO 300 J7=2,J3
      J=J7+4
      R=J7
  300 PO(J)=((2.*R-1.)*Y1*PO(J-1)-(R-1.)*PO(J-2))/R
      IF(E) 305,305,301
  305 YS(1)=0.0
      DO 1150 K=1,MS1
      PS(1,K)=1.5*YS(K)*YS(K)-.5
      PS(2,K)=YS(K)
      PS(3,K)=1.0
      PS(4,K)=1.0
      PS(5,K)=YS(K)
      DO 1130 J7=2,J3
      J=J7+4
      R=J7
 1130 PS(J,K)=((2.*R-1.)*YS(K)*PS(J-1,K)-(R-1.)*PS(J-2,K))/R
 1150 YS(K+1)=YS(K)+B3
      DO 1309 L1=1,J1
      L=L1+2
      R=L1
      GAMA1(2*L-2)=0.0
      GAMA3(2*L-2)=0.0
      DO 1310 K=1,MS
      B1=SQRT(Y1**2-YS(K+1)**2)
      B2=SQRT(Y1**2-YS(K)**2)
      B3=YS(K+1)-YS(K)
      B5=(B1-B2)/B3
      B6=(B2*YS(K+1)-B1*YS(K))/B3
      B7=(B1**3-B2**3)/B3
      B8=((B2**3)*YS(K+1)-(B1**3)*YS(K))/B3
      B9=(B5*YS(K)+B6)*(PS(2*L-1,K)-PS(2*L-3,K))
      B10=(B5*YS(K+1)+B6)*(PS(2*L-1,K+1)-PS(2*L-3,K+1))
      B11=(B7*YS(K+1)+B8)*(PS(2*L-1,K+1)-PS(2*L-3,K+1))
      B12=(B7*YS(K   )+B8)*(PS(2*L-1,K   )-PS(2*L-3,K   ))
       GAMA1(2*L-2)=GAMA1(2*L-2)+B10-B9-B5*((PS(2*L,K+1)-PS(2*L,K))/
     1 (4.*R-1.)+(PS(2*L-4,K+1)-PS(2*L-4,K))/(4.*R-5.)-(8.*R-6.)*(
     2 PS(2*L-2,K+1)-PS(2*L-2,K))/((4.*R-5.)*(4.*R-1.)))
 1310 GAMA3(2*L-2)=GAMA3(2*L-2)+B11-B12-B7*((PS(2*L,K+1)-PS(2*L,K))/
     1 (4.*R-1.)+(PS(2*L-4,K+1)-PS(2*L-4,K))/(4.*R-5.)-(8.*R-6.)*(
```

```
      2 PS(2*L-2,K+1)-PS(2*L-2,K))/((4.*R-5.)*(4.*R-1.)))
 1309   CONTINUE
 1301   THETA=90.0
C       CONSTRUCT THE COEFFICIENT MATRIX FOR HAND CONTACT
        DO 1410 J=1,N
        Y2=COS(THETA*.01745329)
        P(4)=1.
        P(5)=Y2
        DO 1400 L1=2,J3
        L=L1+4
        R=L1
 1400   P(L)=((2.*R-1.)*Y2*P(L-1)-(R-1.)*P(L-2))/R
        DO 1445 L=1,N
 1445   SN(L)=0.0
        SN(1)=GN(0.)
        SN(3)=((Y1**2-1./3.)*GN(0.0)-2./3.*GN(1.)*P(6))/(Y1**2)
        DO 1450 L1=1,J1
        L=L1+2
        R=L1
        SN(1)=SN(1)-GN(R-1.)*(PO(2*L-3  )-PO(2*L-1  ))*P(2*L-2  )
        SN(2)=SN(2)+GN(R-1.)*GAMA1(2*L-2)*P(2*L-2)/Y1
        DN1=  2.*Y1*GN(R-1.)*(PO(2*L-4  )/(4.*R-5.)+PO(2*L  )/(4.*R-1.)
      1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
        DN2=         GN(R-1.)*2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
      1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
      2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
        SN(3)=SN(3)+DN1+DN2
 1450   SN(4)=SN(4)+GN(R-1.)*GAMA3(2*L-2)*P(2*L-2)/(Y1**3)
        DO 1452 L=1,N
 1452   A(J,L)=SN(L)
        B(J)=        .5*(1.-V1)*(2.*S+S2)/(1.+V1)-10.*(1.-V1)*(S-S2)/(7.-
      1 5.*V1)*P(6)
 1410   THETA=THETA-(90.0-BE)/R1
        CALL SOLVE(A,X,B,N,S1,RS1)
        IF(RS1) 1407,1407,1412
 1407   WRITE(6,607) BE
        GO TO 710
 1412   IF(SS2) 650,650,1409
 1409   IF(T) 650,650,1411
 1411   DO 1455 L=1,N
 1455   SN(L)=0.0
        SN(1)=1.
        SN(3)=1.-(Y2**2)/(Y1**2)
        DO 1460 L1=1,J1
        L=L1+2
        R=L1
        SN(1)=SN(1)-         (PO(2*L-3  )-PO(2*L-1  ))*P(2*L-2  )
        SN(2)=SN(2)+         GAMA1(2*L-2)*P(2*L-2)/Y1
        DN1=  2.*Y1*         (PO(2*L-4  )/(4.*R-5.)+PO(2*L  )/(4.*R-1.)
      1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
        DN2=                 2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
      1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
      2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
        SN(3)=SN(3)+DN1+DN2
 1460   SN(4)=SN(4)+         GAMA3(2*L-2)*P(2*L-2)/(Y1**3)
```

```
      GO TO 461
  301 THETA=0.0
C     CONSTRUCT THE COEFFICIENT MATRIX
      DO 410 J=1,M
     Y2= COS(THETA*.01745329)
      P(4)=1.
      P(5)=Y2
      DO 400 L1=2,J3
      L=L1+4
      R=L1
  400 P(L)=((2.*R-1.)*Y2*P(L-1)-(R-1.)*P(L-2))/R
      DO 445 L=1,N
  445 SN(L)=0.0
      DO 450 L1=1,J1
      L=L1+2
      R=L1
      SN(1)=SN(1)+GN(R-1.)*(PO(2*L-3  )-PO(2*L-1  ))*P(2*L-2  )
      SN(2)=SN(2)+GN(R-1.)*GO(2.*R-2.)*P(2*L-2)/Y5
      SN(3)=SN(3)+GN(R-1.)*(PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
    1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**2)
     SN(4)=SN(4)+GN(R-1.)* 1.5*(GO(2.*R-3.)/(4.*R-5.)-GO(2.*R-1.)
    1 /(4.*R-1.))*P(2*L-2)/(Y5**3)
      SN(5)=SN(5)+GN(R-1.)*2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
    1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
    2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y5**4)
  450 SN(6)=SN(6)+GN(R-1.)*3.75*(GO(2.*R-4.)/((4.*R-5.)*(4.*R-7.))
    1 +GO(2.*R)/((4.*R-1.)*(4.*R+1.))-2.*GO(2.*R-2.)
    2 /((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**5)
      DO 452 L=1,N
  452 A(J,L)=SN(L)
      B(J)=        .5*(1.-V1)*(2.*S+SZ)/(1.+V1)-10.*(1.-V1)*(S-SZ)/(7.-
    1 5.*V1)*P(6)
  410 THETA=THETA+BE/R1
      CALL SOLVE(A,X,B,N,51,RS1)
      IF(RS1) 407,407,412
  407 WRITE(6,607) BE
      GO TO 710
  412 IF(SS2) 650,650,409
  409 IF(T) 650,650,411
  411 DO 455 L=1,N
  455 SN(L)=0.0
      DO 460 L1=1,J1
      L=L1+2
      R=L1
      SN(1)=SN(1)+         (PO(2*L-3  )-PO(2*L-1  ))*P(2*L-2  )
      SN(2)=SN(2)+         GO(2.*R-2.)*P(2*L-2)/Y5
      SN(3)=SN(3)+         (PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
    1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**2)
     SN(4)=SN(4)+          1.5*(GO(2.*R-3.)/(4.*R-5.)-GO(2.*R-1.)
    1 /(4.*R-1.))*P(2*L-2)/(Y5**3)
      SN(5)=SN(5)+         2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+
    1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
    2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y5**4)
  460 SN(6)=SN(6)+         3.75*(GO(2.*R-4.)/((4.*R-5.)*(4.*R-7.))
    1 +GO(2.*R)/((4.*R-1.)*(4.*R+1.))-2.*GO(2.*R-2.)
```

```
     2 /((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**5)
  461 TR=0.0
      DO 470 L=1,N
  470 TR=TR+X(L)*SN(L)
  514 IF(90.0*(1.+E)/2.-E*THETA) 653,650,515
  653 WRITE(6,652)
      GO TO 710
  515 IF(-TR) 675,650,600
  600 BF=45.*(1.-E)+E*BE
      C2=BF
      SM2=TR
      BE=BE+E*DBE
      BF=BF+DBE
      GO TO 250
  675 BF=(TR*C2-BF*SM2)/(TR-SM2)
      BE=45.*(1.-E)+E*BF
      BF=BF-E*BF
      IF(SS1) 649,649,677
  677 BE=BE-E*DBE
      DBE=DBE/4.0
      SS1=SS1-2.0
      GO TO 41
C     COMPUTE STRESSES AND DISPLACEMENTS
  649 SS2=-10.
      GO TO 250
  650 THETA=45.0*(1.-E)
      DO 651 L=1,N
  651 WRITE(6,49) L,X(L)
      WRITE(6,704) BE
      WRITE(6,701)
  670 Y2= COS(THETA*.01745329)
      IF(SS3) 672,671,671
  671 Y2=(1.-E)/2.
  672 P(4)=1.0
      P(5)=Y2
      P1(1)=3.0*Y2
      P1(2)=1.0
      P1(3)=0.0
      P1(4)=0.0
      P1(5)=1.0
      DO 680 J7=2,J3
      J=J7+4
      R=J7
      P1(J)=((2.*R-1.)*Y2*P1(J-1)-R*P1(J-2))/(R-1.)
  680 P(J)=((2.*R-1.)*Y2*P(J-1)-(R-1.)*P(J-2))/R
      DO 685 L=1,N
  685 SN(L)=0.0
      IF(E) 1386,1385,1385
 1386 SN(1)=GN(0.)
      SN(3)=((Y1**2-1./3.)*GN(0.0)-2./3.*GN(1.)*P(6))/(Y1**2)
      DO 890 L1=1,J1
      L=L1+2
      R=L1
      SN(1)=SN(1)-GN(R-1.)*(P0(2*L-3  )-P0(2*L-1  ))*P(2*L-2  )
      SN(2)=SN(2)+GN(R-1.)*GAMA1(2*L-2)*P(2*L-2)/Y1
```

```
      DN1=  2.*Y1*GN(R-1.)*(PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
     1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
      DN2=         GN(R-1.)*2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
     1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
     2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
      SN(3)=SN(3)+DN1+DN2
  890 SN(4)=SN(4)+GN(R-1.)*GAMA3(2*L-2)*P(2*L-2)/(Y1**3)
      GO TO 691
 1385 DO 690 L1=1,J1
      L=L1+2
      R=L1
      SN(1)=SN(1)+GN(R-1.)*(PO(2*L-3  )-PO(2*L-1  ))*P(2*L-2  )
      SN(2)=SN(2)+GN(R-1.)*GO(2.*R-2.)*P(2*L-2)/Y5
      SN(3)=SN(3)+GN(R-1.)*(PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
     1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**2)
     SN(4)=SN(4)+GN(R-1.)* 1.5*(GO(2.*R-3.)/(4.*R-5.)-GO(2.*R-1.)
     1 /(4.*R-1.))*P(2*L-2)/(Y5**3)
      SN(5)=SN(5)+GN(R-1.)*2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
     1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
     2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y5**4)
  690 SN(6)=SN(6)+GN(R-1.)*3.75*(GO(2.*R-4.)/((4.*R-5.)*(4.*R-7.))
     1 +GO(2.*R)/((4.*R-1.)*(4.*R+1.))-2.*GO(2.*R-2.)
     2 /((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**5)
  691 UR=0.0
      DO 695 L=1,N
  695 UR=UR+X(L)*SN(L)
      DO 986 L=1,18
  986 SN(L)=0.0
      IF(SS3) 696,694,694
  694 SS3=-10.
      URO=UR
      GO TO 670
  696 IF(E) 896,896,996
  896 SN(1)=1.
      SN(7)=-.5
      SN(13)=-.5
      SN(3)=1.-(Y2**2)/(Y1**2)
      SN(9)=(-.5*(Y1**2-1./3.)+2./3.*(13.+V1-(9.+45.*V1)*Y2*Y2)/(8.*(7.
     1 -5.*V1)))/(Y1**2)
      SN(15)=(-.5*(Y1**2-1./3.)-2./3.*(17.-43.*V1+(87.*V1-21.)*Y2*Y2)/
     1 (8.*(7.-5.*V1)))/(Y1**2)
      DO 1096 L1=1,J1
      L=L1+2
      R=L1
      GN3=(GN1(R-1.)*P(2*L-2)-(R*(R-2.)+.5*(1.+V1))/R*P1(2*L-1)+(R-3.+
     1 2.*V1)*P1(2*L-3))/GN2(R-1.)
      GN4=(-(R-1.)*(4.*R-5.)*(1.-2.*V1)*P(2*L-2)+.5*(2.*R*R-4.*R+1.+V1)
     1  /R*P1(2*L-1)-(R-3.+2.*V1)*P1(2*L-3))/GN2(R-1.)
      SN(1)=SN(1)-          (PO(2*L-3  )-PO(2*L-1  ))*P(2*L-2  )
      D1=-                  (PO(2*L-3  )-PO(2*L-1  ))
      SN(2)=SN(2)+          GAMA1(2*L-2)*P(2*L-2)/Y1
      D2=                   GAMA1(2*L-2)/Y1
      DN1=  2.*Y1*          (PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
     1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
      DN2=                  2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
```

```
  1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
  2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y1**2)
    SN(3)=SN(3)+DN1*DN2
     D3=  2.*Y1*          (PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
  1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))/(Y1**2)
    D33=                  2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
  1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
  2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))/(Y1**2)
    SN(4)=SN(4)+          GAMA3(2*L-2)*P(2*L-2)/(Y1**3)
    D4=                   GAMA3(2*L-2)/(Y1**3)
    SN(7)=SN(7)+D1*GN3
    SN(8)=SN(8)+GN3*D2
    SN(9)=SN(9)+(D3+D33)*GN3
    SN(10)=SN(10)+D4*GN3
    SN(13)=SN(13)+D1*GN4
    SN(14)=SN(14)+D2*GN4
    SN(15)=SN(15)+(D3+D33)*GN4
1096 SN(16)=SN(16)+D4*GN4
    GO TO 1297
 996 DO 697 L1=1,J1
    L=L1+2
    R=L1
    GN3=(GN1(R-1.)*P(2*L-2)-(R*(R-2.)+.5*(1.+V1))/R*P1(2*L-1)+(R-3.+
  1 2.*V1)*P1(2*L-3))/GN2(R-1.)
    GN4=(-(R-1.)*(4.*R-5.)*(1.-2.*V1)*P(2*L-2)+.5*(2.*R*R-4.*R+1.+V1)
  1  /R*P1(2*L-1)-(R-3.+2.*V1)*P1(2*L-3))/GN2(R-1.)
    SN(1)=SN(1)+          (PO(2*L-3  )-PO(2*L-1  ))*P(2*L-2  )
    D1=                   (PO(2*L-3  )-PO(2*L-1  ))
    SN(2)=SN(2)+          GO(2.*R-2.)*P(2*L-2)/Y5
    D2=GO(2.*R-2.)/Y5
    SN(3)=SN(3)+          (PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
  1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**2)
    D3=                   (PO(2*L-4  )/(4.*R-5.)+PO(2*L )/(4.*R-1.)
  1 -2.*(4.*R-3.)*PO(2*L-2  )/((4.*R-5.)*(4.*R-1.)))/(Y5**2)
   SN(4)=SN(4)+           1.5*(GO(2.*R-3.)/(4.*R-5.)-GO(2.*R-1.)
  1 /(4.*R-1.))*P(2*L-2)/(Y5**3)
    D4=                   1.5*(GO(2.*R-3.)/(4.*R-5.)-GO(2.*R-1.)
  1 /(4.*R-1.))/(Y5**3)
    SN(5)=SN(5)+          2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
  1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
  2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))*P(2*L-2)/(Y5**4)
    D5=                   2.*(PO(2*L-5)/((4.*R-5.)*(4.*R-7.))-PO(2*L+1
  1 )/((4.*R-1.)*(4.*R+1.))+3.*PO(2*L-1)/((4.*R-5.)*(4.*R+1.))
  2 -3.*PO(2*L-3)/((4.*R-7.)*(4.*R-1.)))/(Y5**4)
    SN(6)=SN(6)+          3.75*(GO(2.*R-4.)/((4.*R-5.)*(4.*R-7.))
  1 +GO(2.*R)/((4.*R-1.)*(4.*R+1.))-2.*GO(2.*R-2.)
  2 /((4.*R-5.)*(4.*R-1.)))*P(2*L-2)/(Y5**5)
    D6=                   3.75*(GO(2.*R-4.)/((4.*R-5.)*(4.*R-7.))
  1 +GO(2.*R)/((4.*R-1.)*(4.*R+1.))-2.*GO(2.*R-2.)
  2 /((4.*R-5.)*(4.*R-1.)))/(Y5**5)
    SN(7)=SN(7)+D1*GN3
    SN(8)=SN(8)+GN3*D2
    SN(9)=SN(9)+D3*GN3
    SN(10)=SN(10)+D4*GN3
```

```
      SN(11)=SN(11)+D5*GN3
      SN(12)=SN(12)+D6*GN3
      SN(13)=SN(13)+D1*GN4
      SN(14)=SN(14)+D2*GN4
      SN(15)=SN(15)+D3*GN4
      SN(16)=SN(16)+D4*GN4
      SN(17)=SN(17)+D5*GN4
  697 SN(18)=SN(18)+D6*GN4
 1297 TR=0.0
      TTHET=0.0
      TGAMA=0.0
      SN12=1.5/(7.-5.*V1)*(SZ*(9.-5.*V1-10.*Y2*Y2)-2.*S*(1.-5.*Y2*Y2))
      SN13=-1.5/(7.-5.*V1)*(SZ*(1. -5.*V1+10.*Y2*Y2*V1)-2.*S*(4.-5.*V1
     1 +5.*Y2*Y2*V1))
      DO 698 L=1,N
      TTHET=TTHET+SN(L+6)*X(L)
      TGAMA=TGAMA+SN(L+12)*X(L)
  698 TR=TR+X(L)*SN(L)
      TTHET=TTHET+SN12
      TGAMA=TGAMA+SN13
      F8=5.0*(1.5*E-.5)
     UT=.5*(1.-V1)*(2.*S+SZ)/(1.+V1)+ F8*(1.-V1)*(S-SZ)/(7.-5.*V1)
     UK=.5*(1.-V1)*(2.*S+SZ)/(1.+V1)-10.*(1.-V1)*(S-SZ)/(7.-5.*V1)*P(6)
      US=UT-URO
      ERROR=100.*(UK-UR)/US
      WRITE(6,700) THETA,TR,ERROR,TTHET,TGAMA
      IF(SS4) 699,699,709
  699 IF(SS5) 709,708,708
  708 THETA=THETA1
      SS5=SS5-2.0
  709 THETA=THETA+E*DTHET
      IF(SS4) 703,711,711
  711 IF(E*(BE-THETA)) 712,712,703
  712 THETA1=THETA-E*DTHET
      THETA=BE
      SS4=-10.
      SS5=SS5-2.0
  703 IF(E*THETA-90.0*(1.+E)*.5) 670,670,710
  710 CONTINUE
      STOP
      END
```

```
      SUBROUTINE SOLVE (A,X,B,N,M,RS1)
      DIMENSION A(M,M),X(M),B(M),K(160),Y(160)
      DOUBLE PRECISION Y,D1
      DO 14 I = 1,N
   14 K(I) = I
      N1=N-1
      DO  8 L=1,N1
C**** SEARCH FOR LARGEST ELEMENT
      D=0.
      DO 1 L1=L,N
      DO 1 L2=L,N
       IF(ABS(A(L1,L2))-ABS(D)) 1,1,1550
 1550  D=A(L1,L2)
      ID=L1
      JD=L2
    1 CONTINUE
      IF (D) 2,99,2
C**** INTERCHANGE ROWS AND COLUMNS TO PUT LARGEST ELEMENT ON DIAGONAL
    2 TEMP = K(L)
      K(L) = K(JD)
      K(JD) = TEMP
      DO 3 J=L,N
      TEMP=A(L,J)
      A(L,J)=A(ID,J)
    3 A(ID,J)=TEMP
      TEMP=B(L)
      B(L)=B(ID)
      B(ID)=TEMP
      B(L)=B(L)/D
      DO 4 I=1,N
      TEMP=A(I,L)
      A(I,L)=A(I,JD)
    4 A(I,JD)=TEMP
C**** ELIMINATE ELEMENTS IN COLUMN UNDER  LARGEST ELEMENT
      L1=L+1
      DO 5 J=L1,N
    5 A(L,J)=A(L,J)/D
      DO 7 I=L1,N
       IF(A(I,L)) 1515,7,1515
 1515  D1=A(I,L)
      DO 6 J=L1,N
    6 A(I,J)=A(I,J)-D1*A(L,J)
      B(I)=B(I)-D1*B(L)
    7 CONTINUE
    8 CONTINUE
      IF (A(N,N)) 9,99,9
C**** BACK SUBSTITUTE TO SOLVE
    9 Y(N)=B(N)/A(N,N)
      DO 11 L=1,N1
      LL=N-L+1
      D1=B(LL-1)
      DO 10 J=LL,N
   10 D1=D1-A(LL-1,J)*Y(J)
   11 Y(LL-1)=D1
```

```
C**** RE-ORDER ANSWER
      DO 12 I=1,N
      J=K(I)
   12 X(J)=Y(I)
      GO TO 13
   99 WRITE (6,100) L
  100 FORMAT(39HOMATRIX IS SINGULAR  NO SOLUTION GIVEN   I5)
       RS1=-10.0
   13  RETURN
       END
```

AXISYMMETRIC CONTACT STRESSES AROUND A SMOOTH ELASTIC SPHERE
IN AN INFINITE SOLID, UNIFORMLY STRESSED AT INFINITY

6/16/66

```
  S=   .0000
 SZ=  1.0000
 VO=   .3000
 V1=   .3000
 DE=   .0000
 JJ=125.0000
 MS= 30.0000
```

CONTACT EXISTS IN THE FORM OF A BAND

A(1)=   .0632

A(2)= -1.0416

A(3)=   .1822

A(4)= -.1434

THETO= 71.1896

STRESSES AND DISP ERROR ON THE CAVITY

THETA	TR	DISP ERROR	TTHET	TGAMA
.90000000+02	-.93895192-00	-.49902088-06	.16183059+01	.20380616-00
.88000000+02	-.93353231-00	.29442232-03	.16184482+01	.20323962-00
.86000000+02	-.91304785-00	-.24776386-03	.16231421+01	.20405277-00
.84000000+02	-.88049638-00	-.21432947-03	.16295396+01	.20449616-00
.82000000+02	-.83356605-00	-.25949086-04	.16401919+01	.20597997-00
.80000000+02	-.76989618-00	-.97309073-05	.16578049+01	.20995153-00
.77999999+02	-.69176203-00	.10918577-02	.16805667+01	.21511219-00
.76000000+02	-.58496146-00	-.12809866-02	.17232532+01	.22991380-00
.73999999+02	-.44990481-00	-.47998324-02	.17862374+01	.25413932-00
.72000000+02	-.21218877-00	-.24585012-01	.19449419+01	.33196341-00
.71189564+02	.46394841-04	-.00000000	.21238640+01	.43129545-00
.70000000+02	-.54621024-02	.47926153-00	.20699455+01	.39375116-00
.68000000+02	.31522278-02	.18837415+01	.19953862+01	.34856735-00
.66000000+02	-.19799965-02	.38069914+01	.19041597+01	.29972245-00
.64000000+02	.91578253-03	.61276124+01	.18178945+01	.25819680-00
.62000000+02	.27977245-05	.87839099+01	.17248998+01	.21605898-00
.59999999+02	-.64728053-03	.11726562+02	.16294582+01	.17521722-00
.58000000+02	.93163165-03	.14915745+02	.15338257+01	.13652078-00
.55999999+02	-.85290682-03	.18319476+02	.14327422+01	.96478641-01
.54000000+02	.50062419-03	.21904220+02	.13330572+01	.58837865-01
.52000000+02	-.26735084-04	.25642629+02	.12301454+01	.20573208-01
.50000000+02	-.39674911-03	.29506052+02	.11264355+01	-.17110219-01
.48000000+02	.63363793-03	.33467031+02	.10235891+01	-.53462330-01
.45999999+02	-.61918652-03	.37500690+02	.91834938-00	-.90617940-01
.44000000+02	.37975857-03	.41579688+02	.81567628-00	-.12583157-00
.41999999+02	-.15313768-04	.45680273+02	.71237062-00	-.16119441-00
.40000000+02	-.33482226-03	.49777352+02	.61033100-00	-.19574686-00
.38000000+02	.54430910-03	.53846493+02	.51112325-00	-.22874184-00
.36000000+02	-.54048653-03	.57865471+02	.41201904-00	-.26195356-00
.34000000+02	.33097982-03	.61810058+02	.31735989-00	-.29296236-00
.32000000+02	.29753560-05	.65659361+02	.22442014-00	-.32354368-00
.30000000+02	-.33430322-03	.69391650+02	.13478760-00	-.35288355-00
.27999999+02	.53801082-03	.72986005+02	.50084808-01	-.38017096-00
.26000000+02	-.53470244-03	.76424061+02	-.32462724-01	-.40717138-00

.23999999+02	.32120682-03	.79685799+02	-.10862445+00	-.43149541-00
.22000000+02	.27634068-04	.82754853+02	-.18111932-00	-.45488477-00
.20000000+02	-.38322417-03	.85614444+02	-.24860714-00	-.47662713-00
.18000000+02	.60869432-03	.88248786+02	-.30925238-00	-.49575723-00
.16000000+02	-.60692891-03	.90645266+02	-.36639056-00	-.51432909-00
.14000000+02	.35814439-03	.92789590+02	-.41539137-00	-.52967118-00
.12000000+02	.69114533-04	.94671731+02	-.45943699-00	-.54377698-00
.10000000+02	-.53512900-03	.96281451+02	-.49742656-00	-.55600090-00
.80000000+01	.87072348-03	.97609434+02	-.52683018-00	-.56495783-00
.60000000+01	-.92434870-03	.98650662+02	-.55273084-00	-.57367492-00
.40000000+01	.59488104-03	.99397266+02	-.56854931-00	-.57822796-00
.20000000+01	.23202244-03	.99847267+02	-.57921943-00	-.58184820-00
.00000000	-.65107951-02	.99999998+02	-.58818546-00	-.58818554-00