A CONSTRUCTIVE NULLSTELLENSATZ FOR UNIVARIATE POLYNOMIALS

by

ANUPAN NETYANUN

TAVAN TRENT, COMMITTEE CHAIR
JAMES WANG
ROBERT MOORE
HYUN KWON
BEVERLY ROSKOS

A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the Graduate School of
The University of Alabama

TUSCALOOSA, ALABAMA

2014

ABSTRACT

In this dissertation, we will take an effective approach to prove the Hilbert's Nullstellensatz in a special case where we have univariate polynomials $f_i(z)'s$ for $i \in \{1, 2, ..., m\}$. This approach will explicitly construct polynomials $p_i(z)'s$ for $i \in \{1, 2, ..., m\}$. Moreover, we will get the best result on the bounds for the degrees of polynomials $p_i(z)'s$. We then use a similar technique to solve the problems in a matrix case.

Previous work motivated by algebraic techniques are from [2] W.D.Brownawell, [5] J.Kollar. They made a big improvement on the bounded degree of $p_i(z)'s$ in solutions. We are also motivated by works done in analysis from L. Carleson (1962), T. Wolff (1979). These are used to get the best result on the bounds on the degrees of $p_i's$ in the solutions obtained in this dissertation. For the matrix case, we are motivated by [11] T.T. Trent, X. Zhang. This will enable us to derive the results in the matrix case.

ACKNOWLEDGMENTS

I have to thank Professor Tavan Trent. As a teacher, he has shown and taught us the beauty of Analysis. As an advisor, he has introduced and helped me understand the material and guide me through the process of doing this thesis paper.

I have to thank all the committe members, Professor James Wang, Professor Robert Moore, Professor Hyun Kwon and Professor Beverly Roskos. Thank you so much for your time and your interest in this dissertation. I also have to thank all professors, friends and staffs in the mathematics department.

Thanks to my parents who raise this little ε to become a big M and who are mathematics teachers. You are the number one reason why I choose to pursue my study in mathematics. Thanks to my brother and all relatives who always supports me. Thanks to my girl friend, Kunyapass. No matter what happen, she always encourages me and always sends big cheers from another side of the world. I also have to thank my country that gives me a chance to study in the US. Last but not least, I would like to thanks the University of Alabama, the university of champions.

Roll Tide.

CONTENTS

CHAPTER 1

INTRODUCTION

In 1900, David Hilbert gave a proof to an important theorem, called *Hilbert's Nullstellensztz.*

THEOREM A (HILBERT'S NULLSTELLENSATZ). *Let* $f_1(x_1, ..., x_m) = 0$, $f_2(x_1, ..., x_m) = 0, ..., f_n(x_1, ..., x_m) = 0$ *be a system of polynomials in* $\mathbb{C}[x_1, ..., x_m]$, *then the system has no solution if and only if there exist polynomials* $p_1, p_2, ..., p_n$ *such that*

$$1 = \sum_{i=1}^{n} p_i(x_1, ..., x_m) f_i(x_1, ..., x_m). \tag{1.0.1}$$

This theorem give an important condition to tell whether the original system of polynomials has solutions. Hilbert proved the *Hilbert's Nullstellensatz* by using induction on the number of variables. As a result, we have no information for the $p_i's$. We only know if they exist. In 1926, G. Hermann first tried to find a good bound on the degrees of $p_i's$. However, it was doubly exponential in the number of variables. There were some improvement on some following studies. However, the results were still doubly exponential in the number of variables. In 1987, Brownawell made a big improvement on the bound on the degrees of $p_i's$. He established the following theorem:

THEOREM B. *Let* $f_1, ..., f_n \in \mathbb{C}[x_1, ..., x_m]$ *be such that* $f_i's$ *have no common zero. If degree* $f_i's \leq D$, *then there exists* $p_1, ..., p_n \in \mathbb{C}[x_1, ..., x_m]$ *such that*

$$\sum_{i=1}^{n} f_i p_i = 1 \tag{1.0.2}$$

and

$$deg\, g_i \leq m \min(n,m) D^{\min(n,m)} + \min(n,m) D\,. \tag{1.0.3}$$

Later on, in 1988, J. Kollar made some improvement to the previous theorem. He used an algebraic method of eliminating the degree of the $g_i's$ and found that one can drop off the term $m\min(n,m)$ in (1.0.3) in most cases.

To work on the condition, $1 = \sum_{i=1}^{n} p_i(x_1,...,x_m) f_i(x_1,...,x_m)$, we can employ a Linear Algebra approch to help solve the problem in a special case where $n = 2$ and $f_i's$ are univariate polynomials. We can transform the condition $1 = \sum_{i=1}^{n} p_i(x_1,...,x_m) f_i(x_1,...,x_m)$ to a matrix equation and we will check whether the determinant, called *Resultant*, of the matrix we form, called the *Sylvester matrix* is equal to 0. If the *Resultant* is not equal to 0, then $p_i's$ exist and this implies the original system has no solution. Otherwise, the original system has a solution. In this special case, we have an important fact as following:

LEMMA 1.0.1. *Let f and g be polynomials in $\mathbb{C}[x]$ such that f_1 and f_2 are nonzero and $deg(f_1) = c$ and $deg(f_2) = d$,then f_1 and f_2 have a common factor if and only if there exist p_1 and p_2 in $\mathbb{C}[x]$ such that*

(1) $p_1 f_1 + p_2 f_2 = 0$

(2) *$deg(p_1) < d$ and $deg(p_2) < c$.*

(3) *p_1 and p_2 are nonzero.*

We will give an example to illustrate this special case.

Let $f_1 = 2x^3 + 4x - 2$ and $f_2 = 2x^2 + 3x + 9$ so $p_1 = a_2x^2 + a_1x + a_0$ and $p_2 = b_1x + b_0$. We can then set

$$\begin{aligned}0 &= p_1 f_1 + p_2 f_2 \\ &= (a_1 x + a_0)(2x^3 + 4x - 2) + (b_2 x^2 + b_1 x + b_0)(2x^2 + 3x + 9) \\ &= (2a_1 + 2b_2)x^4 + (2a_0 + 3b_2 + 3b_1)x^3 + (4a_1 + 9b_2 + 3b_1 + 2b_0)x^2 + \\ &\quad (-2a_1 + 4a_0 + 9b_1 + 3b_0)x + (-2a_0 + 9b_0).\end{aligned}$$

Then we set the coefficients to be equal to zeros:

$$\begin{aligned}2a_1 + 2b_2 &= 0 \\ 2a_0 + 3b_2 + 3b_1 &= 0 \\ 4a_1 + 9b_2 + 3b_1 + 2b_0 &= 0 \\ -2a_1 + 4a_0 + 9b_1 + 3b_0 &= 0 \\ -2a_0 + 9b_0 &= 0\end{aligned}$$

and form the *Sylvester Matrix*, $S(f_1, f_2)$ and a matrix equation as follow:

$$\begin{bmatrix} 2 & 0 & 2 & 0 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 4 & 0 & 9 & 3 & 2 \\ -2 & 4 & 0 & 9 & 3 \\ 0 & -2 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} a_1 \\ a_0 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We then find the *Resultant*, $res(f_1, f_2) = det(S(f_1, f_2) = 1163 \neq 0$. Hence, f_1 and f_2 have no common factor.

We can apply this approach to the case where we have the number of univariate polynomials $f_i's$ greater than 2. However, we have to know a good bound for the degree of $p_i's$ to construct the *Sylvester matrix* efficiently. This approach still works when we have bivariate polynomials, $f_i's$. We can treat one variable as a constant and form the *Sylvester matrix* in terms of that variable. Finally, we have to check when we substitue values for that variable in the *Sylvester matrix*, whether or not the resultant will be always 0. Again, degrees of $p_i's$ are concerned. Things will be more complex and questionable when the number of variables is greater than 2.

We will introduce a theorem in algebra that is related to the condition of the *Hilbert's Nullstellensatz* and the study in this thesis.

THEOREM C. *Let I be the ideal in $\mathbb{C}[x]$ generated by $\{f_1, f_2, ..., f_n\}$ such that $f_1, f_2, ..., f_n \in \mathbb{C}[x]$ and $0 < |f_1(x)|^2 + |f_2(x)|^2 + ... + |f_n(x)|^2 \leq 1$, for all $x \in \mathbb{C}$, then $I = \mathbb{C}[x]$.*

PROOF. Since I is an ideal in $\mathbb{C}[x]$, and since $\mathbb{C}[x]$ is a *Principal Ideal Domain* , $I = (\mathbb{C}[x])d$ for some fixed $d \in \mathbb{C}[x]$. I is also generated by $\{f_1, f_2, ..., f_n\}$ so $d|f_i$ for each i. By the *Fundamental Theorem of Algebra*, each $f_i = C^i \prod_{j=1}^{N_i}(x - c_j)$ for some $N_i \in \mathbb{N}$ and $c_j, C^i \in \mathbb{C}$. As a result, $d = c$ for some $c \in \mathbb{C}$. This is because $f_i's$ have no zeroes in common, in other words, $f_i's$ have no factor of the form $(x - c)$ in common. Again since $\mathbb{C}$ is a field then d^{-1} exists and hence 1 is in I. This implies that $I = \mathbb{C}[x]$. □

In this thesis, we would like to take an analysis approach to prove the *Hilbert's Nullstellensatz Theorem* in a special case where the polynomials are univariate and the number of polynomials $f_i's$ is any fixed $n \in \mathbb{N}$. This approach can determine the best bound on the degrees of $g_i's$ in the theorem and also give explicit solutions. Also, we can generalize the approach to the case where f_i is a matrix of polynomials and we can determine an effective bound on the degrees of the polynomials for entries of the solution matrix.

The motivation of this thesis started in 1962. Lennart Carleson gave a proof to an important theorem called *Carleson's Corona Theorem*. This theorem answers a question raised by S. Kakutani in 1941 whether the open unit disc in the complex plane is dense in the maximal ideal space of the Banach algebra of bounded analytic functions.

DEFINITION 1.0.1. *The Banach algebra $H^\infty(\mathbb{D})$ is the collection of all analytic functions defined on the disc with*

$$||f||_{H^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

THEOREM D (CARLESON'S CORONA THEOREM). *Let $\{f_j\}_{j=1}^N \in H^\infty(\mathbb{D})$ which satisfy*

$$0 < \varepsilon \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \quad \textit{for all}\, z \in \mathbb{D}.$$

Then there are funciton $\{g_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^{N} f_j(z) g_j(z) = 1 \quad \textit{for all } z \in \mathbb{D}\, \textit{and}\, ||g_j||_\infty \leq C_{\varepsilon,N}.$$

There are serveral proofs to this theorem including Wolff (1979), Tolokonnikov (1980), Rosenblum (1980), Uchiyama (1980) and Trent (1998). In this thesis, we will use some important material from Wolff's proof. In 1979, Thomas Wolff gave a proof to Corona Theorem. However, it was not published but was mentioned in [Koosis 1980] and [Gamelin 1980]. His proof is much simplier than the original one of Carleson. In his proof, he used one differential operation:

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

From the definition, $\bar{\partial} f = 0$ in an open set $\Omega \subset \mathbb{C}$ for some function $f = u + iv$ if and only if

$$\begin{aligned} \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} &= 0 \\ \frac{\partial f}{\partial x} &= -i\frac{\partial f}{\partial y} \\ \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x} &= -i\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y} \end{aligned}$$

Hence

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \text{ and } \frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial x}$$

Therefore, f is analytic by the *Cauchy-Riemann* equations in Ω. The question now is that let $g(z)$ be C^1 with a compact support, can we find $f(z)$ such that

$$\bar{\partial} f(z) = g(z)$$

for all $z \in \mathbb{C}$ and $f(z) \in C^1(\Omega)$. It turns out that there are infinitely many solutions for $f(z)$. The choice that we will pick is call the Cauchy Transform of $g(z)$ and $f(z)$ is defined as:

$$f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta) d\xi d\eta}{z - \zeta},$$

such that $\zeta = \xi + i\eta$.

To show why we pick $f(z)$ to be our candidate for a solution, generalize the Green's formula for a complex case. By Green's formula, let Ω be a bounded region and $P(x,y)$, $Q(x,y) \in C^1(\overline{\Omega})$, we have

$$\int_{\partial\Omega} P\,dx + \int_{\partial\Omega} Q\,dy = \int_{\Omega} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\,dxdy. \tag{1.0.4}$$

Let $P = F$ and $Q = iF$ where $F(z) = F(x+iy)$. From (1.0.4), we have

$$\begin{aligned} \int_{\partial\Omega} F(z)\,dz &= \int_{\partial\Omega} F\,dx + iF dy \\ &= \int_{\Omega} (i\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y})\,dxdy \\ &= 2i \int_{\Omega} \frac{\partial F}{\partial \overline{z}}\,dxdy. \end{aligned} \tag{1.0.5}$$

Now let $F(z) = \frac{f(z)}{z-a}$ where $a \in \Omega$ and $\Lambda_\varepsilon = \Omega - \overline{D_{(a,\varepsilon)}}$ where $D_{(a,\varepsilon)}$ is a ball of radius ε around a. Also, let Γ be the boundary of Ω counterclockwise oriented and β be the boundary of $D_{(a,\varepsilon)}$ clockwise oriented. Then if we apply (1.0.5) to our new $F(z)$ and the region Λ_ε, we have

$$\begin{aligned} \int_{\partial\Lambda_\varepsilon} F(z)\,dz &= \int_{\Gamma} \frac{f(z)}{z-a}dz - \int_{\beta} \frac{f(z)}{z-a}dz \\ &= 2i \int_{\Lambda_\varepsilon} \frac{\partial F}{\partial \overline{z}}\,dxdy \\ &= 2i \int_{\Lambda_\varepsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} + f(z) \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a}\,dxdy \\ &= 2i \int_{\Lambda_\varepsilon} \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a} + f(z) \frac{\partial f}{\partial \overline{z}} \frac{1}{z-a}\,dxdy \end{aligned}$$

so

$$\int_\Gamma \frac{f(z)}{z-a}dz - \int_\beta \frac{f(z)}{z-a}dz = 2i\int_{\Lambda_\varepsilon} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy. \tag{1.0.6}$$

Since f is continuous and if we change the integrand to polar coordinate we have $\int \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a+\varepsilon e^{i\theta})d\theta$ which converges to $2\pi i f(a)$ as $\varepsilon \to 0$. Since $\frac{\partial f}{\partial \bar{z}}$ is continuous hence $f(z)$ is bounded by some $M \in R$ and $\frac{1}{z-a}$ is integrable over $D_{(a,\varepsilon)}$ then

$$\begin{aligned} |\int_{\Omega-\Lambda_\varepsilon} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy| &\leq \int_{\overline{D_{(a,\varepsilon)}}} |\frac{\partial f}{\partial \bar{z}}| \frac{1}{|z-a|}\, dxdy \\ &\leq M \int_{\overline{D_{(a,\varepsilon)}}} \frac{1}{|z-a|}\, dxdy \\ &= 2\pi M\varepsilon. \end{aligned}$$

This implies $|\int_{\Omega-\Lambda_\varepsilon} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy| \longrightarrow 0$ as $\varepsilon \to 0$. Hence

$$\int_\Omega \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy = \int_{\Lambda_\varepsilon} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy.$$

So from (1.0.6), we have

$$f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}dz - \frac{1}{\pi}\int_\Omega \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy. \tag{1.0.7}$$

(1.0.7) is a *General Cauchy's Formula*. Consider the case when $f(z)$ is $C^1(\mathbb{C})$ with compact support. For a fixed $a \in \mathbb{C}$, let Ω be $D_{(0,R)}$ that contains the support and a. So $\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}dz = 0$ and $-\frac{1}{\pi}\int_\Omega \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy$. So in the case where $f(z)$ has compact support we have

$$f(a) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} \frac{1}{z-a}\, dxdy. \tag{1.0.8}$$

If we let $a = z$ and $z = \zeta = \xi + i\eta$ and substitute them in (1.0.8), we will have

$$f(z) = -\frac{1}{\pi}\int_{\Omega} \frac{\partial f}{\partial \bar{z}} \frac{1}{\zeta - z}\, d\xi d\eta. \tag{1.0.9}$$

This suggest that a possible choice for a solution of $\bar{\partial} f(z) = g(z)$ is $f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z}\, d\xi d\eta$.

Then let's verify that it really is the solution.

LEMMA 1.0.2. *If $g \in C^1_{\mathbb{C}}(\mathbb{C})$ with compact support and define*

$$f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)}{z - \zeta} d\xi d\eta,$$

then

$$\bar{\partial} f(z) = g(z)$$

PROOF. Following [3], we will use substitution of variables to modify the equation (1.0.9). Let $z - \zeta = w = u + iv$ then

$$f(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(z-w)}{w}\, dudv.$$

For any fix $z_0 \in \mathbb{C}$ and $D_{(z_0,\varepsilon)}$ where ε is small, consider

$$\frac{g(z_0 - w + h) - g(z_0 - w)}{h} - \frac{\partial g}{\partial x}(z_0 - w) = \frac{1}{h}\int_0^h \left(\frac{\partial g}{\partial x}(z_0 - w + t) - \frac{\partial g}{\partial x}(z_0 - w) \right) dt. \tag{1.0.10}$$

Since $\frac{\partial g}{\partial x}$ is continuous and has compact support then $\frac{\partial g}{\partial x}$ is uniformly continuous on $\mathbb{C}$. So $\frac{1}{h}\int_0^h \left(\frac{\partial g}{\partial x}(z_0 - w + t) - \frac{\partial g}{\partial x}(z_0 - w)\right) dt \longrightarrow 0$ as $h \to 0$. Then from (1.0.10), we multiply both sides by the integrable function $\frac{1}{w}$ and integrate over $\mathbb{C}$ we have

$$\int_{\mathbb{C}} \frac{g(z_0 - w + h) - g(z_0 - w)}{h} \frac{1}{w} dudv - \int_{\mathbb{C}} \frac{\partial g}{\partial x}(z_0 - w)\frac{1}{w} dudv =$$
$$\int_{\mathbb{C}} \frac{1}{h} \int_0^h \left(\frac{\partial g}{\partial x}(z_0 - w + t) - \frac{\partial g}{\partial x}(z_0 - w)\right) dt \frac{1}{w} dudv,$$

$$-\pi \frac{f(z_0 - w + h) - f(z_0 - w)}{h} - \int_{\mathbb{C}} \frac{\partial g}{\partial x}(z_0 - w)\frac{1}{w} dudv =$$
$$\int_{\mathbb{C}} \frac{1}{h} \int_0^h \left(\frac{\partial g}{\partial x}(z_0 - w + t) - \frac{\partial g}{\partial x}(z_0 - w)\right) dt \frac{1}{w} dudv.$$

If we let $h \to 0$,we have

$$\frac{\partial f}{\partial x}(z_0) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial x}(z_0 - w)\frac{1}{w} dudv.$$

Similarly we have,

$$\frac{\partial f}{\partial y}(z_0) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial y}(z_0 - w)\frac{1}{w} dudv.$$

This shows that

$$\frac{\partial f}{\partial \overline{z}}(z_0) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial \overline{z}}(z_0 - w)\frac{1}{w} dudv$$

$$= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial \overline{z}}(\zeta)\frac{1}{\zeta - z_0} d\xi d\eta.$$

□

CHAPTER 2

MAIN THEOREMS, SKETCH OF THE PROOF AND LEMMAS

THEOREM 2.1. *Let* $F(z) = (p_1(z), p_2(z), ..., p_n(z))$ *be an* $1 \times n$ *matrix of polynomials* $p_i(z) \in \mathbb{C}(z)$ *for* $i \in \{1, 2, ..., n\}$ *and suppose that*

$$0 < F(z)F(z)^*.$$

Then there exists $Y(z)$, *an* $n \times 1$ *matrix of polynomials such that*

$$F(z)Y(z) = 1.$$

Moreover, the matrix $Y(z)$ *can be explicitly constructed and the maximum degree of the polynomial entries of* $Y(z)$ *are at most* 1 *less than the maximum degree of the polynomial entries of* $F(z)$.

THEOREM 2.2. *Let* $F(z) = [(p_{i,j}(z))]_{m \times n}$ *be an* $m \times n$ *matrix of polynomials* $p_{i,j}(z) \in \mathbb{C}(z)$ *for* $i \in \{1, 2, ..., m\}$*and* $j \in \{1, 2, ..., n\}$ *and*

$$\varepsilon(z)^2 I_m \leq F(z)F(z)^*,$$

where $\varepsilon(z) > 0$ *for all* $z \in \mathbb{C}$. *Then there exist* $Y(z)$, *an* $n \times m$ *matrix of polynomials such that*

$$F(z)Y(z) = I_m.$$

We also estimate the maximum degree of the polynomial entries of the matrix $Y(z)$.

In this thesis, we will first outline main steps that we will use to prove *Theorem 2.1* for the a special case, where $n = 2$. The case when $n = 2$ will give ideas of how to form $Y(z)$ and then it will give ideas of which properties of $Y(z)$ that we have to verify so that $Y(z)$ will be the matrix we need to solve $F(z)Y(z) = 1$.

Then we will modify some ideas of the Sketch of proof for a general case of *Theorem 2.1* and we will prove *Theorem 2.2* after that.

2.1. Sketch of *Theorem 2.1*'s proof

In this case, $F(z) = (p_1(z), p_2(z))$ and $0 < F(z)F(z)^*$. One possible choice for $Y(z)$ is $\frac{F^*(z)}{F(z)F^*(z)}$ since

$$F(z)(\frac{F^*(z)}{F(z)F^*(z)}) = 1.$$

However, $\frac{F^*(z)}{F(z)F*(z)}$ is not a matrix of analytic functions, much less polynomials. As a result, we want to add some terms to modify $\frac{F^*(z)}{F(z)F^*(z)}$ and hopefully the modified term will be our solution.

In the case when $n = 2$, we let $Q = \begin{bmatrix} p_2(z) \\ -p_1(z) \end{bmatrix}$. Then we notice that

$$F(z)Q(z) = 0.$$

We then modify the term $\frac{F^*}{FF^*}$ to be $\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)$, where $K(z)$ is some function of z. Since we want $\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)$, to be analytic. That means we have to show that $\bar{\partial}\left[\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)\right] = 0$ in $\mathbb{C}$. This will narrow down possible choices for $K(z)$ to be a Cauchy Transform of some functions. We then have to check that the Cauchy Transform is well-defined. Finally, we estimate and apply a *"Liouville type theorem"* to show that $\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)$ is a matrix of polynomials. Hence $Y(z) = \frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)$ will be our solution.

2.2. LEMMAS

We will first establish and prove several lemmas before we prove the main theorems. The following lemma can be found in [9]. For convenience, we will give a proof.

LEMMA 2.2.1. *Let* $\{c_j\}_{j=1}^{\infty} \in l^2$ *and* $C = (c_1, c_2, ...) \in B(l^2, \mathbb{C})$. *Then there exists* Q *such that entries of* Q *are either* 0 *or* $\pm c_j$ *for some* j *and*

$$CC^*I - C^*C = QQ^*.$$

PROOF. For $k \in \mathbb{N}$, define

$$A_k = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ c_{k+1} & c_{k+2} & c_{k+3} & \cdots \\ -c_k & 0 & 0 & \cdots \\ 0 & -c_k & 0 & \cdots \\ 0 & 0 & -c_k & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then by matrix multiplication,

$$A_kA_k^* = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \sum_{j=k+1}^{\infty} |c_j|^2 & -\bar{c}_k c_{k+2} & -\bar{c}_k c_{k+3} & \cdots \\ 0 & \cdots & 0 & -c_k\bar{c}_{k+2} & |c_k|^2 & 0 & \cdots \\ 0 & \cdots & 0 & -c_k\bar{c}_{k+3} & 0 & |c_k|^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It yields

$$\sum_{k=1}^{\infty} A_k A_k^* = \begin{bmatrix} \sum_{k\neq 1}^{\infty} |c_k|^2 & -\bar{c}_1 c_2 & -\bar{c}_1 c_3 & \cdots \\ -\bar{c}_2 c_1 & \sum_{k\neq 2}^{\infty} |c_k|^2 & -\bar{c}_2 c_3 & \cdots \\ -\bar{c}_3 c_1 & -\bar{c}_3 c_2 & \sum_{k\neq 3}^{\infty} |c_k|^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= CC^* I - C^* C.$$

Then we can define Q as

$$Q = [A_1, A_2, ...] \in B(\bigoplus_1^{\infty} l^2,\, l^2).$$

□

By the assumption on $F(z)$ that $0 < F(z)F(z)^*$, and $F(z) \in \mathbb{C}^n$, we have

$$(F(z)F(z)^*)I - F(z)^* F(z) = Q(z)Q(z)^*.$$

Moreover, each entry of $Q(z)$ is 0, $p_j(z)$, or $-p_j(z)$ for some $j = 1, ..., n$. Also for each fixed z, $F(z)^*(F(z)F(z)^*)^{-1}F(z)$ is the projection of $\mathbb{C}^n$ onto the kernel of $F(z)$. Thus, range $Q(z) =$ kernel $F(z)$. For $\underline{c} = (c_1, c_2, ...)$, let S be the backward shift operation where $S(\underline{c}) = (c_2, c_3, ...)$.Define $Q_{\underline{c}}^{(0)} = (c_1, c_2, ...)$.

Also we define

$$H_0 = \mathbb{C}$$

$$H_1 = \bigoplus_1^{\infty} H_0 = l^2$$

$$\vdots$$

$$H_{n+1} = \bigoplus_1^{\infty} H_n.$$

Then we can define

$$A_k^{(j)}(\underline{c}) = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ & & & \cdots \\ & & -Q^{(j-1)}_{S^{(k)}(\underline{c})} & \\ & & c_k I & \end{bmatrix},$$

where the first $k-1$ rows have all zeoro entries. In the k^{th}row is $-Q^{(j-1)}_{S^{(k)}(\underline{c})}$. The rest will be $c_k I$. As a result, we can define

$$Q^{(j)}_{\underline{c}} = (A_1^{(j)}(\underline{c}),\, A_1^{(j)}(\underline{c}),\, ...) \,:\, H_{j+1} \to H_j. \tag{2.2.1}$$

More generally, we have the following lemma. Its proof can be found in [7] J. Ryle and T.T. Trent and [10] T.T. Trent and X. Zhang.

Lemma 2.2.2. *Suppose* $F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$ *and* $G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}$ *where* $f_i,\, g_i \in l^2$ *for* $i \in \{1, 2, ..., m\}$. *Define* $Q_F = Q^{(1)}_{f_1} Q^{(2)}_{f_2} ... Q^{(m)}_{f_m}$. *Similarly, define* $Q_G = Q^{(1)}_{g_1} Q^{(2)}_{g_2} ... Q^{(m)}_{g_m}$. *If* $FG^T = I_m$, *then*

- $G^T F + Q_F Q_G^T = I$
- $-Q^{(k)}_{f_i} Q^{(k+1)}_{f_{i+1}} = Q^{(k)}_{f_{i+1}} Q^{(k+1)}_{f_i}$
- $f_1 Q^{(1)}_{f_2} ... Q^{(j)}_{f_j} Q^{(j)^*}_{f_j} ... f_1^* = det(F E_j F^*)$ *where* $E_j = \begin{bmatrix} I_j & 0 \\ 0 & 0 \end{bmatrix}$.

By using the lemma, one can deduce that

$$range\, Q_{F(z)} = kernel\, F(z)$$

Lemma 2.2.3. *If $\psi \in C^1_{\mathbb{C}}(\mathbb{C})$ with compact support and define*

$$\hat{\psi}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\psi(w)}{z-w} dA(w)$$

then

$$\bar{\partial}\hat{\psi}(z) = \psi(z).$$

The proof of this lemma can be found in the introduction.

Lemma 2.2.4. *For any $r > 0$ and $\varepsilon > 0$, there exists $\varphi_{r,\varepsilon} \in C^{\infty}_{\mathbb{C}}$ such that*

$$\varphi_{r,\varepsilon}(z) = \begin{cases} 0, \, when\, |z| \geq r \\ 1, \, when\, |z| \leq r - \varepsilon \end{cases}.$$

The function in previous Lemma is called a *Partition of Unity Function.* More details can be found in [8] M. Spivak. The next Lemma is a result from Lemma 2.2.3 and Lemma 2.2.4.

Lemma 2.2.5. *For $R > 0$ and $\varepsilon > 0$, and $h(z) \in C^1(D_{(0,R)}) \cap L^1(\mathbb{C})$, define*

$$\hat{h}(z) = -\frac{1}{\pi}\int_{D_{(0,R)}} \frac{h(w)}{z-w} dA(w).$$

Then we have

$$\bar{\partial}\hat{h}(z) = \begin{cases} h(z), \, when\, z \in D_{(0,R)} \\ 0, \, when\, z \notin D_{(0,R)} \end{cases}.$$

Proof. Since

$$h(z) = h(z)\varphi_{R,\varepsilon}(z) + h(z)(1 - \varphi_{R,\varepsilon}(z)),$$

then

$$\hat{h}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(w)\varphi_{R,\varepsilon}(w)}{z-w} dA(w) + -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(w)(1-\varphi_{R,\varepsilon}(w))}{z-w} dA(w).$$

For $z \in D_{(0,R-\varepsilon)}$, we have

$$\bar{\partial}\hat{h}(z) = \bar{\partial}\left[-\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(w)\varphi_{R,\varepsilon}(w)}{z-w} dA(w)\right] + \bar{\partial}\left[-\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(w)(1-\varphi_{R,\varepsilon}(w))}{z-w} dA(w)\right].$$

Since $z \in D_{(0,R-\varepsilon)}$ and $h(w)\varphi_{R,\varepsilon}(w) \in C^1_{\mathbb{C}}(\mathbb{C})$ with a compact support, by Lemma 1.0.2, we have

$$\bar{\partial}\left[-\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(w)\varphi_{R,\varepsilon}(z)}{z-w} dA(w)\right] = h(z)\varphi_{R,\varepsilon}(z) = h(z).$$

On the other hand, $-\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(w)(1-\varphi_{R,\varepsilon}(w))}{z-w} dA(w) = -\frac{1}{\pi}\int_{\mathbb{C}-D_{(0,R-\varepsilon)}} \frac{h(w)}{z-w} dA(w)$.

We have $\mathbb{C}-D_{(0,R-\varepsilon)}$ is compact and $h(z) \in C^1(D_{(0,R)}) \cap L^1(D_{(0,R)})$, which implies that $\int_{\mathbb{C}-D_{(0,R-\varepsilon)}} |h(w)| dA(w) < \infty$. For $z \in D_{(0,R-\varepsilon)}$ we can define a function $f(z)$ as:

$$z \overset{f}{\longmapsto} \int_{\mathbb{C}-D_{(0,R-\varepsilon)}} \frac{h(w)}{z-w} dA(w).$$

Claim that that $f(z)$ is analytic in $D_{(0,R-\varepsilon)}$. Fix $z \in D_{(0,R-\varepsilon)}$,

$$\begin{aligned} \frac{f(z+h)-f(z)}{h} &= \int_{\mathbb{C}-D_{(0,R-\varepsilon)}} h(w)\left[\frac{\frac{1}{z+h-w}-\frac{1}{z-w}}{h}\right] dA(w) \\ &= \int_{\mathbb{C}-D_{(0,R-\varepsilon)}} \frac{h(w)}{(z-w)(z+h-w)} dA(w) \end{aligned}$$

Let $h_n \to 0$ and $0 < |h_n| < \frac{R-\varepsilon-|z|}{2}$, define

$$g_n(w) = \frac{h(w)}{(z-w)(z+h_n-w)}.$$

We have $g_n(w)$ converges pointwise to $\frac{h(w)}{(z-w)^2}$. Since $z \in D_{(0,R-\varepsilon)}$ and $w \in \mathbb{C} - D_{(0,R-\varepsilon)}$, we have, for $0 < |h_n| < \frac{R-\varepsilon-|z|}{2}$,

$$R - \varepsilon - |z| < |z - w| \,\text{and}\, (R - \varepsilon - |z + h|) < |z + h - w|.$$

We then have

$$\begin{aligned}
|g_n(w)| &\leq \frac{|h(w)|}{|z-w||z+h_n-w|} \\
&\leq \frac{|h(w)|}{|R-\varepsilon-|z|||(R-\varepsilon-|z+h_n|)|} \\
&\leq \frac{|h(w)|}{|R-\varepsilon-|z|||(R-\varepsilon-|z+h_n|)|} \\
&\leq \frac{|h(w)|}{|R-\varepsilon-|z|||(R-\varepsilon-|z|-\frac{R-\varepsilon-|z|}{2})|}.
\end{aligned}$$

Since $h(w)$ is integrable so $\frac{|h(w)|}{|R-\varepsilon-|z|||(R-\varepsilon-|z|-\frac{R-\varepsilon-|z|}{2})|}$ is integrable. So by the *Dominated Convergence Theorem*, we have

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \int_{\mathbb{C}-D_{(0,R-\varepsilon)}} g_n(w)\, dA(w)$$

$$= \int_{\mathbb{C}-D_{(0,R-\varepsilon)}} \frac{h(w)}{(z-w)^2}\, dA(w) < \infty$$

Therefore $f'(z)$ exists and $f(z)$ is analytic in $D_{(0,R-\varepsilon)}$.

The claim implies, for $z \in D_{(0,R-\varepsilon)}$,

$$\bar{\partial}\left[-\frac{1}{\pi}\int_{\mathbb{C}-D_{(0,R-\varepsilon)}} \frac{h(w)}{z-w} dA(w)\right] = 0.$$

Since all results are true for all $\varepsilon > 0$, this implies

$$\bar{\partial}\hat{h}(z) = \begin{cases} h(z)\, when\, z \in D_{(0,R)} \\ 0\, when\, z \notin D_{(0,R)} \end{cases} .$$

□

LEMMA 2.2.6. *Let* $p(z) = a_0 + a_1 z ... + a_n z^n$ *where* $a_0, a_1, ..., a_n \in \mathbb{C}$ *and* $a_n \neq 0$. *For* $|z| \geq max\{1, \sqrt{\frac{2\sum_{j=0}^{n-1}|a_j|}{|a_n|}}\}$. *We have, for* $|z| > 1$,

$$|p(z)| \geq |z|^n \left(\frac{|a_n|}{2}\right).$$

PROOF. By the *reverse of triangle inequality*, we have

$$\begin{aligned} |p(z)| &\geq ||a_n||z|^n - |\sum_{j=0}^{n-1} a_j z^j|| \\ &\geq |z|^n \left[|a_n| - \sum_{j=0}^{n-1} \frac{|a_j|}{|z|^{n-j}}\right] \\ &\geq |z|^n \left[|a_n| - \sum_{j=0}^{n-1} \frac{|a_j|}{|z|}\right]. \end{aligned}$$

Since $|z| \geq max\{1, \sqrt{\frac{2\sum_{j=0}^{n-1}|a_j|}{|a_n|}}\}$, we have

$$|p(z)| \geq |z|^n \left(\frac{|a_n|}{2}\right).$$

□

An immediate consequence of Lemma 2.2.6 is as follows.

Let m be a maximum degree of $p_i's$, where each p_i is an entry of F and let a_j denote the coefficient of the term of p_j with maximum degree. For a large $z \in \mathbb{C}$, and $|z| \geq 1$,

$$\begin{aligned} |F(z)F(z)^*| &= \sum_{j=1}^{n} |p_j(z)|^2 \\ &\geq \sum_{j=1}^{n} \frac{|a_j|^2}{4} |z|^{2(degree\, of\, p_j)}. \end{aligned}$$

This is from Lemma 2.2.6. If p_{j_0} has degree m, then

$$|F(z)F(z)^*| \geq \frac{|a_{j_0}|^2}{4} |z|^{2m}. \tag{2.2.2}$$

LEMMA 2.2.7. *If $Y(z)$ is an entire function on $\mathbb{C}$, and if there exists m so that*

$$\lim_{|z| \to \infty} \frac{|Y(z)|}{|z|^m} \longrightarrow 0,$$

then $Y(z)$ is a polynomial of a degree at most $m - 1$.

PROOF. Since $\lim_{|z|\to\infty} \frac{|Y(z)|}{|z|^m} \longrightarrow 0$, for any fixed $\varepsilon \in \mathbb{R}$, there exists R_ε such that

$$\frac{|Y(z)|}{|z|^m} < \varepsilon,$$

for $|z| > R_\varepsilon$. So

$$|Y(z)| < \varepsilon |z|^m. \tag{2.2.3}$$

Since $Y(z)$ is entire then we can express $Y(z)$ as

$$Y(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for $a_n \in \mathbb{C}$ where $n \in \mathbb{N} \cup \{0\}$. Then for any $R \in \mathbb{R}$ and from *Cauchy Integral Formula*, we have that

$$a_n = \frac{Y^n(0)}{n!} = \frac{1}{2\pi i} \int_{\partial D_{(0,R)}} \frac{f(z)}{z^{n+1}} dz.$$

Consider $|a_{m+p}|$. Let $\varepsilon > 0$ and use $R = R_\varepsilon$. Then, from (2.2.3), we have

$$\begin{aligned} |a_{m+p}| &\leq \frac{1}{2\pi}\int_{\partial D_{(0,R)}} \frac{\varepsilon|z|^m}{|z|^{m+p+1}}|dz| \\ &= \frac{\varepsilon}{2\pi R^{p+1}}\int_{\partial D_{(0,R)}} 1|dz| \\ &= \frac{\varepsilon}{R^p}. \end{aligned}$$

Since this is true for all $p \geq 1$, $R > 0$ and for any $\varepsilon > 0$, then

$$a_j = 0$$

for all $j \geq m$. This implies $Y(z)$ is a polynomial of degree at most $m-1$. □

Now we are ready to prove *Theorem 2.1* and *Theorem 2.2*.

2.3. Proof of Theorem 2.1

THEOREM 2.1. *Let $F(z) = (p_1(z), p_2(z), ..., p_n(z))$ be an $1 \times n$ matrix of polynomials $p_i(z) \in \mathbb{C}(z)$ for $i \in \{1, 2, ..., n\}$ and*

$$0 < F(z)F(z)^*.$$

Then there exists $Y(z)$, an $n \times 1$ matrix of polynomials such that

$$F(z)Y(z) = 1.$$

PROOF. Lemma 2.2.1 gives us a choice of $Q(z)$ in which range of $Q(z) =$kernel of $F(z)$. As a result, we will use the fact that the range of $Q(z)$ acting on polynomials is the kenel of $F(z)$ to modify the term $\frac{F^*(z)}{F(z)F^*(z)}$ to be $\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)$. This yields

$$F(z)(\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)) = 1.$$

First we have to check that $\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)$ is analytic. $\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)$ is analytic if and only if

$$\bar{\partial}\left(\frac{F^*(z)}{F(z)F^*(z)} - Q(z)K(z)\right) = 0 \text{ in } \mathbb{C}.$$

Then

$$\frac{F'^*(z)}{F(z)F^*(z)} - \frac{F(z)F'^*(z)F^*(z)}{(F(z)F^*(z))^2} - Q(z)\bar{\partial}K(z) \ = 0$$

and

$$\left(I_2 - \frac{F^*(z)F(z)}{F(z)F^*(z)}\right)\left(\frac{F'^*(z)}{F(z)F^*(z)}\right) - Q(z)\bar{\partial}K(z) \ = 0.$$

Since $\left(I_2 - \frac{F^*(z)F(z)}{F(z)F^*(z)}\right) = \frac{Q(z)Q^*(z)}{F(z)F^*(z)}$, we have

$$Q(z)\left[\left(\frac{Q^*(z)}{F(z)F^*(z)}\right)\left(\frac{F'^*(z)}{F(z)F^*(z)}\right) - \bar{\partial}K(z)\right] = 0. \tag{2.3.1}$$

Then we pick the following solution for the equation (2.3.1):

$$\left(\frac{Q^*(z)}{F(z)F^*(z)}\right)\left(\frac{F'^*(z)}{F(z)F^*(z)}\right) - \bar{\partial}K(z) \ = 0.$$

As a result,

$$\bar{\partial}K(z) \ = \left(\frac{Q^*(z)}{F(z)F^*(z)}\right)\left(\frac{F'^*(z)}{F(z)F^*(z)}\right).$$

By using the idea of *Cauchy Transform*, we let

$$G(z) = \left(\frac{Q^*(z)}{F(z)F^*(z)}\right)\left(\frac{F'^*(z)}{F(z)F^*(z)}\right)$$

and define

$$K(z) = \widehat{G}(z) = \left[\frac{Q^*\widehat{(z)F^{*'}}(z)}{(F(z)F^*(z))^2}\right] = \int_{\mathbb{C}} \frac{Q^*(w)F^{*'}(w)}{(F(w)F^*(w))^2} . \frac{dA(w)}{z-w}$$

All of the above is a formal argument. To make this precise, we must first show that $G \in L^1(dA)$.

Claim: $G \in L^1(dA)$.

PROOF. Clearly, G is smooth on $\mathbb{C}$. Let R be large enough, then by (2.2.2) and for the $(i,j)^{th}$ entry of $G(z)$ and for some $D_0 \in \mathbb{C}$ we have,

$$\begin{aligned} \int_{\mathbb{C}-D_{(0,R)}} |G(z)|_{(i,j)} d(A(z) &\leq \int_{\mathbb{C}-D_{(0,R)}} D_0 \frac{|z|^{2m-1}}{|z|^{4m}} dA(z) \\ &= D_0 \int_{\mathbb{C}-D_{(0,R)}} |z|^{-2m-1} dA(z) \\ &< \infty \end{aligned}$$

since $m \geq 1$. Hence $G \in L^1(dA)$. The claim is established. □

We claim that $\bar{\partial}\widehat{G}(z) = G(z)$.

LEMMA 2.3.1. *Assume* $G \in C^1(\mathbb{C}) \cap L^1(dA)$. *Then* $\bar{\partial}\widehat{G}(z) = G(z)$.

PROOF. For a fix $z \in \mathbb{C}$, we select $1 < R$ to be large enough so that $z \in D_{(0,R)}$. Then

$$\widehat{G}(z) = -\frac{1}{\pi}\int_{D_{(0,R)}} \frac{G(w)}{z-w} dA(w) + -\frac{1}{\pi}\int_{\mathbb{C}-D_{(0,R)}} \frac{G(w)}{z-w} dA(w)$$

and

$$\bar{\partial}\widehat{G}(z) = \bar{\partial}\left[-\frac{1}{\pi}\int_{D_{(0,R)}} \frac{G(w)}{z-w} dA(w)\right] + \bar{\partial}\left[-\frac{1}{\pi}\int_{\mathbb{C}-D_{(0,R)}} \frac{G(w)}{z-w} dA(w)\right].$$

From Lemma 2.2.5, we have

$$\bar{\partial}\left[-\frac{1}{\pi}\int_{D_{(0,R)}}\frac{G(w)}{z-w}dA(w)\right]=G(z).$$

If we use the same technique as in the proof of Lemma 2.2.5, we define the function f as

$$z\overset{f}{\longmapsto}\int_{\mathbb{C}-D_{(0,R)}}\frac{G(w)}{z-w}dA(w).$$

Then using the fact that $G(z)\in L^1(\mathbb{C})$, we see that, as in the proof of Lemma 2.2.5, for $z\in D_{(0,R)}$,

$$f^{'}(z)=-\int_{\mathbb{C}-D_{(0,R)}}\frac{G(w)}{(z-w)^2}dA(w).$$

Hence

$$\bar{\partial}\left[-\frac{1}{\pi}\int_{\mathbb{C}-D_{(0,R)}}\frac{G(w)}{z-w}dA(w)\right]=0.$$

So $\bar{\partial}\widehat{G}(z)=G(z)$ as desired. Lemma 2.3.1 is established.

□

The last part of the proof is to show that our $Y(z)=\frac{F^*(z)}{F(z)F^*(z)}-Q(z)K(z)$ is actually a matrix of polynomials. So we need to use Lemma 2.2.7. That is we must show that

$$\lim_{z\to\infty}\frac{||Y(z)||}{|z|^m}\longrightarrow 0.$$

We will consider each term separately. First, let $\left[\frac{F^*(z)}{F(z)F^*(z)}\right]_{i,j}$ be the $(i,j)^{th}$ entry for $\frac{F^*(z)}{F(z)F^*(z)}$. From Lemma 2.2.6 and when $|z|$ get large enough, we have the upper bound,

$$|\left[\frac{F^*(z)}{F(z)F^*(z)}\right]_{i,j}| \leq C_0|z|^{-m}.$$

for some $C_0 \in \mathbb{R}$. This shows that

$$\lim_{z\to\infty}\left\|\frac{F^*(z)}{F(z)F^*(z)}\right\| = 0.$$

Secondly, we will consider $\lim_{z\to\infty}\|K(z)\|$. For $z \in \mathbb{C}$ where $|z|$ large enough, we will divide $\mathbb{C}$ into 3 regions which are

- $D_{(0,1)}$
- $D_{(z,\frac{|z|}{10})}$
- $\mathbb{C} - D_{(0,1)} - D_{(z,\frac{|z|}{10})}$.

Then we consider

- $\left\|\int_{D_{(0,1)}}\frac{G(w)}{z-w}dA(w)\right\|$
- $\left\|\int_{D_{(z,\frac{|z|}{10})}}\frac{G(w)}{z-w}dA(w)\right\|$
- $\left\|\int_{\mathbb{C}-D_{(0,1)}-D_{(z,\frac{|z|}{10})}}\frac{G(w)}{z-w}dA(w)\right\|$.

The $(i,j)^{th}$ entry of $\left[\int_{D_{(0,1)}}\frac{G(w)}{z-w}dA(w)\right]$ is $\int_{D_{(0,1)}}\frac{[G(w)]_{(i,j)}}{z-w}dA(w)$. Then for some $M \in \mathbb{R}$, we have

$$\begin{aligned}
|\int_{D_{(0,1)}}\frac{[G(w)]_{(i,j)}}{z-w}dA(w)| &\leq \int_{D_{(0,1)}}\frac{|[G(w)]_{(i,j)}|}{|z-w|}dA(w) \\
&\leq \frac{M}{|z|-1}\int_{D_{(0,1)}}1\,dA(w) \\
&= \frac{M\pi}{|z|-1}.
\end{aligned}$$

As a result, $\lim_{z\to\infty}\left\|\int_{D_{(0,1)}}\frac{G(w)}{z-w}dA(w)\right\| = 0.$

Next, the $(i,j)^{th}$ entry of $\left[\int_{\mathbb{C}-D_{(0,1)}-D_{(z,\frac{|z|}{10})}} \frac{G(w)}{z-w} dA(w)\right]$ is

$$\int_{\mathbb{C}-D_{(0,1)}-D_{(z,\frac{|z|}{10})}} \frac{[G(w)]_{(i,j)}}{z-w} dA(w).$$

From Lemma 2.2.6, we know that $|[G(w)]_{(i,j)}| \leq C_1|z|^{-2m-1}$for some $C_1 \in \mathbb{R}$ and $m \geq 1$ so $\int_{\mathbb{C}-D_{(0,1)}-D_{(z,R)}} |[G(w)]_{(i,j)}| \, dA(w) \leq \int_{\mathbb{C}-D_{(0,1)}} |[G(w)]_{i,j}| dA(w) = C_2 \in \mathbb{C}$.

$$\begin{aligned} \left| \int_{\mathbb{C}-D_{(0,1)}-D_{(z,\frac{|z|}{10})}} \frac{[G(w)]_{(i,j)}}{z-w} dA(w) \right| &\leq \int_{\mathbb{C}-D_{(0,1)}-D_{(z,\frac{|z|}{10})}} \frac{|[G(w)]_{(i,j)}|}{|z-w|} dA(w) \\ &\leq \frac{10}{|z|} \int_{\mathbb{C}-D_{(0,1)}-D_{(z,R)}} |[G(w)]_{(i,j)}| \, dA(w) \\ &\leq \frac{10C_2}{|z|}. \end{aligned}$$

As a result, $\lim_{z\to\infty} \left\| \int_{\mathbb{C}-D_{(0,1)}-D_{(z,\frac{|z|}{10})}} \frac{G(w)}{z-w} dA(w) \right\| = 0$.

Lastly, the $(i,j)^{th}$ entry of $\left[\int_{D_{(z,\frac{|z|}{10})}} \frac{G(w)}{z-w} dA(w)\right]$ is $\int_{D_{(z,\frac{|z|}{10})}} \frac{[G(w)]_{(i,j)}}{z-w} dA(w)$. From Lemma 2.2.6, we know that $|[G(w)]_{(i,j)}| \leq C_3|z|^{-2m-1}$for some $C_3 \in \mathbb{R}$ and $m \geq 1$ so

$$
\begin{aligned}
\left| \int_{D_{(z,\frac{|z|}{10})}} \frac{[G(w)]_{(i,j)}}{z-w} dA(w) \right| &\leq \left| \int_{D_{(z,\frac{|z|}{10})}} \frac{|[G(w)]_{(i,j)}|}{|z-w|} dA(w) \right| \\
&\leq \int_{D_{(z,\frac{|z|}{10})}} \frac{C_3 |w|^{-2m-1}}{|z-w|} dA(w) \\
&\leq \int_{D_{(z,\frac{|z|}{10})}} \frac{C_3 (\frac{9|z|}{10})^{-2m-1}}{|z-w|} dA(w) \\
&\leq 2\pi C_3 (\frac{9|z|}{10})^{-2m-1} \int_0^{\frac{|z|}{10}} 1 \, dr \\
&= 2\pi C_3 (\frac{9|z|}{10})^{-2m-1} (\frac{|z|}{10}) \\
&\leq \frac{2\pi 10^{2m} C_3}{9^{2m+1} |z|^{2m}}.
\end{aligned}
$$

Since $m \geq 1$, $\lim_{z\to\infty} \int_{D_{(z,\frac{|z|}{10})}} \left| \frac{[G(w)]_{(i,j)}}{z-w} dA(w) \right| = 0$. This again implies that $\lim_{z\to\infty} \left\| \int_{D_{(z,\frac{|z|}{10})}} \frac{G(w)}{z-w} dA(w) \right\| = 0$.

From $Y(z) = \frac{F^*}{FF^*} - QK$, we consider

$$
\begin{aligned}
\lim_{z\to\infty} \frac{\|Y(z)\|}{|z|^m} &\leq \lim_{z\to\infty} \frac{\left\| \frac{F^*(z)}{F(z)F^*(z)} \right\| + \|Q(z)\| \, \|K(z)\|}{|z|^m} \\
&\leq \lim_{z\to\infty} \frac{\left\| \frac{F^*(z)}{F(z)F^*(z)} \right\|}{|z|^m} + \lim_{z\to\infty} \frac{\|Q(z)\|}{|z|^m} \lim_{z\to\infty} \|K(z)\|
\end{aligned}
$$

The previous steps shows that $\lim_{z\to\infty} \frac{\left\| \frac{F^*(z)}{F(z)F^*(z)} \right\|}{|z|^m} = 0$ and $\lim_{z\to\infty} \|K(z)\| = 0$ this forces $\lim_{z\to\infty} \frac{\left\| \frac{F^*(z)}{F(z)F^*(z)} \right\|}{|z|^m} + \lim_{z\to\infty} \frac{\|Q(z)\|}{|z|^m} \lim_{z\to\infty} \|K(z)\| = 0$. Hence

$$
\lim_{z\to\infty} \frac{\|Y(z)\|}{|z|^m} = 0.
$$

□

Since $\lim_{z \to \infty} \frac{\|Y(z)\|}{|z|^m} = 0$, from Lemma 2.2.7 , the entries of $Y(z)$ are polynomials of a degree at most $m-1$.

CHAPTER 3

Theorem 2.2 (Matrix Case)

In this chapter, we will use the result from *Theorem 2.1* and apply to a matrix case. Since matrix multiplication is not commutative. We will consider the solution for a right multiplication.

3.1. Existance of a solution $Y(z)$

THEOREM 2.2. *Let* $F(z) = [(p_{i,j}(z))]_{m\times n}$ *be an* $m \times n$ *matrix of polynomials* $p_{i,j}(z) \in \mathbb{C}(z)$ *for* $i \in \{1, 2, ..., m\}$*and* $j \in \{1, 2, ..., n\}$ *and*

$$\varepsilon(z)^2 I_m \leq F(z)F(z)^*,$$

where $\varepsilon(z) > 0$ *for all* $z \in \mathbb{C}$. *Then there exist* $Y(z)$, *an* $n \times m$ *matrix of polynomials such that*

$$F(z)Y(z) = I_m.$$

We also get an estimate for the maximum degree of any entry of $Y(z)$.

PROOF. We have $F(z) = \begin{bmatrix} p_1(z) \\ \vdots \\ p_m(z) \end{bmatrix}$ where each $p_i(z)$ is a row vector consisting of n polynomial entries for $i \in \{1, 2, ..., m\}$. We want to find $Y(z) = \begin{bmatrix} y_1(z) & \dots & y_m(z) \end{bmatrix}$where each y_i is a column vector consisting of n polynomial entries for $i \in \{1, 2, ..., m\}$. Moreover, $Y(z)$ satisfies

$$F(z)Y(z) = I_m$$

To find $Y(z)$, it suffices to figure out what each $y_i(z)$ would be. We can work with the first column vector $y_1(z)$ of $Y(z)$ and we take the same approach to the rest of the column vectors $y_i(z)'s$.

Since $y_1(z)$ is the first column of the solution $Y(z)$, we have

$$F(z)y_1(z) = \begin{bmatrix} p_1(z) \\ p_2(z) \\ \vdots \\ p_m(z) \end{bmatrix} y_1(z) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

That is

$$p_1(z)y_1(z) = 1 \tag{3.1.1}$$

$$p_2(z)y_1(z) = 0 \tag{3.1.2}$$

$$\vdots$$

$$p_m(z)y_1(z) = 0. \tag{3.1.3}$$

Since for all $z \in \mathbb{C}$, we have $F(z)F(z)^* > 0$, so $det(F(z)F(z)^* > 0$. This implies $F(z)$ has a full rank. So, from Lemma 2.2.2, $det(F(z)E_1F(z)^*) > 0$. Thus $p_1(z)p_1(z)^* > 0$. So to find a possible choice for $y_1(z)$ that satisfies (3.1.1), we can use *Theorem 2.1.* That is there exists a vector of polynomial $v_1(z)$ such that

$$p_1(z)v_1(z) = 1$$

So at this point, a possible choice for $y_1(z)$ is $v_1(z)$.

However, $y_1(z)$ needs to satisfy (3.1.2) also. So we have to do some adjustment to $v_1(z)$ to be a new candidate for $y_1(z)$ that satisfies both (3.1.1) and (3.1.2). Set

$$v_2(z) = v_1(z) - Q_{p_1}(z)x_1(z)$$

where $Q_{p_1}(z)$ is defined in (2.2.1) and $x_1(z)$, to be determined, is a column vector that has n polynomial entries.

$v_2(z)$ statisfies (3.1.1) becasue v_1 does and $v_2(z)Q_{p_1}(z)x_1(z) = 0$. So we have to find $x_1(z)$ that makes $v_2(z)$ satisfies (3.1.2). That is

$$p_2(z)v_2(z) = p_2(z)[v_1(z) - Q_{p_1}(z)x_1(z)] = 0.$$

We then have

$$p_2(z)v_1(z) = p_2(z)Q_{p_1}(z)x_1(z).$$

Since $p_2(z)Q_{p_1}(z)$ is a row vector of polynomials and from $F(z)F(z)^* > 0$, we know that $F(z)$ has full rank m. But then $\begin{bmatrix} p_1(z) \\ p_2(z) \end{bmatrix}$ must have rank 2 for all z. Thus from Lemma 2.2.2, $p_2(z)Q_{p_1}(z)Q_{p_1}(z)^*p_2(z)^* > 0$. Now *Theorem 2.1* applies again. So we can find a column vector consisting of polynomials $w_1(z)$ such that

$$1 = p_2(z)Q_{p_1}(z)w_1(z) \tag{3.1.4}$$

Multiply both side of (3.1.4) by $p_2(z)v_1(z)$ we have

$$p_2(z)v_1(z) = p_2(z)Q_{p_1}(z)[w_1(z)p_2(z)v_1(z)].$$

We can then set

$$x_1(z) = w_1(z)p_2(z)v_1(z).$$

Then v_1 has polynomial entries and solves the first 2 equations. We continue the process by using induction on the number of rows. That is if we work with the first

$m-1$ equations. We will get

$$v_{m-1}(z) = v_{m-2}(z) - Q_{p_1}(z)Q^{(2)}_{p_2}(z)Q^{(3)}_{p_3}(z)...Q^{(m-2)}_{p_{m-2}}(z)x_{m-2}(z).$$

Then, to satisfy all m equations, we must have

$$y_1(z) = v_m(z) = v_{m-1}(z) - Q_{p_1}(z)Q^{(2)}_{p_2}(z)Q^{(3)}_{p_3}(z)...Q^{(m-2)}_{p_{m-2}}(z)Q^{(m-1)}_{p_{m-1}}(z)x_{m-1}(z).$$

From $F(z)F(z)^* > 0$ and lemma 2.2.2 again, we have

$$p_m(z)Q_{p_1}(z)Q^{(2)}_{p_2}Q^{(3)}_{p_3}...Q^{(m-2)}_{p_{m-2}}Q^{(m-1)}_{p_{m-1}}p_m(z)^* > 0,$$

since $\begin{bmatrix} p_1(z) \\ \vdots \\ p_{m-1}(z) \end{bmatrix}$ must have rank $m-1$ for all $z \in \mathbb{C}$. Then we can find $x_{m-1}(z)$ by again applying *Theorem 2.1* to

$$p_m(z)Q_{p_1}(z)Q^{(2)}_{p_2}(z)Q^{(3)}_{p_3}(z)...Q^{(m-2)}_{p_{m-2}}(z)Q^{(m-1)}_{p_{m-1}}(z).$$

That is there exists a polynomial $w_{m-1}(z)$ such that

$$1 = p_m(z)Q_{p_1}(z)Q^{(2)}_{p_2}(z)Q^{(3)}_{p_3}(z)...Q^{(m-2)}_{p_{m-2}}(z)Q^{(m-1)}_{p_{m-1}}(z)w_{m-1}(z). \tag{3.1.5}$$

Multiply both sides of (3.1.5) by $p_m(z)v_{m-2}(z)$. So we get

$$x_{m-1}(z) = w_{m-1}(z)p_m(z)v_{m-1}(z).$$

As a result, this shows the method of finding $y_1(z)$. We can use a similar process to find $y_2(z), y_3(z), ...,$ and $y_m(z)$. This implies $Y(z)$ exists and $F(z)Y(z) = I_m$ $\square$

3.2. Determine the Bound on the Degree of a Solution

THEOREM 2.2. *Let* $F(z) = [(p_{i,j}(z))]_{m\times n}$ *be an* $m \times n$ *matrix of polynomials* $p_{i,j}(z) \in \mathbb{C}(z)$ *for* $i \in \{1, 2, ..., m\}$*and* $j \in \{1, 2, ..., n\}$ *and*

$$\varepsilon(z)^2 I_m \leq F(z)F(z)^*,$$

where $\varepsilon(z) > 0$ *for all* $z \in \mathbb{C}$. *Then there exist* $Y(z)$, *an* $n \times m$ *matrix of polynomials such that*

$$F(z)Y(z) = I_m.$$

Let $k_i > 0$ *and* k_i *be a maximum degree of polynomial entries of the* i^{th} *row of* $F(z)$ *for* $i \in \{1, ..., m\}$. *Then, without loss of generality, permute the rows of* $F(z)$ *and assume that* $1 \leq k_1 \leq k_2 \leq ... \leq k_m$. *We will still denote the srsulting matrix by* $F(z)$. *We then have*

$$\textit{degree}[Y(z)]_{i,j} \leq 2k_{m-1} + 4k_{m-2} + ... + 2(m-1)k_1 + (2(m-1)+1)k_m - m$$

PROOF. Let k_i satisfy the hypothesis above. Then the bound on the degree of $v_1(z)$ is $(k_1 - 1)$. This is the result from the proof of *Theorem 2.1*. Since $v_2(z) = v_1(z) - Q_{f_1}(z)x_1(z)$ and $x_1(z) = w_1(z)p_2(z)v_1(z)$. Then the bound on the degree of $v_2(z)$ is $\max\{(k_1 - 1), (k_1 + k_2 + k_1 - 1 + k_2 + k_1 - 1)\} = k_1 + k_2 + k_1 - 1 + k_2 + k_1 - 1 = 2k_2 + 3k_1 - 2$.

Notice that $(k_1 + k_2 + k_1 - 1 + k_2 + k_1 - 1)$ will be always greater than $(k_1 - 1)$. This is because $v_1(z)$ is a factor of $Q_{f_1}(z)x_1(z)$ and $k_i \geq 1$. Similarly, the bound on the degree of $v_3(z)$ is $k_1 + k_2 + k_3 + k_2 + k_1 - 1 + k_3 + k_1 + k_2 + k_1 - 1 + k_2 + k_1 - 1 = 2k_3 + 4k_2 + 5k_1 - 3$. Similarly, the bound on the degree of $v_4(z)$ is $k_1 + k_2 + k_3 + k_4 + k_3 + k_2 + k_1 - 1 + k_4 + 5k_1 + 4k_2 + 2k_3 - 3 = 2k_4 + 4k_3 + 6k_2 + 7k_1 - 4$.

So from this pattern, we know that if we work on the first $m-1$ rows, we have that the bound on the degree of $v_{m-1}(z)$ is $2k_{m-1}+4k_{m-2}+6k_{m-3}+...+2(m-2)k_2+(2(m-2)+1)k_1-(m-1)$. Then if we work on the first m rows, the bounded degree for $v_m(z)$ is $k_1+k_2+...+k_{m-1}+k_m+k_1+...+k_{m-1}-1+k_m+(2k_{m-1}+4k_{m-2}+6k_{m-3}+...+2(m-2)k_2+(2(m-2)+1)k_1-(m-1)) = 2k_m+4k_{m-1}+6k_{m-2}+...+2(m-1)k_2+(2(m-1)+1)k_1-(m)$.

Let e_p denote the p^{th} element of the standard basis for $\mathbb{C}^m$. Then, in a similar way, we get

$$F(z)y_p(z) = e_p$$

where the maximum degree of the entries of $y_p(z)$ is

$$2k_m+4k_{m-1}+...+2(m-p)k_{p+1}+2(m-p+1)k_{p-1}+...+2(m-1)k_1+(2(m-1)+1)k_p-m.$$

By our assumption that $1 \le k_1 \le k_2 \le ... \le k_p$, we see that the maximum degree of the entries of the polynomials in $Y(z) = [y_1(z), ..., y_m(z)]$ is

$$2k_{m-1}+4k_{m-2}+...+2(m-1)k_1+(2(m-1)+1)k_m-m.$$

□

We conclude with the following observation, when some of the rows of $F(z)$ are composed entirely of constants. If we have $k_m \ge k_{m-1} \ge ... \ge k_{n+1} > 0$ and $k_n = k_{n-1} = ... = k_1 = 0$. We will consider 2 cases which are the bound on the degree of $y_1(z), ..., y_n(z)$ and the bound degree of $y_{n+1}(z), ..., y_m(z)$. Then we will compare these two values to get the bound degree of $Y(z)$.

1. The bound on the degree of $y_1(z), ..., y_n(z)$.

We first consider the bound on the degree of $y_1(z)$. We can use the same method as in section 3.1 to find $v_1(z), v_2(z), ..., v_m(z)$. As a result, the bound on the degree of $v_1(z), ..., v_n(z)$ is 0. Then the bound on the degree of $v_{n+1}(z)$ is $2k_{n+1} - 1$. If we continue the process, we get the bound on the degree of $v_m(z) = y_1(z)$ is $2k_m + 4k_{m-1} + ... + 2(m-n)k_{n+1} - (m-n)$. If we process the same method to $y_2(z), ..., y_n(z)$, we see that $2k_m + 4k_{m-1} + ... + 2(m-n)k_{n+1} - (m-n)$ is the bound on the degree of $y_1(z), ..., y_n(z)$.

2. The bound on the degree of $y_{n+1}(z), ..., y_m(z)$.

We first consider the bound on the degree of $y_{n+1}(z)$. To get the best bound on the degree of $y_{n+1}(z)$, we must have $v_{n+1}(z) = Q^1_{p_1}(z)Q^2_{p_2}(z)...Q^n_{p_n}(z)x_n(z)$. Also, $p_{n+1}(z)v_{n+1}(z) = p_{n+1}(z)Q^1_{p_1}(z)Q^2_{p_2}(z)...Q^n_{p_n}(z)x_n(z) = 1$. Since $F(z)$ has full rank, we can apply *Theorem 2.1* and we have the degree of $x_n(z)$ is bounded by $k_{n+1} - 1$. Hence the bound on the degree of $v_{n+1}(z)$ is also $k_{n+1} - 1$. Then we can continue the similar process as in the proof of *Theorem 2.2* in this section and we will have that the bound on the degree of $v_m(z) = y_{n+1}(z)$ is $2k_m + 4k_{m-1} + ... + 2(m-n-1)k_{n+2} + (2(m-n-1)+1)k_{n+1} - (m-n)$. Then the bound on the degree of $y_{n+1}(z), ..., y_m(z)$ is $2k_{m-1} + 4k_{m-2} + ... + 2(m-n-1)k_{n+1} + (2(m-n-1)+1)k_m - (m-n)$.

From the two cases, we have the bound on the degree of $Y(z)$ is $\max\{2k_m + 4k_{m-1} + ... + 2(m-n)k_{n+1} - (m-n),\ 2k_{m-1} + 4k_{m-2} + ... + 2(m-n-1)k_{n+1} + (2(m-n-1)+1)k_m - (m-n)\}$.

CHAPTER 4

FUTURE STUDY

After working through this dissertation, there are some questions that arose up and these can lead to future study.

In *Theorem 2.1*, we are working on univariate polynomials. The immediate question is if we can apply the method in proving the existence of solution for *Theorem 2.1* to the case when we have a case of multivariate polynomials. The multivariate polynomial case seems to be more complicated. In the proof of the single variable case, we can appoximate the growth of each polynomials by its leading term. However, in 2 or more variable case, it is hard to know and approximate the growth of each polynomials.

Also, it is more complicated to compute the Cauchy Transform in multivariable case. In some cases, the number of variables can be collapsed down to a smaller number of variables and this will make the original integration of Cauchy Transform undefined.

In *theorem 2.2*, the condition of $F(z)F(z)^* > 0$ guarantees that the matrix $F(z)$ has full rank, we can then continuously use the result from *Theorem 2.1* to get existence of solution. In the future study, we can consider the case when $F(z)F(z)^* \geq \varepsilon^2 P_{ranF(z)}$, for some $\varepsilon \in \mathbb{R}$ and $P_{ranF(z)}$ is a projection onto the range of $F(z)$ where $F(z)$ does not neccessary have a full rank.

References

[1] M. Andersson(1996), *Topics In Complex Analysis*, Springer, New York

[2] W.D. Brownawell(1987), *Bounds for the degrees in nullstellensatz*, Ann. of Math 126, 577-592

[3] J. B. Garnett(1981), *Bounded Analytic Functions*, Academic Press, New York

[4] D. Hilbert(1893), *ber die vollen Invariantensysteme*, Math. Ann. 42, 313-373

[5] J. Kollar(1988), *Sharp effective nullstellensatz*, J. Amer. Math. Soc. 1, 963-975

[6] P. Koosis(1998), *Introduction to H_p Spaces*, Cambridge University Press, New York

[7] J. Ryle, T. T. Trent(2012), *A corona theorm for certain subalgebras of $H^\infty(\mathbb{D})$ II*, Houston J. math 38, no. 4, 1277-1295

[8] M.Spivak(1971), *Calculus On Manifolds:A Modern Approach To Classical Theorems Of Advanced Calculus*, Westview Press, Boulder

[9] T.T. Trent(2002), *A new estimate for the vector valued corona problem*, J.Func.Anal. 189, 267-282

[10] T.T. Trent, X. Zhang(2006), *A matricial corona theorem*,Proc. Amer. Math. Soc.,134, 2549-2558

[11] T.T. Trent, X. Zhang(2007), *A matricial corona theorem II*, Proc. Amer. Math. Soc., 135, 2845-2854