WOLFF'S THEOREM ON IDEALS

FOR MATRICES

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A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the Graduate School of
The University of Alabama

TUSCALOOSA, ALABAMA

2012

ABSTRACT

Wolff's theorem on ideals in $H^\infty(\mathbb{D})$ states that, given a set of functions $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ and a function $h \in H^\infty(\mathbb{D})$, the function h^3 is a member of the ideal generated by the functions f_i if $|f_1| + \ldots |f_n| \geq |h|$ everywhere on the unit disc. The hypotheses can be reformulated to give sufficient conditions for when h itself is in the ideal.

In this paper we prove a matricial analogue of Wolff's theorem. Instead of a set of functions, we take a (possibly infinite) matrix of functions in $H^\infty(\mathbb{D})$, $F(z)$, and a vector of H^∞-functions, H, and prove sufficient conditions for the existence of a vector G of H^∞-functions such that $FG = H$. Although we focus on the space $H^\infty(\mathbb{D})$, our methods actually apply to any algebra of functions that satisfies a Wolff-type theorem.

At the end of the paper, we list some applications and extensions of our theorem.

DEDICATION

I dedicate this dissertation to my cat, Leibniz, the most mathematically inclined cat I've ever known.

LIST OF SYMBOLS

$\mathbb{D}$	The unit disc
$H^\infty(\mathbb{D})$	The space of bounded analytic functions on the unit disc
$\mathbb{C}$	The set of complex numbers
M_ϕ	The multiplication operator defined by ϕ
$B(H)$	The set of bounded linear operators from H to H
A^*	The adjoint of the operator A
$H^2(\mathbb{D})$	Hardy space: the space of square-summable analytic functions on the unit disc
$\mathrm{rad}(\mathcal{I})$	The radical of the ideal $\mathcal{I}$
$\mathbb{R}_+$	The set of positive real numbers
$\det_k(A)$	The k-determinant of the matrix A
l^2	The space of square-summable sequences
$M_m(\mathbb{C})$	The space of $m \times m$ matrices with entries in $\mathbb{C}$
$\Pi_k(m)$	The set of increasing k-tuples on the integers $1, \ldots, m$
E_π	The $m \times m$ matrix whose entries on the diagonal that are indexed by π are 1, and for which all other entries are 0
$\Lambda_n(e)$	The exterior algebra on n generators $e_1, \ldots, e_n$
$H \wedge K$	The wedge product of H and K
$l^2_{(n)}$	The wedge product of n copies of l^2
$M(H)$	The set of multiplication operators on the space H
$Q_j^{(n)}$	The Q-operator with on $l^2_{(n+1)}$ with symbol j
$S^\perp$	The space of elements perpendicular to a subset S of a Hilbert space H
$B(H, K)$	The set of bounded linear operators from H to K
$P(n)$	The set of permutations on $\{1, \ldots, n\}$

$\mathrm{sgn}(\sigma)$	The sign of the permutation σ
$\mathcal{D}^2(\mathbb{D})$	Dirichlet space: the space of analytic functions on the unit disc whose complex derivatives are square-integrable

ACKNOWLEDGMENTS

I would like to thank the members of my dissertation committee for setting aside time to prepare for and participate in my defense. I would like to give special thanks to my advisor, Dr. Tavan Trent, for agreeing to see me through my goal of attaining my degree, for his invaluable aid in my research, and for putting up with the questions that I probably should have known the answers to.

I would like to thank all the professors who have helped me in my nine and a half years of college education, both at Alabama and at UNA. It's thanks to them that I developed my love of mathematics, and because of them that I was able to develop my understanding and broaden my appreciation for it.

I would like to thank my parents, who decided to educate me themselves when I was still young. Through their upbringing I developed an inquisitive mind, a love of learning, and an understanding of the value of good work.

I would also like to thank my wife, Rebecca, for being a constant encouragement, both with regard to my work and in all other areas. She has truly been everything I could have hoped for, and more besides.

Finally, I would like to thank God for giving me the gift of life and the ability to learn and discover new knowledge.

CONTENTS

CHAPTER 1

INTRODUCTION

Carleson's famous corona theorem states that, given a set of functions $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ such that $\sum_{i=1}^n |f_i(z)| \geq \epsilon > 0 \; \forall \; z \in \mathbb{D}$, that the ideal generated by $\{f_i\}_{i=1}^n$ is the entire space $H^\infty(\mathbb{D})$. Wolff attempted to answer a more general question than Carleson, namely, given a particular function $h \in H^\infty(\mathbb{D})$, when is h contained in the ideal generated by $\{f_i\}$? In [**6**] Wolff proves that the function h^3 is contained in the ideal whenever $\sum_{i=1}^n |f_i(z)| \geq |h(z)| \; \forall \; z \in \mathbb{D}$. This result can be reformulated to give sufficient conditions for h itself being in the ideal.

In 1989, Andersson [**1**] proved an extension (of sorts) of Carleson's corona theorem to a matrix case. His theorem resulted in a vector G of H^∞-functions that satisfied $FG = H$, where F is a finite matrix and H is a vector, both having H^∞-functions as entries. Thus, the result of Andersson's theorem resembles that of Wolff's in a matrix situation, but he assumed a uniform boundedness in the matrix F more in keeping with Carleson's theorem. In 2007, Trent and Zhang [**15**] generalized Andersson's result to apply to any algebra that satisfies a corona theorem, as well as allowing for one-sided infinite matrices.

Given these results, a question that naturally arises is whether Wolff's theorem on ideals can be similarly extended. Essentially, we wish to replace Andersson's hypothesis on the boundedness of the matrix F with a condition that relates F to the vector H, similarly to the conditions in Wolff's theorem. In this paper we prove that Wolff's problem can be so extended.

We begin in the next chapter with a discussion on reproducing kernel Hilbert spaces. Following that we state and discuss Carleson's corona theorem and its matricial extensions. We also state Wolff's theorem and discuss improvements that have been made to it in recent years.

In chapter three we state the theorem that is the object of this paper. Chapters four through six consist of background information that must be covered before the theorem is proved: chapter four covers exterior algebras, chapter five, Q-operators, and chapter six consists of several lemmas instrumental to our proof.

Then, in chapter seven, we give the proof of Wolff's theorem for matrices. Finally, in chapter eight, we give several extensions and applications of this theorem.

CHAPTER 2

PRELIMINARIES

2.1. Reproducing Kernel Hilbert Spaces

DEFINITION 2.1. *Let H be a Hilbert space whose elements are functions on a set E , satisfying the following properties:*

(i) $\forall\, e \in E,\ \exists\, c_e$ *such that* $|h(e)| \leq c_e \|h\|_H\ \forall\, h \in H$

(ii) *if* $h(e) = 0\ \forall\, e \in E$, *then h is the zero element in H.*

We then say that H is a reproducing kernel Hilbert space on E .

For a fixed $e \in E$, define $l(h) = h(e)\ \forall\ h \in H$. Then $l(h) \leq c_e \|h\|_H$, so l is a bounded linear functional on H. By the Riesz Representation Theorem, there exists a unique element $k_e \in H$ such that

$$h(e) = l(h) = \langle h, k_e \rangle_H$$

The function $k : E \times E \to \mathbb{C}$, $k(e, z) = k_e(z)$ is called the reproducing kernel for H. Thus, for a reproducing kernel Hilbert space H, evaluation of an element $h \in H$ at e is equivalent to taking the inner product of h with the reproducing kernel.

DEFINITION 2.2. *Let $H(E)$ be a reproducing kernel Hilbert space on a set E, and let $\phi \in H(E)$. For all $h \in H(E)$, define an operator M_ϕ by*

$$M_\phi(h) = \phi h$$

If $M_\phi \in B(H)$, then we say that M_ϕ is a multiplication operator (or "multiplier") for $H(E)$.

We will now examine some properties of the reproducing kernel.

First, the reproducing kernel is unique. Observe that if k is a reproducing kernel for $H(E)$, then

$$k(w,z) = \langle k_w, k_z \rangle = \overline{\langle k_z, k_w \rangle} = \overline{k(z,w)} \tag{1}$$

Suppose k and K are both reproducing kernels for $H(E)$. Let $w, z \in E$. Then

$$K(w,z) = \langle K_w, k_z \rangle = \overline{\langle k_z, K_w \rangle} = \overline{k(z,w)} = k(w,z) \tag{2}$$

Thus $k = K$.

By (1), we observe that

$$K(w,w) = \|K_w\|^2 \tag{3}$$

Now, suppose M_ϕ is a multiplication operator on H. Let $f \in H$. Then

$$\langle M_\phi^*(k_w), f \rangle = \langle k_w, M_\phi(f) \rangle = \overline{\langle M_\phi(f), k_w \rangle} = \overline{\phi(w) f(w)} =$$

$$\overline{\phi(w) \langle f, k_w \rangle} = \overline{\phi(w)} \langle k_w, f \rangle = \langle \overline{\phi(w)} k_w, f \rangle$$

Since this holds for all $f \in H(E)$, we have

$$M_\phi^* k_w = \overline{\phi(w)} k_w \tag{4}$$

Example 1. *Consider the space* $H^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} | f \textit{ is analytic on } \mathbb{D}, f(z) = \sum_{n=0}^{\infty} f_n z^n, \|f\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |f_n|^2 < \infty\}$, *the so-called Hardy space on the unit disc. For* $f, g \in H^2(\mathbb{D})$, *the inner product is defined by*

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f_n \overline{g_n}$$

Let $f \in H^2(\mathbb{D})$, $z \in \mathbb{D}$. *Then*

$$|f(z)| = |\sum_{n=0}^{\infty} f_n z^n| \leq \sum_{n=0}^{\infty} |f_n||z^n| \leq (\sum_{n=0}^{\infty} |f_n|^2)^{\frac{1}{2}} (\sum_{n=0}^{\infty} |z^{2n}|)^{\frac{1}{2}} = c_z \|f\|,$$

where $c_z = (\sum_{n=0}^{\infty} |z^{2n}|)^{\frac{1}{2}}$. *Thus* $H^2(\mathbb{D})$ *is a reproducing kernel Hilbert space.*

That being the case, what is the reproducing kernel for $H^2(\mathbb{D})$*? We know that for* $z \in \mathbb{D}\ \exists\ k_z \in H^2(\mathbb{D})$ *such that* $\forall\ f \in H^2(\mathbb{D})$, $f(z) = \langle f, k_z \rangle$. *Thus*

$$\sum_{n=0}^{\infty} f_n \overline{(k_z)_n} = \langle f, k_z \rangle = f(z) = \sum_{n=0}^{\infty} f_n z^n$$

Let $h_m(z) = z^m \in H^2(\mathbb{D})$. *Then* $h_m(z) = \sum_{n=0}^{\infty} (h_m)_n z^n$, *so* $(h_m)_n = \delta_{nm}$. *Thus* $z^m = h_m(z) = \sum_{n=0}^{\infty} (h_m)_n \overline{(k_z)_n} = \overline{(k_z)_m}$, *that is,* $(k_z)_m = \overline{z^m}$. *Thus*

$$k(w, z) = k_w(z) = \sum_{n=0}^{\infty} (k_w)_n z^n = \sum_{n=0}^{\infty} (\overline{w} z)^n = \frac{1}{1 - \overline{w} z} \tag{5}$$

Before leaving the topic of reproducing kernel Hilbert spaces, we should mention that the algebra of multiplier operators on $H^2(\mathbb{D})$ is the Banach space $H^\infty(\mathbb{D})$, the space of bounded analytic functions on the unit disc. This fact is well known.

2.2. The Carleson Corona Theorem

In 1962, Lennart Carleson [**3**] solved the Corona Problem, which asked the following question: given a set of functions in $H^\infty(\mathbb{D})$, when does the ideal generated by those functions generate the entire space? Carleson showed that this occurs when the sum of the moduli of those functions is uniformly bounded below.

Carleson Corona Theorem. *Let* $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$, *and suppose there exists* $\delta > 0$ *such that*

$$\sum_{i=1}^{n} |f_i(z)|^2 \geq \delta^2$$

for all $z \in \mathbb{D}$. *Then there exists* $\{g_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ *such that*

$$\sum_{i=1}^{n} f_i g_i = 1$$

With an adjustment in the hypotheses, the result applies for infinitely many functions (see [**12**]). Fuhrmann [**5**] established that the corona theorem on H^∞ also holds for finite matrices. That is, given a matrix F of functions in $H^\infty(\mathbb{D})$, there exists a matrix G of functions in $H^\infty(\mathbb{D})$ such that $FG = I$. This result was extended to one-sided infinite matrices by Vasyunin (see Nikolski [**7**]). However, Treil [**9**] showed that a complete extension of the corona theorem is not possible in the two-sided infinite matrix case. Trent and Zhang [**14**] proved that the result of Furmann and Vasyunin can be extended to any algebra that satisfies a corona theorem.

In 1989, Andersson [**1**] proved another extension of the corona theorem to matrices. Andersson sought to find an H^∞-solution G to the equation $FG = H$, where H is a given vector of H^∞-functions. In his paper, Andersson weakened the hypothesis on the lower boundedness of the matrix F (while maintaining a uniform type of lower estimate) but added the condition that an arbitrary solution u exists to the equation $Fu = I$. He then showed that, under these hypotheses, one can find an H^∞-solution G. Trent and Zhang [**15**] also extended this theorem to any algebra that satisfies a corona theorem.

2.3. Wolff's Theorem on Ideals

Given Carleson's result, one might naturally wonder under what conditions a particular function $h \in H^\infty(\mathbb{D})$ might be found in the ideal generated by the functions f_i. The original result, proved by Thomas Wolff [**16**], follows:

Wolff's Theorem on Ideals. *Let $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ and $h \in H^\infty(\mathbb{D})$. Suppose that, for all $z \in \mathbb{D}$,*

$$\sum_{i=1}^n |f_i(z)| \geq |h(z)| \tag{6}$$

Then there exists $\{g_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ *such that*

$$\sum_{i=1}^{n} f_i g_i = h^3 \tag{7}$$

The question that immediately arises is whether the exponent on h can be improved upon. Rao (see Garnett [**6**]) proved that the result fails when "3" is replaced by "1". More recently, Treil [**11**] showed that it also fails when the "3" is replaced with "2".

Note that if we replace the ideal $\mathcal{I}$ generated by f_1 and f_2 with the radical of $\mathcal{I}$, $\mathrm{Rad}(\mathcal{I}) = \{h \in H^\infty(\mathbb{D}) \mid h^p \in \mathcal{I} \text{ for some positive integer } p\}$, then Wolff's condition is necessary *and* sufficient, up to a constant:

$$h \in \mathrm{Rad}(\mathcal{I}) \leftrightarrow \ \exists\, N \in \mathbb{N},\, C > 0 \text{ such that } C(|f_1(z)| + |f_2(z)|) \geq |h(z)|$$

$\forall\, z \in \mathbb{D}$. We will revisit this idea later.

Although Wolff's theorem on ideals fails for integers smaller than 3, it is still possible to improve the estimate for the bound on h.First, we will state an alternate formulation which implies the above theorem; we will refer to this result for the remainder of the paper as "Wolff's Theorem."

Wolff's Theorem. *Let* $\{f_i\}_{i=1}^\infty \subset H^\infty(\mathbb{D})$ *and* $h \in H^\infty(\mathbb{D})$, *and let* $F(z) = (f_1(z), f_2(z), \dots)$. *Suppose that, for all* $z \in \mathbb{D}$,

(i) $F(z)F(z)^* \leq 1$, *and*

(ii) $[F(z)F(z)^*]^{\frac{3}{2}} \geq |h(z)|$.

Then there exists $\{g_i\}_{i=1}^\infty \subset H^\infty(\mathbb{D})$ *with* $G(z) = (g_1(z), g_2(z), \dots)^T$ *such that*

(a) $F(z)G(z) = h(z)\ \forall\, z \in \mathbb{D}$, *and*

(b) $\sup_{z \in \mathbb{D}} G(z)G(z)^* < \infty$.

To improve the estimate, we wish to replace $[F(z)F(z)^*]^{\frac{3}{2}}$ in the above theorem with $F(z)F(z)^*\alpha(F(z)F(z)^*)$, where $\alpha : [0,1] \to [0,1]$ is a strictly increasing, onto, C^1-smooth function with $\alpha(0) = 0$. Cegrell [**4**] showed that the problem is solvable for

$$\alpha(t) = A_0(\ln\frac{c}{t})^{-\frac{3}{2}}(\ln\ln\frac{c}{t})^{-\frac{3}{2}}(\ln\ln\ln\frac{c}{t})^{-1},$$

for $t \in (0,1]$ and $\alpha(0) = 0$, in the case where F has finite length. Here c is chosen so that all log terms are positive and A_0 is chosen so that $\alpha(1) = 1$. Trent [**13**] extended this result to the case where F has infinite length, and improved the estimate to

$$\alpha(t) = A_0(\ln\frac{c}{t})^{-\frac{3}{2}}(\ln\ln\frac{c}{t})^{-1}\dots(\underbrace{\ln\ln\dots\ln}_{m-1}\frac{c}{t})^{-1}(\underbrace{\ln\ln\dots\ln}_{m}\frac{c}{t})^{-1-\epsilon}. \qquad (8)$$

Currently, the best estimate is due to Treil [**11**]. He showed that, for a bounded non-increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \psi(x)dx < \infty$, the function

$$\phi(t) = \psi(\ln(t^{-2}))$$

satisfies the problem.

CHAPTER 3

WOLFF'S THEOREM FOR MATRICES

In this section we will state a theorem which extends Wolff's theorem on ideals to the matrix case. In addition, this theorem is stronger than Andersson's results.

WOLFF'S THEOREM FOR MATRICES. *Let $F(z)$ be an $m \times \infty$ matrix of functions in $H^\infty(\mathbb{D})$ with* $\max\{\operatorname{rank} F(z) \mid z \in \mathbb{D}\} = k \leq m$. *Let $H(z)$ be an $m \times 1$ vector of functions in $H^\infty(\mathbb{D})$. Suppose*

(i) $[\det_k(F(z)F(z)^*)]^{\frac{3}{2}} \geq |h_i(z)|\ \forall\ z \in \mathbb{D}$, $i = 1, \ldots, m$

(ii) $\|M_F\| = 1$

(iii) *there exists an arbitrary function $u : \mathbb{D} \to l^2$ such that $Fu = H$ everywhere on $\mathbb{D}$.*

Then there exists an $\infty \times 1$ vector $G(z)$ of functions in $H^\infty(\mathbb{D})$ such that

(a) $F(z)G(z) = H(z)\ \forall\ z \in \mathbb{D}$, *and*

(b) $\|M_G\| < \infty$.

NOTE 2. *If $k = m$, then (iii) follows from (i).*

We should explain here what we mean by "$\det_k(F(z)F(z)^*)$".

DEFINITION 3.1. *Let $B \in M_m(\mathbb{C})$. For $1 \leq k \leq m$, define*

$$\det_k(B) = \sum_{\pi \in \Pi_k(m)} \det(E_\pi B E_\pi)$$

where $\Pi_k(m)$ denotes the increasing k-tuples of integers in $\{1, 2, \ldots, m\}$.

Here E_π is the $m \times m$ matrix whose ith column is the ith column of the $m \times m$ identity matrix if $i \in \pi$, and is zero otherwise. When taking the determinant of $E_\pi B E_\pi$ in the above definition, we delete those columns and rows consisting of all zeros.

The first two premises give are similar to the conditions in Wolff's theorem, and in fact allow us to invoke that theorem in our proof. The condition that $Fu = H$ guarantees that the equation is at least solvable.

Before giving the proof of our theorem, some groundwork must be laid.

CHAPTER 4

EXTERIOR ALGEBRAS

DEFINITION 4.1. *The exterior algebra on n generators $e_1, \ldots, e_n$, denoted $\Lambda = \Lambda_n(e)$, is the algebra of forms in $e_1, \ldots, e_n$ with identity $e_0 = 1$, subject to the collapsing property $e_i e_j + e_j e_i = 0$.*

In particular, we have $e_i e_i = 0$.

If we declare $\{e_1, \ldots, e_n\}$, $\{e_1e_2, e_1e_3, \ldots, e_1e_n, e_2e_3, \ldots, e_2e_n, \ldots, e_{n-1}e_n\}$, etc., to be orthonormal bases for

$$\begin{aligned}
&\Lambda^1 = \operatorname{span}\{e_1, e_2, \ldots, e_n\},\\
&\Lambda^2 = \operatorname{span}\{e_1e_2, e_1e_3, \ldots, e_1e_n, e_2e_3, \ldots, e_2e_n, \ldots, e_{n-1}e_n\},\\
&\vdots\\
&\Lambda^n = \operatorname{span}\{e_1e_2 \ldots e_n\},
\end{aligned}$$

then Λ becomes a Hilbert space with orthogonal decomposition $\Lambda = \oplus_{k=1}^n \Lambda^k$.

By the kth exterior product on n generators, we mean the space $\Lambda^k = \Lambda^k_n$. Thus the kth exterior product of l^2 would be denoted by

$$\underbrace{l^2 \wedge l^2 \wedge \cdots \wedge l^2}_{k \text{ times}} = \Lambda^k_\infty = l^2_{(k)}.$$

In keeping with this notation, $l^2_{(0)} = \mathbb{C}$. Of course, $l^2_{(1)} = l^2$.

Let $\{e_i\}_{i=1}^\infty$ denote the standard basis in l^2. If I_n denotes increasing n-tuples of positive integers and if $(i_1, i_2, \ldots, i_n) \in I_n$, we let $\pi_n = \{i_1, i_2, \ldots, i_n\}$ and, abusing notation, we write $\pi \in I_n$. If we define $e_{\pi_n} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, then $\{e_{\pi_n}\}_{\pi_n \in I_n}$ is defined to be the standard basis for $l^2_{(n)}$.

CHAPTER 5

Q-OPERATORS

Let $H(E)$ be a reproducing kernel Hilbert space on a set E, and let $\mathcal{A} = M(H(E))$, the multiplier algebra on $H(E)$. Let $f_j(z) = (v_1(z), v_2(z), \dots)$, where $\{v_n\}_{n=1}^{\infty} \subset \mathcal{A}$, such that $f_j(z)f_j(z)^* \leq 1 \ \forall \ z \in E$. Fix $z \in E$, and for $n = 0, 1, \dots$ define

$$Q_j^{(n)*}(z) : l^2_{(n)} \to l^2_{(n+1)}$$

by

$$Q_j^{(n)*}(z)(w_n) = \overline{f_j(z)} \wedge w_n,$$

where $w_n \in l^2_{(n)}$. Note that $Q_j^{(0)*}(z) = \overline{f_j(z)}$. We observe a few properties.

First, we observe that $\operatorname{ran} Q_j^{(n)*}(z) \subset \ker Q_j^{(n+1)*}(z)$. Since, for any bounded linear operator L, $(\operatorname{ran} L)^{\perp} = \ker L^*$, we have $\operatorname{ran} Q_j^{(n+1)}(z) \subset \ker Q_j^{(n)}(z)$ for $n = 0, 1, \dots$. Furthermore, equality can be shown if we stipulate that $f_j(z)f_j(z)^* \geq \delta > 0$. (This follows from (11) below.)

Secondly, since

$$Q_j^{(n+1)*}(z)Q_k^{(n)*}(z)(w_n) = \overline{f_j(z)} \wedge \overline{f_k(z)} \wedge w_n =$$

$$-\overline{f_k(z)} \wedge \overline{f_j(z)} \wedge w_n = -Q_k^{(n+1)*}(z)Q_j^{(n)*}(z)(w_n),$$

we see that

$$Q_j^{(n)}Q_k^{(n+1)} = -Q_k^{(n)}Q_j^{(n+1)}. \tag{9}$$

For $e_{\pi_n} \in l^2_{(n)}$, we have

$$Q_j^{(n)*}(z)(e_{\pi_n}) = \overline{f_j(z)} \wedge e_{\pi_n} = (\sum_{p=1}^{\infty} \overline{v_p(z)}) \wedge e_{\pi_n},$$

so with respect to the standard basis, the entries in $Q_j^{(n)*}(z)$ are 0 or $\pm\overline{v_n(z)}$ for some n. Thus $Q_j^{(n)}(\cdot)$ has entries belonging to $\mathcal{A}$ with respect to the standard basis.

Assume that there exists a fixed $z \in E$ such that $f_j(z) \neq 0$, and let $a = \overline{f_j(z)}$, for fixed j. Also, let $Q_n = Q_j^{(n)}(z)$. Choose an orthonormal basis $\{u_n\}_{n=1}^{\infty}$ of l^2 with $u_1 = \frac{a}{\|a\|}$. Then for $\pi_n \in \mathcal{I}_n$ and $u_{\pi_n} = u_{i_1} \wedge \cdots \wedge u_{i_n}$, we have $\{u_{\pi_n}\}_{\pi_n \in \mathcal{I}_n}$ is an orthonormal basis for $l^2_{(n)}$.

Thus for $w \in l^2_{(n+1)}$,

$$\begin{aligned} Q_n(w) &= \sum_{\pi_n \in \mathcal{I}_n} \langle Q_n(w), u_{\pi_n} \rangle u_{\pi_n} \\ &= \sum_{\pi_n \in \mathcal{I}_n} \langle w, Q_n^*(u_{\pi_n}) \rangle u_{\pi_n} \\ &= \sum_{\pi_n \in \mathcal{I}_n} \langle w, a \wedge u_{\pi_n} \rangle u_{\pi_n} \\ &= \|a\| \sum_{\pi_n \in \mathcal{I}_n} \langle w, u_1 \wedge u_{\pi_n} \rangle u_{\pi_n}. \end{aligned} \tag{10}$$

We wish to show that for $n = 0, 1, \ldots,$

$$Q_n^* Q_n + Q_{n+1} Q_{n+1}^* = \|a\| I_{l^2_{n+1}}. \tag{11}$$

To do this, we need only show that, for $w \in l^2_{(n+1)}$,

$$\|Q_n(w)\|^2 + \|Q_{n+1}^*(w)\|^2 = \|a\|^2 \|w\|^2. \tag{12}$$

By (10),

$$\|Q_n(w)\|^2 = \|a\|^2 \sum_{\substack{\pi_n \in \mathcal{I}_n \\ 1 \notin \pi_n}} |\langle w, u_1 \wedge u_{\pi_n}\rangle|^2$$

$$= \|a\|^2 \sum_{\substack{\pi_{n+1} \in \mathcal{I}_{n+1} \\ 1 \in \pi_{n+1}}} |\langle w, u_{pi_{n+1}}\rangle|^2.$$

Also, since

$$Q^*_{n+1}(w) = a \wedge w = \|a\| u_1 \wedge \sum_{\pi_{n+1} \in \mathcal{I}_{n+1}} |\langle w, u_{pi_{n+1}}\rangle|^2 u_{\pi_{n+1}}$$

$$= \|a\| \sum_{\substack{\pi_{n+1} \in \mathcal{I}_{n+1} \\ 1 \notin \pi_{n+1}}} \langle w, u_{pi_{n+1}}\rangle u_1 \wedge u_{\pi_{n+1}},$$

we have

$$\|Q^*_{n+1}(w)\|^2 = \|a\|^2 \sum_{\substack{\pi_{n+1} \in \mathcal{I}_{n+1} \\ 1 \notin \pi_{n+1}}} |\langle w, u_{\pi_{n+1}}\rangle|^2.$$

Thus (11) holds.

If $a_i = f_i(z)$, then for $k = 2, \ldots, m$, let $a'_k = P^{\perp}_{\mathrm{sp}\{a_1,\ldots,a_{k-1}\}}(a_k)$. Then

$$a_1 a_1^* = \|a_1\|^2,$$

$$a_1 Q^{(1)}_{a_2} Q^{(1)*}_{a_1} a_1^* = a_1 Q^{(1)}_{a_2} (a_1 Q^{(1)}_{a_2})^* = a_2 Q^{(1)}_{a_1} (a_2 Q^{(1)}_{a_1})^*$$

$$= a_2 Q^{(1)}_{a_1} a_1^* Q^{(1)*}_{a_2} = a_2 (\|a_1\| I_{l^2} - a_1^* a_1) a_2^*$$

$$= \|a_1\|^2 a_2 (P_{\mathrm{ran}\, a_1^{\perp}}) a_2^* = \|a_1\|^2 \|a'_2\|^2,$$

where we have used (9) and (11) in the second result, and repeating this process, we obtain

$$\begin{aligned} a_1 Q_{a_2}^{(1)} \dots Q_{a_m}^{(m-1)} Q_{a_m}^{(m-1)*} \dots Q_{a_2}^{(1)*} a_1^* &= \|a_1\|^2 \prod_{j=2}^{m} \|a_j'\|^2 \\ &= \det(F(z)F(z)^*). \end{aligned} \tag{13}$$

The last equality is obtained by a straightforward computation, using the fact that $\frac{a_i^* a_i}{\|a_i\|^2}$ is the rank one projection of l^2 onto a_i.

Further discussion can be found in [**14**] and [**8**].

CHAPTER 6

SOME LEMMAS

The following lemmas are due to Trent and Zhang [**15**].

Lemma 6.1. *Let* $A = (a_1, a_2, \dots)$, $a_i \in \mathcal{A}$, *and assume that*

$$\max\{\|M_A\|, \|M_A^T\|\} < \infty.$$

Fix $z \in E$, *and let*

$$Q^{(j)}(z) = Q_A^{(j)}(z).$$

Then for $1 \leq j$,

$$\|M_{Q^{(j)}}\| \leq (j+1)\|M_A\| and \|M_{Q^{(j)}}\| \leq (j+1)\|M_A^T\|.$$

Proof. We will show that the first inequality holds. The second follows similarly. Let $\Pi(j)$ denote increasing j-tuples of positive integers. Now

$$\begin{aligned} Q^{(j)*}(z)\underline{x}(z) &= \overline{A(z)} \wedge \underline{x}(z) \\ &= \left(\sum_{j=1}^{\infty} \overline{a_j(z)} e_j\right) \wedge \left(\sum_{\pi \in \Pi(j)} x_\pi(z) e_\pi\right) \\ &= \sum_{j=1}^{\infty} \sum_{\pi \in \Pi(j)} \overline{a_j(z)} x_\pi(z) e_j \wedge e_\pi \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_{j+1}} \sum_{r=1}^{j+1} (-1)^{r-1} \overline{a_r(z)} x_{i_1, \dots, \hat{i_r}, \dots, i_{j+1}} e_{i_1, \dots, i_{j+1}}. \end{aligned}$$

Thus

$$\begin{aligned}
\|M^*_{Q^{(j)}}\underline{x}\|^2 &= \sum_{\substack{\sigma\in\Pi(j+1)\\ \sigma=(i_1,\dots,i_{j+1})}} \|\sum_{r=1}^{j+1}(-1)^{r-1}M^*_{a_r}x_{\sigma-\{i_r\}}\|^2 \\
&\le (j+1)\sum_{i_1<\dots i_{j+1}}\sum_{r=1}^{j+1}\|M^*_{a_{i_r}}x_{\sigma-\{i_r\}}\|^2 \\
&= (j+1)\sum_{i_1<\dots i_{j+1}}(\|M^*_{a_{i_1}}x_{\sigma-\{i_1\}}\|^2+\cdots+\|M^*_{a_{i_{j+1}}}x_{\sigma-\{i_{j+1}\}}\|^2) \\
&\le (j+1)\|M^*_A\|^2\left(\sum_{i_2<\dots i_{j+1}}\|x_{i_2,\dots,i_{j+1}}\|^2+\cdots+\sum_{i_1<\dots i_j}\|x_{i_1,\dots,i_j}\|^2\right) \\
&\le (j+1)^2\|M_A\|^2\|x\|^2.
\end{aligned}$$

Thus

$$\|M_{Q^{(j)}}\| \le (j+1)\|M_A\|.$$

□

We should note that we get a better estimate in the case where $\mathcal{A} = H^\infty(\mathbb{D})$. From (12) we have the following estimate on the operator norms (for fixed $z \in \mathbb{D}$):

$$\|Q^{(j)}\| \le \|A\|.$$

If $\mathcal{A} = H^\infty(\mathbb{D})$, then

$$\|M_{Q^{(j)}}\| = \sup_{z\in\mathbb{D}}\|Q^{(j)}(z)\| \le \sup_{z\in\mathbb{D}}\|A(z)\| = \|M_A\|.$$

(Here $j \ge 1$. If $j = 0$, then $Q^{(j)} = A$, so the result is trivial.) Thus

$$\|M_{Q^{(j)}}\| \le \|M_A\|. \tag{14}$$

We will need 6.1 if we wish to extend our main theorem to spaces besides $H^\infty(\mathbb{D})$, however.

DEFINITION 6.1. *Let* $\{X_j\}_{j=1}^{n+1}$ *be Banach spaces and let* $\{T_{jk}\}_{j,k=1}^{n}$ *denote operators such that* $T_{jk} \in B(X_{j+1}, X_j)$. *Let*

$$\text{“}det\text{”} \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & & & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix} = \sum_{\substack{\sigma \in P(n) \\ \sigma = \{i_1, \dots, i_n\}}} (-1)^{sgn(\sigma)} T_{1i_1} T_{2i_2} \dots T_{ni_n}$$

where the products are given in the order indicated and $P(n)$ *denotes the permutations of* $\{1, \dots, n\}$ *and* $sgn(\sigma)$ *denotes the sign of the permutation of* σ.

LEMMA 6.2.

$$\text{“}det\text{”} \begin{pmatrix} \underline{f}_1 & \cdots & \underline{f}_p \\ Q_1^{(1)} & \cdots & Q_p^{(1)} \\ \vdots & & \vdots \\ Q_1^{(p-1)} & \dots & Q_p^{(p-1)} \end{pmatrix} = p! \underline{f}_1 Q_2^{(1)} \dots Q_p^{(p-1)}.$$

PROOF.

$$\text{“det”} \begin{pmatrix} \underline{f}_1 & \cdots & \underline{f}_p \\ Q_1^{(1)} & \cdots & Q_p^{(1)} \\ \vdots & & \vdots \\ Q_1^{(p-1)} & \dots & Q_p^{(p-1)} \end{pmatrix} = \sum_{\substack{\sigma \in P(p) \\ \sigma = (i_1, \dots, i_p)}} (-1)^{\text{sgn}(\sigma)} \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_p}^{(p-1)}$$

$$= p! \underline{f}_1 Q_2^{(1)} \dots Q_p^{(p-1)}.$$

The last line follows by (9). ☐

LEMMA 6.3.

$$\text{“}\mathit{det}\text{”}\begin{pmatrix} H_1 & H_{i_1} & \dots & H_{i_p} \\ \underline{f}_1 & \underline{f}_{i_1} & \cdots & \underline{f}_{i_p} \\ Q_1^{(1)} & Q_{i_1}^{(1)} & \dots & Q_{i_p}^{(1)} \\ \vdots & & & \vdots \\ Q_1^{(p-1)} & Q_{i_1}^{(p-1)} & \dots & Q_{i_p}^{(p-1)} \end{pmatrix}$$

$$= p![H_1 \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_p}^{(p-1)} + \sum_{l=1}^{p} (-1)^l \underline{f}_1 H_{i_l} Q_{i_1}^{(1)} \dots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \dots Q_{i_p}^{(p-1)}].$$

PROOF. By definition,

$$\text{“det”}\begin{pmatrix} H_1 & H_{i_1} & \dots & H_{i_p} \\ \underline{f}_1 & \underline{f}_{i_1} & \cdots & \underline{f}_{i_p} \\ Q_1^{(1)} & Q_{i_1}^{(1)} & \dots & Q_{i_p}^{(1)} \\ \vdots & & & \vdots \\ Q_1^{(p-1)} & Q_{i_1}^{(p-1)} & \dots & Q_{i_p}^{(p-1)} \end{pmatrix}$$

$$= \sum_{\substack{\sigma \in P(p+1) \\ \sigma = \{j_1, \dots, j_{p+1}\} \\ j_k \in \{1, i_1, \dots, i_p\}}} (-1)^{\mathrm{sgn}(\sigma)} H_{j_1} \underline{f}_{j_2} Q_{j_3}^{(1)} \dots Q_{j_{p+1}}^{(p-1)}.$$

For a given term, if $j_1 = 1$, the term becomes $H_1 f_{i_1} \dots Q_{i_p}^{(p-1)}$ by (9). Else, $j_1 = i_l$ for some $l = 1, \dots, p$, so the term becomes

$$(-1)^{\mathrm{sgn}(\sigma)} H_{j_1} \underline{f}_{j_2} Q_{j_3}^{(1)} \dots Q_{j_{p+1}}^{(p-1)} = (-1)^l \underline{f}_1 H_{i_l} Q_{i_1}^{(1)} \dots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \dots Q_{i_p}^{(p-1)}.$$

□

LEMMA 6.4. *Suppose $F(\cdot)$ is $m \times \infty$ with $F(z)F(z)^*$ having maximum rank $p < m$. Suppose that for some function $\underline{u} : E \to l^2$, $F(z)\underline{u}(z) = H(z)$, where $H = (h_1, \dots, h_m)$ with $h_j \in M(H(E))$. Then for any $\pi \in \Pi_{(p+1)}(n)$ with $\pi = (j_1, \dots, j_{p+1})$, we have for* *each* $z \in E$

$$\text{“}det\text{”}\begin{pmatrix} H_{i_1} & \dots & H_{i_{p+1}} \\ \underline{f}_{i_1} & \dots & \underline{f}_{i_{p+1}} \\ Q^{(1)}_{i_1} & \dots & Q^{(1)}_{i_{p+1}} \\ \vdots & & \vdots \\ Q^{(p-1)}_{i_1} & \dots & Q^{(p-1)}_{i_{p+1}} \end{pmatrix} = 0.$$

PROOF. For ease of notation, let $i_1 = 1, \dots, i_{p+1} = p+1$. Fix $z \in E$ and choose a $(p+1)$-row vector of scalars, $\underline{a}$, so that

$$\underline{a} \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_{p+1} \end{pmatrix} = 0, \text{ where } F = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_{p+1} \end{pmatrix}.$$

Since $\max \operatorname{rank} F(z) = p$, we may assume that not all the scalars in $\underline{a}$ are zero. Then

$$0 = \underline{a} \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_{p+1} \end{pmatrix} \underline{u} = \underline{a} \begin{pmatrix} h_1 \\ \vdots \\ h_{p+1} \end{pmatrix}.$$

Also, $[Q^{(k)}_1, \dots, Q^{(k)}_{p+1}]\underline{a}^T = Q^{(k)}_{\sum_{j=1}^{p+1} a_j \underline{f}_j} = Q^{(k)}_0 = 0$, so we have

$$\begin{pmatrix} H_{i_1} & \dots & H_{i_{p+1}} \\ \underline{f}_{i_1} & \dots & \underline{f}_{i_{p+1}} \\ Q^{(1)}_{i_1} & \dots & Q^{(1)}_{i_{p+1}} \\ \vdots & & \vdots \\ Q^{(p-1)}_{i_1} & \dots & Q^{(p-1)}_{i_{p+1}} \end{pmatrix} \underline{a}^T = 0$$

The result follows since “det” is multilinear in its columns. □

CHAPTER 7

PROOF OF WOLFF'S THEOREM FOR MATRICES

We restate our main theorem and then give its proof.

Wolff's Theorem for Matrices. *Let $F(z)$ be an $m \times \infty$ matrix of functions in $H^\infty(\mathbb{D})$ with $\max\{\operatorname{rank} F(z) \mid z \in \mathbb{D}\} = k \leq m$. Let $H(z)$ be an $m \times 1$ vector of functions in $H^\infty(\mathbb{D})$. Suppose*

(i) $[\det_k(F(z)F(z)^*)]^{\frac{3}{2}} \geq |h_i(z)| \ \forall\, z \in \mathbb{D},\ i = 1, \ldots, m$

(ii) $\|M_F\| = 1$

(iii) *there exists an arbitrary function $u : \mathbb{D} \to l^2$ such that $Fu = H$ everywhere on $\mathbb{D}$.*

Then there exists an $\infty \times 1$ vector $G(z)$ of functions in $H^\infty(\mathbb{D})$ such that

(a) $F(z)G(z) = H(z) \ \forall\, z \in \mathbb{D}$, *and*

(b) $\|M_G\| < \infty$.

Proof. Denote the rows of $F(z)$ by $f_1(z), f_2(z), \ldots, f_m(z)$. We divide the proof into two cases. In each case, we find a vector A_1 such that $f_1 A_1 = h_1$ and $f_i A_1 = 0$ for $i \neq 1$. The vectors $A_2, \ldots, A_m$ are found similarly. Summing the vectors A_1 through A_m gives the solution G.

Suppose first that $k = m$. By (13), this means

$$[f_1(z)Q_2^{(1)}(z)\ldots Q_m^{(m-1)}(z)Q_m^{(m-1)*}(z)\ldots Q_2^{(1)*}(z)f_2^*(z)]^{\frac{3}{2}} \geq |h_i(z)|$$

for all $z \in \mathbb{D}$ and for $i = 1, \ldots, m$. By (14),

$$\|M_{f_1}M_{Q_2^{(1)}}\ldots M_{Q_m^{(m-1)}}\| \leq \|M_{f_1}\|\ldots\|M_{Q_m^{(m-1)}}\| < \infty,$$

so by Wolff's theorem there exists an $\infty \times 1$ vector $v_1(z)$ of functions in $H^\infty(\mathbb{D})$ such that

$$f_1 Q_2^{(1)} \ldots Q_m^{(m-1)} v_1 = h_1$$

and $\|M_{v_1}\| < \infty$.Thus $Q_2^{(1)} \ldots Q_m^{(m-1)} v_1$ is the desired vector, since

$$f_i Q_2^{(1)} \ldots Q_m^{(m-1)} v_1 = 0 \text{ for } i \neq 1.$$

Also, by (14),

$$\begin{aligned} \|M_{G_1}\| &\leq \|M_{f_2}\| \|M_{f_3}\| \ldots \|M_{f_m}\| \|M_{v_1}\| \\ &\leq \underbrace{\|M_F\| \ldots \|M_F\|}_{m-1 \text{ times}} \|M_{v_1}\| \\ &\leq \|M_{v_1}\|. \end{aligned}$$

Using the estimates in [**13**] for $\alpha(t) = t^{\frac{1}{2}}$, we obtain

$$\|M_{v_1}\| \leq 1 + 4\sqrt{e} + 8\sqrt{2}e + 72e^{\frac{3}{2}} = K < 362$$

so

$$\|M_G\| \leq mK.$$

Now suppose $k < m$. By the same process we used to obtain (13), we see that

$$[\sum_{\substack{\pi \in \Pi_k(m) \\ \pi = \{i_1, \ldots i_k\}}} f_{i_1}(z) Q_{i_2}^{(1)}(z) \ldots Q_{i_k}^{(k-1)}(z) Q_{i_k}^{(k-1)*}(z) \ldots Q_{i_2}^{(1)*}(z) f_{i_1}^*(z)]^{\frac{3}{2}} \geq | h_i(z) |$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, m$. Again using (14),

$$\| M_{f_{i_1}} M_{Q_{i_2}^{(1)}} \ldots M_{Q_{i_k}^{(k-1)}} \| < \infty,$$

so by Wolff's Theorem there exists, for each $\pi \in \Pi_k(m)$, a vector v_π with entries in $H^\infty(\mathbb{D})$ such that

$$k! \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = \{i_1, \ldots i_k\}}} f_{i_1} Q_{i_2}^{(1)} \ldots Q_{i_k}^{(k-1)} v_\pi = h_1$$

and

$$\|M_{v_\pi}\| < \infty.$$

We can rewrite this equation in terms of exterior algebras as

$$\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1 = h_1$$

where v_1 is a vector with $\begin{pmatrix} m \\ k \end{pmatrix}$ entries v_π for $\pi \in \Pi_k(m)$. Then we have $\|M_{v_i}\| < \infty$.

We claim the vector

$$G_1 = k \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1$$

is the vector we seek. To prove this, we will consider a more general vector,

$$A = k \begin{pmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_m I \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix}$$

where $\alpha_1, \ldots, \alpha_m \in H^\infty(\mathbb{D})$. Now

$$f_1 A = k \begin{pmatrix} \alpha_1 f_1 \\ \vdots \\ \alpha_m f_1 \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix}$$

$$= k! \begin{pmatrix} \alpha_1 f_1 \\ \vdots \\ \alpha_m f_1 \end{pmatrix} \wedge \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma=(j_2,\ldots,j_k)}} Q_{j_2}^{(1)} \ldots Q_{j_k}^{(k-1)} e_\sigma$$

$$= k! \sum_{j=1}^{m} \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma=(j_2,\ldots,j_k) \\ 1 \notin \sigma}} \alpha_j f_1 Q_{j_2}^{(1)} \ldots Q_{j_k}^{(k-1)} e_j \wedge e_\sigma$$

$$= k! \alpha_1 \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma=(j_2,\ldots,j_k) \\ 1 \notin \sigma}} f_1 Q_{j_2}^{(1)} \ldots Q_{j_k}^{(k-1)} e_1 \wedge e_\sigma$$

$$+ k! \sum_{j=2}^{m} \sum_{\substack{\sigma \in \Pi_{k-1}(m) \\ \sigma=(j_2,\ldots,j_k) \\ 1,j \notin \sigma}} \alpha_j f_1 Q_{j_2}^{(1)} \ldots Q_{j_k}^{(k-1)} e_j \wedge e_\sigma$$

$$= k! \alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=(1,i_2,\ldots,i_k)}} f_1 Q_{i_2}^{(1)} \ldots Q_{i_k}^{(k-1)} e_\pi$$

$$+ k! \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=(i_1,\ldots,i_k) \\ 1 \notin \pi}} \sum_{l=1}^{k} (-1)^{l-1} \alpha_{i_l} f_1 Q_{i_1}^{(1)} \ldots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \ldots Q_{i_k}^{(k-1)} e_\pi.$$

For the second and third equalities, we simply applied the definition of the exterior product. For the fourth, we broke the summation into two. The last inequality is obtained by renaming the indices and using (9).

By Lemma 6.3,

$$k! \sum_{l=1}^{k} (-1)^l \alpha_{i_l} f_1 Q_{i_1}^{(1)} \ldots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \ldots Q_{i_k}^{(k-1)}$$

$$= \text{“det”} \begin{pmatrix} \alpha_1 & \alpha_{i_1} & \dots & \alpha_{i_k} \\ f_1 & f_{i_1} & \dots & f_{i_k} \\ \vdots & \vdots & \ddots & \vdots \\ Q_1^{(k-1)} & Q_{i_1}^{(k-1)} & \dots & Q_{i_k}^{(k-1)} \end{pmatrix} - k!\alpha_1 f_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)}.$$

But rank $F(z)F(z)^* = k$, so by Lemma 6.4, “det”$() = 0$. Thus

$$k! \sum_{l=1}^{k} (-1)^{l-1} \alpha_{i_l} f_1 Q_{i_1}^{(1)} \dots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \dots Q_{i_k}^{(k-1)} = k!\alpha_1 f_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)}.$$

Now we have

$$\begin{aligned} f_1 A &= k!\alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=(1,i_2,\dots,i_k)}} f_1 Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)} e_\pi \\ &+ k!\alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=(i_1,i_2,\dots,i_k) \\ 1 \notin \pi}} f_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)} e_\pi \\ &= k!\alpha_1 \sum_{\substack{\pi \in \Pi_k(m) \\ \pi=(i_1,i_2,\dots,i_k)}} f_{i_1} Q_{i_2}^{(1)} \dots Q_{i_k}^{(k-1)} e_\pi \\ &= \alpha_1 \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} f_1 G_1 &= 1 \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1 \\ &= h_1 \end{aligned}$$

and

$$f_i G_1 = 0 \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1$$

$$= 0$$

for $i \neq 1$.

We need only show that $\|M_G\| < \infty$. By (14),

$$\|M_{G_1}\| = \|k \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \cdot v_1\|$$

$$\leq k\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ \vdots \\ Q_m^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(k-1)} \\ Q_2^{(k-1)} \\ \vdots \\ Q_m^{(k-1)} \end{pmatrix} \|\|M_{v_1}\|$$

$$= k!\| \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = 1, i_2, \ldots, i_k}} Q_{i_2}^{(1)} \ldots Q_{i_k}^{(k-1)} \|\|M_{v_1}\|$$

$$= k!\| \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = 1, i_2, \ldots, i_k}} Q_{i_2}^{(1)} \ldots Q_{i_k}^{(k-1)} \|\|M_{v_1}\|$$

$$\leq k! \left(\sum_{\substack{\pi \in \Pi_k(m) \\ \pi = 1, i_2, \ldots, i_k}} \|M_{Q_{i_2}^{(1)}}\| \ldots \|M_{Q_{i_k}^{(k-1)}}\| \right) \|M_{v_1}\|$$

$$\leq k! \left(\sum_{\substack{\pi \in \Pi_k(m) \\ \pi = 1, i_2, \ldots, i_k}} \|M_{f_{i_2}}\| \ldots \|M_{f_{i_k}}\| \right) \|M_{v_1}\|$$

$$\leq k! \sum_{\substack{\pi \in \Pi_k(m) \\ \pi = 1, i_2, \ldots, i_k}} \|M_{v_1}\|$$

$$= k! \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} \|M_{v_1}\|.$$

Again using the esimates in [**13**] for $\alpha(t) = t^{\frac{1}{2}}$, we get

$$\|M_G\| \leq mk! \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} \|M_{v_1}\| \leq mk! \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} \frac{K}{k!} \leq m \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} K.$$

This concludes our proof.

□

CHAPTER 8

FURTHER RESULTS

8.1. Improved Estimates

As noted in Section 2.3, one can improve the estimate in Wolff's theorem. The exponent "$\frac{3}{2}$" used in our hypotheses isn't optimal, but was used for convenience. Our theorem still holds when

$$[F(z)F(z)^*]^{\frac{3}{2}}$$

is replaced by

$$F(z)F(z)^*\alpha(F(z)F(z)^*,$$

where α is as in (8). In this case we would use lemma 6.1 instead of (14) to estimate $\|M_G\|$. Further improvements automatically extend to our theorem.

8.2. Extensions to Other Spaces

Although we restricted our attention to functions in $H^\infty(\mathbb{D})$, the methods used in the proof of Wolff's theorem for matrices apply to any algebra of functions that satisfies a Wolff theorem. (Note that some hypotheses may have to be changed. For example, on $H^\infty(\mathbb{D})$, $\|M_F\| = \|M_F^T\|$, but on other spaces we may have to stipulate that $\max\{\|M_F\|, \|M_F^T\|\} < \infty$.)

As an example, consider Dirichlet space, $\mathcal{D}^2(\mathbb{D})$, defined by

$$\begin{aligned}\mathcal{D}^2(\mathbb{D}) =&\{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic on } \mathbb{D},\\ &f(z) = \sum_{n=0}^{\infty} a_n z^n, \|f\|^2 = \sum_{n=0}^{\infty}(n+1)|a_n|^2 < \infty\}.\end{aligned}$$

Banjade [**2**] has recently proved that the algebra of multipliers on Dirichlet space, $M(\mathcal{D}^2(\mathbb{D}))$, satisfies a Wolff theorem: given $\{f_j\}_{j=1}^{\infty} \subset M(\mathcal{D}^2(\mathbb{D}))$ and $h \in M(\mathcal{D}^2(\mathbb{D}))$ such that

$$F(z)F(z)^*\alpha(F(z)F(z)^*) \geq |h(z)| \ \forall\ z \in \mathbb{D},$$

$$|F'(z)F^*(z)|\alpha(F(z)F(z)^*) \geq |h'(z)| \ \forall\ z \in \mathbb{D},$$

and

$$\|M_F\| \leq 1,$$

where $F(z) = (f_1(z), f_2(z), \dots)$ and α is as in (8), then there exists $\{g_j\}_{j=1}^{\infty} \subset M(\mathcal{D}^2(\mathbb{D}))$ with $G(z) = (g_1(z), g_2(z), \dots)$ such that $F(z)G(z)^T = h(z) \ \forall\ z \in \mathbb{D}$, and $\|M_G\| < \infty$.

Thus, given an $m \times \infty$ matrix $\mathcal{F}(z)$ of functions in $M(\mathcal{D}^2(\mathbb{D}))$ with $\max\{\operatorname{rank} \mathcal{F}(z) \mid z \in \mathbb{D}\} = k \leq m$ and an $m \times 1$ vector of functions in $M(\mathcal{D}^2(\mathbb{D}))$ such that

(i) $\det_k(\mathcal{F}(z)\mathcal{F}(z)^*)\alpha(\det_k(\mathcal{F}(z)\mathcal{F}(z)^*)) \geq |h_i(z)| \ \forall\ z \in \mathbb{D}$, $i = 1, \dots, m$

(ii) $\det_k(|\mathcal{F}'(z)\mathcal{F}(z)^*|)\alpha(\det_k(\mathcal{F}(z)\mathcal{F}(z)^*)) \geq |h'_i(z)| \ \forall\ z \in \mathbb{D}$, $i = 1, \dots, m$

(iii) $\|M_{\mathcal{F}}\| = 1$

(iv) there exists an arbitrary function $u : \mathbb{D} \to l^2$ such that $\mathcal{F}u = H$ everywhere on $\mathbb{D}$

then there exists an $\infty \times 1$ vector $\mathcal{G}(z)$ of functions in $M(\mathcal{D}^2(\mathbb{D}))$ such that

(a) $\mathcal{F}(z)\mathcal{G}(z)^T = H(z) \ \forall\ z \in \mathbb{D}$, and

(b) $\|M_{\mathcal{G}}\| < \infty$.

8.3. Radicals

As previously noted, Wolff's condition (6) is not sufficient to show (7) when "3" is replaced by "1". However, it is necessary *and* sufficient to show that h is contained in the radical of the ideal generated by the functions f_i.

We would like to show a similar result for the matrix case. Let F and H be as before, with $det_k(F(z)F(z)^*) \geq |h_i(z)|^n \; \forall \; z \in \mathbb{D}, i = 1, \ldots, m$, and for some $n \in \mathbb{N}$, where $k = \max\{\operatorname{rank} F(z) | z \in \mathbb{D}\}$. (Here H^n is the vector obtained by raising each entry of H to the nth power.) Suppose also that $\|M_F\| = 1$ and that we can find an arbitrary u such that $Fu = H$ on $\mathbb{D}$, as before. Then by Wolff's Theorem for Matrices, there exists an $\infty \times 1$-vector G with entries in $H^\infty(\mathbb{D})$ such that $FG = H^{3n}$ everywhere on $\mathbb{D}$.

On the other hand, suppose we have $FG = H^n$ for some $n \in \mathbb{N}$. Then

$$FG(G^*F^*) = H^n(H^n)^* \Rightarrow$$

$$F\|M_G\|^2 IF^* \geq H^n(H^n)^*$$

which, by a lemma in [**14**], implies

$$\det_1(\|M_G\|^2 FF^*) \geq \det_1(H^n(H^n)^*) = \sum_{i=1}^{m} |h_i|^{2n} \Rightarrow$$

$$C\det_1(FF^*) \geq |h_i|^{2n}$$

$\forall \; z \in \mathbb{D}, i = 1, \ldots, m$, where $C = \|M_G\|^{2m}$.

Note that if $k = \max\{\operatorname{rank} F(z) | n \in \mathbb{D}\} = 1$, then this second statement is the converse of the first. It is currently unknown whether the converse holds for $k > 1$.

8.4. When H Is a Matrix

Our theorem extends easily to the case where H is an $m \times n$ matrix. If F is $m \times \infty$, we seek an $\infty \times n$ matrix G such that $FG = H$. Wolff's Theorem for Matrices allows us to find G by finding its n columns $g_1, \ldots, g_n$.

What if we wish to solve an equation involving two (or more) matrices F_1 and F_2? That is, we wish to find G_1 and G_2 such that

$$F_1 G_1 + F_2 G_2 = H.$$

This is handled easily if we define $\mathcal{F} = [F_1\, F_2]$; that is, $\mathcal{F}$ is obtained by concatenating F_1 with F_2 (rearranging the entries in the case where F_1 and F_2 are $m \times \infty$). Then, provided hypotheses *(i)*, *(ii)*, and *(iii)* of our main theorem hold on $\mathcal{F}$, there exists $\mathcal{G}$ such that

$$\mathcal{F}\mathcal{G} = H,$$

and from $\mathcal{G}$ we obtain G_1 and G_2 such that

$$\mathcal{F}\mathcal{G} = F_1 G_1 + F_2 G_2.$$

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