GROUPS WITH CONDITIONS ON NON-PERMUTABLE SUBGROUPS

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ABSTRACT

In this dissertation, we study the structure of groups satisfying the weak minimal condition and weak maximal condition on non-permutable subgroups. In Chapter 1, we discuss some definitions and well-known results that we will be using during the dissertation. In Chapter 2, we establish some preliminary results which will be useful during the proof of the main results. In Chapter 3, we express our main results, one of which states that a locally finite group satisfying the weak minimal condition on non-permutable subgroups is either Chernikov or quasihamiltonian. We also prove that, a generalized radical group satisfying the weak minimal condition on non-permutable subgroups is either Chernikov or is soluble-by-finite of finite rank.

In the Final Chapter, we will discuss the class of groups satisfying the weak maximal condition on non-permutable subgroups.

DEDICATION

I would like to dedicate this dissertation to my parents. First, to my deceased mother who taught me invaluable lessons in life. Second to my father, who always provided me unconditional moral as well as financial support.

LIST OF ABBREVIATIONS AND SYMBOLS

$\langle x\rangle$	Subgroup generated by x
$\operatorname{Aut} G$	The automorphism group of G
$C_G(H)$	Centralizer of the subgroup H in G
$N_G(H)$	Normalizer of H in G
$\|G:H\|$	Index of H in G
$G' = [G,G]$	Derived subgroup of group G
$H \lhd G$	H is a normal subgroup of the group G
$T(G)$	The torsion subgroup of the group G
G_p	The p-primary component of the group G
$\underset{\lambda\in\Lambda}{\mathrm{Dr}}\, A_\lambda$	The direct product of the groups $\{A_\lambda \| \lambda\in\Lambda\}$
$\prod_{\lambda\in\Lambda} H_\lambda$	The product of the groups $\{H_\lambda \| \lambda\in\Lambda\}$
$r_p(G)$	The p-rank of the group G
$r_0(G)$	The torsion-free rank of the group G
$r(G)$	The rank of the group G
$\mathrm{Core}_H(G)$	Core of H in G
$H \operatorname{per} G$	H is permutable in G
$\rtimes$	Semi direct product
$\zeta(G)$	Center of the group G
C_{p^∞}	Quasicyclic p-group, or Prüfer p-group

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CONTENTS

CHAPTER 1

INTRODUCTION

The structure of infinite groups satisfying the minimal (respectively maximal) condition or the weak minimal (respectively weak maximal) condition has been one of the important aspects for the development of infinite group theory.

If $\mathcal{P}$ is a subgroup theoretical property or class of groups then $\bar{\mathcal{P}}$ denotes the class of all groups that are either not $\mathcal{P}$-groups or are trivial. In 1964, S. N. Chernikov started the investigation of groups satisfying the minimal condition on subgroups which do not have the property $\mathcal{P}$, denoted by min-$\bar{\mathcal{P}}$, in a series of articles (for a general reference, see [**4**]). In particular, he studied groups satisfying the minimal condition on non-abelian subgroups and groups satisfying the minimal condition on non-normal subgroups. He proved that if such a group has a series with finite factors, then either the group satisfies the minimal condition on all subgroups or all the subgroups have the prescribed property $\mathcal{P}$. Later, Philips and Wilson [**19**] investigated the same type of problem for several different choices of the property $\mathcal{P}$, and proved in particular that a locally graded group satisfying the minimal condition on non-normal subgroups either is a Chernikov group or has all its subgroups normal.

Taking motivation from such work, in their paper [**3**], M.R. Celentani and Antonella Leone considered the class of groups satisfying the minimal condition on non-quasinormal subgroups. The structure of such groups was given by the following result:

Theorem 1.1. [**3**, Theorem C] *Let G be a group which either is non-periodic or locally graded. If G satisfies the minimal condition on non-quasinormal subgroups, then either G is quasihamiltonian or it is a Chernikov group.*

The concept of the weak minimal condition was introduced by D. I. Zaitsev [**24**]. In [**25**], he investigated the class of groups satisfying the weak minimal condition on non-abelian subgroups where he was able to show that in the case of locally soluble-by-finite (also known as locally almost soluble) groups the weak minimal condition on non-abelian subgroups is equivalent to the weak minimal condition on subgroups. By virtue of the result in [**25**] such a group is then a soluble-by-finite minimax group. Groups with the weak minimal condition on non-normal subgroups and non-subnormal subgroups were the subject of interest in [**13**] and [**14**] respectively. In [**14**], the authors were able to establish the following theorem:

THEOREM 1.2. [**14**, Theorem C] *Let G be a generalized radical group with weak minimal condition on non-subnormal subgroups. Then either G is soluble-by-finite and minimax or every subgroup of G is subnormal.*

The results in these papers and similar other papers were enough to motivate us to investigate the structure of groups satisfying the weak minimal condition on non-permutable subgroups. The main purpose of this dissertation is to study the structure of generalized radical groups satisfying the weak minimal condition on non-permutable subgroups, which we will discuss in Chapter 3.

In the coming sections of this chapter, we will go through some important definitions, and results which will be useful in the further chapters and which will also explain some of the terminology used in the Introduction.

1.1 Permutable Subgroups

In this first section we discuss the notion of permutable subgroups. We begin the section with the definition of permutable subgroup.

DEFINITION 1.1. *A subgroup H of a group G is said to be* permutable *or* quasinormal *in G if $HK = KH$ for every subgroup K of G. This is equivalent to affirming that HK is a subgroup of G. We often write H* per *G for H is permutable in G.*

The concept of *permutability* was introduced by Ore [**17**], who called permutable subgroups *quasinormal*. Later S. E. Stonehewer introduced the term permutable and the behavior of permutable subgroups was later investigated by various authors. Obviously, every normal subgroup is permutable but the converse is not true. This can be seen by letting the group G be the semidirect product of a cyclic group $\langle x \rangle$ of order p^2 and a cyclic group $\langle y \rangle$ of order p acting non-trivially. In other words: G is a group of order p^3 generated by x and y satisfying the relations $x^{p^2} = y^p = 1$ and $y^{-1}xy = x^{p+1}$. That is, $G = \langle x, y \,|\, x^{p^2} = y^p = 1, \quad y^{-1}xy = x^{p+1} \rangle$. We claim that the subgroup $\langle y \rangle$ is permutable in G, but it is not normal. Let H be the subgroup generated by the elements of order p. Then, $x \notin H$, hence $H \neq G$. Moreover, $x^p, y \in H$ together imply that H has order p^2. So, H is elementary abelian. Now, any cyclic subgroup of H permutes with $\langle y \rangle$. Also, if we pick an element $a \in G \setminus H$ then $\langle a \rangle$ is of order p^2, hence it is maximal in G, so $\langle a \rangle$ is normal in G and permutes with $\langle y \rangle$. So every cyclic subgroup of G permutes with $\langle y \rangle$, hence $\langle y \rangle$ is permutable. However, $\langle y \rangle$ is not normal as $x^{-1}yx = yx^{-p}$.

An alternate proof for the permutability of $\langle y \rangle$ can be given when p is an odd prime. If G is a finite p-group which is the product of two cyclic subgroups where p is an odd prime, B. Huppert showed in [**10**] that every subgroup of G is permutable. Hence, in our example, if p is odd, then every subgroup of G is permutable, hence $\langle y \rangle$ is permutable.

However, Ore [**18**] proved that in a finite group permutable subgroups are subnormal. In the same paper, he was also able to show that a maximal permutable subgroup of a finite group is normal. We also have the following interesting result of Stonehewer for another point of view.

Theorem 1.3. [**21**, 13.2.3] *Let $G = HK$ where H per G and $K = \langle k \rangle$ is an infinite cyclic group. Assume that $H \cap K = 1$. Then $H \lhd G$.*

One of the many consequences of this fundamental result is a further result of Stonehewer [**23**].

Lemma 1.1. *A simple group cannot have a proper, non-trivial permutable subgroup.*

Now we prove one more lemma related to the properties of permutable subgroups.

Lemma 1.2. *Let G be a group and let $N \lhd G$. If H/N is permutable in G/N then H is permutable in G.*

Proof. For each subgroup K of G, we need to show that $HK = KH$. Since, H/N is permutable in G/N we have

$$\left(\frac{H}{N}\right)\left(\frac{KN}{N}\right) = \left(\frac{KN}{N}\right)\left(\frac{H}{N}\right).$$

Then $HKN = KHN$. Since N is normal in G and $N \leq H$, it follows that $HK = KH$ so H is permutable in G.

□

The converse of Lemma 1.2 also holds.

Lemma 1.3. *Let G be a group and let $N \lhd G$. If H is permutable in G then $\frac{HN}{N}$ is permutable in $\frac{G}{N}$.*

Proof. Since H is permutable in G then, by definition of permutability, we have $HK = KH$ for any subgroup K of G. This implies

$$\frac{HN}{N} \cdot \frac{K}{N} = \frac{HKN}{N} = \frac{KHN}{N} = \frac{K}{N} \cdot \frac{HN}{N}$$

for any subgroup $\frac{K}{N}$ of $\frac{G}{N}$. Hence by definition $\frac{HN}{N}$ is permutable in $\frac{G}{N}$.

□

Definition 1.2. *Groups in which every subgroup is normal are called* Dedekind groups.

Abelian groups are trivial examples of Dedekind groups. A non-abelian Dedekind group is called a *Hamiltonian group.* The quaternion group of order 8 is an example of a non-abelian Dedekind group. Now the more general case, when all subgroups of a group are permutable, is rather interesting. Clearly, abelian groups and Dedekind groups are examples of such groups.

Definition 1.3. *A group is called* quasihamiltonian *if all its subgroups are permutable.*

It was proved by Stonehewer [**23**] that *permutable* subgroups of arbitrary groups are ascendant, a far-reaching generalization of subnormality, so the *quasihamiltonian* groups are locally nilpotent. The structure of *quasihamiltonian* groups was described by Iwasawa [**11**]. Here we give a flavor of the results obtained.

THEOREM 1.4. [**11**, Iwasawa] *Let G be a non-abelian group with all subgroups permutable. Then*

(i) *G is locally nilpotent*

(ii) *G is metabelian*

(iii) *If $T(G)$ is the torsion subgroup of G and if $G \neq T(G)$ then $T(G)$ is abelian and $G/T(G)$ is a torsion-free abelian group of rank 1.*

Finally we give another well-known criterion for a group to be Quasihamiltonian.

LEMMA 1.4. *Let G be a group. Then every subgroup of G is permutable if and only if $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$ for all $a, b \in G$.*

PROOF. Suppose every subgroup of G is permutable. Then by definition, $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$ for all $a, b \in G$. Conversely, assume that $\langle a\rangle\langle b\rangle = \langle b\rangle\langle a\rangle$ for all $a, b \in G$. Now to show every subgroup of G is permutable, we need to show that $BC = CB$ for all subgroups B, C. Let B, C be subgroups of G. Let $b^i c^j \in BC$ where $i, j \in \mathbb{Z}$ and $b \in B$, $c \in C$. Then by our assumption, $\langle b\rangle\langle c\rangle = \langle c\rangle\langle b\rangle$. Hence $b^i c^j = c^k b^l$ for some $k, l \in \mathbb{Z}$ and so $b^i c^j \in CB$. Thus, $BC \subseteq CB$. Similarly, if $c^i b^j \in CB$ where $i, j \in \mathbb{Z}$, then $\langle c\rangle\langle b\rangle = \langle b\rangle\langle c\rangle$ by our assumption, so $c^i b^j \in BC$ and $CB \subseteq BC$. Therefore $BC = CB$ which completes the proof.

□

For more information about permutable subgroups, we refer the reader to [**22**].

1.2 Min-$\mathcal{P}$ and Max-$\mathcal{P}$

This dissertation is concerned with the weak minimal and weak maximal conditions, so in this section we first discuss the minimum and maximum conditions. We begin with the definition of subgroup theoretical property and finiteness conditions on subgroups.

DEFINITION 1.4. *A* subgroup theoretical property $\mathcal{P}$ *is a property of certain subgroups of a group G so that always the identity subgroup of G has property $\mathcal{P}$ and whenever $H \leq G$*

has the property $\mathcal{P}$ then $H\theta$ also has property $\mathcal{P}$, whenever θ is an isomorphism of G with some other group.

Let $\mathcal{P}$ be a class of groups. We shall say that a group H is a $\mathcal{P}$-group if it belongs to $\mathcal{P}$ and that $H \leq G$ is a $\mathcal{P}$-subgroup of G if H is a $\mathcal{P}$-group. For a class of groups we write $\bar{\mathcal{P}}$ for the class that consists of all groups that do not belong to $\mathcal{P}$ together with all trivial groups. Similarly, if $\mathcal{P}$ is a property of groups, $\bar{\mathcal{P}}$ will denote the property not-$\mathcal{P}$.

A *finiteness condition* in the theory of groups is a property satisfied by all finite groups. It is well known that finiteness conditions have played a great role in the study of infinite subgroups. Two of the first finiteness conditions defined in the theory of groups were the minimal and the maximal conditions for subgroups. The minimal condition consists of the requirements of finiteness of descending chains of subgroups of the group whereas the maximal condition requires the finiteness of ascending chains of subgroups. Now we define those two finiteness conditions on subgroups with the property $\mathcal{P}$.

Definition 1.5. *Let $\mathcal{P}$ be a subgroup theoretical property. The group G is said to satisfy the* minimal *condition on $\mathcal{P}$-subgroups (min-$\mathcal{P}$) if every nonempty set S of subgroups of G with property $\mathcal{P}$ contains a subgroup H with the property $\mathcal{P}$ such that if $K \leq H$ and $K \in S$ then $K = H$;*

For example, if $\mathcal{P}$ is simply the property of being a subgroup, then the condition min-$\mathcal{P}$ is called the minimum condition and often denoted by *min*. Similarly, if $\mathcal{P}$ represents the property of being an abelian subgroup, then the group G has *min-ab*, the minimum condition on abelian subgroups. We recall that a group G is said to have *min-ab*, if every abelian subgroup has the minimum condition. If $\mathcal{P}$ is the property of being a permutable subgroup, then min-$\bar{\mathcal{P}}$ is called the minimum condition on non-permutable subgroups and so on. For any subgroup theoretical property $\mathcal{P}$, we have the following easily proven result for the condition min-$\mathcal{P}$.

Lemma 1.5. *Let $\mathcal{P}$ be a subgroup theoretical property. Then the group G has min-$\mathcal{P}$ if and only if every descending chain of $\mathcal{P}$-subgroups terminates in finitely many steps.*

Proof. Suppose that G has min-$\mathcal{P}$ and $H_1 \geq H_2 \geq H_3 \geq \cdots$ is a descending chain of $\mathcal{P}$-subgroups. Then $\{H_1, H_2, H_3, \ldots\}$ is a non-empty set of $\mathcal{P}$-subgroups so there is a minimal element, H_k, say. Then $H_k = H_{k+1} = H_{k+2} = \ldots$ so the chain terminates in finitely many steps.

Conversely, let $\mathcal{S}$ be a non-empty set of $\mathcal{P}$-subgroups of G and suppose that $\mathcal{S}$ has no minimal element. If $H_i \in \mathcal{S}$ for some $i \in \mathbb{N}$, then, by assumption, there is a subgroup $H_{i+1} \in \mathcal{S}$ such that $H_{i+1} \lneqq H_i$. In this way, since $\mathcal{S} \neq \phi$, we obtain an infinite descending chain

$$H_1 \gneqq H_2 \gneqq H_3 \gneqq H_4 \gneqq \cdots$$

of $\mathcal{P}$-subgroups contrary to the hypothesis. This completes the proof.

□

In this dissertation a group is called *quasicyclic* if it is of the type C_{p^∞} for some prime p. We note that all proper subgroups of a quasicyclic group are finite and cyclic.

Theorem 1.5. [**8**, 1.6] *Suppose G is an abelian group. Then G has the minimum condition if and only if G is a finite direct product of quasicyclic p-groups and finite cyclic subgroups.*

Definition 1.6. *A group which is finite extension of an abelian group satisfying the minimum condition is called a* Chernikov *group.*

Such groups have also been called *extremal* and they were named in honor of S. N. Chernikov, who made an extensive study of groups with the minimum condition. From the structure Theorem 1.5, it follows that a group G is Chernikov if and only if it has a normal divisible abelian subgroup N of finite index, and N is a direct product of finitely many quasicyclic groups. Obviously, all finite groups are Chernikov and every Chernikov group satisfies the minimum condition. The group $C_{p^\infty} \rtimes C_2$ is an example of a Chernikov group, where C_{p^∞} has an automorphism of order 2, namely the inversion automorphism.

Next we discuss locally nilpotent groups satisfying the minimal condition on abelian subgroups (min-ab) via the following Theorem.

Theorem 1.6. [**8**, 3.7] *Suppose that G is a locally nilpotent group satisfying min-ab. Then G is Chernikov and hence hypercentral.*

For locally finite groups with the minimal condition on abelian subgroups, we have the following famous result of Shunkov [**12**].

Theorem 1.7. [**12**, Theorem 5.8] *For the locally finite group G the following are equivalent.*

(i) *G is a Chernikov group.*

(ii) *G satisfies min.*

(iii) *G satisfies min-ab*

(iv) *The centralizer of every non-identity element of G has min.*

Next we define the maximal condition on subgroups with the property $\mathcal{P}$ exactly in the same way we defined the minimal condition.

Definition 1.7. *Let $\mathcal{P}$ be a subgroup theoretical property. The group G is said to satisfy the* maximal *condition on $\mathcal{P}$-subgroups (max-$\mathcal{P}$) if every nonempty set S of subgroups of G with property $\mathcal{P}$, contains a subgroup H with the property $\mathcal{P}$ such that if $H \leq K$ and $K \in S$ then $K = H$.*

Here we note that *min* and *max* both are extension closed properties. Also, analogous to Lemma 1.5 for *min*-$\mathcal{P}$, we have an easy to prove, corresponding lemma for *max*-$\mathcal{P}$ given by:

Lemma 1.6. *Let $\mathcal{P}$ be a subgroup theoretical property. Then the group G has* max-$\mathcal{P}$ *if and only if every ascending chain of $\mathcal{P}$-subgroups terminates in finitely many steps.*

Earlier we mentioned that Phillips and Wilson obtained the structure of locally graded groups with the minimal condition on non-normal subgroups. In [**6**], Giovannni Cutolo studied the structure of locally graded groups satisfying the maximal condition on non-normal subgroups (*max*-$\bar{n}$), where n represents the property of being a normal subgroup and proved the following:

Theorem 1.8. *A locally graded group G satisfies max-$\bar{n}$ if and only if it is one of the following types:*

(a) *G has the maximum condition.*

(b) *G is a Dedekind group.*

(c) *G is a central extension of C_{p^∞} by a finitely generated Dedekind group.*

(d) *G is the direct product of $\mathbb{Q}_2$ and a finite Hamiltonian group.*

We recall that a group G is *locally graded* if every finitely generated non-trivial subgroup of G has a non-trivial finite image. The class of locally graded groups contains the classes of locally finite groups, locally soluble groups and locally-(soluble by finite) groups.

In above theorem $\mathbb{Q}_2$ is defined as follows.

Let π be a non-empty set of primes. Then let $\mathbb{Q}_\pi$ denote the additive group of rational numbers whose denominators are π-numbers. In particular, if $\pi = \{2\}$, then $\mathbb{Q}_2$ represents the additive group of all rational numbers whose denominators are a power of 2. For more about the maximal condition on subgroups, we refer the reader to [**20**].

The class of locally graded groups also contains the class of generalized radical groups, which we now discuss. It is well-known that the product of two normal nilpotent subgroups is again nilpotent which is Fitting's Theorem. The following well-known result due to Hirsch and Plotkin shows that the same result holds for normal locally nilpotent subgroups. It is known as the Hirsch-Plotkin Theorem.

Theorem 1.9. [**21**, 12.1.2] *Let H and K be normal locally nilpotent subgroups of a group G. Then the product $J = HK$ is a normal locally nilpotent subgroup of G.*

As a consequence of this theorem, we conclude that in any group G there is a unique maximal normal locally nilpotent subgroup containing all normal locally nilpotent subgroups of G, which is called the *Hirsch-Plotkin radical* of G. We denote it by ρ(G) and it is a characteristic subgroup of G. Next, we can define the *upper Hirsch-Plotkin series* $\{\rho_\alpha(G)\}$ of the group G by

$$\begin{aligned}
\rho_0(G) &= 1 \\
\rho_1(G) &= \text{ Hirsch-Plotkin Radical of } G \\
\rho_{\alpha+1}(G)/\rho_\alpha(G) &= \rho(\mathrm{G}/\rho_\alpha(G)) \text{ for ordinals } \alpha \\
\rho_\gamma(G) &= \bigcup_{\beta<\gamma} \rho_\beta(G) \text{ for limit ordinals } \gamma
\end{aligned}$$

Clearly, $\{\rho_\alpha(G)\}$ is an ascending locally nilpotent series of characteristic subgroups. Now, we are ready to define an important class of groups.

DEFINITION 1.8. *A group G which has an ascending locally nilpotent series terminating in G is said to be a* radical group.

The class of radical groups is an important class as it contains the locally nilpotent groups and the soluble groups. However, it does not contain the class of locally soluble groups. Note that a group G is radical if and only if its upper Hirsch-Plotkin series terminates at G. So, we conclude that a radical group has at least one ascending locally nilpotent series of characteristic subgroups.

Next, we prove the following useful lemma which concludes that the product of normal radical subgroups is again radical.

LEMMA 1.7. *Let G be a group and let H be a subgroup of G. Assume that H is a product of proper normal radical subgroups. Then H is a radical, normal subgroup of G.*

PROOF. Let H be a product of proper normal radical subgroups, that is, $H = \prod\limits_{\alpha} H_\alpha$ where H_α is a proper normal radical subgroup of G for each α. Let H_β be any proper normal radical subgroup of G which occurs as a factor in the product $H = \prod\limits_{\alpha} H_\alpha$. Then H_β has an ascending series of characteristic subgroups $1 = H_{\beta,0} \leq H_{\beta,1} \leq H_{\beta,2} \leq \ldots \leq H_{\beta,\lambda_\beta} = H_\beta$ such that $H_{\beta,\theta+1}/H_{\beta,\theta}$ is locally nilpotent for any ordinal θ and $H_{\beta,\gamma} = \cup_{\theta<\gamma} H_{\beta,\theta}$ if γ is a limit ordinal. Note that each term of this series is a normal subgroup of G as H_α is normal in G for each α and in particular $\alpha = \beta$. Consider the ascending series

$$1 = H_{1,0} \leq H_{1,1} \leq H_{1,2} \leq \ldots \leq H_{1,\lambda_1} = H_1 = H_1H_{2,0} \leq H_1H_{2,1} \leq \ldots \leq H_1H_{2,\lambda_2} = H_1H_2 =$$
$$H_1H_2H_{3,0} \leq H_1H_2H_{3,1} \leq \ldots \leq \prod_{\alpha} H_\alpha = H.$$

It is clearly an ascending series of H and each factor of this series is locally nilpotent.

□

Next we discuss the class of generalized radical groups in brief.

DEFINITION 1.9. *A group G is said to be* generalized radical *if it has an ascending series of normal subgroups terminating in G, the factors of which are locally nilpotent or locally finite.*

The subgroups, quotients, and extensions of generalized radical groups are generalized radical. Moreover, the class of generalized radical groups contains the class of radical groups and locally finite groups.

In Theorem 1.1 we gave a structure theorem for groups with the minimal condition on non-permutable subgroups. Continuing with this theme, Maria De Falco and Carmella Musella in [**7**] investigated the structure of generalized radical groups satisfying the maximal condition on non-permutable subgroups. They introduced the class of $\mathfrak{L}_1$-group defined as:

Let A be a torsion-free abelian group of finite rank r (The definition is given in Section 1.4). We will say that A is an $\mathfrak{L}$-*group* if it is not finitely generated while all its subgroups of rank less than r are finitely generated. Obviously, every torsion-free abelian group of rank 1 which is not finitely generated is an $\mathfrak{L}$-group. An $\mathfrak{L}$-group will be called an $\mathfrak{L}_1$-*group* if it is an extension of a finitely generated group by a group of type p^∞ for some prime p.

THEOREM 1.10. [**7**, Theorem B] *A generalized radical group G satisfies the maximal condition on non-permutable subgroups if and only if one of the following condition holds:*

(i) *G is polycyclic-by-finite.*

(ii) *G is quasihamiltonian.*

(iii) *G contains a central subgroup P of type p^∞ (p prime) and a subgroup E with the maximal condition such that $G = PE$; moreover, either $G/(P \cap E)$ is quasihamiltonian or*

$$G/(P \cap E) = P/(I \cap E) \times E/(P \cap E),$$

where $p = 2$ *and* $E/(P \cap E)$ *is the direct product of a finite Dedekind* 2*-group and a finite quasihamiltonian* $2'$*-group.*

(iv) G *contains a central* $\mathfrak{L}_1$*-subgroup* J*, and for every subgroup* I *of finite index of* J*, there exists a subgroup* E *of* G *with the maximal condition such that* $G = IE$*; moreover, either* $G/(I \cap E)$ *is quasihamiltonian or*

$$G/(I \cap E) = I/(I \cap E) \times E/(I \cap E),$$

where $I/(I \cap E)$ *is a group of type* 2^{∞} *and* $E/(I \cap E)$ *is the direct product of a finite Dedekind* 2*-group and a finite quasihamiltonian* $2'$*-group.*

Now we introduce the finiteness property known as *minimax* which generalizes both minimality (min) and maximality (max). The term minimax was introduced by R. Baer [**2**] which he introduced in connection with abelian minimax groups. A detailed study of minimax groups was done later by several authors including R. Baer, D. Robinson, D. I. Zaitsev and others.

Definition 1.10. *A group* G *is called* minimax *if it has a finite series of subgroups* $1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$*, each factor of which satisfies either the minimal condition (min) or the maximal condition (max).*

This important class of groups has received much attention in the case of soluble groups. The length of a shortest minimax series in a minimax group G is called the *minimax length* and written as *m(G)*. The class of minimax groups is closed under taking subgroups, and homomorphic images. It is also closed under extensions. Now we give an example of an abelian minimax group;

Example 1.1. *If* π *is a finite set of primes, let* $\mathbb{Q}_\pi$ *denote the additive group of rational numbers whose denominators are* π*-numbers. Then* $\mathbb{Q}_\pi$ *is an example of an abelian minimax group. This is because* $\mathbb{Q}_\pi$ *is an extension of the integers* $\mathbb{Z}$ *and* $\mathbb{Q}_\pi/\mathbb{Z}$ *is a group with min.*

Concerning minimax groups we have

THEOREM 1.11. [**21**, 15.2.9] *Let $A \lhd G$ where A is abelian. If every abelian subgroup of G is minimax, then the same is true of the abelian subgroups of G/A.*

The relation between radical groups and minimax group is given by the following famous theorem of Baer and Zaitsev.

THEOREM 1.12. [**20**, 10.35 Baer-Zaitsev] *A radical group, all of whose abelian subgroups are minimax groups, is itself a soluble minimax group.*

1.3 Weak-Min-$\mathcal{P}$ and Weak-Max-$\mathcal{P}$

In this section we will discuss two further finiteness conditions, namely, the weak minimal and weak maximal conditions. The concept of the weak minimal condition on $\mathcal{P}$-subgroups, denoted by min-∞-$\mathcal{P}$, was introduced by R. Baer [**2**] and D. I. Zaitsev [**24**]. Using the notation of D. I. Zaitsev, we have the following definition.

DEFINITION 1.11. *For a property $\mathcal{P}$ of groups, a group G is said to satisfy the* weak minimal condition for $\mathcal{P}$-subgroups *(min-∞-$\mathcal{P}$) if there is no infinite descending chain $H_1 > H_2 > H_3 > \cdots$ of $\mathcal{P}$-subgroups of G with each $|H_i : H_{i+1}|$ infinite. Equivalently G has min-∞-$\mathcal{P}$ if, for every descending chain $H_1 > H_2 > H_3 > \cdots$ of $\mathcal{P}$-subgroups of G, $|H_i : H_{i+1}|$ is infinite only for finitely many i.*

If $\mathcal{P}$ is the class of all subgroups, then the condition is simply known as the weak minimal condition (min-∞). Every subgroup and factor group of a group satisfying the weak minimal condition for subgroups, satisfies, obviously, also the weak minimal condition for subgroups. We have the following lemma, which will be generalized in Lemma 2.3

LEMMA 1.8. [**24**, Lemma 1] *Let G be a group and let $C \lhd G$. Suppose $A, B \leq G$ and $B \leq A$. Let $\{a_\alpha CB \,|\alpha \in \Lambda\}$ be a set of distinct cosets of CB in CA, where $a_\alpha \in A$ for all $\alpha \in \Lambda$, and let $\{c_\beta(C \cap B) \,|\beta \in \Gamma\}$ be a set of distinct cosets of $C \cap B$ in $C \cap A$. Then $\{a_\alpha c_\beta B | \alpha \in \Lambda,\, \beta \in \Gamma\}$ is a set of distinct cosets of B in A for all $\alpha \in \Lambda$ and $\beta \in \Gamma$.*

Using Lemma 1.8 we prove that the weak minimal condition on subgroups is an extension closed property. This follows quite easily using Lemma 1.8.

Theorem 1.13. *The weak minimal condition on subgroups (min-∞) is closed under extensions.*

Proof. Let $N \lhd G$ be such that both N and $\dfrac{G}{N}$ have the weak minimal condition on subgroups. Then we need to show that the group G also has the property min-∞. Suppose $G_1 \gneq G_2 \gneq G_3 \gneq \cdots$ is a descending chain of subgroups in G. Form descending chains of subgroups of N and $\dfrac{G}{N}$ as follows:

$$N \cap G_1 \geq N \cap G_2 \geq \cdots \geq N \cap G_k \geq \cdots$$

$$\frac{G_1N}{N} \geq \frac{G_2N}{N} \geq \frac{G_3N}{N} \gneq \cdots \geq \frac{G_kN}{N} \gneq \cdots$$

Since both N and $\dfrac{G}{N}$ have min-∞, we can choose an integer k such that, $|N \cap G_i : N \cap G_{i+1}|$ is finite for all $i \geq k$ and also $\left|\dfrac{G_iN}{N} : \dfrac{G_{i+1}N}{N}\right|$ is finite for all $i \geq k$. Now set $C = N, B = G_{i+1}, A = G_i$ in the statement of Lemma 1.8. Then $|A : B| = |AC : BC||C \cap A : C \cap B|$ implies that

$$|G_i : G_{i+1}| = |G_iN : G_{i+1}N||N \cap G_i : N \cap G_{i+1}|$$

is finite for all $i \geq k$. So, $G_1 \gneq G_2 \gneq G_3 \gneq \cdots$ is a descending chain with min-∞. Hence G has the weak minimal condition on subgroups.

□

We also note that a group satisfying the weak minimal condition for subgroups need not be periodic as is the case of a group satisfying the minimal condition on subgroups, as the infinite cyclic group shows. Next we define the weak maximal condition for $\mathcal{P}$-subgroups.

Definition 1.12. *For a property $\mathcal{P}$ of groups, a group G is said to satisfy the* weak maximal condition for $\mathcal{P}$-subgroups *(max-∞-$\mathcal{P}$) if there is no infinite ascending chain $H_1 < H_2 < H_3 < \cdots$ of $\mathcal{P}$-subgroups of G with each $|H_{i+1} : H_i|$ infinite. Equivalently G has max-∞-$\mathcal{P}$ if, for every ascending chain $H_1 < H_2 < H_3 < \cdots$ of $\mathcal{P}$-subgroups of G, $|H_{i+1} : H_i|$ is infinite only for finitely many i.*

Like the weak minimal condition on subgroups, the weak maximal condition on subgroups is closed under taking subgroups, and homomorphic images. It is also closed under extensions.

For a property $\mathcal{P}$ of groups, if $\bar{\mathcal{P}}$ denotes the class of non-$\mathcal{P}$ groups or all trivial groups, we can speak of the weak minimal condition for subgroups which do not have property $\mathcal{P}$ (min-∞-$\bar{\mathcal{P}}$) or the weak maximal condition on subgroups which do not have the property $\mathcal{P}$ (max-∞-$\bar{\mathcal{P}}$). For example, on letting $\mathcal{P}$ denote the class of permutable subgroups, we may speak of groups satisfying min-∞-$\bar{\mathcal{P}}$ or max-∞-$\bar{\mathcal{P}}$, the weak minimal condition on non-permutable subgroups or the weak maximal condition on non-permutable subgroups, which are a subject of concern in this dissertation. Next we prove the following lemma.

Lemma 1.9. (i) *If G is a group satisfying the weak minimal condition on non-normal subgroups then G has the weak minimal condition on non-permutable subgroups.*

(ii) *If G is a group satisfying the weak maximal condition on non-normal subgroups then G has the weak maximal condition on non-permutable subgroups*

Proof. (i) Suppose

$$H_1 \gneqq H_2 \gneqq H_3 \gneqq \cdots \qquad (1)$$

is a descending chain of non-permutable subgroups in G. Then each H_i is non-normal. So $H_1 \gneqq H_2 \gneqq H_3 \gneqq \cdots$ is a descending chain of non-normal subgroups in G. Since G has the weak minimal condition on non-normal subgroups then, by definition, $|H_i : H_{i+1}|$ is infinite for only finitely many i. Therefore G has the weak minimal condition on non-permutable subgroups as each H_i in (1) is non-permutable and $|H_i : H_{i+1}|$ is infinite for only finitely many i.

(ii) Suppose

$$K_1 \lneqq K_2 \lneqq K_3 \lneqq \cdots \qquad (2)$$

is an ascending chain of non-permutable subgroups in G. Then each K_i is non-normal. So $K_1 \lneqq K_2 \lneqq K_3 \lneqq \cdots$ is an ascending chain of non-normal subgroups in G. Since G has the weak maximal condition on non-normal subgroups then, by definition, $|K_{i+1} : K_i|$ is infinite for only finitely many i. Therefore G has the weak maximal condition on non-permutable subgroups, as each K_i in (2) is non-permutable and $|K_{i+1} : K_i|$ is infinite for only finitely many i. □

In [**24**], Zaitsev studied the weak minimal condition for the classes of locally finite and locally solvable groups. He proved that in the case of locally finite groups, the weak minimal condition for subgroups, min-∞, is equivalent to the usual minimal condition for subgroups, min. Hence a locally finite group with min-∞ is a Chernikov group. In the same paper, he was also able to prove that a locally soluble group satisfying the weak minimal condition on subgroups is in fact soluble. In the case of soluble groups we have:

THEOREM 1.14. [**16**, 5.1.5] *The following properties of a soluble group G are equivalent:*

(i) *G is a minimax group;*

(ii) *G satisfies max-∞;*

(iii) *G satisfies min-∞*

Zaitsev proved in [**26**] that a locally (soluble-by-finite) group G satisfying either the weak maximal or the weak minimal condition for all subgroups is a soluble-by-finite minimax group, that is, G has a normal soluble subgroup H of finite index which in turn has a finite normal series whose factors are abelian and satisfy either *max* or *min*. The structure of groups satisfying the weak minimal condition on non-normal subgroups (min-∞-$\bar{n}$) and weak maximal condition on non-normal subgroups (max-∞-$\bar{n}$) was investigated by L. A. Kurdachenko and V. E. Goreteskii [**13**], where the following result was established.

THEOREM 1.15. *A locally (soluble-by-finite) group G satisfies the condition min-∞-$\bar{n}$ (respectively, max-∞-$\bar{n}$) if and only if G is either Dedekind or almost soluble and minimax.*

Continuing with this theme, L. A. Kurdachenko and Howard Smith studied the structure of groups satisfying the weak minimal and weak maximal condition [**14, 15**] on non-subnormal subgroups. In their paper [**15**] they proved the following:

Theorem 1.16. *Let G be a locally finite group with the weak maximal condition for non-subnormal subgroups. Then either G is a Chernikov group or G has all subgroups subnormal. In either case, G is soluble-by-finite.*

All these papers motivated us to study groups satisfying the weak minimal condition (and weak maximal condition) on non-permutable subgroups and some results will be given in Chapter 3.

1.4 Groups of Finite Rank

As will become apparent we will need to refer to the notion of rank in a group. For most of the definitions in this section, we use [**8**] and [**21**]. We start with the following definition.

Definition 1.13. *A group G is said to have* finite (Prüfer) rank *r if every finitely generated subgroup can be generated by r elements and r is the least such integer. If there is no such integer r, the group is said to have* infinite rank. *We denote the rank of G by $r(G)$.*

It is easy to see that every subgroup and quotient group of a group of rank at most r also has rank at most r. The class of groups with finite rank is closed under forming extensions and if $H \lhd G$ then $r(G) \leq r(H) + r(G/H)$, where it is easy to see that inequality holds in general. Both C_{p^∞} and $\mathbb{Q}$ are locally cyclic, in the sense that every finitely generated subgroup is cyclic, and hence have rank 1. A finite direct product of quasicyclic groups has finite rank, so a Chernikov group is of finite rank. Similarly, polycyclic groups and soluble minimax groups are other examples of groups of finite rank.

Definition 1.14. *Let G be an abelian group. The number of elements in a maximal independent subset consisting of element of infinite order is called the 0-rank of the group G, denoted by $r_0(G)$. The number of elements in a maximal independent subset consisting of elements of p-power order is called the p-rank of G and denoted by $r_p(G)$.*

Now we can give the following well-known lemma.

Lemma 1.10. *An abelian group G is a direct sum of cyclic groups if and only if it is generated by an independent set.*

It is easy to show that if G is abelian then $r_p(G) = r_p(T(G))$ and that $r_0(G) = r_0(G/T(G))$, where $T(G)$ is the set of elements of finite order in G. It can be also shown that for each abelian group G, $r(G) = r_0(G) + \max_p\{r_p(G)\}$. Moreover, in an abelian group G, two maximal independent subsets consisting of elements with order a power of the prime p have the same cardinality, and the same is true of maximal independent subsets consisting of elements of infinite order as stated in [**21**, 4.2.1].

LEMMA 1.11. *Let G be an abelian group of infinite rank. Then G has a proper subgroup N of infinite rank which is a direct sum of cyclic subgroups.*

PROOF. Let G be an abelian group of infinite rank. Then G has a linearly independent subset S of infinite cardinality. If $\langle S\rangle \neq G$, then $\langle S\rangle$ is a proper normal subgroup of G of infinite rank, and by Lemma 1.10, it is a direct sum of cyclic subgroups. Therefore we assume that $\langle S\rangle = G$. Let $x \in S$. Consider $T = S \setminus \{x\}$. Then, clearly $\langle T\rangle \lneqq \langle S\rangle = G$ and $\langle T\rangle$ has infinite rank. Thus, $\langle T\rangle$ is a proper subgroup of infinite rank. By Lemma 1.10, it is a direct sum of cyclic subgroups.

$\square$

Next we proceed with the lemma which describes the structure of abelian p-groups with finite rank.

LEMMA 1.12. [**8**, 1.12] *The abelian p-group G satisfies the minimum condition if and only if G has finite rank. In this case G is Chernikov and $G = D \oplus F$ for some divisible subgroup D and finite subgroup F.*

In [**1**], Baer and Heineken studied radical groups with finite rank and obtained the following result which discusses the effect of the abelian subgroups on the structure of a radical group G.

THEOREM 1.17 (Baer-Heineken Theorem [**1**]). *Let G be a radical group. Then G has finite rank if and only if the abelian subgroups of G have finite rank.*

From the above theorem, we can conclude that if a radical group G has infinite rank then it contains an abelian subgroup A of infinite rank. Hence by Lemma 1.11, A contains a subgroup of infinite rank which is direct sum of cyclic subgroups.

We conclude the section with the structure of generalized radical group of finite rank. In [**9**], the authors investigated generalized radical groups of finite rank and established the following theorem:

THEOREM 1.18. [**9**, Theorem A] *Let G be a generalized radical group of finite rank. Then G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that*

(1) *T is locally finite and G/T is soluble-by-finite of finite rank,*

(ii) *L/T is a torsion-free nilpotent group,*

(iii) *K/L is a finitely generated torsion-free abelian group,*

(iv) *G/K is finite and S/T is the soluble radical of G/T.*

CHAPTER 2

PRELIMINARY RESULTS

In this chapter, we obtain various preliminary results which arise during the proof of the main results. We start with the following example.

EXAMPLE 2.1. *There is a group with the weak minimal condition on non-permutable subgroups which does not satisfy the minimal condition on non-permutable subgroups.*

Consider the infinite dihedral group $G = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle a \rangle \rtimes \langle x \rangle$ where x acts as a power automorphism of $\langle a \rangle$. Then $a^x = a^{-1}$. Since $\langle a \rangle$ is normal in G it follows that $\langle a^m \rangle$ is also normal in G for all $m \in \mathbb{Z}$. So we can write $G/\langle a^4 \rangle = \langle a \rangle \langle x \rangle / \langle a^4 \rangle$.

Since $\langle a^4 \rangle \langle x \rangle / \langle a^4 \rangle$ is not permutable in $G/\langle a^4 \rangle$ it follows that $\langle a^4 \rangle \langle x \rangle$ is not permutable in G. Similarly, $\langle a^8 \rangle \langle x \rangle$ is not permutable in G and in general $\langle a^{2^i} \rangle \langle x \rangle$ is not permutable in G for $i \geq 2$. Thus we have a descending chain of non-permutable subgroups

$$\langle a^4 \rangle \langle x \rangle \gneq \langle a^8 \rangle \langle x \rangle \gneq \langle a^{16} \rangle \langle x \rangle \gneq \cdots$$

which never terminates. Thus G does not have the minimal condition on non-permutable subgroups. On the other hand the infinite dihedral group is a soluble minimax group, so it has the weak minimal condition on subgroups and hence the weak minimal condition on non-permutable subgroups.

We recall that an automorphism of a group G that leaves every subgroup invariant is called a *power automorphism*. Note that such an automorphism maps each element to one of its powers. Clearly, the set of power automorphisms of G is a subgroup of $\operatorname{Aut} G$. In [**5**], Cooper proved that each power automorphism of a group is central. As a consequence of this, power automorphisms fix the elements of the derived subgroup. Moreover, a power automorphism of the form $x \to x^n$ for some fixed integer n is said to be *universal*. For more information about power automorphisms, we refer the reader to [**5**].

Next, we prove that the class of groups with the weak minimal condition on non-permutable subgroups is closed under taking subgroups and homomorphic images. We have the following proposition:

PROPOSITION 2.1. *Every subgroup and factor group of a group satisfying the weak minimal condition on non-permutable subgroups satisfies the weak minimal condition on non-permutable subgroups.*

PROOF. Let G be a group satisfying the weak minimal condition on non-permutable subgroups. Let $H \leq G$. Suppose that

$$H_1 \gneqq H_2 \gneqq H_3 \gneqq \cdots \gneqq H_i \gneqq \cdots$$

is an infinite descending chain in H and suppose these subgroups are not permutable in H. Then the H_i are not permutable in G either. However G has the weak minimal condition on non-permutable subgroups so there is an integer k such that, for all $j \geq k$, $|H_{j+1} : H_j|$ is finite. Hence H has the weak minimal condition on non-permutable subgroups.

Let N be a normal subgroup of the group G. Suppose there is a descending chain $H_1/N \geq H_2/N \geq H_3/N \geq \cdots$ of non-permutable subgroups of G/N. Then $H_1 \geq H_2 \geq H_3 \geq \cdots$ is a descending chain of non-permutable subgroups of G. Since G has the weak minimal condition on non-permutable subgroups, $|H_i : H_{i+1}|$ is infinite for only finitely many i and hence $|H_i/N : H_{i+1}/N|$ is infinite for only finitely many i. Thus the group G/N also satisfies the weak minimal condition on non-permutable subgroups. □

In a group G, if all the subgroups of infinite index are permutable, then the group G has the weak minimal condition on non-permutable subgroups as we prove in our next lemma.

LEMMA 2.1. *Let G be a group in which all subgroups of infinite index are permutable in G. Then G has the weak minimal condition on non-permutable subgroups.*

Proof. Let $H_1 \gneq H_2 \gneq H_3 \gneq H_4 \cdots$ be a descending chain of subgroups of G, and suppose that $|G : H_i|$ is infinite for some i. Then, by hypothesis, H_i per G. If $K \leq H_i$ then $|G : K|$ is also infinite so K per G. In particular, for $k \geq i$, H_k per G. Hence G has the weak minimal condition on non-permutable subgroups.

□

We note that in a non-periodic group if the set of all elements of finite order forms a subgroup then the group is generated by its elements of infinite order.

Proposition 2.2. *Let G be a non-periodic, non-quasihamiltonian infinite group. Then every subgroup of infinite index is permutable in G if and only if the following conditions hold:*

(a) *$T = \{x \in G \,|\, x$ has finite order$\}$ is a Dedekind group, and indeed if $x \in T$ then $\langle x \rangle \lhd G$.*

(b) *T is finite.*

(c) *G/T is abelian and isomorphic to the infinite cyclic group.*

(d) *$G = T \rtimes \langle z \rangle$ for some element z of infinite order, which acts as a group of power automorphisms on T.*

Proof. First we suppose that every subgroup of infinite index in G is permutable and prove $(a) - (d)$.

(a) We first note that T is a subgroup of G. If $x \in T$ then x has finite order and hence $|G : \langle x \rangle|$ is infinite. Therefore, by hypothesis, $\langle x \rangle$ per G. Hence if $y \in T$, we have $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$, which is finite. Thus T is a normal, being the unique maximal locally finite subgroup of G. Also, every subgroup of T is permutable in T and hence T is quasihamiltonian.

Let g be an element of infinite order. Then $\langle g \rangle \cap T = 1$. If $H \leq T$ then H is permutable, since T and therefore H has infinite index in G. Thus, $H\langle g \rangle \leq G$. Now $H = H\langle g \rangle \cap T \lhd H\langle g \rangle$. Therefore $\langle g \rangle$ normalizes H. Since G is non-periodic and the set of elements of finite order forms a subgroup, then G is generated by its elements of infinite order. It follows that

$G \leq N_G(H)$ and hence $H \lhd G$. Therefore T is Dedekind, and indeed every subgroup of T is G-invariant.

(b) Since G is non-quasihamiltonian there exists an element $g \in G$ of infinite order such that $\langle g \rangle$ is not permutable in G. Then, by hypothesis, $|G : \langle g \rangle|$ is finite and consequently $|G : \mathrm{Core}_G(\langle g \rangle)|$ is also finite. Thus G is cyclic-by-finite. Hence there exists $n \in \mathbb{Z}$ such that $\langle g^n \rangle \lhd G$ and also $|G : \langle g^n \rangle|$ is finite. But $|T\langle g^n \rangle : \langle g^n \rangle| = |T|$, so T is finite.

(c) Let $g \in G$ have infinite order and suppose $\langle g \rangle$ is not permutable in G. We have $|G : \langle g \rangle|$ is finite, so G is finitely generated. Since G is generated by its elements of infinite order, we have $G = \langle g_1, g_2, \cdots, g_k \rangle$, where the order of each g_i is infinite. Then $|G : \langle g_i \rangle|$ is finite, as is $|G : C_G(g_i)|$. Since $\zeta(G) = \cap_{i=1}^{k} C_G(g_i)$, we have $|G : \zeta(G)|$ is finite. Then, by Schur's Theorem [**21**, 10.1.4], G' is finite. Thus $G' \leq T$ so G/T is abelian. Hence $G/T \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ copies}}$ for some n.

If $n \geq 2$ then $|G : \langle x \rangle|$ is infinite for all $x \in G$ and hence G is quasihamiltonian, contrary to our hypothesis. Therefore $n = 1$ and hence $G/T \cong \mathbb{Z}$.

(d) Since G/T is isomorphic to the infinite cyclic group, there exist $z \in G$ so that $G/T = \langle zT \rangle$. Then $|z|$ is infinite and $G = T \rtimes \langle z \rangle$. Since every subgroup of T is normal in G, $\langle z \rangle$ acts as a power automorphism of T.

Next we prove the converse, and assume $(a) - (d)$. Suppose $G = T \rtimes \langle z \rangle$, for some element z of infinite order. Since T is finite, $G/C_G(T)$, which is isomorphic to a subgroup of $\mathrm{Aut}\, T$, is also finite. Hence there exists an integer k such that $z^k \in C_G(T)$, which implies $z^k \in \zeta(G)$. Hence $|G : \zeta(G)|$ is finite and moreover $|G : \langle z^k \rangle|$ is also finite.

Let $x \in G$ be such that $|G : \langle x \rangle|$ is infinite. Then we need to prove that $\langle x \rangle$ is permutable in G. For this it suffices to show that $\langle x \rangle$ has finite order. Suppose for a contradiction that $|\langle x \rangle|$ is infinite. Then $\langle x \rangle \langle z^k \rangle / \langle z^k \rangle \leq G/\langle z^k \rangle$ implies that $|\langle x \rangle \langle z^k \rangle / \langle z^k \rangle|$ is finite. Thus $|\langle x \rangle : \langle x \rangle \cap \langle z^k \rangle|$ is finite as well. However, then $\langle x \rangle \cap \langle z^k \rangle \neq 1$ so $|\langle z^k \rangle : \langle x \rangle \cap \langle z^k \rangle|$ is finite. Then $|G : \langle x \rangle \cap \langle z^k \rangle|$ is finite since $|G : \langle z^k \rangle|$ is finite. Therefore $|G : \langle x \rangle|$ is finite, a contradiction. Thus x has finite order, so $x \in T$ and hence $\langle x \rangle \lhd G$ and consequently every subgroup of infinite index is permutable in G.

□

Here we note that, in this case, every subgroup of infinite index is normal in G. Moreover, in the case of an infinite periodic group, if every subgroup of infinite index is permutable then the group is quasihamiltonian. Now we proceed with the following lemma.

LEMMA 2.2. *Let G be a group and let C* per *G. Suppose $A, B \leq G$ and $B \leq A$. Then there is a transversal $X = \{a_\alpha \,|\, \alpha \in \Lambda\}$ to CB in CA such that $X \subseteq A$.*

PROOF. To prove the lemma, we first note that CB and CA are subgroups of G. Let xCB be an arbitrary coset of CB in CA. Since $x = ca$, for some $c \in C, a \in A$ and since C is permutable in G, we have $xCB = caCB = a'c'CB = a'CB$ for some $a' \in A$ and $c' \in C$. Therefore $xCB = a'CB$ and it can be assumed that $a_\alpha \in A$ for all $\alpha \in \Lambda$.

□

Now using Lemma 2.2, we can give the generalization of Lemma 1.8 as follows.

LEMMA 2.3. *Let A, B be two permutable subgroups of a group G with $B \leq A$. Suppose C is any subgroup of G. Let $\{a_\alpha CB \,|\, \alpha \in \Lambda\}$ be a set of all the distinct cosets of CB in CA, where $a_\alpha \in A$ and let $\{c_\beta(C \cap B) \,|\, \beta \in \Gamma\}$ be a set of all the distinct cosets of $C \cap B$ in $C \cap A$, where $c_\beta \in C$. Then $\{a_\alpha c_\beta B \,|\, \alpha \in \Lambda, \beta \in \Gamma\}$ is a set of all distinct cosets of B in A for all $\alpha \in \Lambda$ and $\beta \in \Gamma$.*

PROOF. We remark that by the previous lemma we are justified in our selection of the $a_\alpha \in A$. Let xB be an arbitrary coset of B in A, where $x \in A$. For some $\alpha \in \Lambda$ we have $xCB = a_\alpha CB$ and this implies that $x = a_\alpha y$ for some $y \in CB$. Then, y can be written in the form $y = cb$ for some $c \in C$, $b \in B$, and hence $x = a_\alpha cb$. Then $c = a_\alpha^{-1} x b^{-1}$, so $c \in A \cap C$. Also, by assumption, $c(C \cap B) = c_\beta(C \cap B)$ is true for some $\beta \in \Gamma$. Hence $c = c_\beta z$, where $z \in C \cap B$. Thus, $x = a_\alpha c_\beta z b$. Hence $xB = a_\alpha c_\beta z b B = a_\alpha c_\beta B$.

Now we will show that all the cosets $a_\alpha c_\beta B$ are distinct. Suppose that $a_\alpha c_\beta B = a_\delta c_\gamma B$ where a_α, $a_\delta \in A$ and $c_\beta, c_\delta \in C$. Then $a_\alpha c_\beta BC = a_\delta c_\gamma BC$. Since $c_\beta \in BC$, we have $a_\alpha c_\beta BC = a_\alpha CB$ and, by the same argument, we also have, $a_\delta c_\gamma BC = a_\delta CB$. Consequently

$a_\alpha CB = a_\delta CB$. By the hypothesis of the lemma, $\alpha = \delta$. But in this case $c_\beta B = c_\gamma B$, which implies that $c_\gamma^{-1} c_\beta \in B$. Hence $c_\gamma^{-1} c_\beta \in B \cap C$, that is $c_\beta(C \cap B) = c_\gamma(C \cap B)$ and again by the hypothesis of the lemma, $\beta = \gamma$. Hence all the cosets are distinct. This completes the proof.

□

Lemma 2.3 has the following interesting consequence.

LEMMA 2.4. *Let $C \leq B \leq A$ and suppose A, B, C are permutable where $|A : B|$, $|B : C|$ are infinite. Let $x \in G$. Then at least one of $|A\langle x\rangle : B\langle x\rangle|$, $|B\langle x\rangle : C\langle x\rangle|$ is infinite.*

PROOF. Since $B \leq A$ and both A, B are permutable, $B\langle x\rangle$ is a subgroup of $A\langle x\rangle$. Let $\{a_\alpha B\langle x\rangle\}$ be a set of all distinct cosets of $B\langle x\rangle$ in $A\langle x\rangle$ where $a_\alpha \in A$ and $\{c_\beta(B \cap \langle x\rangle)\}$ be a set of all distinct cosets of $B \cap \langle x\rangle$ in $A \cap \langle x\rangle$ where $c_\beta \in \langle x\rangle$. Then, by Lemma 2.3, $\{a_\alpha c_\beta B\}$ is a set of all distinct cosets of B in A. Therefore $|A : B| = |A\langle x\rangle : B\langle x\rangle|\,|A \cap \langle x\rangle : B \cap \langle x\rangle|$. Since $|A : B|$ is infinite, the quantities on the right hand side can not both be finite. If $|A\langle x\rangle : B\langle x\rangle|$ is finite then $|A \cap \langle x\rangle : B \cap \langle x\rangle|$ is infinite, so $B \cap \langle x\rangle = 1$. Similarly, $|B\langle x\rangle : C\langle x\rangle||B \cap \langle x\rangle : C \cap \langle x\rangle| = |B : C|$. Since $B \cap \langle x\rangle = 1$ we have $|B\langle x\rangle : C\langle x\rangle|$ infinite. In either case, the result holds.

□

As a consequence of the above lemma, we have the following frequently used corollary.

COROLLARY 2.1. (i) *Suppose $A_1 \geq A_2 \geq A_3 \geq \cdots$ is a descending chain of permutable subgroups of G with $|A_i : A_{i+1}|$ infinite for all i. Let $x \in G$. Then there is a subsequence $\{i_j\}_{j\geq 1}$ such that $|A_{i_j}\langle x\rangle : A_{i_{j+1}}\langle x\rangle|$ is infinite for all $j \geq 1$.*

(ii) *Suppose $A_1 \leq A_2 \leq A_3 \leq \cdots$ is an ascending chain of permutable subgroups of G with $|A_{i+1} : A_i|$ infinite for all i. Let $x \in G$. Then there is a subsequence $\{i_j\}_{j\geq 1}$ such that $|A_{i_{j+1}}\langle x\rangle : A_{i_j}\langle x\rangle|$ is infinite for all $j \geq 1$.*

PROOF. (i) For each $i \geq 1$, we consider $A_{3i} \geq A_{3i+1} \geq A_{3i+2}$ and for any $x \in G$ form $A_{3i}\langle x\rangle \geq A_{3i+1}\langle x\rangle \geq A_{3i+2}\langle x\rangle$. Then by Lemma 2.4, either $|A_{3i}\langle x\rangle : A_{3i+1}\langle x\rangle|$ is infinite or $|A_{3i+1}\langle x\rangle : A_{3i+2}\langle x\rangle|$ is infinite.

(ii) The proof of this part is analogous to part (i).

□

Elements of infinite order often create difficulties. One difficulty is removed by the following lemma.

LEMMA 2.5. *Suppose* $A_1 \gneq A_2 \gneq A_3 \gneq \cdots$, *where* $A_i = B_i \times B_{i+1} \times B_{i+2} \times \cdots$ *and* $B_i \leq G$ *for all* i. *If* x *is an element of* G *of infinite order then, for some* j, $\langle x\rangle \cap A_j = 1$.

PROOF. Here we may assume that $A_1 \cap \langle x\rangle \neq 1$. Then for some $i \in \mathbb{Z}$,
$x^i = (b_1, b_2, b_3, \cdots, b_{i_1}, 1, 1, \cdots)$ for some integer i_1. Now for $j > i_1$, we have
$(B_1 \times B_2 \times \cdots \times B_{i_1}) \cap A_j = 1$. If $A_j \cap \langle x\rangle \neq 1$ we can find $l \in \mathbb{Z}$ such that $x^l \in A_j$. Then
$x^l = (\underbrace{1, 1, ..1}_{i_1}, 1, 1, .., b_{i_2}, ..., b_{i_3}, 1, ...)$ and hence

$$x^{il} = (b_1^l, b_2^l, \cdots, b_{i_1}^l, 1, 1, \cdots)$$

$$x^{li} = (1, 1, \cdots, 1, 1, \cdots, 1, b_{i_2}^i, \cdots, b_{i_3}^i, 1, 1, \cdots).$$

But $x^{il} = x^{li}$ so $(b_k)^l = 1$ for all k, such that $1 \leq k \leq i_1$. Therefore, $x^{il} = x^{li} = 1$ which implies that x has finite order, a contradiction. Therefore $\langle x\rangle \cap A_j = 1$ for some j.

□

The next lemma is a very important observation which will be used in the proof of some of our other results later in the dissertation.

LEMMA 2.6. *Let* G *be a group with the weak minimal condition on non-permutable subgroups and suppose that* G *contains subgroups* X, Y *with* $Y \lhd X$ *such that* $X/Y = \underset{i\geq 1}{\mathrm{Dr}}\,(A_i/Y)$ *for certain* A_i *such that* $A_i/Y \neq 1$. *Then*

(i) X *is permutable in* G.

(ii) *For any* $x \in G$, $X\langle x\rangle$ *is permutable in* G.

Proof. (i) Since $X/Y = \operatorname*{Dr}_{i\geq 1}(A_i/Y)$, we can relabel X/Y and rewrite it as $X/Y = \operatorname*{Dr}_{\substack{i\geq 1\\ j\geq 1}}(B_{ij}/Y)$. For fixed j, let us define $C_j/Y = \operatorname*{Dr}_{\substack{i\geq 1\\ j \text{ fixed}}}(B_{ij}/Y)$. Therefore, we can write $X/Y = \operatorname*{Dr}_{i\geq 1}(C_i/Y) \cong \operatorname*{Dr}_{i\geq 1}(A_i/Y)$ where each C_i/Y is an infinite direct product and $X = \prod_{i\geq 1} C_i$. Now we have a descending chain $D_1 \gneqq D_2 \gneqq D_3 \gneqq \cdots$ of subgroups of G with $|D_i : D_{i+1}|$ infinite for all i, where $D_i = C_i C_{i+1} C_{i+2} \cdots$. Since G has the weak minimal condition on non-permutable subgroups, there is a positive integer k such that D_k per G. Thus $C_k C_{k+1} C_{k+2} \cdots$ is permutable in G. Now construct a descending chain of subgroups of G

$$C_1C_2C_3\cdots \gneqq C_1C_2\cdots C_kC_{k+2}\cdots \gneqq C_1C_2\cdots C_kC_{k+3}\cdots \gneqq C_1C_2\cdots C_kC_{k+4} \gneqq \cdots$$

with $|C_1C_2C_3\cdots C_rC_{r+1}C_{r+2}\cdots : C_1C_2C_3\cdots C_rC_{r+2}\cdots|$ infinite for all r. Since the group G has the weak minimal condition on non-permutable subgroups, there exists a positive integer l such that $C_1C_2C_3\cdots C_kC_{k+l}C_{k+l+1}\cdots$ is permutable in G. Since the product of two permutable subgroups is permutable therefore $C_1C_2\cdots C_kC_{k+1}\cdots = X$ is permutable in G.

(ii) By part (i) each D_k is permutable in G and we have a descending chain $D_1 \gneqq D_2 \gneqq D_3 \gneqq \cdots$ of permutable subgroups of G with $|D_k : D_{k+1}|$ is infinite for all k.
Fix $x \in G$. We have $D_1 \gneqq D_2 \gneqq D_3 \gneqq \cdots$. By Corollary 2.1 there is a subsequence $D_{k_1} \gneqq D_{k_2} \gneqq D_{k_3} \gneqq \cdots$ such that $|D_{k_l}\langle x\rangle : D_{k_{l+1}}\langle x\rangle|$ is infinite for all l. Since G has the weak minimal condition on non-permutable subgroups, there exists a positive integer m such that $D_{k_m}\langle x\rangle$ is permutable in G, that is $C_{k_m}C_{k_{m+1}}C_{k_{m+3}}\cdots\langle x\rangle$ is permutable in G. Since each C_i is permutable in G by the first part therefore $C_1C_2C_3\cdots C_{k_{m-1}}$ is permutable in G. Since the product of two permutable subgroups is permutable therefore $C_1C_2C_3\cdots C_{k_{m-1}}C_{k_m}C_{k_{m+1}}\cdots\langle x\rangle$ is permutable in G and equivalently $X\langle x\rangle$ is permutable in G.

□

Here we note that, in the above lemma, if Y is trivial then every subgroup X of the form $X = \operatorname*{Dr}_{i\geq 1} A_i$ is permutable in G provided G satisfies the weak minimal condition on non-permutable subgroups. We also observe that the proof of Lemma 2.6 shows that if $X = \underset{i\geq 1}{\mathrm{Dr}}\, C_i$ then we can write $X = \underset{i\geq 1}{\mathrm{Dr}}\, B_i$ where $|\underset{i\geq j}{\mathrm{Dr}}\, B_i : \underset{i\geq j+1}{\mathrm{Dr}}\, B_i|$ is infinite for all j, and each B_i is permutable in G. As a corollary we have

Corollary 2.2. *Let G be a group satisfying the weak minimal condition on non-permutable subgroups. Suppose that G contains a subgroup A of the form $B_1 \times B_2 \times B_3 \times \cdots$ where $|\underset{i\geq j}{\mathrm{Dr}}\, B_i : \underset{i\geq j+1}{\mathrm{Dr}}\, B_i|$ is infinite for all j, and B_i* per *G. Let x be any element of infinite order such that $A \cap \langle x\rangle = 1$. Then $x \in N_G(B_k)$ for all k.*

The proof of the corollary follows by Theorem 1.3. Now we continue this section with another lemma which is interesting in its own right.

Lemma 2.7. *Let G be a group satisfying the weak minimal condition on non-permutable subgroups. Suppose that G contains a subgroup A of the form $B_1 \times B_2 \times B_3 \times \cdots$, with $B_i \neq 1$. Let $x \in G$ be any element of infinite order such that $A \cap \langle x\rangle = 1$. Then $x \in N_G(B_k)$ for all k.*

Proof. Since $A = B_1 \times B_2 \times B_3 \times \cdots$. We can write $A = B_k \times \underset{i\geq 1}{\mathrm{Dr}}\, C_i$ where $|\underset{i\geq j}{\mathrm{Dr}}\, C_i : \underset{i\geq j+1}{\mathrm{Dr}}\, C_i|$ is infinite. Then $B_k \times C_i$ per G by Lemma 2.6 and so $x \in N_G(B_k \times C_i)$ for all i by Theorem 1.3. Hence $x \in N_G(\bigcap_{i\geq 1}(B_k \times C_i)) = N_G(B_k)$ □

We proceed the section with the following useful lemma.

Lemma 2.8. *Let $C_1 \geq C_2 \geq C_3 \geq \cdots$ be a descending chain of subgroups of a group G satisfying the weak minimal condition on non-permutable subgroups such that $\bigcap_{i\geq 1} C_i = L \lhd C_i$ and $C_i/L = \underset{j\geq i}{\mathrm{Dr}}\,(D_j/L)$ where $|C_i : C_{i+1}|$ is infinite. Then $\bigcap_{i\geq 1} C_i\langle x\rangle = L\langle x\rangle$ for all $x \in G$. In particular, $L\langle x\rangle \leq G$, for all $x \in G$ and L is permutable in G.*

Proof. We note that, by Lemma 2.6, C_i per G. To prove the lemma, we first prove that $\bigcap_{i\geq 1} C_i\langle x\rangle = L\langle x\rangle$ for all $x \in G$. Since $L\langle x\rangle \subseteq C_i\langle x\rangle$, for all i, it follows that $L\langle x\rangle \subseteq \bigcap_{i\geq 1} C_i\langle x\rangle$. For the converse, let $d \in \bigcap_{i\geq 1} C_i\langle x\rangle$. Then $d \in C_i\langle x\rangle$ for all i. Thus, we have

$$d = c_1 x^{i_1} = c_2 x^{i_2} = c_3 x^{i_3} = \cdots \qquad (1)$$

where $c_i \in C_i$ and $i_k \in \mathbb{Z}$ for each k. This implies $c_{k+1}^{-1} c_k = x^{i_{k+1}} x^{-i_k} \in C_1 \cap \langle x \rangle$ for all k. Suppose $C_1 \cap \langle x \rangle = 1$. Then $c_{k+1} = c_k$ for all k, so $c_k \in \bigcap_{k \geq 1} C_k = L$ and hence $d \in L\langle x \rangle$. Therefore $L\langle x \rangle = \bigcap_{i \geq 1} C_i \langle x \rangle$ in this case.

On the other hand when $C_1 \cap \langle x \rangle \neq 1$ then for some l, $x^l \in C_1$. This implies that $x^l L \in C_1/L$ and hence $x^l L = (a_1 L, a_2 L, a_3 L, \cdots, a_r L, L, L, \cdots)$ for certain elements $a_k \in D_k$. Suppose that $x^m \in C_n$ for some $n > r$, and some $m \in \mathbb{Z}$, so that $x^m L \in C_n/L$. Then $x^m L = (L, L, L, \cdots, L, a_{r+1} L, a_{r+2} L, a_{r+3} L \cdots)$ for some $a_k \in D_k$. We have,

$$x^{lm} L = (a_1^m L, a_2^m L, a_3^m L, \cdots, a_r^m L, L, L, \cdots) \tag{2}$$

$$x^{ml} L = (L, L, L, \cdots, L, a_{r+1}^l L, a_{r+2}^l L, a_{r+3}^l L \cdots) \tag{3}$$

Therefore $a_i^m,\ a_{r+i}^l \in L$ for all i. Hence $x^{lm} \in L = \bigcap_{i \geq 1} C_i$, which implies that $L \cap \langle x \rangle = \langle x^\mu \rangle$ for some $\mu \in \mathbb{Z}$. Thus $x^\mu \in C_i$ for all i. Now, in equation (1) we write $i_j = \mu q_j + r_j$ with $0 < r_j < \mu$. Therefore

$$d = c_1 x^{q_1 \mu + r_1} = c_2 x^{q_2 \mu + r_2} = c_3 x^{q_3 \mu + r_3} = \cdots$$

and hence,

$$d = c_1 x^{q_1 \mu} x^{r_1} = c_2 x^{q_2 \mu} x^{r_2} = c_3 x^{q_3 \mu} x^{r_3} = \cdots \tag{4}$$

Now we can write equation (4) in the form

$$d = d_1 x^{r_1} = d_2 x^{r_2} = d_3 x^{r_3} = \cdots$$

where $d_i = c_i x^{q_i \mu} \in C_i$. Since $0 < r_i < \mu$, we can find a subset $\{l_1, l_2, l_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ with $l_1 \leq l_2 \leq l_3 \leq \cdots$ such that $r_{l_0} = r_{l_1} = r_{l_2} = r_{l_3} = \cdots$. Therefore

$$d = d_{l_1} x^{l_{r_0}} = d_{l_2} x^{l_{r_0}} = d_{l_3} x^{l_{r_0}} = \cdots$$

and consequently, $d_{l_1} = d_{l_2} = d_{l_3} = \cdots$. Thus $d_{l_1} \in \bigcap_{i \geq 1} C_{l_i} = L$ and $d = d_{l_1} x^{l_{r_0}} \in L\langle x \rangle$. Therefore, $\bigcap_{i \geq 1} C_i \langle x \rangle = L\langle x \rangle$. Since each C_i per G it follows that $C_i \langle x \rangle \leq G$ for all i and hence $\bigcap_{i\geq 1} C_i\langle x\rangle \leq G$. Therefore $L\langle x\rangle \leq G$. Since $L\langle x\rangle$ is a subgroup for each $x \in G$, $L\langle x\rangle = \langle x\rangle L$ so L is permutable in G.

□

We conclude the section with two similar lemmas.

LEMMA 2.9. *Let G be a group and let $x, y \in G$ be elements of finite order. Suppose $\{A_k\}_{k\geq 1}$ is a collection of subgroups of G such that $\bigcap_{k\geq 1} A_k = 1$. Suppose further $A_k\langle x\rangle\langle y\rangle$is a subgroup for all k. Then $\langle x\rangle\langle y\rangle$ is a subgroup of G.*

PROOF. Since $A_k\langle x\rangle\langle y\rangle$ is a subgroup for all k so $\bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$ is a subgroup. To show $\langle x\rangle\langle y\rangle$ is a subgroup of G it suffices to to show that $\bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$. Since $\langle x\rangle\langle y\rangle \subseteq A_k\langle x\rangle\langle y\rangle$ for all k we have $\langle x\rangle\langle y\rangle \subseteq \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$. Conversely, suppose $d \in \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$. Then $d \in A_k\langle x\rangle\langle y\rangle$ for all k. So we have the equations;

$$d = a_1 x^{i_1} y^{j_1} = a_2 x^{i_2} y^{j_2} = a_3 x^{i_3} y^{j_3} = \cdots \qquad (1)$$

where $a_k \in A_k$ and i_k, j_k are non-negative integers for all k. Since both x and y have finite order, there is a subset $\{k_1, k_2, k_3, \dots\}$ of $\{1, 2, 3, \dots\}$ with $k_1 \leq k_2 \leq k_3 \leq \cdots$ such that $i_0 = i_{k_1} = i_{k_2} \cdots$ and $j_0 = j_{k_1} = j_{k_2} = \cdots$ where i_0, $j_0 \in \mathbb{Z}$. Then, from (1), we have,

$$d = a_{k_1} x^{i_0} y^{j_0} = a_{k_2} x^{i_0} y^{j_0} = a_{k_3} x^{i_0} y^{j_0} = \cdots \qquad (2)$$

where $a_{k_i} \in A_{k_i}$. So, $a_{k_1} = a_{k_2} = a_{k_3} = \cdots$. Thus, $a_{k_1} \in \bigcap_{i\geq 1} A_{k_i} = 1$. So, $a_{k_i} = 1$ for all i and hence $d = x^{i_0} y^{j_0}$. Thus $d \in \langle x\rangle\langle y\rangle$. Therefore, $\bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle \subseteq \langle x\rangle\langle y\rangle$. So, $\langle x\rangle\langle y\rangle = \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$ is a subgroup of G. □

We can prove a similar lemma in the case when x is an element of infinite order and y is an element of finite order.

LEMMA 2.10. *Let G be a group and let x be an element of infinite order and y be an element of finite order. Suppose $\{A_k\}_{k\geq 1}$ is a collection of subgroups of G with $A_k = \operatorname{Dr}_{j\geq k} C_j$ and $\bigcap_{k\geq 1} A_k = 1$. Suppose further $A_k\langle x\rangle\langle y\rangle$is a subgroup for all k. Then $\langle x\rangle\langle y\rangle$ is a subgroup of G.*

Proof. Suppose $d \in \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$. Then $d \in A_k\langle x\rangle\langle y\rangle$ for all k. So,

$$d = a_1x^{i_1}y^{j_1} = a_2x^{i_2}y^{j_2} = a_3x^{i_3}y^{j_3} = \cdots \qquad (1)$$

where $a_k \in \mathrm{A}_k$ and i_k is an integer, j_k is a non-negative integer for all k. Since y has finite order, there is a subset $\{k_1, k_2, k_3, \ldots\}$ of $\{1, 2, 3, \ldots\}$ with $k_1 \leq k_2 \leq k_3 \leq \cdots$ such that $j_0 = j_{k_1} = j_{k_2} = j_{k_3} \cdots$, where $j_0 \in \mathbb{Z}$. Then, from (1),

$$d = a_{k_1}x^{i_{k_1}}y^{j_0} = a_{k_2}x^{i_{k_2}}y^{j_0} = a_{k_3}x^{i_{k_3}}y^{j_0} = \cdots \qquad (2)$$

where, $a_{k_i} \in A_{k_i}$ for all i. So, $a_{k_1}x^{i_{k_1}} = a_{k_2}x^{i_{k_2}} = a_{k_3}x^{i_{k_3}} \cdots$. Therefore,

$$a_{k_m}^{-1}\, a_{k_n} = x^{i_{k_m}}\, x^{-i_{k_n}} \in A_1 \cap \langle x\rangle \text{ for all } m,\, n \in \mathbb{N}.$$

Using Lemma 2.5, we may assume that $A_1 \cap \langle x\rangle = 1$, so $a_{k_m}^{-1}\ a_{k_n} = x^{i_{k_m}}\ x^{-i_{k_n}} = 1$. Therefore, we have $a_{k_1} = a_{k_2} = a_{k_3} = \cdots$ and $i_0 = i_{k_1} = i_{k_2} = \cdots$ for some $i_0 \in \mathbb{Z}$. Thus, $a_{k_1} \in \bigcap_{i\geq 1} A_{k_i}$. But $\bigcap_{i\geq 1} A_{k_i} = 1$ and thus $d = x^{i_0}y^{j_0}$. This implies that $d \in \langle x\rangle\langle y\rangle$ and hence $\bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle \subseteq \langle x\rangle\langle y\rangle$. It follows that $\bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$ is a subgroup of G. $\square$

CHAPTER 3

PROOF OF MAIN THEOREMS

In this chapter, we prove the main results of our research. The proof of our first main result is very similar to that given in [**3**], where it is proved that if G is non-periodic or locally graded group satisfying the minimal condition on non-quasinormal subgroups then either it is quasihamiltonian or it is a Chernikov group. The first main result is as follows.

THEOREM 3.1. *Let G be a locally finite group satisfying the weak minimal condition on non-permutable subgroups, then either G is quasihamiltonian or it is a Chernikov group.*

PROOF. Suppose that G is not a Chernikov group. Then, by Theorem 1.7, G does not satisfy the minimal condition on abelian subgroups. Therefore G contains an infinite abelian subgroup A, which is not Chernikov. Since A is an abelian, locally finite group, it follows that $A = \operatorname{Dr}_p A_p$, where A_p is the p-component of A. Let $\pi(A)$ denote the set of primes dividing the orders of the elements of A. If all A_p have finite rank then, by Lemma 1.12, each A_p is Chernikov and consequently A is Chernikov if $\pi(A)$ is finite. Therefore $\pi(A)$ is infinite and hence A contains an infinite direct product of the form $\operatorname{Dr}_{i\geq 1} \langle a_i \rangle$, where every a_i has prime power order. On the other hand, if some A_p has infinite rank then A_p contains an infinite direct product of the form $\operatorname{Dr}_{i\geq 1} \langle a_i \rangle$, by Lemma 1.11. Hence, in either case, we may assume A is of the form $A = \operatorname{Dr}_{i\geq 1} \langle a_i \rangle$, where every a_i has prime power order. For our convenience, let us write $A_i = \langle a_i \rangle$, so $A = \operatorname{Dr}_{i\geq 1} A_i$.

We can rewrite A as $A = \operatorname{Dr}_{\substack{i\geq 1\\ j\geq 1}} B_{ij}$ where $B_{ij} \neq 1$ for all i, j and hence assume $A = \operatorname{Dr}_{j\geq 1} C_j$ where each $C_j = \operatorname{Dr}_{\substack{i\geq 1\\ j \text{ fixed}}} B_{ij}$ and $|\operatorname{Dr}_{j\geq i} C_j : \operatorname{Dr}_{j\geq i+1} C_j|$ is infinite. Also, by Lemma 2.6, each C_j is permutable in G.

Define $B_m = \operatorname*{Dr}_{j \geq m} C_j$ so that $B_1 = A$. Let x, y be fixed elements of G. Let $T = \langle x, y \rangle$, a finite group. Then we may assume that $T \cap B_1 = 1$ (on replacing A with some suitable subgroup if necessary). Since each B_i is permutable in G, by Lemma 2.6, we have a descending chain

$$B_1 \gneq B_2 \gneq B_3 \gneq B_4 \gneq \cdots$$

of permutable subgroups with $|B_i : B_{i+1}|$ infinite for all i.

For every non-negative integer n, let $H_n = \langle x \rangle B_n$. Then, there is a descending chain of subgroups

$$H_1 \gneq H_2 \gneq H_3 \gneq H_4 \cdots$$

with $|H_i : H_{i+1}|$ infinite for all $i \geq 1$, upon passing to a subsequence, by Corollary 2.1. Since G has the weak minimal condition on non-permutable subgroups, there is an integer $r \in \mathbb{Z}$ such that H_r is permutable. Then, $\langle y \rangle \langle x \rangle B_r = \langle x \rangle B_r \langle y \rangle = \langle x \rangle \langle y \rangle B_r$. In particular, there exist $s, t \in \mathbb{Z}$ and an element $z \in B_1$ such that $yx = x^s y^t z$. Then $z \in \langle x, y \rangle \cap B_1$. But $\langle x, y \rangle \cap B_1 = 1$, and hence $z = 1$. Thus we have $x^s y^t = yx$. It follows that $\langle x \rangle$ and $\langle y \rangle$ permute. Since x and y were arbitrary elements of G, it follows that G is quasihamiltonian. □

Now using some of the preliminary results that we obtained in Chapter 2, we prove the next theorem:

THEOREM 3.2. *Let G be a group satisfying the weak minimal condition on non-permutable subgroups. Suppose that G contains a subgroup B of the form $B = B_1 \times B_2 \times B_3 \times \cdots$ where each $B_i \neq 1$. Then G is quasihamiltonian.*

PROOF. Since B is an infinite direct product we can write it as $B = \operatorname*{Dr}_{\substack{i \geq 1 \\ j \geq 1}} B_{ij}$ and hence assume $B = \operatorname*{Dr}_{j \geq 1} C_j$ where each $C_j = \operatorname*{Dr}_{\substack{i \geq 1 \\ j \text{ fixed}}} B_{ij}$ and $|\operatorname*{Dr}_{j \geq i} C_j : \operatorname*{Dr}_{j \geq i+1} C_j|$ is infinite. Also, by Lemma 2.6, each C_j is permutable in G.

Define $A_i = \operatorname*{Dr}_{j \geq i} C_j$ so that $A_1 = B$. Clearly $\bigcap_{i \geq 1} A_i = 1$. Since each A_i is permutable in G, by Lemma 2.6, we have a descending chain

$$A_1 \gneq A_2 \gneq A_3 \gneq A_4 \gneq \cdots$$

of permutable subgroups with $|A_i : A_{i+1}|$ infinite for all i.

Fix $x, y \in G$. Then by Lemma 2.6, $A_k\langle x\rangle$, $A_k\langle y\rangle$ per G for all $k \geq 1$. Therefore $(A_k\langle x\rangle)\langle y\rangle = \langle x\rangle\langle y\rangle A_k = \langle y\rangle(A_k\langle x\rangle) = \langle y\rangle(\langle x\rangle A_k)$ for all k.

Claim: $\langle x\rangle\langle y\rangle = \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$

Since $\langle x\rangle\langle y\rangle \subseteq A_k\langle x\rangle\langle y\rangle$ for all k it follows that, $\langle x\rangle\langle y\rangle \subseteq \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$. To prove the converse, $\bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle \subseteq \langle x\rangle\langle y\rangle$, we consider the following different cases:

Case (1): When x and y both are elements of finite order.

By the above construction, Lemma 2.9 implies that $\langle x\rangle\langle y\rangle$ is a subgroup of G for all elements $x, y \in$ of finiter order. Hence all finite cyclic subgroups of G permute.

Case (2): When x is an element of infinite order and y is an element of finite order.

By the above construction, Lemma 2.10 implies that $\langle x\rangle\langle y\rangle$ is a subgroup. Hence $\langle x\rangle$ and $\langle y\rangle$ permute, in this case.

Case (3): When both x and y are elements of infinite order.

Since both x and y are of infinite order, using Lemma 2.5, we may assume that $A_k \cap \langle x\rangle = 1 = A_k \cap \langle y\rangle = 1$ for all k. Here we note that each A_k is permutable in G. Then, by Theorem 1.3, $x, y \in N_G(A_k)$. Therefore, $A_k \lhd A_k\langle x\rangle\langle y\rangle$ for all k. Now we have the following sub-cases to consider:

Case 3(a): $A_k\langle x\rangle \cap \langle y\rangle = 1$ for all k.

Since $A_k\langle x\rangle$ is permutable and $A_k\langle x\rangle \cap \langle y\rangle = 1$ for all k, then, by Theorem 1.3, $A_k\langle x\rangle \lhd A_k\langle x\rangle\langle y\rangle$ for all k. Therefore, $\bigcap_{k\geq 1} A_k\langle x\rangle \lhd \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$. But, by Lemma 2.8, $\bigcap_{k\geq 1} A_k\langle x\rangle = \langle x\rangle$. It follows that $\langle x\rangle \lhd \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$ is a subgroup and hence $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ in this case.

Case 3(b): $A_k\langle x\rangle \cap \langle y\rangle \neq 1$ for all k and $\langle x\rangle \cap \langle y\rangle \neq 1$.

Again, in this case, for any $d \in \bigcap_{k\geq 1} A_k\langle x\rangle\langle y\rangle$, we have

$$d = a_1x^{i_1}y^{j_1} = a_2x^{i_2}y^{j_2} = a_3x^{i_3}y^{j_3} = \cdots . \qquad (1)$$

Here $A_k \cap \langle x\rangle = 1 = A_k \cap \langle y\rangle$ for all k. Since $\langle x\rangle \cap \langle y\rangle \neq 1$, we have $x^r = y^s$ for some r, $s \in \mathbb{Z}$. Also, $j_n = sq_n + r_n$ where $0 \leq r_n < s$. So

$$x^{i_n}y^{j_n} = x^{i_n}y^{q_ns+r_n} = x^{i_n}x^{rq_n}y^{r_n} = x^{i_n+rq_n}y^{r_n}.$$

Thus, from the above observation, we can write equation (1) in the form

$$d = a_1 x^{l_1} y^{r_1} = a_2 x^{l_2} y^{r_2} = a_3 x^{l_3} y^{r_3} = \cdots , \qquad (2)$$

where $r_i < s$ and $l_n = i_n + r\,q_n$. Since, $0 \le r_i < s$ there are only finitely many powers of y occurring. Therefore we can find a subset $\{k_1, k_2, k_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ with $k_1 \le k_2 \le k_3 \le \cdots$ (possibly after renumbering or re-indexing if necessary) such that $r_0 = r_{k_1} = r_{k_2} = \cdots$, with $r_0 \in \mathbb{Z}$. Equation (2), now reduces to

$$d = a_{k_1} x^{l_{k_1}} y^{r_0} = a_{k_2} x^{l_{k_2}} y^{r_0} = a_{k_3} x^{l_{k_3}} y^{r_0} = \cdots , \qquad (3)$$

where, r_0, $l_{k_i} \in \mathbb{Z}$ and $a_{k_i} \in A_{k_i}$ for all i. Now, from equation (3), we have

$$dy^{-r_0} = a_{k_1} x^{l_{k_1}} = a_{k_2} x^{l_{k_2}} = a_{k_3} x^{l_{k_3}} = \cdots$$

which implies in particular that $a_{k_m}^{-1} a_{k_1} = x^{l_{k_m}} x^{-l_{k_1}} \in \langle x \rangle \cap A_{k_1} = 1$. So, $a_{k_m} = a_{k_1}$ and $x^{l_{k_1}} = x^{l_{k_m}}$. Thus $a_{k_1} \in \bigcap_{i \ge 1} A_{k_i} = 1$ and hence $a_{k_1} = a_{k_2} = a_{k_3} = \cdots = 1$. Therefore, from equation (2),

$$d = x^{l_1} y^{r_1} = x^{l_2} y^{r_2} = x^{l_3} y^{r_3} = \cdots .$$

This implies that $d \in \langle x \rangle \langle y \rangle$ and consequently, $\langle x \rangle \langle y \rangle = \bigcap_{k \ge 1} A_k \langle x \rangle \langle y \rangle$, which is a subgroup of G. Hence $\langle x \rangle$ and $\langle y \rangle$ permute in this case as well.

Case 3(c): $A_k \langle x \rangle \cap \langle y \rangle \ne 1$ for all k and $\langle x \rangle \cap \langle y \rangle = 1$.

Since x and y both have infinite order then, using Lemma 2.5 we can assume that $A_k \cap \langle x \rangle = 1 = A_k \cap \langle y \rangle$ for all k. Also from Lemma 2.7, we have $B_k \lhd (B_1 \times B_2 \times \dots) \langle x \rangle \langle y \rangle$ for all k. To prove our claim in this case, suppose $d \in \bigcap_{k \ge 1} A_k \langle x \rangle \langle y \rangle$. Then $d \in A_k \langle x \rangle \langle y \rangle$ for all k. Since $A_k \langle x \rangle \langle y \rangle = \langle x \rangle A_k \langle y \rangle$, we may write

$$d = x^{i_1} a_1 y^{j_1} = x^{i_2} a_2 y^{j_2} = x^{i_3} a_3 y^{j_3} = \cdots . \qquad (4)$$

Suppose $a_1 = (b_1, b_2, \cdots b_s, 1, 1, \cdots)$ and $a_{s+1} = (1, 1, \cdots, 1, b_{s+1}, b_{s+2}, \cdots, b_t, 1, \cdots)$; for some s and t. Then we have,

$x^{i_1}(b_1, b_2, \cdots, b_s, 1, 1, \cdots) y^{j_1} = x^{i_2}(1, 1, \cdots, 1, b_{s+1}, b_{s+2}, \cdots, b_t, 1, \cdots) y^{j_2}$.

Hence $x^{(i_1-i_2)}(b_1, b_2, \cdots b_s, 1, 1, \cdots) = (1, 1, \cdots, 1, b_{s+1}, b_{s+2}, \cdots, b_t, 1, \cdots)y^{(j_2-j_1)}$.
Using Lemma 2.7, we get,

$$x^{(i_1-i_2)} = (1, 1, \cdots 1, b_{s+1}, \cdots b_t, 1, \cdots)(b'_1, b'_2, \cdots, b'_s, 1, 1, \cdots)y^{(j_2-j_1)},$$

and consequently,

$$x^{(i_1-i_2)} = (b'_1, b'_2, \cdots b'_s, b_{s+1}, \cdots b_t, 1, \cdots)y^{(j_2-j_1)} \qquad (5)$$

Similarly, repeating the above procedure for large enough k, we have

$$x^{(i_k-i_{k+1})} = (1, \cdots, 1, b'_{t+1}, b'_{t+2} \cdots, b'_u, b_{u+1}, \cdots, b_v, 1, \cdots)y^{j_{k+1}-j_k} \qquad (6)$$

Combining equations (5) and (6) and, then using Lemma 2.7 we have,

$$x^{(i_1-i_2)(i_k-i_{k+1})} = [(b'_1, b'_2, \cdots b'_s, b_{s+1}, \cdots, b_t, \cdots)y^{(j_2-j_1)}]^{(i_k-i_{k+1})}$$

or

$$x^{(i_1-i_2)(i_k-i_{k+1})} = (c_1, c_2, c_3, \cdots c_t, 1, 1, \cdots)y^{(j_2-j_1)(i_k-i_{k+1})} \qquad (7)$$

Similarly,

$$x^{(i_k-i_{k+1})(i_1-i_2)} = (1, 1, \cdots, 1, c_{t+1}, \cdots c_v, 1, \cdots)y^{(j_{k+1}-j_k)(i_1-i_2)} \qquad (8)$$

Thus from equations (7) and (8), we have

$$(c_1, c_2, c_3, \cdots, c_t, 1, \cdots)y^{(j_2-j_1)(i_k-i_{k+1})} = (1, 1, \cdots, 1, c_{t+1}, \cdots c_v, 1, \cdots)y^{(j_{k+1}-j_k)(i_1-i_2)},$$

which implies that,

$$(c_1, c_2, \cdots, c_t, c_{t+1}^{-1}, \cdots, c_v^{-1}, 1, \cdots) = y^{(j_{k+1}-j_k)(i_1-i_2)-(j_2-j_1)(i_k-i_{k+1})} \qquad (9)$$

Since $A_1 \cap \langle y \rangle = 1$ it follows that, $c_1 = c_2 = c_3 = \cdots = 1$ and hence

$$x^{(i_1-i_2)(i_k-i_{k+1})} = y^{(j_2-j_1)(i_k-i_{k+1})}$$

Also, $\langle x\rangle \cap \langle y\rangle = 1$, so it follows that, $(i_1 - i_2)(i_k - i_{k+1}) = (j_2 - j_1)(i_k - i_{k+1}) = 0$.

If $i_1 = i_2$, then from equation (4) it follows that $a_2^{-1}\, a_1 = y^{j_2 - j_1} \in A_1 \cap \langle y\rangle = 1$ Thus we have $a_1 = a_2$, which contradicts our choice of a_1 and a_2.
If $i_1 \neq i_2$ then for all large k, $i = i_k = i_{k+1}$. Therefore,

$$d = x^{i_k} a_k y^{j_k} = x^{i_{k+1}} a_{k+1} y^{j_{k+1}} = x^{i_{k+2}} a_{k+2} y^{j_{k+2}} = \cdots$$

or,

$$x^{-i} d = a_k y^{j_k} = a_{k+1} y^{j_{k+1}} = a_{k+2} y^{j_{k+2}} = \cdots \qquad (10)$$

which implies $a_{k+l}^{-1}\, a_k = y^{(j_{k+l} - j_k)} \in A_1 \cap \langle y\rangle = 1$. Therefore, $a_k = a_{k+1} = a_{k+1} = \cdots$, and hence $a_k \in \bigcap_{k \geq 1} A_k = 1$. Thus, $a_k = 1$ for all k and consequently,

$$d = x^{i_1} y^{j_1} = x^{i_2} y^{j_2} = x^{i_3} y^{j_3} = \cdots$$

Thus $d \in \langle x\rangle\langle y\rangle$ and hence $\bigcap_{k \geq 1} A_k \langle x\rangle\langle y\rangle \subseteq \langle x\rangle\langle y\rangle$. Therefore $\langle x\rangle\langle y\rangle$ is a subgroup and hence $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. It follows that for all $x, y \in G$, $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$. Thus G is quasihamiltonian.

□

Lemma 3.1. *Let G be a generalized radical group with the weak minimal condition on non-permutable subgroups. Suppose G contains an abelian subgroup A that has a normal subgroup K such that A/K is periodic and $\pi(A/K)$ is infinite. Then $K\langle x\rangle\langle y\rangle$ is a subgroup for all $x, y \in G$.*

Proof. Since $\pi(A/K)$ is infinite and A/K is periodic we can write $A/K = \operatorname{Dr}_{i \geq 1} (A_i/K)$, with $\bigcap_{i \geq 1} A_i = K$, and each A_i/K is the p-component of A/K. Moreover each $A_i/K \neq 1$. We can rewrite A/K as $A/K = \operatorname{Dr}_{\substack{i \geq 1 \\ j \geq 1}} B_{ij}/K$ and hence assume $A = \operatorname{Dr}_{j \geq 1} C_j/K$ where each $C_j/K = \operatorname{Dr}_{\substack{i \geq 1 \\ j \text{ fixed}}} B_{ij}/K$ and $|\operatorname{Dr}_{j \geq i} C_j/K : \operatorname{Dr}_{j \geq i+1} C_j/K|$ is infinite. Also, by Lemma 2.6, each C_j is permutable in G.

Define $B_m/K = \operatorname{Dr}_{j \geq m} C_j/K$. Since each B_i is permutable in G, by Lemma 2.6, we have a descending chain

$$B_1 \gneq B_2 \gneq B_3 \gneq B_4 \gneq \cdots$$

of permutable subgroups with $|B_i : B_{i+1}|$ infinite for all i.

Fix $x, y \in G$. Then by Lemma 2.6, $B_k\langle x\rangle$, $B_k\langle y\rangle$ per G for all $k \geq 1$. Therefore $(B_k\langle x\rangle)\langle y\rangle = \langle x\rangle\langle y\rangle B_k = \langle y\rangle(B_k\langle x\rangle) = \langle y\rangle(\langle x\rangle B_k)$ for all k. To prove $K\langle x\rangle\langle y\rangle$ is a subgroup of G, we have the following different cases:

Case 1: When x and y are both elements of finite order.

Clearly, $K\langle x\rangle\langle y\rangle \subseteq \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$. For the converse, suppose $d \in \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$. Then $d \in B_i\langle x\rangle\langle y\rangle$ for all i. So we have

$$d = b_1 x^{i_1} y^{j_1} = b_2 x^{i_2} y^{j_2} = b_3 x^{i_3} y^{j_3} = \cdots \qquad (1)$$

where $b_i \in B_i$ and i_k, j_k are non-negative integers for all k. Since both x and y have finite order, we can find a subset $\{k_1, k_2, k_3, \dots\}$ of $\{1, 2, 3, \dots\}$ with $k_1 \leq k_2 \leq k_3 \leq \cdots$ such that $i_0 = i_{k_1} = i_{k_2} \cdots$ and $j_0 = j_{k_1} = j_{k_2} = \cdots$ where i_0, $j_0 \in \mathbb{Z}$. Then from (1), we have

$$d = b_{k_1} x^{i_0} y^{j_0} = b_{k_2} x^{i_0} y^{j_0} = b_{k_3} x^{i_0} y^{j_0} \cdots \qquad (2)$$

where $b_{k_i} \in B_{k_i}$. Hence, $b_{k_1} = b_{k_2} = b_{k_3} = \cdots$. Thus, $b_{k_1} \in \bigcap_{i\geq 1} B_{k_i} = K$. So, $b_{k_i} \in K$ for all i and hence $d \in K\langle x\rangle\langle y\rangle$. Therefore, $\bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle \subseteq K\langle x\rangle\langle y\rangle$, and $K\langle x\rangle\langle y\rangle = \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$. Hence $K\langle x\rangle\langle y\rangle$ is a subgroup of G.

Case 2 : Suppose x is an element of infinite order and y is an element of finite order.

Suppose $d \in \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$. Then $d \in B_i\langle x\rangle\langle y\rangle$ for all i and hence we can write d as

$$d = b_1 x^{i_1} y^{j_1} = b_2 x^{i_2} y^{j_2} = b_3 x^{i_3} y^{j_3} = \cdots \qquad (3)$$

where $b_i \in B_i$, $i_k \in \mathbb{Z}$, j_k is a positive integer. Since y has finite order, we can find a subset $\{k_1, k_2, k_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ with $k_1 \leq k_2 \leq k_3 \leq \cdots$ such that $j_0 = j_{k_1} = j_{k_2} = j_{k_3} \cdots$, where $j_0 \in \mathbb{Z}$. Then from (3),

$$d = b_{k_1} x^{i_{k_1}} y^{j_0} = b_{k_2} x^{i_{k_2}} y^{j_0} = b_{k_3} x^{i_{k_3}} y^{j_0} = \cdots \qquad (4)$$

where, $b_{k_i} \in B_{k_i}$ for all i. So,

$$dy^{-j_0} = b_{k_1}x^{i_{k_1}} = b_{k_2}x^{i_{k_2}} = b_{k_3}x^{i_{k_3}} = \cdots \in \bigcap_{i\geq 1} B_{k_i}\langle x\rangle$$

But $\bigcap_{i\geq 1} B_{k_i}\langle x\rangle = K\langle x\rangle$ by Lemma 2.8. Therefore, $dy^{-j_0} \in \bigcap_{i\geq 1} B_{k_i}\langle x\rangle = K\langle x\rangle$. So $d \in K\langle x\rangle\langle y\rangle$ and hence $K\langle x\rangle\langle y\rangle$ is subgroup.

Case 3: When x and y both are elements of infinite order.

We have the following sub-cases to prove here.

Case 3(a): $B_i\langle x\rangle \cap \langle y\rangle = 1$ for all $i \geq 1$.

Since $B_i\langle x\rangle$ per G and $B_i\langle x\rangle \cap \langle y\rangle = 1$ for all i then, by Theorem 1.3, $B_i\langle x\rangle \lhd B_i\langle x\rangle\langle y\rangle$ for all i. Therefore, $\bigcap_{i\geq 1} B_i\langle x\rangle \lhd \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$. Since $\bigcap_{i\geq 1} B_i\langle x\rangle = K\langle x\rangle$ by Lemma 2.8 we have, $K\langle x\rangle \lhd \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$, so $K\langle x\rangle\langle y\rangle$ is a subgroup.

Case 3(b): $B_i\langle x\rangle \cap \langle y\rangle \neq 1$ and $\langle x\rangle \cap \langle y\rangle \neq 1$ for all i.

Since $\langle x\rangle \cap \langle y\rangle \neq 1$, we have $x^r = y^s$ for some $r,\ s \in \mathbb{Z}$. Also, for some $d \in \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$, we have

$$d = b_1x^{i_1}y^{j_1} = b_2x^{i_2}y^{j_2} = b_3x^{i_3}y^{j_3} = \cdots, \qquad (5)$$

where $b_i \in B_i, i_k,\ j_k \in \mathbb{Z}$. Here we can write $j_k = q_ks + r_k,\ 0 \leq r_k < s$. Therefore,

$$d = b_1x^{i_1}y^{q_1s}y^{r_1} = b_2x^{i_2}y^{q_2s}y^{r_2} = \cdots$$

or

$$d = b_1x^{i_1}x^{q_1r}y^{r_1} = b_2x^{i_2}x^{q_2r}y^{r_2} = \cdots.$$

Hence

$$d = b_1x^{l_1}y^{r_1} = b_2x^{l_2}y^{r_2} = b_3x^{l_3}y^{r_3} = \cdots \qquad (6)$$

where $l_k = i_k + rq_k$. Since $r_k < s$, we can find a subset $\{k_1, k_2, k_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ such that $r_0 = r_{k_1} = r_{k_2} = \cdots$. So, equation (6) reduces to

$$d = b_{k_1}x^{l_{k_1}}y^{r_0} = b_{k_2}x^{l_{k_2}}y^{r_0} = b_{k_3}x^{l_{k_3}}y^{r_0}$$

and, consequently

$$dy^{-r_0} = b_{k_1}x^{l_{k_1}} = b_{k_2}x^{l_{k_2}} = b_{k_3}x^{l_{k_3}} = \cdots . \qquad (7)$$

Thus, $dy^{-r_0} \in \bigcap_{i\geq 1} B_{k_i}\langle x\rangle$. Now Lemma 2.8 implies that $dy^{-r_0} \in K\langle x\rangle$ and therefore $d \in K\langle x\rangle\langle y\rangle$. Hence $\bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle = K\langle x\rangle\langle y\rangle$ is a subgroup in this case.

Case 3(c): When $B_i\langle x\rangle \cap \langle y\rangle \neq 1$, $\langle x\rangle \cap \langle y\rangle = 1$ for all i.

Sub-case 1: $B_i \cap \langle y\rangle \neq 1$, $B_1 \cap \langle x\rangle = 1$.

Suppose $d \in \bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle$. Then $d \in B_i\langle x\rangle\langle y\rangle$ for all i and hence d can be written in the form

$$d = x^{i_1}b_1y^{j_1} = x^{i_2}b_2y^{j_2} = x^{i_3}b_3y^{j_3} = \cdots \qquad (8)$$

Since $B_1 \cap \langle y\rangle \neq 1$, then $y^k \in B_1$ and hence $y^kK \in B_1/K$. Therefore, $y^kK = (b_1K, b_2K, b_3K, \cdots, b_rK, K, K, \cdots)$ where $b_i \in B_i$. If $\langle y\rangle \cap B_{r+1} \neq 1$, then for some l, $y^l \in B_{r+1}$ and hence $y^lK \in B_{r+1}/K$. Thus, $y^lK = (K, K, K, \cdots K, b_{r+1}K, b_{r+2}K, \cdots)$. So, $y^{kl}K = (b_1^lK, b_2^lK, \cdots b_r^lK, K, K, \cdots)$ and $y^{lk}K = (K, K \cdots K, b_{r+1}^kK, b_{r+2}^kK, \cdots)$. Therefore $b_i^l \in K$, $b_{r+i}^k \in K$ for all i. Thus $y^{kl} \in K = \bigcap_{i\geq 1} B_i$, and hence $K \cap \langle y\rangle = \langle y^\mu\rangle$ for some $\mu \in \mathbb{Z}$. So $y^\mu \in B_i$ for all i. We have $j_k = q_k\mu + r_k$ where $0 < r_k < \mu$, for some $q_k \in \mathbb{Z}$. Therefore, from equation (8),

$$d = x^{i_1}b_1y^{q_1\mu}y^{r_1} = x^{i_2}b_2y^{q_2\mu}y^{r_2} = x^{i_3}b_3y^{q_3\mu}y^{r_3} \cdots$$

Since $y^\mu \in B_i$ for all i, $y^{q_i\mu} \in B_i$ for all i. Thus we can write,

$$d = x^{i_1}c_1y^{r_1} = x^{i_2}c_2y^{r_2} = x^{i_3}c_3y^{r_3} = \cdots \qquad (9)$$

where $c_k = b_ky^{\mu q_k}$. Since, $0 < r_k < \mu$, we can find a subset $\{l_1, l_2, l_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ with $l_1 \leq l_2 \leq l_3 \leq \cdots$ such that $r_0 = r_{l_1} = r_{l_2} = r_{l_3} = \cdots$. Therefore,

$$d = x^{i_{l_1}}c_{l_1}y^{r_0} = x^{i_{l_2}}c_{l_2}y^{r_0} = x^{i_{l_3}}c_{l_3}y^{r_0} = \cdots$$

and

$$dy^{-r_0} = x^{i_{l_1}} c_{l_1} = x^{i_{l_2}} c_{l_1} = x^{i_{l_3}} c_{l_3} = \cdots \qquad (10)$$

which implies $dy^{-r_0} \in \bigcap_{i \geq 1} B_i\langle x\rangle$ and hence by Lemma 2.8, $dy^{-r_0} \in K\langle x\rangle$. Thus, $d \in K\langle x\rangle\langle y\rangle$ and henc $K\langle x\rangle\langle y\rangle$ is a subgroup of G.

Sub-Case 2: $B_1 \cap \langle x\rangle = 1 = B_1 \cap \langle y\rangle$.

Since $B_1 \cap \langle x\rangle = 1 = B_1 \cap \langle y\rangle$, therefore by Theorem 1.3 $x, y \in N_G(B_i)$ for all i. So $x, y \in N_G(K)$. Hence we can form the groups $B_i\langle x\rangle\langle y\rangle/K$. In particular by Theorem 3.2, we have $\bigcap_{i\geq 1} B_i\langle x\rangle\langle y\rangle/K = \langle Kx\rangle\langle Ky\rangle$. Therefore $K\langle x\rangle\langle y\rangle/K$ is a subgroup of $B_i\langle x\rangle\langle y\rangle/K$ and hence $K\langle x\rangle\langle y\rangle$ is a subgroup of G. □

Now we prove a proposition which will be useful in the proof of our main theorem.

PROPOSITION 3.1. *Let G be a generalized radical group satisfying the weak minimal condition on non-permutable subgroups. Then G is radical-by-finite.*

PROOF. Let R be the maximal normal radical subgroup of G. The existence of such a group is a consequence of Lemma 1.7. Suppose $R \neq G$. Then there exists $N \lhd G$ such that $R \lneq N$ and N/R is either locally finite or locally nilpotent. The choice of R implies that N/R is locally finite. Let $L = G/R$ and suppose K is the maximal normal locally finite subgroup of L. Since K satisfies the weak minimal condition on non-permutable subgroups then, by Theorem 3.1, K is either quasihamiltonian or Chernikov. If K is quasihamiltonian then, by Theorem 1.4, K is locally nilpotent and therefore K is trivial by the choice of R. Thus K is Chernikov. Since a radical-by-abelian group it follows that K must be finite.

If L is infinite then $K \neq L$. Since L/K is a generalized radical group and since an extension of a locally finite group by a locally finite group is locally finite it follows that the locally finite radical of L/K is trivial. Hence there is a normal locally nilpotent subgroup M/K of L/K. Furthermore M/K must be torsion-free. Therefore, L/K must contain a torsion free normal locally nilpotent subgroup M/K. Note that M/K is infinite. Moreover $L/C_L(K)$ is finite since K is finite and $K \cap C_M(K) = \zeta(K) = 1$, by the choice of R. The isomorphisms

$$C_M(K) \simeq C_M(K)/C_M(K) \cap K \simeq C_M(K)K/K \leq M/K$$

imply that $C_M(K)$ is locally nilpotent. Also $L/C_L(K)$ is finite so that $MC_L(K)/C_L(K) \simeq M/M \cap C_L(K)$ is finite and hence $M \cap C_L(K) = C_M(K)$ is infinite. This contradicts the choice of R. Hence the result follows.

□

THEOREM 3.3. *Let G be a generalized radical group satisfying the weak minimal condition on non-permutable subgroups. Then either*

(i) *G is quasihamiltonian, or*

(ii) *G is soluble-by-finite of finite rank.*

PROOF. Since G is a generalized radical group with the weak minimal condition on non-permutable subgroups it follows, by Proposition 3.1, that G is radical-by-finite. Let N be a normal radical subgroup of G such that G/N is finite. Now, we consider the abelian subgroups of G. If G contains an abelian subgroup A of infinite rank, then A contains a subgroup of the form $A_1 \times A_2 \times A_3 \times \cdots$, with $A_i \neq 1$, by Lemma 1.11. Since G satisfies the weak minimal condition on non-permutable subgroups and it has a subgroup of the form $A_1 \times A_2 \times A_3 \times \cdots$, it follows, by Theorem 3.2, that G is quasihamiltonian.

If all the abelian subgroups of G have finite ranks then, by Theorem 1.17, G also has finite rank. Next we prove that G is soluble-by-finite in this case. To prove this, we may assume that G is non-quasihamiltonian. Since G has finite rank then, by Theorem 1.18, there exist normal subgroups $1 \leq T \leq L \leq M \leq G$ such that T is locally finite and G/M is finite. Moreover, L/T is a torsion-free nilpotent group and M/L is a finitely generated torsion-free abelian group. Since T satisfies the weak minimal condition on non-permutable subgroups then, by Theorem 3.1, T is either quasihamiltonian or Chernikov. If T is quasihamiltonian then T is locally nilpotent, by Theorem 1.4. Hence by the structure theorem of periodic locally nilpotent groups, $T = \operatorname{Dr}_{p \in \pi} T_p$, where π is a set of primes. If $|\pi|$ is infinite then G is quasihamiltonian by Theorem 3.2. Therefore $|\pi|$ is finite. Since a locally finite p-group of finite rank is Chernikov, by Theorem 1.6, each T_p is Chernikov and hence so is T.

Since the class of soluble groups is closed with respect to formation of extensions, it follows by Theorem 1.18, that M/T is soluble. We also note that T is Chernikov, therefore it is either finite or has a non-trivial normal abelian subgroup of finite index. Hence without loss of generality, we may assume that T is finite. This implies that $M/C_M(T)$ is finite. The isomorphism

$$C_M(T)/\zeta(T) = C_M(T)/C_M(T) \cap T \simeq C_M(T)T/T \leq M/T$$

implies that $C_M(T)/\zeta(T)$ is soluble and hence by the extension property of soluble group, $C_M(T)$ is soluble. Moreover, G/M and $M/C_M(T)$ both are finite and $C_M(T)$ is soluble and consequently G is soluble-by-finite. This completes the proof. □

THEOREM 3.4. *Let G be a generalized radical group satisfying the weak minimal condition on non-permutable subgroups. If G is neither quasihamiltonian nor minimax then G has a torsion subgroup T, consisting of the set of all elements of finite order, such that T is Chernikov and G/T is torsion-free.*

PROOF. Since G is not quasihamiltonian, G is soluble-by-finite of finite rank, by Theorem 3.3. Furthermore, if all abelian subgroups of G are minimax then G is a soluble-by-finite minimax group, by Theorem 1.12. Hence we may assume that G has an abelian non-minimax subgroup A. Thus A contains a free abelian subgroup K of the form $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}$, such that A/K is periodic.

If $\pi(A/K)$ is finite then $A/K = \underset{\text{finite}}{\mathrm{Dr}}\, A_p$ where A_p is a p-component of A/K. Since each A_p is Chernikov, by Lemma 1.12, it follows that A/K is Chernikov and thus has the minimum condition. But K has the maximum condition, So A is a minimax group, which is a contradiction. Therefore, $\pi(A/K)$ is infinite and $A/K = \underset{i\geq 1}{\mathrm{Dr}}\,(A_i/K)$ where $\underset{i\geq 1}{\cap} A_i = K$ and each $A_i/K \neq 1$. Hence, by Lemma 3.1, $K\langle x\rangle\langle y\rangle$ is a subgroup for any $x, y \in G$.

Let $K^n = \{k^n | k \in K\}$. Then K/K^n is finite. Moreover, A/K^n is also periodic and we may replace K by K^n in Lemma 3.1 to deduce that $K^n\langle x\rangle\langle y\rangle$ is also a subgroup, for each

n and for each $x, y \in G$. Let $H_n = K^n$. For any two elements x, y of finite order, we claim that

$$\langle x\rangle\langle y\rangle = \bigcap_{n\geq 1} H_n\langle x\rangle\langle y\rangle$$

Clearly $\langle x\rangle\langle y\rangle \subseteq H_n\langle x\rangle\langle y\rangle$ for all n. So, $\langle x\rangle\langle y\rangle \subseteq \bigcap_{n\geq 1} H_n\langle x\rangle\langle y\rangle$. Conversely, suppose $d \in \bigcap_{n\geq 1} H_n\langle x\rangle\langle y\rangle$. Then $d \in H_n\langle x\rangle\langle y\rangle$ for all n. So we have

$$d = h_1 x^{i_1} y^{j_1} = h_2 x^{i_2} y^{j_2} = h_3 x^{i_3} y^{j_3} = \cdots \qquad (1)$$

where each $h_n \in H_n$. Since both x and y have finite order, we can find a subset $\{k_1, k_2, k_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ such that $i_0 = i_{k_1} = i_{k_2} = \cdots$ and $j_0 = j_{k_1} = j_{k_2} = \cdots$, where $i_0, j_0 \in \mathbb{Z}$. Then from equation (1), we have

$$d = h_{k_1} x^{i_0} y^{j_0} = h_{k_2} x^{i_0} y^{j_0} = h_{k_3} x^{i_0} y^{j_0} = \cdots \qquad (2)$$

Thus, $h_{k_1} = h_{k_2} = h_{k_3} = \cdots$, which implies, $h_{k_1} \in \bigcap_{i\geq 1} H_{k_i} = 1$. So $h_{k_i} = 1$ for all i and hence $d = x^{i_0} y^{j_0}$ and thus $d \in \langle x\rangle\langle y\rangle$. Therefore, $\bigcap_{n\geq 1} H_n\langle x\rangle\langle y\rangle \subseteq \langle x\rangle\langle y\rangle$ and hence $\langle x\rangle\langle y\rangle = \bigcap_{n\geq 1} H_n\langle x\rangle\langle y\rangle$. This implies that $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ for all x, y of finite order.

Let T be the set of elements of finite order. Then T is a characteristic subgroup of G and G/T is torsion-free.

But, by Theorem 1.4, T is locally nilpotent. Therefore, $T = \underset{p\in\pi}{Dr}\, T_p$, where π is a set of primes and T_p is the p-component of T. If π is infinite, then G is quasihamiltonian by Theorem 3.2. Therefore π is finite. Since each T_p is Chernikov, by Theorem 1.6, T is Chernikov.

□

It seems to be a very difficult problem to prove that if G is generalized radical group with weak minimal condition on non-permutable subgroups then either G is quasihamiltonian or minimax. If G is a generalized radical group of finite rank with the weak minimal condition on non-permutable subgroups then there are normal subgroups T, L, M of G such that $1 \leq T \leq L \leq M \leq G$, where T is locally finite, L/T is torsion-free nilpotent, M/L is a finitely generated torsion-free abelian group and G/M is finite. By Theorem 3.4, T is Chernikov so minimax, and G/L is likewise minimax. Thus whether G is minimax is dependent upon whether L/T is minimax or not.

CHAPTER 4

GROUPS WITH THE WEAK MAXIMAL CONDITION ON NON-PERMUTABLE SUBGROUPS

In this chapter we give some results related to the groups satisfying the weak maximal condition on non-permutable subgroup. We begin the chapter with an easy to prove lemma:

LEMMA 4.1. *Every subgroup and factor group of a group satisfying the weak maximal condition on non-permutable subgroups satisfies the weak maximal condition on non-permutable subgroups.*

Next we give a lemma, which connects the weak maximal condition on non-permutable subgroups with the maximal condition on non-normal subgroups.

LEMMA 4.2. *If G is a group satisfying the weak maximal condition on non-normal subgroups then G has the weak maximal condition on non-permutable subgroups.*

The proof of the above lemma is analogous to the proof of the Lemma 1.9. Next we establish another lemma which will be useful later in this chapter.

LEMMA 4.3. *Let G be a group satisfying the weak maximal condition on non-permutable subgroups. Suppose G contains a subgroup B of the form $B_1 \times B_2 \times B_3 \times \cdots$, with $B_i \neq 1$. Then B is permutable in G.*

PROOF. Since B is an infinite direct product we can relabel B and rewrite it as $B = \operatorname*{Dr}_{\substack{i\geq 1\\ j\geq 1}} B_{ij}$ and hence assume $B = \operatorname*{Dr}_{j\geq 1} C_j$ where each $C_j = \operatorname*{Dr}_{\substack{i\geq 1\\ j \text{ fixed}}} B_{ij}$ and $|\operatorname*{Dr}_{j\leq i+1} C_j : \operatorname*{Dr}_{j\leq i} C_j|$ is infinite.

Let us suppose $H = C_1 \times C_3 \times C_5 \times \cdots$ and $K = C_2 \times C_4 \times C_6 \times \cdots$ so that $B = H \times K$. Now we can construct an ascending chain

$$H \leq H \times C_2 \leq H \times C_2 \times C_4 \leq H \times C_2 \times C_4 \times C_6 \leq \cdots$$

of subgroups. Since G has the weak maximal condition on non-permutable subgroups, there exists a positive integer k such that $H \times C_2 \times C_4 \times C_6 \times \cdots \times C_{2k}$ is permutable in G. Similarly, for an ascending chain

$$K \leq K \times C_1 \leq K \times C_1 \times C_3 \leq K \times C_1 \times C_3 \times C_5 \leq \cdots$$

there exists a positive integer l such that $K \times C_1 \times C_3 \times C_5 \times \cdots \times C_{2l+1}$ is permutable in G. Since the product of two permutable subgroups is permutable it follows that B is permutable.

□

Analogous to Theorem 3.1, we have the corresponding theorem for group satisfying the weak maximal condition on non-permutable suubgroups.

THEOREM 4.1. *Let G be a locally finite group satisfying the weak maximal condition on non-permutable subgroups, then either G is quasihamiltonian or it is a Chernikov group.*

PROOF. Suppose that G is not a Chernikov group. Then, by Theorem 1.7, G does not satisfy the minimal condition on abelian subgroups. Therefore G contains an infinite abelian subgroup A, which is not Chernikov. Since A is an abelian, locally finite group, it follows that $A = \underset{p}{\mathrm{Dr}}\, A_p$, where A_p is the p-component of A. Let $\pi(A)$ denote the set of primes dividing the orders of the elements of A. If all A_p have finite rank then, by Lemma 1.12, each A_p is Chernikov and consequently A is Chernikov if $\pi(A)$ is finite. Therefore $\pi(A)$ is infinite and hence A contains an infinite direct product of the form $\underset{i\geq 1}{\mathrm{Dr}}\, \langle a_i \rangle$, where every a_i has prime power order. On the other hand, if some A_p has infinite rank then A_p contains an infinite direct product of the form $\underset{i\geq 1}{\mathrm{Dr}}\, \langle a_i \rangle$. Hence, in either case, we may assume A is of the form $A = \underset{i\geq 1}{\mathrm{Dr}}\, \langle a_i \rangle$, where every a_i has prime power order. For our convenience, let us write $A_i = \langle a_i \rangle$, so $A = \underset{i\geq 1}{\mathrm{Dr}}\, A_i$.

We can rewrite A as $A = \underset{\substack{i\geq 1\\ j\geq 1}}{\mathrm{Dr}}\, B_{ij}$ and hence assume $A = \underset{j\geq 1}{\mathrm{Dr}}\, C_j$ where each $C_j = \underset{\substack{i\geq 1\\ j \text{ fixed}}}{\mathrm{Dr}}\, B_{ij}$ and $|\underset{j\leq i+1}{\mathrm{Dr}}\, C_j : \underset{j\leq i}{\mathrm{Dr}}\, C_j|$ is infinite. Also, by Lemma 4.3, each C_j is permutable in G.

Define $B_n = \operatorname*{Dr}_{j\leq n} C_j$ so that $B_1 = C_1$. Let x, y be fixed elements of G. Let $T = \langle x, y\rangle$, a finite group. Then we may assume that $T \cap B_n = 1$ for all n. Since each B_i is permutable in G, by Lemma 4.3, we have an ascending chain

$$B_1 \lneqq B_2 \lneqq B_3 \lneqq \cdots$$

of permutable subgroups with $|B_{i+1} : B_i|$ infinite for all i.

For every non-negative integer n, let $H_n = \langle x\rangle B_n$. Then, there is an ascending chain

$$H_1 \lneqq H_2 \lneqq H_3 \lneqq H_4 \cdots$$

of subgroups with $|H_{i+1} : H_i|$ infinite for all $i \geq 1$, by Corollary 2.1. Since G has the weak maximal condition on non-permutable subgroups, there is an integer $r \in \mathbb{Z}$ such that H_r is permutable. Then, $\langle y\rangle\langle x\rangle B_r = \langle x\rangle B_r\langle y\rangle = \langle x\rangle\langle y\rangle B_r$. In particular, there exist $s, t \in \mathbb{Z}$ and an element $z \in B_r$ such that $yx = x^s y^t z$. Then $z \in \langle x, y\rangle \cap B_r$. But $\langle x, y\rangle \cap B_r = 1$, and hence $z = 1$. Thus we have $x^s y^t = yx$. It follows that $\langle x\rangle$ and $\langle y\rangle$ permute. Since x and y were arbitrary elements of G, it follows that G is quasihamiltonian.

□

Using Theorem 4.1, we have the following easy to prove proposition.

PROPOSITION 4.1. *Let G be a generalized radial group satisfying the weak maximal condition on non-permutable subgroups. Then G is radical-by-finite.*

The proof of the above proposition is analogous to the proof of the Proposition 3.1.

THEOREM 4.2. *Let G be a group satisfying the weak maximal condition on non-permutable subgroups. Suppose G contains a subgroup B of the form $B_1 \times B_2 \times B_3 \times \cdots$, with $B_i \neq 1$. Then G is quasihamiltonian.*

PROOF. Since B is an infinite direct product we can write it as $B = \operatorname*{Dr}_{\substack{i\geq 1\\ j\geq 1}} B_{ij}$ and hence assume $B = \operatorname*{Dr}_{j\geq 1} C_j$ where each $C_j = \operatorname*{Dr}_{\substack{i\geq 1\\ j \text{ fixed}}} B_{ij}$ and $|\operatorname*{Dr}_{j\leq i+1} C_j : \operatorname*{Dr}_{j\leq i} C_j|$ is infinite. Also, by Lemma 4.3, each C_j is permutable in G. Clearly $\bigcap_{j\geq 1} C_j = 1$.

Since C_1 is an infinite direct product, we can construct an ascending chain

$$D_1 \lneq D_2 \lneq D_3 \lneq \cdots \lneq C_1$$

of permutable subgroups D_i such that $|D_{i+1} : D_i|$ is infinite for all i

Fix $x, y \in G$. We have $D_1 \lneq D_2 \lneq D_3 \lneq \cdots$. By Corollary 2.1 there is a subsequence $D_{i_1} \lneq D_{i_2} \lneq D_{i_3} \lneq \cdots$ such that $|D_{i_{j+1}}\langle x\rangle : D_{i_j}\langle x\rangle|$ is infinite for all j. By relabeling we may assume that $D_1 \lneq D_2 \lneq D_3 \lneq \cdots$ and $|D_{i+1}\langle x\rangle : D_i\langle x\rangle|$ is infinite for all i. Then there exists a subsequence $D_{n_1} \lneq D_{n_2} \lneq D_{n_3} \lneq \cdots$ such that $|D_{n_{i+1}}\langle y\rangle : D_{n_i}\langle y\rangle|$ is infinite. Relabeling then we have $D_1 \lneq D_2 \lneq D_3 \lneq \cdots$, with $|D_{i+1} : D_i|$ infinite. Also $|D_{i+1}\langle x\rangle : D_i\langle x\rangle|$, $|D_{i+1}\langle y\rangle : D_i\langle y\rangle|$ are both infinite for all i. In this manner we construct ascending chains of subgroup of G

$$D_1\langle x\rangle \lneq D_2\langle x\rangle \lneq D_3\langle x\rangle \lneq \cdots$$

$$D_1\langle y\rangle \lneq D_2\langle y\rangle \lneq D_3\langle y\rangle \lneq \cdots$$

of subgroups of G with $|D_{i+1}\langle x\rangle : D_i\langle x\rangle|$, $|D_{i+1}\langle y\rangle : D_i\langle y\rangle|$ both infinite for all i and each D_i is permutable in G. Since G has the weak maximal condition on non-permutable subgroups, there exist a positive integer k such that $D_k\langle x\rangle$, $D_k\langle y\rangle$ per G. Let us write $D_k = E_1$ so $E_1\langle x\rangle$, $E_1\langle y\rangle$ per G. Again repeating the above procedure inside C_k for all $k \geq 1$, we can find $E_k \leq C_k$ such that $E_k\langle x\rangle$, $E_k\langle y\rangle$ per G. Therefore we may assume that $E_k\langle x\rangle, E_k\langle y\rangle$ per G for all k. This implies that $E_k\langle x\rangle\langle y\rangle \leq G$ for all k and hence $\bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle \leq G$. Notice that $\bigcap_{k\geq 1} E_k = 1$.

Now we claim $\bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$. Since $\langle x\rangle\langle y\rangle \subseteq E_k\langle x\rangle\langle y\rangle$ for all k we have $\langle x\rangle\langle y\rangle \subseteq \bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$. Conversely, for $d \in \bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$, we consider the following different cases:

Case 1: When x and y are both elements of finite order.

By the above construction, Lemma 2.9 implies that $\langle x\rangle\langle y\rangle$ is a subgroup of G for all elements $x, y \in G$ of finite order. Hence all finite cyclic subgroups of G permute.

Case 2: When x is an element of infinite order and y is an element of finite order.

Suppose $d \in \bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$. Then $d \in E_k\langle x\rangle\langle y\rangle$ for all k. So,

$$d = e_1 x^{i_1} y^{j_1} = e_2 x^{i_2} y^{j_2} = e_3 x^{i_3} y^{j_3} = \cdots \qquad (1)$$

where $e_k \in E_k$ and i_k is an integer, j_k are non-negative integers for all k. Since y has finite order, we can find a subset $\{k_1, k_2, k_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ with $k_1 \leq k_2 \leq k_3 \leq \cdots$ such that $j_0 = j_{k_1} = j_{k_2} = j_{k_3} \cdots$, where $j_0 \in \mathbb{Z}$. Then from (1),

$$d = e_{k_1} x^{i_{k_1}} y^{j_0} = e_{k_2} x^{i_{k_2}} y^{j_0} = e_{k_3} x^{i_{k_3}} y^{j_0} = \cdots \qquad (2)$$

where, $e_{k_i} \in E_{k_i}$ for all k. Now we can write equation (2) in the form

$$dy^{-j_0} = e_{k_1} x^{i_{k_1}} = e_{k_2} x^{i_{k_2}} = e_{k_3} x^{i_{k_3}} = \cdots$$

or

$$dy^{-j_0} = e_{k_1} x^{i_{k_1}} = e_{k_2} x^{i_{k_2}} = e_{k_3} x^{i_{k_3}} = \cdots \qquad (3)$$

Let us define $E = E_1 \times E_2 \times E_3 \times \cdots$. Then from (3), we have

$$e_{k_2}^{-1}\, e_{k_1} = x^{i_{k_2}}\, x^{-i_{k_1}} \in E \cap \langle x\rangle.$$

If $E \cap \langle x\rangle = 1$, then $e_{k_1} = e_{k_2} = \cdots$ and this implies that $e_{k_1} \in \bigcap_{i\geq 1} E_{k_i} = 1$. Therefore $e_{k_1} = e_{k_2} = e_{k_3} = \cdots$ and $i_0 = i_{k_1} = i_{k_2} = \cdots$. Thus $d \in \langle x\rangle\langle y\rangle$. It follows that $\bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$.

When $E \cap \langle x\rangle \neq 1$, then for some k, $x^k = (e_1, e_2, e_3, \cdots, e_r, 1, 1, \cdots)$ and suppose that $x^l = (1, 1, \cdots, 1, e_{r+1}, e_{r+2}, \cdots)$. Then,
$x^{kl} = (e_1^l, e_2^l, \cdots, e_r^l, 1, 1, \cdots)$ and $x^{lk} = (1, 1, 1, \cdots, 1, e_{r+1}^k, e_{r+2}^k, \cdots)$ which implies that $e_i^l = 1$ and $e_{r+i}^k = 1$ for all i. Therefore, $x^{kl} = 1$ and thus x has finite order, contrary to our assumption. Hence $\bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle = \langle x\rangle\langle y\rangle$.

Case (3): When both x and y are element of infinite order.

In this case, we may assume that $\langle x\rangle \cap E_k = \langle y\rangle \cap E_k = 1$ for all k. Since each E_k is permutable then by Theorem 1.3, $x, y \in N_G(E_k)$ for all k. Now we have the following subcases to prove:

Case 3(a): $E_k\langle x\rangle \cap \langle y\rangle = 1$ for all k.

Since $E_k\langle x\rangle$ is permutable and $E_k\langle x\rangle \cap \langle y\rangle = 1$ for all k. Then by Theorem 1.3, $E_k\langle x\rangle \lhd E_k\langle x\rangle\langle y\rangle$ for all k. Therefore, $\bigcap_{k\geq 1} E_k\langle x\rangle \lhd \bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$. Next we claim $\bigcap_{k\geq 1} E_k\langle x\rangle = \langle x\rangle$.

Clearly, $\langle x\rangle \subseteq E_k\langle x\rangle$ for all k. So, $\langle x\rangle \subseteq \bigcap_{k\geq 1} E_k\langle x\rangle$. Conversely, suppose $d \in \bigcap_{k\geq 1} E_k\langle x\rangle \backslash \langle x\rangle$, then

$$d = e_1x^{i_1} = e_2x^{i_2} = e_3x^{i_3} = \cdots$$

This implies that $e_2^{-1}e_1 = x^{i_2-i_1} \in E \cap \langle x\rangle$. If $E \cap \langle x\rangle = 1$ then $e_1 = e_2 = e_3 =$. Therefore, $e_1 \in \bigcap_{k\geq 1} E_k = 1$. Thus $d \in \langle x\rangle$. If we have $E \cap \langle x\rangle \neq 1$ then x has finite order as in the earlier case, a contradiction. Therefore $\bigcap_{k\geq 1} E_k\langle x\rangle = \langle x\rangle$. However, $\bigcap_{k\geq 1} E_k\langle x\rangle \lhd \bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$. This implies $\langle x\rangle \lhd E_k\langle x\rangle\langle y\rangle$, so $\langle x\rangle\langle y\rangle$ is a subgroup in this case. Thus $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$ in this case.

Case 3(b): $E_k\langle x\rangle \cap \langle y\rangle \neq 1$ for all k and $\langle x\rangle \cap \langle y\rangle \neq 1$.

In this case, for any $d \in \bigcap_{k\geq 1} E_k\langle x\rangle\langle y\rangle$, we have

$$d = e_1x^{i_1}y^{j_1} = e_2x^{i_2}y^{j_2} = e_3x^{i_3}y^{j_3} = \cdots \qquad (4)$$

Here $E_k \cap \langle x\rangle = 1 = E_k \cap \langle y\rangle$. Since $\langle x\rangle \cap \langle y\rangle \neq 1$, we have $x^r = y^s$ for some r, $s \in \mathbb{Z}$. Also, $j_1 = sq_1 + r_1$ where $0 \leq r_1 < s$. So

$$x^{i_1}y^{j_1} = x^{i_1}y^{q_1 s+r_1} = x^{i_1}x^{rq_1}y^{r_1} = x^{i_1+rq_1}y^{r_1}$$

Thus, from the above observation, we can write equation (4) in the form

$$d = e_1x^{l_1}y^{r_1} = e_2x^{l_2}y^{r_2} = e_3x^{l_3}y^{r_3} = \cdots (5)$$

where $r_i < s$ and $l_n = i_n + r\,q_n$. Since, $r_i < s$, we can find a subset $\{k_1, k_2, k_3, \cdots\}$ of $\{1, 2, 3, \cdots\}$ with $k_1 \leq k_2 \leq k_3 \cdots$ (possibly after renumbering or re-indexing if necessary) such that $r_0 = r_{k_1} = r_{k_2} = \cdots$, with $r_0 \in \mathbb{Z}$. So,

$$d = e_{k_1} x^{l_{k_1}} y^{r_0} = e_{k_2} x^{l_{k_2}} y^{r_0} = e_{k_3} x^{l_{k_3}} y^{r_0} = \cdots \qquad (6)$$

Here, r_0, i_{j_1}, $i_{j_2}, \cdots \in \mathbb{Z}$ and $e_{k_i} \in E_{k_i}$ for all i. Now, from equation (6), we have

$$dy^{-r_0} = e_{k_1} x^{l_{k_1}} = e_{k_2} x^{l_{k_2}} = e_{k_3} x^{l_{k_3}} = \cdots$$

which implies in particular that $e_{k_2}^{-1} e_{k_1} = x^{l_{k_2}} x^{-l_{k_1}} \in \langle x \rangle \cap E_k = 1$. So, $e_{k_2} = e_{k_1}$. Hence, $e_{k_1} \in \bigcap_{i \geq 1} E_{k_i} = 1$. Therefore, $e_{k_1} = e_{k_2} = e_{k_3} = \cdots = 1$. Thus,

$$d = x^{l_1} y^{r_1} = x^{l_2} y^{r_2} = x^{l_3} y^{r_3} = \cdots$$

which implies that $d \in \langle x \rangle \langle y \rangle$ and hence, $\langle x \rangle \langle y \rangle = \bigcap_{k \geq 1} E_k \langle x \rangle \langle y \rangle$, which is a subgroup of G. Hence $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$.

Case 3(c): $E_k \langle x \rangle \cap \langle y \rangle \neq 1$ for all k and $\langle x \rangle \cap \langle y \rangle = 1$.

Note that x and y both have infinite order and also we can assume that $E_k \cap \langle x \rangle = 1 = E_k \cap \langle y \rangle$. Using Theorem 1.3, we also get

$$E_k \lhd (E_1 \times E_2 \times E_3 \times \cdots) \langle x \rangle \langle y \rangle \qquad (7)$$

for all k. Suppose $d \in \bigcap_{k \geq 1} E_k \langle x \rangle \langle y \rangle$. Then $d \in E_k \langle x \rangle \langle y \rangle$ for all k. Since $E_k \langle x \rangle \langle y \rangle = \langle x \rangle E_k \langle y \rangle$, we can write

$$d = x^{i_1} e_1 y^{j_1} = x^{i_2} e_2 y^{j_2} = x^{i_3} e_3 y^{j_3} = \cdots \qquad (8)$$

where $e_k \in E_k$ and $\bigcap_{k \geq 1} E_k$=1. Suppose $e_1 = (b_1, b_2, \cdots b_s, 1, 1, \cdots)$ and $e_{s+1} = (1, 1, \cdots, 1, b_{s+1}, b_{s+2}, \cdots, b_t, 1, \cdots)$ for some s and t. Then we have,

$$x^{i_1}(b_1, b_2, \cdots, b_s, 1, 1, \cdots) y^{j_1} = x^{i_2}(1, 1, \cdots, 1, b_{s+1}, b_{s+2}, \cdots, b_t, 1, \cdots) y^{j_2}$$

$$x^{(i_1 - i_2)}(b_1, b_2, \cdots b_s, 1, 1, \cdots) = (1, 1, \cdots, 1, b_{s+1}, b_{s+2}, \cdots, b_t, 1, \cdots) y^{(j_2 - j_1)}$$

From equation 7,

$$x^{(i_1 - i_2)} = (1, 1, \cdots 1, b_{s+1}, \cdots b_t, 1, \cdots)(b'_1, b'_2, \cdots, b'_s, 1, 1, \cdots) y^{(j_2 - j_1)}.$$

This implies that $x^{(i_1-i_2)} = (b_1', b_2', \cdots b_s', b_{s+1}, \cdots b_t, 1, \cdots)y^{(j_2-j_1)}$ (9)

Similarly repeating the above procedure for large enough k, we have

$$x^{(i_k-i_{k+1})} = (1, \cdots, 1, b_{t+1}', b_{t+2}' \cdots, b_u', b_{u+1}, \cdots, b_v, 1, \cdots)y^{j_{k+1}-j_k} \tag{10}$$

Now combining (9) and (10), we can write

$$x^{(i_1-i_2)(i_k-i_{k+1})} = [(b_1', b_2', \cdots b_s', b_{s+1}, \cdots, b_t, \cdots)y^{(j_2-j_1)}]^{(i_k-i_{k+1})}$$

and then by (7), we have

$$x^{(i_1-i_2)(i_k-i_{k+1})} = (c_1, c_2, c_3, \cdots c_t, 1, 1, \cdots)y^{(j_2-j_1)(i_k-i_{k+1})}.$$

Again, from (8) and (9) and then using (7),

$$x^{(i_k-i_{k+1})(i_1-i_2)} = (1, 1, \cdots, 1, c_{t+1}, \cdots c_v, 1, \cdots)y^{(j_{k+1}-j_k)(i_1-i_2)}.$$

Therefore,

$$(c_1, c_2, c_3, \cdots, c_t, 1,)y^{(j_2-j_1)(i_k-i_{k+1})} = (1, 1, \cdots, 1, c_{t+1}, \cdots c_v, 1, \cdots)y^{(j_{k+1}-j_k)(i_1-i_2)}.$$

This implies that,

$$(c_1, c_2, \cdots, c_t, c_{t+1}^{-1}, \cdots, c_v^{-1}, 1, \cdots) = y^{(j_{k+1}-j_k)(i_1-i_2)-(j_2-j_1)(i_k-i_{k+1})} \in E \cap \langle y \rangle.$$

Since $E \cap \langle y \rangle = 1$ it follows that $c_1 = c_2 = c_3 = \cdots = 1$ and hence

$$x^{(i_1-i_2)(i_k-i_{k+1})} = y^{(j_2-j_1)(i_k-i_{k+1})}$$

Also, $\langle x \rangle \cap \langle y \rangle = 1$ it follows that, $(i_1 - i_2)(i_k - i_{k+1}) = (j_2 - j_1)(i_k - i_{k+1}) = 0$.
When $i_1 = i_2$, then from equation (10) it follows that $e_2^{-1} e_1 = y^{j_2-j_1} \in E \cap \langle y \rangle = 1$
Thus we have $e_1 = e_2$, which contradicts our choice of e_1 and e_2.

If $i_1 \neq i_2$ then for all large k, $i = i_k = i_{k+1}$. Therefore,

$$d = x^{i_k} e_k y^{j_k} = x^{i_{k+1}} e_{k+1} y^{j_{k+1}} = x^{i_{k+2}} e_{k+2} y^{j_{k+2}} = \cdots, \tag{11}$$

which implies that

$$x^{-i}d = e_k y^{j_k} = e_{k+1} y^{j_{k+1}} = e_{k+2} y^{j_{k+2}} = \cdots$$

and thus $e_{k+1}^{-1}\ e_k = y^{(j_{k+1}-j_k)} \in E \cap \langle y \rangle = 1$. Therefore, $e_k = e_{k+1} = e_{k+1} = \cdots,$

Thus, $e_k \in \bigcap\limits_{k\geq 1} E_k = 1$. Therefore, $e_k = 1$ for all k. Hence

$$d = x^{i_1} y^{j_1} = x^{i_2} y^{j_2} = x^{i_3} y^{j_3} = \cdots$$

and thus $d \in \langle x \rangle \langle y \rangle$. Hence $\bigcap\limits_{k\geq 1} E_k \langle x \rangle \langle y \rangle \subseteq \langle x \rangle \langle y \rangle$. Therefore $\langle x \rangle \langle y \rangle = \bigcap\limits_{k\geq 1} E_k \langle x \rangle \langle y \rangle$, is a subgroup in this case also. It follows that G is quasihamiltonian.

□

Finally, using the above theorem and other preliminary results, we can prove a theorem analogous to Theorem 3.3.

THEOREM 4.3. *Let G be a generalized radical group satisfying the weak maximal condition on non-permutable subgroups. Then either*

(i) *G is quasihamiltonian, or*

(ii) *G is soluble-by-finite of finite rank.*

The proof of the above theorem is analogous to the proof of the Theorem 3.3.

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