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Curved Surfaces of Right Cones.

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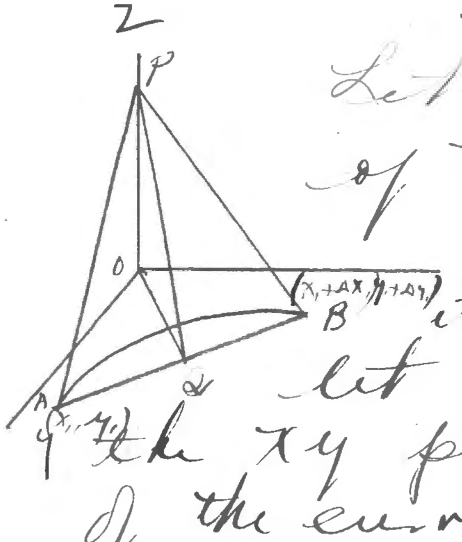
It is the purpose of this article to derive a formula for finding the curved surface of a right cone with any plane curve as a base.

The usual method of finding such a surface is by projecting an element of the tangent plane on one of the coördinate planes and evaluating the double integral

$$\int_a^b \int \sec \gamma \, dy \, dx,$$

where γ is the angle which the tangent plane, at any point, makes with the xy plane.

In many cases the evaluation of this integral is difficult, and requires transformations which are complicated. It will be shown that the area of the curved surface of any right cone may be found by evaluating a single integral.



Let us take the altitude of the cone as the Z-axis, and denote its altitude by c, and let the base lie in the XY plane. The equation of the curve determining the base may then be denoted by $y = f(x)$. Let (x_1, y_1) be any point on this curve and $(x_1+\Delta x, y_1+\Delta y_1)$ be any point close to (x_1, y_1). Call the line AB. Draw PA, PB and PQ. PQ being the altitude of the triangle PAB. The area of this triangle is the element of surface, and we may call it ΔS. Its area is equal to $\frac{1}{2} AB \times PQ$.

The equation of the line AB is $y - y_1 = \frac{\Delta y}{\Delta x}(x - x_1)$, or

$$x \Delta y - y \Delta x + y_1 \Delta x - x_1 \Delta y = 0.$$

OQ is found by throwing the equation into normal form and substituting $(0, 0)$ for x and y.

$\therefore OQ = \dfrac{y_1 \Delta x - x_1 \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$ where x_1 and y_1 are the coordinates of a point on the curve $y = f(x)$.

$$PQ = \sqrt{OQ^2 + c^2}$$

$$= \sqrt{c^2 + \frac{(y_1 \Delta x - x_1 \Delta y)^2}{\Delta x^2 + \Delta y^2}}$$

$$AB = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\Delta S = \tfrac{1}{2}\sqrt{c^2 + \frac{(y_1\Delta x - x_1\Delta y)^2}{\Delta x^2 + \Delta y^2}}\ \sqrt{\Delta x^2 + \Delta y^2}$$

$$= \tfrac{1}{2}\sqrt{(y_1\Delta x - x_1\Delta y)^2 + c^2(\Delta x^2 + \Delta y^2)}$$

$$= \tfrac{1}{2}\sqrt{\left(y_1 - x_1\frac{\Delta y}{\Delta x}\right)^2 + c^2\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]}\ \Delta x$$

$$\sum \Delta S = \tfrac{1}{2}\sum_a^b \sqrt{\left(y_1 - x_1\frac{\Delta y}{\Delta x}\right)^2 + c^2\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]}\ \Delta x$$

$$S = \mathop{L}_{\Delta x \doteq 0} \tfrac{1}{2}\sum_a^b \sqrt{\left(y_1 - x_1\frac{\Delta y}{\Delta x}\right)^2 + c^2\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]}\ \Delta x$$

$$S = \tfrac{1}{2}\int_a^b \sqrt{\left(y_1 - x\frac{dy}{dx}\right)^2 + c^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\ dx$$

The advantage in using the above formula may be shown by solving a problem, Let us find the area of the curved surface of a ~~&~~ right cone whose base is the astroid $x^{2/3} + y^{2/3} = \frac{a^{2/3}}{c^{2/3}} z^{2/3}$.

This solution follows the suggestion given in Byerly's Integral Calculus, and the work is here given in detail.

$$\sec\gamma = \frac{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}}{\frac{\partial f}{\partial z}}$$

$$\frac{\partial f}{\partial x} = \tfrac{2}{3}x^{-1/3}, \quad \frac{\partial f}{\partial y} = \tfrac{2}{3}y^{-1/3}, \quad \frac{\partial f}{\partial z} = \tfrac{2}{3}\frac{a^{2/3}}{c^{2/3}}z^{-1/3},$$

$$\sec\gamma = \frac{\tfrac{2}{3}\sqrt{y^{-2/3}+x^{-2/3}+\frac{a^{4/3}}{c^{4/3}}z^{-2/3}}}{-\tfrac{2}{3}a^{2/3}z^{-1/3}},$$

$$= -\frac{\sqrt{\frac{1}{y^{2/3}}+\frac{1}{x^{2/3}}+\frac{a^{4/3}}{c^{4/3}z^{2/3}}}}{\frac{a^{2/3}}{c^{2/3}z^{1/3}}},$$

$$= -\frac{1}{x^{1/3}c^{2/3}y^{1/3}z^{1/3}}\;\frac{\sqrt{x^{2/3}z^{2/3}c^{4/3}+c^{4/3}y^{2/3}z^{2/3}+a^{4/3}x^{2/3}y^{2/3}}}{\frac{a^{2/3}}{c^{2/3}z^{1/3}}},$$

$$= -\frac{1}{x^{1/3}y^{1/3}a}\sqrt{c^2\left(x^{2/3}+y^{2/3}\right)^2+a^2x^{2/3}y^{2/3}}.$$

Since we are seeking only the numerical value of the area we may drop the negative sign, and the entire area is

$$A = \frac{4}{a}\int_0^a\int_0^{\left(a^{2/3}-x^{2/3}\right)^{3/2}} x^{-1/3}y^{-1/3}\sqrt{c^2\left(x^{2/3}+y^{2/3}\right)^2+a^2x^{2/3}y^{2/3}}\;dy\,dx.$$

Let $v^3 = x$, and $w^3 = y$, then $dx = 3v^2\,dv$, $dy = 3w^2\,dw$.
Making these substitutions, the integral becomes,

$$A = \frac{3c}{a}\int_0^{a^{1/3}}\int_0^{(a^{2/3}-v^2)^{2/3}} vw\sqrt{c^2(v^2+w^2)+a^2v^2w^2}\;dw\,dv$$

For convenience we may put x for v and y for w, if we remember that these letters are no longer the x and y in the original integral;

$$A = \frac{3c}{a}\int_0^{a^{1/3}}\int_0^{(a^{2/3}-x^2)^{1/2}} xy\sqrt{c^2(x^2+y^2)^2+a^2x^2y^2}\;dy\,dx.$$

The above integral is what we would have to find if we were seeking the area of such a surface that its projection on the xy plane was a quadrant of the circle $x^2+y^2=a^{2/3}$, and the secant of the angle which the tangent plane, at any point on the surface, makes with the xy plane was equal to $xy\sqrt{a^2x^2y^2+c^2(x^2+y^2)^2}$.

Let us divide the projection into polar elements and replace x and y by their equals $x = r\cos\theta$, $y = r\sin\theta$. Let us remember that the polar element on the xy plane is $r\,d\theta\,dr$. Making these substitutions, the integral becomes

$$\frac{36}{a}\int_0^{\pi/2}\int_0^{a/3} r^2\sin\theta\cos\theta\sqrt{a^2r^4\sin^2\theta\cos^2\theta + c^2(r^2\cos^2\theta + r^2\sin^2\theta)^2}\; r\,dr\,d\theta$$

$$A = \frac{36}{a}\int_0^{\pi/2}\int_0^{a/3} r^5\sin\theta\cos\theta\sqrt{a^2\sin^2\theta\cos^2\theta + c^2}\;dr\,d\theta$$

$$= 6a\int_0^{\pi/2}\sin\theta\cos\theta\sqrt{a^2\sin^2\theta\cos^2\theta + c^2}\;d\theta$$

$$= 3a\int_0^{\pi/2}\sqrt{a^2\sin^2\theta\cos^2\theta + c^2}\;\sin 2\theta\,d\theta$$

$$= \frac{3a}{2}\int_0^{\pi/2}\sqrt{a^2\sin^2 2\theta + 4c^2}\;\sin 2\theta\,d\theta$$

$$= \frac{3a}{2}\int_0^{\pi/2}\sqrt{a^2\sin^2 2\theta + 4c^2}\;\sin 2\theta\,d\theta$$

$$= \frac{3a^2}{2}\int_0^{\pi/2}\sqrt{\frac{a^2+4c^2}{a^2} - \cos^2 2\theta}\;\sin 2\theta\,d\theta$$

$$= -\frac{3a^2}{4}\int_0^{\pi/2}\sqrt{\frac{a^2+4c^2}{a^2}-\cos^2 2\theta}\; d\cos 2\theta$$

(Form of $\int\sqrt{a^2-x^2}\,dx$)

$$= \frac{3}{4}\left[2ac + (a^2+4c^2)\arcsin\frac{a^2}{a^2+4c^2}\right]$$

$$= \frac{3}{4}\left[2ac + (a^2+4c^2)\arctan\frac{a}{2c}\right],$$

The above problem admirably shows up the difficulties which confront one in solving for the area of a comparatively simple surface. The obvious advantage of using the formula derived in this article may be shown by solving this same problem.

The formula is

$$A = \frac{1}{2}\int_a^b \sqrt{\left(y - x\frac{dy}{dx}\right)^2 + c^2\left[1+\left(\frac{dy}{dx}\right)^2\right]}\; dx.$$

For the astroid, let $x = a\cos^3\theta$, $y = a\sin^3\theta$

$dx = -3a\cos^2\theta\sin\theta\,d\theta$, $dy = 3a\sin^2\theta\cos\theta\,d\theta$, $\frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta}$

and the limits of integration are obviously 0 and $\pi/2$.

The entire area is therefore

$$2\int_{\pi/2}^{0}\sqrt{(a\sin^3\theta + a\sin\theta\cos^2)^2 + c^2\sec^2\theta}\;(-3a\sin\theta\cos^2\theta\, d\theta)$$

$$= -6a\int_{\pi/2}^{0}\sqrt{a^2\sin^2\theta(\sin^2\theta + \cos^2\theta)^2 + c^2\sec^2\theta}\;\sin\theta\cos^2\theta\, d\theta$$

$$= -6a\int_{\pi/2}^{0}\sqrt{a^2\sin^2\theta\cos^2\theta + c^2}\;\sin\theta\cos\theta\, d\theta$$

$$= -\frac{3}{2}\int_{\pi/2}^{0}\sqrt{a^2\sin^2 2\theta + 4c^2}\;\sin 2\theta\, d\theta$$

$$= \frac{3a^2}{4}\int_{\pi/2}^{0}\sqrt{\frac{a^2+4c^2}{a^2} - \cos^2 2\theta}\;\;d\cos 2\theta,$$

It will be noticed that this is precisely the integral derived in the preceding solution whose value is

$$3/4\left[2ac + (a^2+4c^2)\,\mathrm{arc\,tan}\,a/2c\right].$$