

GRAPHS OF GROUPS

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ABSTRACT

Graphs of groups were first introduced by Jean-Pierre Serre in his book entitled *Arbres, Amalgames, SL_2* (1977), whose first English translation was in 1980 [5]. In 1993, Hyman Bass wrote a paper [1] which discussed such concepts in the category of Graphs of Groups as morphisms, fundamental groups, and infinite covers. Hence, this area of geometric group theory is typically referred to as Bass-Serre Theory. The contents of this dissertation lie within this broad area of study. The main focus of the research is to try to apply to the category of Graphs of Groups what John Stallings did in the category of Graphs [6]. In that paper, he explored in graphs a vast number of topics such as pullbacks, paths, stars, coverings, and foldings. The goal of this dissertation is to apply many of those concepts to the category of Graphs of Groups. In this work, we develop our notion of paths, links, maps of graphs of groups, and coverings. We then explore the resultant path-lifting properties.

DEDICATION

This work is dedicated to my lovely wife, Barbra. What a blessing my God bestowed upon me when he gave me you. Thank you for your steady, invaluable companionship.

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I would like to first acknowledge and thank my Lord and savior, Jesus Christ. He has been with me since I was a boy and through Him all things truly are possible. Next, I would like to thank my wife and children. Accomplishing this goal was truly a team achievement. Thank you for your unwavering support and sacrifice. We did it! Without the love and support of my mother and father, I do not know what my life would be like today. Thank you for instilling in me a desire to work hard and excel in academics. I am forever indebted to you. To my friend Kevin Bowling, your constant encouragement has meant the world to me and has truly helped me to press on through the tough times I have endured during this process. I am pretty sure that you were calling me “Dr. Green” my first semester. I would also like to thank my high school math teacher, Mr. James Rogers. You opened my eyes to the beautiful mathematical theory behind the applications and I have remained enamoured with it. I am greatly indebted to Dr. Vivian Wright for her constant tutelage and assistance in my career pursuits, and more importantly, life in general. You are truly a light. Thank you to Dr. Dixon for the countless hours you spent in your office with me getting my feet wet in the world of abstract algebra and the wonderful in-class instruction that you provided. You were extremely patient with me, and I greatly appreciate it. Dr. Corson, the same is true for you. I will have very fond memories of the time that we worked together on our paper and my dissertation. It was truly an exciting time in my life, and I thank you so much for being willing to be my research advisor. It was my pleasure to be able to work with you. Finally, to my committee, I say thank you for the time and effort that you have invested in this work. All of you have taught me at some point, and I am definitely the better for it.

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CHAPTER 1

INTRODUCTION

We begin in Chapter 2 with listing the basic graph theory definitions and conventions that underly our work throughout the remainder of the paper. Upon completion of our basic graph theory discussion, we give the definition of graphs of groups that we will employ along with a discussion of the canonical example of graphs of groups, a group acting on a graph. The chapter concludes with a definition of the fundamental groupoid of a graph of groups and Proposition 2.4 which states that the map $E(\Gamma, A) \rightarrow \pi(\Gamma, A)$ given by $[aeb] \mapsto aeb$ is injective.

In Chapter 3, we turn our attention toward maps of graphs of groups, which we define simply as homomorphisms of the fundamental groupoids of graphs of groups in Definition 3.1. We then explore the necessary structure embedded in a map of graphs of groups. This work is primarily found in Lemma 3.3 and Proposition 3.4, where we recover Bass' notion of a map of graphs of groups.

The defining of paths in a graph of groups and the notion of homotopy of paths are the main topics explored in Chapter 4. Paths are defined in Definition 4.1. From there, we discuss round-trips, elementary reductions, and homotopy. Properties preserved under homotopy are then discussed and the chapter concludes with a proof that every homotopy class contains a unique reduced path.

In Chapter 5, we start by defining our notion of a link in Definition 5.1. From there, we define a covering in Definition 5.2 and proceed to delve into various path lifting conditions and properties. These include the unique path-lifting property stated

in Theorem 5.5 and the homotopy-lifting property found in Theorem 5.7. The concepts of conjugates of maps of graphs of groups, equivalence of coverings, and the existence of coverings round out the chapter.

Chapter 6 begins with the \sharp -construction. This is a necessary step in the proof of Theorem 5.10 which states that if $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a covering map, $G: (\Sigma, C) \rightarrow (\Delta, B)$ is a map of graphs of groups, where Σ is a connected graph, and u, v are vertices of Γ, Σ such that $f(u) = g(v)$, then there exists a map $\tilde{G}: (\Sigma, C) \rightarrow (\Gamma, A)$ such that $\tilde{g}(v) = u$ and $F\tilde{G}$ is conjugate to G if and only if $[G\pi(\Sigma, C, v)]^{b_0} \subseteq F\pi(\Gamma, A, u)$ for some $b_0 \in B_{f(u)} = B_{g(v)}$. Furthermore, the theorem states that if \tilde{G} exists, it is unique up to conjugacy. The chapter concludes with a detailed proof of the theorem.

Covering transformations of maps of graphs of groups are defined in Definition 7.1 at the outset of Chapter 7. This is followed by Theorem ??, which deals with the existence and uniqueness of covering transformations. From there, in Lemma ??, we see the relationship between regular coverings of a graph of groups and regular coverings of the \sharp -construction. The chapter concludes with Theorem 7.4 which states that if $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a regular covering of connected graphs of groups, then the group of covering transformations is isomorphic to the quotient group $\pi(\Delta, B, f(v))/F\pi(\Gamma, A, v)$ for any vertex v of Γ .

In Chapter 8, we conclude with a generalized Bass-Serre theory. Along the way, we discuss covering maps associated to group actions. Then, in Theorem 8.1, the relationship between quotient graphs of groups $(\Gamma/H, H_\tau)$ and $(\Gamma/G, G_\sigma)$, where $H \leq G$ and G is acting on Γ , is explored. The main theorem of Bass-Serre Theory is then stated in Corollary 8.3 with a subsequent discussion of the uniqueness of the associated covering and a proof of the existence of coverings.

CHAPTER 2

PRELIMINARIES

The notion of a graph that we prefer to use is more or less standard. Namely, a *graph* Γ consists of two disjoint sets $V(\Gamma)$ and $E(\Gamma)$, two functions $s, t: E(\Gamma) \rightarrow V(\Gamma)$, and a fix-point free involution on $E(\Gamma)$ denoted by $e \mapsto \bar{e}$ satisfying the following conditions: $s(\bar{e}) = t(e)$ and $t(\bar{e}) = s(e)$ for all $e \in E(\Gamma)$. The elements of $V(\Gamma)$ are called *vertices*. An element $e \in E(\Gamma)$ is called a (directed) *edge*; the vertices $s(e)$ and $t(e)$ are called the *source* and *target* of e , respectively; and the directed edge \bar{e} is called the *reverse* or *inverse* of e . (By assumption, $\bar{e} \neq e$ and $\bar{\bar{e}} = e$.)

A *map of graphs* $f: \Gamma \rightarrow \Delta$ consists of two functions

$$f: V(\Gamma) \rightarrow V(\Delta) \quad \text{and} \quad f: E(\Gamma) \rightarrow E(\Delta)$$

(both denoted by f) that preserve the source, target, and reverse of each edge $e \in E(\Gamma)$.

A non-trivial *path* in a graph Γ is an element $e_1 e_2 \cdots e_n$ ($n \geq 1$) of the free monoid $E(\Gamma)^*$ such that $t(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$. There is an obvious extension of the source and target functions to paths: If $p = e_1 \cdots e_n$ is a path in Γ , then define $s(p) = s(e_1)$ and $t(p) = t(e_n)$. In addition, we include a trivial path ϵ_x (with no edges) for each vertex $x \in V(\Gamma)$, where $s(\epsilon_x) = t(\epsilon_x) = x$.

A transformation taking a path p to a path q is called an *elementary homotopy* if q is obtained from p by the deletion or insertion of a subword of the form $e\bar{e}$ for some $e \in E(\Gamma)$. We say that paths p and q are *homotopic* if there is a finite sequence of elementary homotopies that carries p to q . Homotopy is an equivalence relation, denoted by \simeq , on the set of all paths of Γ . Note that if $p \simeq q$, then $s(p) = s(q)$

and $t(p) = t(q)$ and hence it makes sense to talk about the source and target of a homotopy class. The homotopy class of a path p is denoted by $[p]$ and the set of all homotopy classes of paths in Γ is denoted by $\pi(\Gamma)$.

If p and q are paths in Γ and $t(p) = s(q)$, then the concatenation pq is again a path and $s(pq) = s(p)$ and $t(pq) = t(q)$. Concatenation respects homotopy of paths and therefore determines a (partial) product on the set $\pi(\Gamma)$ of homotopy classes of paths given by $[p][q] = [pq]$, provided that $t(p) = s(q)$. The set $\pi(\Gamma)$ with this operation is called the *fundamental groupoid* of Γ .

If $f: \Gamma \rightarrow \Delta$ is a map of graphs, then the induced homomorphism of monoids $f: E(\Gamma)^* \rightarrow E(\Delta)^*$ obviously takes paths to paths and homotopic paths to homotopic paths. Thus f determines a homomorphism of fundamental groupoids which we also denote by $f: \pi(\Gamma) \rightarrow \pi(\Delta)$. Note that f maps trivial paths to trivial paths and paths of length one to paths of length one. Conversely, every homomorphism of groupoids $\pi(\Gamma) \rightarrow \pi(\Delta)$ with this property arises from a map of graphs $\Gamma \rightarrow \Delta$.

We will now define our notion of a graph of groups. The definition below is a departure from the traditional Bass-Serre Theory in that we require that our edge groups be subgroups of the corresponding source vertex groups. Our definition agrees with the one given in [2] and is obviously equivalent to Serre's definition [5].

DEFINITION 2.1. A *graph of groups* (Γ, A) consists of

- (i) a graph Γ ;
- (ii) for each $x \in V(\Gamma)$, a group A_x ;
- (iii) for each $e \in E(\Gamma)$, a subgroup $A_e \leq A_{s(e)}$ and a monomorphism $A_e \rightarrow A_{t(e)}$ denoted by $c \mapsto c^e$; and
- (iv) the edge monomorphisms are required to be such that for each edge e , the image of the map $A_e^e = A_{\bar{e}}$ and the maps e, \bar{e} are inverses of one another—i.e., $(c^e)^{\bar{e}} = c$ and $(d^{\bar{e}})^e = d$ for all $c \in A_e$ and all $d \in A_{\bar{e}}$.

EXAMPLE 2.2. [*Main example.*] Let G be a group acting on a graph Γ without edge inversions, on the left. Note here that it is also convenient to write the group action on the right by letting

$$x \cdot g = g^{-1} \cdot x$$

for all vertices or edges x and $g \in G$.

Let $\Delta = \Gamma/G$ be the quotient graph and choose a *section* σ of the orbit map $\Gamma \rightarrow \Gamma/G = \Delta$; for each vertex and edge z of Δ , $\sigma(z)$ is a choice of a member in the orbit of z . We require that for each $e \in E(\Delta)$,

$$s(\sigma(e)) = \sigma(s(e)).$$

Also, for each $e \in E(\Delta)$, choose an element $g_e \in G$ such that

$$\overline{\sigma(e)} \cdot g_e = \sigma(\bar{e}).$$

We require that $g_{\bar{e}} = g_e^{-1}$.

Construct a graph of groups (Δ, B) as follows. For each vertex or edge z of Δ , let

$$B_z = G_{\sigma(z)} \text{ (the stabilizer subgroup).}$$

Since G acts without inversions, then for every $g \in B_e = G_{\sigma(e)}$ and $e \in E(\Delta)$, we have that

$$\begin{aligned} s(\sigma(e) \cdot g) &= s(\sigma(e)) \\ \Leftrightarrow s(\sigma(e)) \cdot g &= s(\sigma(e)) \\ \Leftrightarrow \sigma(s(e)) \cdot g &= \sigma(s(e)). \end{aligned}$$

Thus, $B_e \leq B_{s(e)}$ and we define $B_e \rightarrow B_{t(e)}$ via $b \mapsto b^e$ where

$$b^e = b^{g_e} = g_e^{-1} b g_e.$$

Note that

$$\begin{aligned}
\sigma(t(e)) \cdot b^e &= \sigma(s(\bar{e})) \cdot g_e^{-1} b g_e = s(\sigma(\bar{e})) \cdot g_e^{-1} b g_e = s(\sigma(\bar{e})) \cdot g_e^{-1} b g_e \\
&= s(\overline{\sigma(e)}) \cdot b g_e = s(\overline{\sigma(e)}) \cdot g_e = s(\sigma(\bar{e})) = \sigma(s(\bar{e})) \\
&= \sigma(t(e)).
\end{aligned}$$

Hence, $b^{g_e} \in G_{\sigma(t(e))} = B_{t(e)}$ and we have that $b \mapsto b^e$ is well-defined. It is a monomorphism as well.

We thus have a graph of groups (Δ, B) determined by the action of G on Γ , the section σ of the orbit map $\Gamma \rightarrow \Gamma/G = \Delta$, and the choices of $g_e \in G$ ($e \in E(\Delta)$) such that $g_{\bar{e}} = g_e^{-1}$ and $\overline{\sigma(e)} \cdot g_e = \sigma(\bar{e})$.

The fundamental groupoid of our graph of groups can be defined in terms of generators and relations as is done in [3].

DEFINITION 2.3. The *fundamental groupoid* of (Γ, A) , denoted as $\pi(\Gamma, A)$, has vertex set $V(\Gamma)$ and is generated by the edges of Γ together with the elements of the vertex groups A_x for $x \in V(\Gamma)$, where for each $a \in A_x$ we define $s(a) = t(a) = x$. The defining relations are:

- (i) $aa' = b$ whenever $b = aa'$ in A_x for some $x \in V(\Gamma)$, and
- (ii) $\bar{e}ce = c^e$ for all $c \in A_e$, where $e \in E(\Gamma)$.

Note that as a consequence of (ii), $\bar{e} = e^{-1}$ in $\pi(\Gamma, A)$. From the normal form theorem for graphs of groups discussed in [3], it follows that the natural map $A_x \rightarrow \pi(\Gamma, A)$ is an embedding for each $x \in V(\Gamma)$. Thus, to generalize the notion of vertices of graphs to the category of graphs of groups, we define

$$V(\Gamma, A) = \bigcup_{x \in V(\Gamma)} A_x \hookrightarrow \pi(\Gamma, A).$$

Next, we consider the set $E_0 = \{aeb : e \in E(\Gamma), a \in A_{s(e)}, b \in A_{t(e)}\}$ of words, and introduce the following equivalence relation: $aeb \equiv a'e'b'$ whenever $e' = e$, $a' = ac^{-1}$, and $b' = c^e b$ for some $c \in A_e$. This leads to our idea of generalized edges:

$$E(\Gamma, A) = \{[aeb] : e \in E(\Gamma), a \in A_{s(e)}, b \in A_{t(e)}\} = E_0 / \equiv.$$

PROPOSITION 2.4. *The map $E(\Gamma, A) \rightarrow \pi(\Gamma, A)$ given by $[aeb] \mapsto aeb$ is injective.*

PROOF. To see that the map is well-defined, suppose $a_1 e_1 b_1 \equiv a_2 e_2 b_2$. Then, $e_1 = e_2$ and there exists $c \in A_{e_1} = A_{e_2}$ such that $a_2 = a_1 c^{-1}$ and $b_2 = c^{e_1} b_1$. Thus, $a_1 e_1 b_1 = a_1 c^{-1} e_1 c^{e_1} b_1 = a_2 e_2 b_2$ in the fundamental groupoid. Hence, the map is well-defined.

In order to see that the map is injective, suppose that $[a_1 e_1 b_1]$ and $[a_2 e_2 b_2]$ have the same image in $\pi(\Gamma, A)$. We apply the normal form theorem [3]: Let T_e be a choice of left transversal for A_e in $A_{s(e)}$ for each $e \in E(\Gamma)$. Write a_i as its unique product in $A_{s(e_i)}$ of the form $a_i = r_i c_i$, where $r_i \in T_{e_i}$ and $c_i \in A_{e_i}$. Then, working in $\pi(\Gamma, A)$, we see that

$$r_1 e_1 c_1^{e_1} b_1 = a_1 e_1 b_1 = a_2 e_2 b_2 = r_2 e_2 c_2^{e_2} b_2.$$

So by the uniqueness of normal forms, $r_1 = r_2$, $e_1 = e_2$, and $c_1^{e_1} b_1 = c_2^{e_2} b_2$. Hence,

$$a_2 = r_2 c_2 = r_1 (c_1 c_1^{-1}) c_2 = a_1 (c_2^{-1} c_1)^{-1}$$

and

$$b_2 = (c_2^{e_2})^{-1} c_1^{e_1} b_1 = (c_2^{-1})^{e_1} c_1^{e_1} b_1 = (c_2^{-1} c_1)^{e_1} b_1.$$

Therefore, $[a_1 e_1 b_1] = [a_2 e_2 b_2]$. □

CHAPTER 3

MAPS OF GRAPHS OF GROUPS

Now, we turn our attention toward defining a map between two graphs of groups, (Γ, A) and (Δ, B) , say. As in the situation of maps of graphs, we would like such a map $F: (\Gamma, A) \rightarrow (\Delta, B)$ to consist of two functions

$$F: V(\Gamma, A) \rightarrow V(\Delta, B) \quad \text{and} \quad F: E(\Gamma, A) \rightarrow E(\Delta, B)$$

that are compatible (in an obvious sense) so as to determine a homomorphism of groupoids, which we also denote by $F: \pi(\Gamma, A) \rightarrow \pi(\Delta, B)$. Thus, we can avoid worrying about these compatibility conditions all together and define maps of graphs of groups in the following very simple way.

DEFINITION 3.1. A *map of graphs of groups* $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a homomorphism of the fundamental groupoids, that we also denote by $F: \pi(\Gamma, A) \rightarrow \pi(\Delta, B)$, that maps $V(\Gamma, A)$ to $V(\Delta, B)$ and $E(\Gamma, A)$ to $E(\Delta, B)$.

REMARKS 3.2. (1) *Maps of trivial graphs of groups.* Regarding a graph as a graph of groups with a trivial group assigned to each vertex and each edge, we see that every map of graphs $f: \Gamma \rightarrow \Delta$ is a map of graphs of (trivial) groups.

(2) *The identity map.* For any graph of groups (Γ, A) , the identity maps on $V(\Gamma, A)$ and on $E(\Gamma, A)$ determine a map of graphs of groups $\text{Id}: (\Gamma, A) \rightarrow (\Gamma, A)$.

(3) *Compositions.* If $F: (\Gamma, A) \rightarrow (\Gamma', A')$ and $G: (\Gamma', A') \rightarrow (\Gamma'', A'')$ are maps of graphs of groups, then the composite homomorphism $GF: \pi(\Gamma, A) \rightarrow \pi(\Gamma'', A'')$ is a map of graphs of groups $GF: (\Gamma, A) \rightarrow (\Gamma'', A'')$. Thus, graphs of groups with maps of graphs of groups form a category.

(4) *Projection onto the underlying graph.* Given a graph of groups (Γ, A) , define mappings $q: V(\Gamma, A) \rightarrow V(\Gamma)$ and $q: E(\Gamma, A) \rightarrow E(\Gamma)$ by $q(a) = x$ for all $x \in V(\Gamma)$ and all $a \in A_x$ and $q([aeb]) = e$ for all $[aeb] \in E(\Gamma, A)$. These mappings induce a homomorphism $q: \pi(\Gamma, A) \rightarrow \pi(\Gamma)$ which is a map of graphs of groups. We call this map $q: (\Gamma, A) \rightarrow \Gamma$ the projection of (Γ, A) onto its underlying graph Γ .

(5) *Inclusion of the underlying graph.* The mappings $i: V(\Gamma) \rightarrow V(\Gamma, A)$ and $i: E(\Gamma) \rightarrow E(\Gamma, A)$ given by $i(x) = 1_x$ (the identity element in A_x) for all $x \in V(\Gamma)$ and $i(e) = [e]$ for all $e \in E(\Gamma)$ induce a homomorphism $i: \pi(\Gamma) \rightarrow \pi(\Gamma, A)$ which is a splitting of the projection map $q: (\Gamma, A) \rightarrow \Gamma$. That is, the composition $\Gamma \xrightarrow{i} (\Gamma, A) \xrightarrow{q} \Gamma$ is the identity map on Γ .

(6) *The underlying map of graphs.* A map of graphs of groups $F: (\Gamma, A) \rightarrow (\Delta, B)$ projects to a map of the underlying graphs $f: \Gamma \rightarrow \Delta$ that makes the following diagram commutative:

$$\begin{array}{ccc} (\Gamma, A) & \xrightarrow{F} & (\Delta, B) \\ p \downarrow & & \downarrow q \\ \Gamma & \xrightarrow{f} & \Delta \end{array}$$

where p and q are the projections onto the underlying graphs. The map f is given by the composition $f = qFi$, where $i: \Gamma \rightarrow (\Gamma, A)$ is the inclusion map.

(7) *Restrictions to homomorphisms of vertex groups.* It follows from the above commutative diagram that F maps the vertex group A_x to the vertex group $B_{f(x)}$ for each $x \in V(\Gamma)$. Hence for each $x \in V(\Gamma)$, F restricts to a homomorphism of groups $F|_{A_x}: A_x \rightarrow B_{f(x)}$. On the other hand, in general F does not map edge groups to edge groups. However the image of an edge group A_e is a conjugate of $B_{f(e)}$ in the vertex group $B_{s(f(e))}$; see the lemma below.

Recall that the fundamental groupoid of a graph of groups (Γ, A) can be defined by generators and relations, where the generators consist of the elements of A_x (for each $x \in V(\Gamma)$) and the elements of $E(\Gamma)$. We next address the question of

when a function defined on this generating set for $\pi(\Gamma, A)$ induces a homomorphism $\pi(\Gamma, A) \rightarrow \pi(\Delta, B)$ which is a map of graphs of groups. More precisely, let (Γ, A) and (Δ, B) be graphs of groups and assume that the following data is given:

- a map of the underlying graphs $f: \Gamma \rightarrow \Delta$;
- a homomorphism $f_x: A_x \rightarrow B_{f(x)}$, for each $x \in V(\Gamma)$;
- an element $\alpha(e) \in B_{s(f(e))}$ for each $e \in E(\Gamma)$.

Write $\beta(e) = \alpha(\bar{e})$ for each $e \in E(\Gamma)$ and define a function $F: V(\Gamma, A) \cup E(\Gamma) \rightarrow \pi(\Delta, B)$ by $F|_{A_x} = f_x$, for each $x \in V(\Gamma)$, and $F(e) = \alpha(e)f(e)\beta(e)^{-1}$, for each $e \in E(\Gamma)$.

LEMMA 3.3 (Induced maps). *With this setup, F induces a homomorphism $F: \pi(\Gamma, A) \rightarrow \pi(\Delta, B)$ which is a map of graphs of groups if for all $e \in E(\Gamma)$ and all $c \in A_e$,*

$$f_x(c)^{\alpha(e)} \in B_{f(e)} \quad \text{and} \quad f_x(c)^{\alpha(e)f(e)} = f_y(c^e)^{\beta(e)}$$

where $x = s(e)$ and $y = t(e)$.

The conditions in the lemma can be stated as the existence of a commutative diagram:

$$\begin{array}{ccccc}
 A_y & \xrightarrow{f_y} & B_{f(y)} & \xrightarrow{\text{ad } \beta(e)} & B_{f(y)} \\
 \uparrow e & & & & \uparrow f(e) \\
 A_e & \xrightarrow{(\text{ad } \alpha(e)) \circ f_x} & & & B_{f(e)} \\
 \downarrow & & & & \downarrow \\
 A_x & \xrightarrow{f_x} & B_{f(x)} & \xrightarrow{\text{ad } \alpha(e)} & B_{f(x)}
 \end{array}$$

for each $e \in E(\Gamma)$, where $x = s(e)$ and $y = t(e)$. In the above diagram, we make use of the somewhat standard notation: If G is a group and $g \in G$, then $\text{ad } g$ denotes the inner automorphism of G given by $(\text{ad } g)(x) = x^g = g^{-1}xg$.

PROOF OF THE LEMMA. First of all, let $aa' = b$ in a vertex group A_x . Then,

$$\begin{aligned} F(a)F(a') &= f_x(a)f_x(a') \\ &= f_x(aa') \\ &= f_x(b) = F(b). \end{aligned}$$

Also, for any $e \in E(\Gamma)$ and $c \in A_e$, we have

$$\begin{aligned} F(\bar{e})F(c)F(e) &= \alpha(\bar{e})f(\bar{e})\beta(\bar{e})^{-1}f_x(c)\alpha(e)f(e)\beta(e)^{-1} \\ &= \beta(e)\overline{f(e)}\alpha(e)^{-1}f_x(c)\alpha(e)f(e)\beta(e)^{-1} \\ &= \beta(e)f_x(c)^{\alpha(e)f(e)}\beta(e)^{-1} \\ &= \beta(e)f_y(c^e)^{\beta(e)}\beta(e)^{-1} \\ &= f_y(c^e) = F(c^e) \end{aligned}$$

where $x = s(e)$ and $y = t(e)$.

We have shown that F preserves the defining relations of $\pi(\Gamma, A)$, and hence induces a homomorphism $F: \pi(\Gamma, A) \rightarrow \pi(\Delta, B)$. Furthermore, it follows immediately from the way F was defined that it maps $V(\Gamma, A)$ to $V(\Delta, B)$ and $E(\Gamma, A)$ to $E(\Delta, B)$. Therefore F is a map of graphs of groups. \square

Conversely, we next observe that every map of graphs of groups arises as in Lemma 3.3. It also follows from this that our notion of a map of graphs of groups agrees with Bass' notion of a morphism [1].

PROPOSITION 3.4. *Associated to every map $F: (\Gamma, A) \rightarrow (\Delta, B)$ of graphs of groups is the following data:*

- (i) a map of graphs $f: \Gamma \rightarrow \Delta$,
- (ii) a homomorphism $f_x = F|_{A_x}: A_x \rightarrow B_{f(x)}$ for each $x \in V(\Gamma)$,

(iii) a function $\alpha = \alpha_F: E(\Gamma) \rightarrow V(\Delta, B)$

such that for each $e \in E(\Gamma)$, $F([e]) = [\alpha(e)f(e)\alpha(\bar{e})^{-1}]$. Moreover this data completely determines F , satisfies the conditions of Lemma 3.3, and is unique up to the equivalence on α_F discussed below.

The proposition asserts that there is a one-to-one correspondence between maps $F: (\Gamma, A) \rightarrow (\Delta, B)$ and equivalence classes of triples $(f, \{f_x\}_{x \in V(\Gamma)}, \alpha)$ that satisfy the conditions of Lemma 3.3, where $f: \Gamma \rightarrow \Delta$ is a map of graphs, each $f_x: A_x \rightarrow B_{f(x)}$ is a homomorphism, and $\alpha: E(\Gamma) \rightarrow V(\Delta, B)$ is a function such that $\alpha(e) \in B_{f(s(e))}$ for all $e \in E(\Gamma)$. Two such triples $(f, \{f_x\}_{x \in V(\Gamma)}, \alpha)$ and $(f', \{f'_x\}_{x \in V(\Gamma)}, \alpha')$ are *equivalent* if $f = f'$, $f_x = f'_x$ for each $x \in V(\Gamma)$, and there exist elements $c_e \in B_{f(e)}$ associated to the edges of Γ such that $c_{\bar{e}} = c_e^{f(e)}$ and $\alpha(e) = \alpha'(e)c_e$ for all $e \in E(\Gamma)$.

PROOF OF THE PROPOSITION. First of all, the methods for divulging $f: \Gamma \rightarrow \Delta$ and $f_x = F|_{A_x}: A_x \rightarrow B_{f(x)}$ are discussed in Remarks (6) and (7) above. To construct the function α , choose an orientation $E^+(\Gamma)$ of the graph Γ ; i.e., $E^+(\Gamma)$ consists of a choice of exactly one edge in each pair $\{e, \bar{e}\}$. Let $e \in E^+(\Gamma)$ such that $x = s(e)$ and $y = t(e)$. Since F maps $E(\Gamma, A)$ to $E(\Delta, B)$, we have that $F([e]) = [af(e)b]$ for some $a \in B_{f(x)}$ and $b \in B_{f(y)}$. We define $\alpha(e) = a$ and $\beta(e) = b^{-1}$. Then

$$F([e]) = [\alpha(e)f(e)\beta(e)^{-1}]$$

and by Proposition 2.4, we see that $F([\bar{e}]) = [\beta(e)f(\bar{e})\alpha(e)^{-1}]$. Defining $\alpha(\bar{e}) = \beta(e)$ and $\beta(\bar{e}) = \alpha(e)$, we see that α has the desired property for all edges of Γ .

Next, suppose there exists $\alpha': E(\Gamma) \rightarrow V(\Delta, B)$ also satisfying this condition. Let $e \in E^+(\Gamma)$. Since $\alpha(e)f(e)\beta(e)^{-1} \equiv \alpha'(e)f(e)\beta'(e)^{-1}$, there exists $c_e \in B_{f(e)}$ such that $\alpha(e)c_e^{-1} = \alpha'(e)$ and $c_e^{f(e)}\beta(e)^{-1} = \beta'(e)^{-1}$; whence

$$\alpha(e) = \alpha'(e)c_e \text{ and } \beta(e) = \beta'(e)c_e^{f(e)}.$$

Setting $c_{\bar{e}} = c_e^{f(e)}$, we see that $c_{\bar{e}}^{f(\bar{e})} = c_e^{f(e)f(\bar{e})} = c_e$ and we also have

$$\alpha(\bar{e}) = \alpha'(\bar{e})c_{\bar{e}} \text{ and } \beta(\bar{e}) = \beta'(\bar{e})c_{\bar{e}}^{f(\bar{e})}.$$

Hence α' is equivalent to α in the above sense.

It remains to show that the conditions of Lemma 3.3 hold. To this end, let $e \in E(\Gamma)$ and $c \in A_e$. Then since $c^{-1}ec^e \equiv e$, we must have

$$f_x(c^{-1})\alpha(e)f(e)\beta(e)^{-1}f_y(c^e) \equiv \alpha(e)f(e)\beta(e)^{-1}.$$

Hence, there exists $d \in B_{f(e)}$ such that

$$f_x(c^{-1})\alpha(e) = \alpha(e)d^{-1} \text{ and } \beta(e)^{-1}f_y(c^e) = d^{f(e)}\beta(e)^{-1}.$$

The former yields that

$$f_x(c)^{\alpha(e)} = d \in B_{f(e)},$$

and the latter that

$$f_y(c^e)^{\beta(e)} = d^{f(e)} = f_x(c)^{\alpha(e)f(e)},$$

exactly the two conditions of Lemma 3.3. □

REMARK 3.5 (Notational conventions). From here on, we will adhere to the notation and conventions in the previous proposition for maps of graphs of groups. That is, a map of graphs of groups will be denoted by a capital letter, such as $H: (\Gamma, A) \rightarrow (\Delta, B)$. Its underlying map of graphs will be denoted by the corresponding lower case letter, such as $h: \Gamma \rightarrow \Delta$. The vertex homomorphisms will be denoted by this lower case letter with a vertex as a subscript, such as $h_x: A_x \rightarrow B_{h(x)}$. The edge labeling function (which is only unique up to equivalence) will be denoted by α , with a subscript if there is possible ambiguity, such as $\alpha = \alpha_H$. Finally, for each

edge e in the underlying graph of the domain, we will write $\beta(e) = \alpha(\bar{e})$, possibly with a subscript, such as $\beta = \beta_H$.

CHAPTER 4

PATHS

Let n be a positive integer and I_n be the graph obtained by subdividing the interval $[0, n]$ at the integer points. As always, we view graphs as graphs of trivial groups.

DEFINITION 4.1. A *path* of length $n \geq 1$ in a graph of groups (Γ, A) is a map of graphs of groups

$$P: I_n \rightarrow (\Gamma, A).$$

For $n = 0$, the elements of the vertex groups $a \in A_x$, for all $x \in V(\Gamma)$, are regarded as paths of length zero in (Γ, A) .

The underlying map of graphs of a path $P: I_n \rightarrow (\Gamma, A)$ is a path $p: I_n \rightarrow \Gamma$. Let $\alpha = \alpha_P: E(I_n) \rightarrow V(\Gamma, A)$ be an edge labeling function as in Proposition 3.4. Denote the edges of I_n by $[i - 1, i]^{\pm 1}$, where $1 \leq i \leq n$ is an integer. As before, write $\beta(e) = \alpha(\bar{e})$. The path P determines a word $a_0 e_1 a_1 \cdots e_n a_n$ where

- $a_0 = \alpha([0, 1])$;
- $a_i = \beta([i - 1, i])^{-1} \alpha([i, i + 1])$ for $1 \leq i \leq n - 1$;
- $a_n = \beta([n - 1, n])^{-1}$; and
- $e_i = p([i - 1, i])$ for all $1 \leq i \leq n$, the image of $[i - 1, i]$ under the underlying map of graphs $p: I_n \rightarrow \Gamma$.

Note that by Proposition 3.4, this representation is unique up to the following equivalence relation: $a_0 e_1 a_1 \cdots e_n a_n \equiv a'_0 e'_1 a'_1 \cdots e'_m a'_m$ if $m = n$, and for each $1 \leq i \leq n$ we have that $e'_i = e_i$ and there exists $c_i \in A_{e_i}$ such that

- $a'_0 = a_0 c_1^{-1}$,
- $a'_i = c_i^{e_i} a_i c_{i+1}^{-1}$ for $1 \leq i \leq n-1$, and
- $a'_n = c_n^{e_n} a_n$.

That is, the second word is obtained from the first by replacing each e_i by $c_i^{-1} e_i c_i^{e_i}$.

Conversely, every word of the form $a_0 e_1 a_1 \cdots e_n a_n$ where $n \geq 1$, $e_1 e_2 \cdots e_n$ is a path in Γ , and each $a_i \in A_{t(e_i)} = A_{s(e_{i+1})}$ is a representation of some path $P: I_n \rightarrow (\Gamma, A)$. We write $P \equiv a_0 e_1 a_1 \cdots e_n a_n$ to indicate that P is represented by this word and we identify $P = [a_0 e_1 a_1 \cdots e_n a_n]$, the \equiv -class of the word $a_0 e_1 a_1 \cdots e_n a_n$. Note that when $n = 0$ this agrees with our definition of paths of length zero. Thus $V(\Gamma, A)$ consists of all paths of length zero and $E(\Gamma, A)$ consists of all paths of length one in (Γ, A) .

The source and target of a path P in (Γ, A) are the obvious vertices of Γ : if $P \equiv a_0 e_1 a_1 \cdots e_n a_n$, then $s(P) = s(e_1)$ and $t(P) = t(e_n)$. In the case of $n = 0$, $s(P) = t(P) = x$ where $a_0 \in A_x$.

If $P: I_n \rightarrow (\Gamma, A)$ and $Q: I_m \rightarrow (\Gamma, A)$ are paths such that $t(P) = s(Q)$, then the product PQ is the path

$$PQ: I_{m+n} \rightarrow (\Gamma, A)$$

defined using Lemma 3.3 as follows. Its underlying map of graphs is pq , where p, q are the underlying paths of P, Q in Γ , and $\alpha: E(I_{m+n}) \rightarrow V(\Gamma, A)$ is given by

$$\alpha([i, i+1]) = \alpha_P([i, i+1]) \text{ for } 0 \leq i \leq n-1$$

and

$$\alpha([i, i+1]) = \alpha_Q([i-n, i-n+1]) \text{ for } n \leq i \leq n+m-1.$$

Define β the same way. This data determines PQ by Lemma 3.3. It is easy to see that if $P \equiv a_0 e_1 a_1 \cdots e_n a_n$ and $Q \equiv a'_0 e'_1 a'_1 \cdots e'_m a'_m$, then

$$PQ \equiv a_0 e_1 a_1 \cdots e_n (a_n a'_0) e'_1 a'_1 \cdots e'_m a'_m.$$

Note that $|PQ| = |P| + |Q|$, $s(PQ) = s(P)$, and $t(PQ) = t(Q)$.

Composition of paths is an associative operation, provided that the products are all defined. Thus, the set of all paths in (Γ, A) forms a small category, denoted $\mathcal{P}(\Gamma, A)$. A map $F : (\Gamma, A) \rightarrow (\Delta, B)$ of graphs of groups induces a natural length preserving homomorphism (or functor) also denoted $F : \mathcal{P}(\Gamma, A) \rightarrow \mathcal{P}(\Delta, B)$. If $P \in \mathcal{P}(\Gamma, A)$, then $P : I_n \rightarrow (\Gamma, A)$ and $F(P)$ is the composite map

$$F \circ P : I_n \rightarrow (\Delta, B).$$

Suppose that $P \equiv a_0 e_1 a_1 \cdots e_n a_n$ and write $x_i = t(e_i) = s(e_{i+1})$ for $0 \leq i \leq n$. Then,

$$F(P) \equiv f_{x_0}(a_0) \alpha_F(e_1) f(e_1) \beta_F(e_1)^{-1} f_{x_1}(a_1) \cdots f(e_n) \beta_F(e_n)^{-1} f_{x_n}(a_n)$$

DEFINITION 4.2. A *round-trip* in a path is a subpath of the form $\bar{e}ce$, where $e \in E(\Gamma)$ and $c \in A_e$.

First note that the existence of a round-trip in a path P is independent of the representation of P . Specifically, if $P \equiv a_0 e_1 a_1 \cdots e_n a_n$ and $e_i a_i e_{i+1}$ is a round-trip (i.e., $e_i = \bar{e}_{i+1}$ and $a_i \in A_{e_{i+1}}$), then for any representation $P \equiv b_0 e_1 b_1 \cdots e_n b_n$, we have that $e_i b_i e_{i+1}$ is a round-trip. To see this, note that from the definition of \equiv we have that

$$b_i = c_i^{e_i} a_i c_{i+1}^{-1},$$

where $c_i \in A_{e_i}$. Additionally, the fact that $e_i a_i e_{i+1}$ is a round-trip yields

$$e_i = \bar{e}_{i+1} \text{ and } a_i \in A_{e_{i+1}}.$$

Hence,

$$e_i b_i e_{i+1} = \bar{e}_{i+1} c_i^{e_i} a_i c_{i+1}^{-1} e_{i+1}.$$

Now, since $c_i \in A_{e_i}$, we have that $c_i^{e_i} \in A_{e_{i+1}}$. Furthermore, we already have that $a_i \in A_{e_{i+1}}$ and $c_{i+1} \in A_{e_{i+1}}$. Thus, $b_i = c_i^{e_i} a_i c_{i+1}^{-1} \in A_{e_{i+1}}$. Therefore, $e_i b_i e_{i+1}$ is a

round-trip. If a path P contains a round-trip $\bar{e}ce$, then by replacing the round-trip by $c^e \in A_{t(e)}$ we get a path P' with the same source and target vertices as P and with $|P'| = |P| - 2$.

It should be noted that this operation is also independent of the representation of P . If $P \equiv a_0 e_1 a_1 \cdots e_n a_n \equiv b_0 e_1 b_1 \cdots e_n b_n$ and $e_i a_i e_{i+1}$ is a round-trip in the first word, then $e_i b_i e_{i+1}$ is a round-trip in the second word and

$$a_0 \cdots e_{i-1} (a_{i-1} a_i^{e_{i+1}} a_{i+1}) e_{i+2} \cdots a_n \equiv b_0 \cdots e_{i-1} (b_{i-1} b_i^{e_{i+1}} b_{i+1}) e_{i+2} \cdots b_n.$$

To see this fact, since $a_0 e_1 a_1 \cdots e_n a_n \equiv b_0 e_1 b_1 \cdots e_n b_n$, we have that

$$\begin{aligned} b_{i-1} b_i^{e_{i+1}} b_{i+1} &= c_{i-1}^{e_{i-1}} a_{i-1} c_i^{-1} (c_i^{e_i} a_i c_{i+1}^{-1})^{e_{i+1}} c_{i+1}^{e_{i+1}} a_{i+1} c_{i+2}^{-1} \\ &= c_{i-1}^{e_{i-1}} a_{i-1} c_i^{-1} (c_i^{e_i})^{e_{i+1}} a_i^{e_{i+1}} (c_{i+1}^{-1})^{e_{i+1}} c_{i+1}^{e_{i+1}} a_{i+1} c_{i+2}^{-1} \\ &= c_{i-1}^{e_{i-1}} a_{i-1} c_i^{-1} (c_i^{\bar{e}_{i+1}})^{e_{i+1}} a_i^{e_{i+1}} (c_{i+1}^{-1} c_{i+1})^{e_{i+1}} a_{i+1} c_{i+2}^{-1} \\ &= c_{i-1}^{e_{i-1}} a_{i-1} c_i^{-1} c_i a_i^{e_{i+1}} a_{i+1} c_{i+2}^{-1} \\ &= c_{i-1}^{e_{i-1}} a_{i-1} a_i^{e_{i+1}} a_{i+1} c_{i+2}^{-1}. \end{aligned}$$

The above calculation combined with our definition of \equiv yields the fact that

$$a_0 \cdots e_{i-1} (a_{i-1} a_i^{e_{i+1}} a_{i+1}) e_{i+2} \cdots a_n \equiv b_0 \cdots e_{i-1} (b_{i-1} b_i^{e_{i+1}} b_{i+1}) e_{i+2} \cdots b_n,$$

the desired result. We write that $P \searrow P'$ and say that P' is obtained from P by an *elementary reduction*. The equivalence relation on the set of paths in (Γ, A) that is generated by \searrow is called *homotopy* and is denoted by \simeq .

The product of paths is compatible with homotopy. If $P_1 \simeq P_2$, $Q_1 \simeq Q_2$, and $t(P_i) = s(Q_i)$, then $P_1 Q_1 \simeq P_2 Q_2$. To see this, suppose that

$$a_0 \cdots a_{i-1} e_i a_i e_{i+1} a_{i+1} \cdots a_m \equiv P_1 \simeq P_2 \equiv a_0 \cdots e_{i-1} (a_{i-1} a_i^{e_{i+1}} a_{i+1}) e_{i+2} \cdots a_m$$

and

$$b_0 \cdots b_{j-1} d_j b_j d_{j+1} b_{j+1} \cdots b_n \equiv Q_1 \simeq Q_2 \equiv b_0 \cdots d_{j-1} \left(b_{j-1} b_j^{d_{j+1}} b_{j+1} \right) d_{j+2} \cdots b_n.$$

Then,

$$\begin{aligned} P_1 Q_1 &\equiv a_0 \cdots a_{i-1} e_i a_i e_{i+1} a_{i+1} \cdots (a_m b_0) \cdots b_{j-1} d_j b_j d_{j+1} b_{j+1} \cdots b_n \\ &\simeq a_0 \cdots e_{i-1} (a_{i-1} a_i^{e_{i+1}} a_{i+1}) e_{i+2} \cdots (a_m b_0) \cdots b_{j-1} d_j b_j d_{j+1} b_{j+1} \cdots b_n \\ &\simeq a_0 \cdots e_{i-1} (a_{i-1} a_i^{e_{i+1}} a_{i+1}) e_{i+2} \cdots (a_m b_0) \cdots d_{j-1} \left(b_{j-1} b_j^{d_{j+1}} b_{j+1} \right) d_{j+2} \cdots b_n \\ &\equiv P_2 Q_2. \end{aligned}$$

Therefore, $P_1 Q_1 \simeq P_2 Q_2$. Additionally, every path P in (Γ, A) has a homotopy inverse. If $s(P) = x$ and $t(P) = y$, then there exists a path P^{-1} with $s(P^{-1}) = y$ and $t(P^{-1}) = x$ such that $PP^{-1} \simeq 1_x$ and $P^{-1}P \simeq 1_y$, where 1_x and 1_y are paths of length zero corresponding to the identity elements of A_x and A_y . In order to see this fact, first note that if $s(e) = x$ and $t(e) = y$, then the operation \searrow whereby a round-trip $\bar{e}ce$ is replaced with c^e also yields that $\bar{e}e$ is replaced by 1_y , since $\bar{e}e \equiv \bar{e}1_x e$ where $1_x \in A_e$ and we know that $1_x^e = 1_y$. Analogously, $e\bar{e}$ is replaced by 1_x .

Now, consider the path $P \equiv a_0 e_1 a_1 \cdots e_n a_n$ in (Γ, A) . Define

$$P^{-1} \equiv a_n^{-1} \bar{e}_n a_{n-1}^{-1} \cdots \bar{e}_1 a_0^{-1}.$$

Note that P^{-1} is a path in (Γ, A) and

$$s(P^{-1}) = s(\bar{e}_n) = t(e_n) = t(P).$$

Thus PP^{-1} is well-defined and

$$\begin{aligned}
PP^{-1} &\equiv a_0 \cdots a_{n-2} e_{n-1} a_{n-1} e_n (a_n a_n^{-1}) \bar{e}_n a_{n-1}^{-1} \bar{e}_{n-1} a_{n-2}^{-1} \cdots a_0^{-1} \\
&\equiv a_0 \cdots a_{n-2} e_{n-1} a_{n-1} e_n 1_{t(e_n)} \bar{e}_n a_{n-1}^{-1} \bar{e}_{n-1} a_{n-2}^{-1} \cdots a_0^{-1} \\
&\simeq a_0 \cdots a_{n-2} e_{n-1} a_{n-1} 1_{s(e_n)} a_{n-1}^{-1} \bar{e}_{n-1} a_{n-2}^{-1} \cdots a_0^{-1} \\
&\equiv a_0 \cdots a_{n-2} e_{n-1} 1_{t(e_{n-1})} \bar{e}_{n-1} a_{n-2}^{-1} \cdots a_0^{-1} \\
&\simeq a_0 \cdots a_{n-2} 1_{s(e_{n-1})} a_{n-2}^{-1} \cdots a_0^{-1} \\
&\vdots \\
&\simeq a_0 1_{s(e_1)} a_0^{-1} \\
&\equiv 1_x,
\end{aligned}$$

where $x = s(e_1) = s(P)$. A similar calculation yields the fact that $P^{-1}P \simeq 1_y$, where $y = t(e_n) = t(P)$. Hence, if $P \equiv a_0 e_1 a_1 \cdots e_n a_n$, then $P^{-1} \equiv a_n^{-1} \bar{e}_n a_{n-1}^{-1} \cdots \bar{e}_1 a_0^{-1}$. Thus, the set of homotopy classes of paths in (Γ, A) forms a groupoid. This groupoid is naturally isomorphic to the fundamental groupoid, $\pi(\Gamma, A)$. To demonstrate this point, first note that it has already been shown that given paths P and Q in (Γ, A) , where $t(P) = s(Q)$, we have that PQ is a path in (Γ, A) as well. Multiplication at the homotopy level is thus defined by $[P][Q] = [PQ]$. Additionally, it has already been described how to form the inverse of a given path P , and hence $[P]^{-1}$ is defined to be $[P^{-1}]$. Thus, we need to prove the following statements are true for all paths P, Q , and R in (Γ, A) where $t(P) = s(Q)$ and $t(Q) = s(R)$:

- (i) $(PQ)R \simeq P(QR)$;
- (ii) $(PQ)Q^{-1} \simeq P$; and
- (iii) $P^{-1}(PQ) \simeq Q$.

To see (i), suppose that $P \equiv a_0 e_1 \cdots e_n a_n$, $Q \equiv a'_0 e'_1 \cdots e'_m a'_m$, and $R \equiv a''_0 e''_1 \cdots e''_r a''_r$. Then,

$$\begin{aligned}
(PQ)R &\equiv (a_0 e_1 \cdots e_n (a_n a'_0) e'_1 \cdots e'_m a'_m) (a''_0 e''_1 \cdots e''_r a''_r) \\
&\equiv a_0 e_1 \cdots e_n (a_n a'_0) e'_1 \cdots e'_m (a'_m a''_0) e''_1 \cdots e''_r a''_r \\
&\equiv (a_0 e_1 \cdots e_n a_n) (a'_0 e'_1 \cdots e'_m (a'_m a''_0) e''_1 \cdots e''_r a''_r) \\
&\equiv P(QR).
\end{aligned}$$

Thus, we have that $(PQ)R \equiv P(QR)$. Hence, at the homotopy level, we have

$$([P][Q])[R] = ([PQ])[R] = [(PQ)R] = [P(QR)] = [P]([QR]) = [P]([Q][R]).$$

Therefore, $(PQ)R \simeq P(QR)$.

For (ii), we employ (i) to see that

$$\begin{aligned}
(PQ)Q^{-1} &\simeq P(QQ^{-1}) \\
&\simeq P(1_{s(Q)}) \\
&\equiv a_0 e_1 \cdots e_n a_n (1_{s(Q)}) \\
&\equiv a_0 e_1 \cdots e_n (a_n 1_{s(Q)}) \\
&\equiv a_0 e_1 \cdots e_n a_n \\
&\equiv P.
\end{aligned}$$

A similar calculation yields (iii) as well. Therefore, the set of homotopy classes of paths in (Γ, A) does in fact form a groupoid.

Now, it remains to be seen that this groupoid is naturally isomorphic to the fundamental groupoid, $\pi(\Gamma, A)$. To this end, if $P \in \mathcal{P}(\Gamma, A)$ such that $P \equiv a_0 e_1 \cdots e_n a_n$, we define the evaluation map

$$\begin{aligned} \eta : \mathcal{P}(\Gamma, A) &\rightarrow \pi(\Gamma, A) \text{ via} \\ \eta(P) &= a_0 e_1 \cdots e_n a_n. \end{aligned}$$

It is easy to see that this map is well defined. Furthermore, suppose that $e_{i-1} a_{i-1} e_i$ is a round-trip below so that

$$a_0 e_1 \cdots a_{i-2} e_{i-1} a_{i-1} e_i a_i \cdots e_n a_n \equiv P \simeq Q \equiv a_0 e_1 \cdots a_{i-2} a_{i-1}^{e_i} a_i \cdots e_n a_n.$$

Then,

$$\begin{aligned} \eta(P) &= a_0 e_1 \cdots a_{i-2} e_{i-1} a_{i-1} e_i a_i \cdots e_n a_n \\ &= a_0 e_1 \cdots a_{i-2} a_{i-1}^{e_i} a_i \cdots e_n a_n \\ &= \eta(Q). \end{aligned}$$

Thus, η induces a well-defined homomorphism also denoted $\eta : \mathcal{P}(\Gamma, A) / \simeq \rightarrow \pi(\Gamma, A)$.

Now, define ϕ on the generators of $\pi(\Gamma, A)$ by

$$\begin{aligned} \phi : \pi(\Gamma, A) &\rightarrow \mathcal{P}(\Gamma, A) / \simeq \text{ via} \\ \phi(a) &= [a] \text{ and } \phi(e) = [e] \pmod{\simeq} \end{aligned}$$

for all $a \in A_x$ and $e \in E(\Gamma)$ where x runs through the vertices of Γ .

It remains to be seen that the appropriate relations hold. To this end, if $aa' = b$ in A_x for some $x \in V(\Gamma)$, we have that

$$\phi(a)\phi(a') = [a][a'] = [b] = \phi(b)$$

in $\mathcal{P}(\Gamma, A) / \simeq$. In addition, if $e \in E(\Gamma)$ and $c \in A_e$, then

$$\phi(\bar{e})\phi(c)\phi(e) = [\bar{e}][c][e] = [\bar{e}ce] = [c^e] = \phi(c^e) \pmod{\simeq}$$

in $\mathcal{P}(\Gamma, A) / \simeq$. Thus, ϕ extends to a well-defined homomorphism.

Lastly, we need to show that $\eta\phi$ and $\phi\eta$ are the respective identity maps. To this end,

$$\eta\phi(a) = \eta([a]) = a \text{ and } \eta\phi(e) = \eta([e]) = e$$

and

$$\begin{aligned} \phi\eta([P]) &= \phi(a_0 e_1 \cdots e_n a_n) \\ &= \phi(a_0) \phi(e_1) \cdots \phi(e_n) \phi(a_n) \\ &= [a_0] [e_1] \cdots [e_n] [a_n] \\ &= [P]. \end{aligned}$$

Therefore, $\mathcal{P}(\Gamma, A) / \simeq$ is isomorphic to $\pi(\Gamma, A)$.

Finally, we say that a path P in (Γ, A) is *reduced* if it contains no round-trips. Note that each homotopy class of paths contains a unique reduced path. For, by definition every path in (Γ, A) is of finite length. Hence, each path can contain at most finitely many round-trips. Therefore, every homotopy class of paths does contain a reduced path. Uniqueness is handled below in Theorem 4.3.

THEOREM 4.3 (Normal Form Theorem). *Every path P in $\mathcal{P}(\Gamma, A)$ is homotopic to a unique reduced path and the evaluation map $\mathcal{P}(\Gamma, A) \rightarrow \pi(\Gamma, A)$ maps the set of reduced paths in $\mathcal{P}(\Gamma, A)$ bijectively onto $\pi(\Gamma, A)$.*

PROOF. The first part follows immediately from the second part, which is a generalization of Proposition 2.4; and the proof given there extends as follows, using induction on the lengths of reduced paths. Let $P \equiv a_0 e_1 a_1 \cdots e_m a_m$ and $Q \equiv b_0 e'_1 b_1 \cdots e'_n b_n$ be reduced paths with the same evaluation in $\pi(\Gamma, A)$. By the normal form theorem of Higgins [3], we must have that $e_1 = e'_1$, $a_0 = r_0 c_1$, and $b_0 = r_0 d_1$, where $r_0 \in T_{e_1}$ and $c_1, d_1 \in A_{e_1}$ (with the setup as in the proof of Proposition 2.4). Thus, in $\pi(\Gamma, A)$

we have

$$r_0 e_1 (c_1^{e_1} a_1) e_2 \cdots e_m a_m = r_0 e_1 (d_1^{e_1} b_1) e'_2 \cdots e'_n b_n.$$

Cancelling $r_0 e_1$ from the left sides of this equation and using induction on the sum of the lengths of the reduced paths, we conclude that

$$(c_1^{e_1} a_1) e_2 a_2 \cdots e_m a_m \equiv (d_1^{e_1} b_1) e'_2 b_2 \cdots e'_n b_n.$$

Note also that $b_0 = r_0 d_1 = (a_0 c_1^{-1}) d_1 = a_0 (d_1^{-1} c_1)^{-1}$ and that $d_1^{-1} c_1 \in A_{e_1}$. Thus

$$\begin{aligned} a_0 e_1 a_1 \cdots e_m a_m &\equiv a_0 (d_1^{-1} c_1)^{-1} e_1 (d_1^{-1} c_1)^{e_1} a_1 e_2 a_2 \cdots e_m a_m \\ &= b_0 e_1 (d_1^{-1})^{e_1} (c_1^{e_1} a_1) e_2 a_2 \cdots a_m e_m \\ &\equiv b_0 e_1 (d_1^{-1})^{e_1} (d_1^{e_1} b_1) e'_2 b_2 \cdots e'_n b_n \\ &= b_0 e'_1 b_1 e'_2 b_2 \cdots e'_n b_n, \end{aligned}$$

yielding that $P = Q$, as required. □

CHAPTER 5

COVERINGS

Now, in an effort to develop some aspects of covering space theory in the category of graphs of groups, we turn our attention toward defining the notion of a link in (Γ, A) .

DEFINITION 5.1. Let (Γ, A) be a graph of groups and $x \in V(\Gamma)$. Then, the *link of x* in (Γ, A) is defined as

$$L_x = \{(aA_e, e) : a \in A_x, e \in E(\Gamma), s(e) = x\}.$$

If $P \equiv a_0e_1a_1 \cdots e_na_n$ is a path of length $n \geq 1$, then define $P'(0) \in L_x$ where $x = s(P)$ by

$$P'(0) = (a_0A_{e_1}, e_1).$$

Note that $P'(0)$ does not depend on the word representing P . This is easily demonstrated. Suppose that $P \equiv a_0e_1a_1 \cdots e_na_n \equiv b_0e_1b_1 \cdots e_nb_n$ is a path in (Γ, A) . Then, by our definition of \equiv , we have that $b_0 = a_0c^{-1}$ for some $c \in A_{e_1}$. Thus, using the representative $a_0e_1a_1 \cdots e_na_n$ we have that $P'(0) = (a_0A_{e_1}, e_1)$ and using the representative $b_0e_1b_1 \cdots e_nb_n$ we have that

$$P'(0) = (b_0A_{e_1}, e_1) = (a_0c^{-1}A_{e_1}, e_1) = (a_0A_{e_1}, e_1)$$

since $c \in A_{e_1}$. Thus $P'(0)$ does not depend on the representation of P .

Now, suppose that $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a map of graphs of groups. Write $f: \Gamma \rightarrow \Delta$ for the underlying map of graphs. For each vertex $x \in V(\Gamma)$, F induces a map

$$F_x : L_x(\Gamma, A) \rightarrow L_{f(x)}(\Delta, B)$$

given by $(aA_e, e) \mapsto (f_x(a)\alpha(e)B_{f(e)}, f(e))$. Note that F_x is well-defined. For if $aA_e = bA_e$ for some $a, b \in A_x$, then $b = ac$ for some $c \in A_e$, and we have that

$$\begin{aligned} F_x(bA_e, e) &= (f_x(b)\alpha(e)B_{f(e)}, f(e)) \\ &= (f_x(ac)\alpha(e)B_{f(e)}, f(e)) \\ &= (f_x(a)f_x(c)\alpha(e)B_{f(e)}, f(e)) \\ &= (f_x(a)\alpha(e)f_x(c)^{\alpha(e)}B_{f(e)}, f(e)) \\ &= (f_x(a)\alpha(e)B_{f(e)}, f(e)) \\ &= F_x(aA_e, e) \end{aligned}$$

since $f_x(c)^{\alpha(e)} \in B_{f(e)}$ by Proposition 3.4. Furthermore, $F_x(P'(0)) = (FP)'(0)$. In order to see this, let $P \equiv a_0e_1a_1 \cdots e_n a_n$ be a path in (Γ, A) . Then,

$$F_x(P'(0)) = F_x(a_0A_{e_1}, e_1) = (f_x(a_0)\alpha(e_1)B_{f(e_1)}, f(e_1)),$$

and

$$FP \equiv f_{s(e_1)}(a_0)\alpha(e_1)f(e_1)\beta(e_1)^{-1}f_{s(e_2)}(a_1) \cdots f(e_n)\beta(e_n)^{-1}f_{t(e_n)}.$$

Thus,

$$\begin{aligned} (FP)'(0) &= (f_{s(e_1)}(a_0)\alpha(e_1)B_{f(e_1)}, f(e_1)) \\ &= (f_x(a_0)\alpha(e_1)B_{f(e_1)}, f(e_1)) \\ &= F_x(P'(0)). \end{aligned}$$

Now that we have a well-defined map on the links in (Γ, A) we can explore the notion of a covering map in the category of Graphs of Groups, as Bass did in [1].

DEFINITION 5.2. A map $F: (\Gamma, A) \rightarrow (\Delta, B)$ is called a *covering* if for all $x \in V(\Gamma)$,

- (i) $f_x = F|_{A_x}: A_x \rightarrow B_{f(x)}$ is injective, and
- (ii) $F_x: L_x \rightarrow L_{f(x)}$ is bijective.

5.1. Path-lifting

We first consider the path lifting property for coverings of graphs of groups. From here forward in this section we will assume that $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a covering.

LEMMA 5.3 (Uniqueness of lifts of paths). *If $P \equiv a_0e_1a_1 \cdots e_ma_m$ and $Q \equiv b_0e'_1b_1 \cdots e'_nb_n$ are paths in (Γ, A) such that $s(P) = s(Q)$ and $F(P) = F(Q)$, then $P = Q$.*

PROOF. First of all, since $F(P) = F(Q)$, we have that $(FP)'(0) = (FQ)'(0)$. But, since $F_x(P'(0)) = (FP)'(0)$ and $F_x(Q'(0)) = (FQ)'(0)$ where $x = s(P) = s(Q)$, we also have that $F_x(P'(0)) = F_x(Q'(0))$. The fact that F is a covering then yields that $P'(0) = Q'(0)$ which implies that $(a_0A_{e_1}, e_1) = (b_0A_{e'_1}, e'_1)$. Hence, $e'_1 = e_1$ and $b_0 = a_0c$ for some $c \in A_{e_1}$.

Now, we have that $P \equiv a_0e_1a_1 \cdots e_ma_m$ and $Q \equiv b_0e'_1b_1 \cdots e'_nb_n \equiv a_0e_1c^{e_1}b_1e'_2 \cdots e'_nb_n$. Note here that paths given by $a_1e_2a_2 \cdots e_ma_m$ and $c^{e_1}b_1e'_2b_2 \cdots e'_nb_n$ have the same initial vertex, namely $t(e_1)$, and map to the same path under F . So, by induction on the length of paths,

$$a_1e_2a_2 \cdots e_ma_m \equiv c^{e_1}b_1e'_2b_2 \cdots e'_nb_n.$$

Hence, we have that

$$(a_0e_1)a_1e_2a_2 \cdots e_ma_m \equiv (a_0e_1)c^{e_1}b_1e'_2b_2 \cdots e'_nb_n.$$

Thus, $P = Q$. For paths of length 0, the result follows immediately from (i) in Definition 5.2. \square

LEMMA 5.4. *Let P be a path in (Δ, B) from a vertex x to a vertex y in Δ and let \tilde{x} be a vertex in Γ with $f(\tilde{x}) = x$. Then, there exists a path \tilde{P} in (Γ, A) with source vertex \tilde{x} and an element $b \in B_y$ such that*

$$F(\tilde{P}) = Pb.$$

PROOF. This will be a proof by induction on $|P|$. To begin, if $|P| = 0$, let $\tilde{P} = 1_{\tilde{x}}$ and let $b = P^{-1}$. Then, $F(\tilde{P}) = 1_x = Pb$. Now, assume that $|P| = n \geq 1$. Then, $P'(0) \in L_x$ and $F_{\tilde{x}} : L_{\tilde{x}} \rightarrow L_x$ is a bijection. So, there exists $(\tilde{a}_0 A_{\tilde{e}_1}, \tilde{e}_1) \in L_{\tilde{x}}$ such that

$$F_{\tilde{x}}(\tilde{a}_0 A_{\tilde{e}_1}, \tilde{e}_1) = (f_{\tilde{x}}(\tilde{a}_0) \alpha(\tilde{e}_1) B_{f(\tilde{e}_1)}, f(\tilde{e}_1)) = P'(0).$$

Thus, P has a representation

$$P \equiv a_0 e_1 a_1 e_2 \cdots e_n a_n$$

where $a_0 = f_{\tilde{x}}(\tilde{a}_0) \alpha(\tilde{e}_1)$ and $e_1 = f(\tilde{e}_1)$. Let $\tilde{P}_0 \equiv \tilde{a}_0 \tilde{e}_1$ be a path in (Γ, A) from $s(\tilde{e}_1) = \tilde{x}$. Then,

$$F(\tilde{P}_0) \equiv f_{\tilde{x}}(\tilde{a}_0) \alpha(\tilde{e}_1) f(\tilde{e}_1) \beta(\tilde{e}_1)^{-1} = a_0 e_1 \beta(\tilde{e}_1)^{-1}.$$

Now, since $P_1 \equiv \beta(\tilde{e}_1) a_1 e_2 \cdots e_n a_n$ is a path of length $n - 1$ starting at $s(e_2) = t(e_1) = f(\tilde{x}_1)$, by induction there exists a path \tilde{P}_1 in (Γ, A) with source vertex $\tilde{x}_1 = t(\tilde{e}_1) = s(\tilde{e}_2)$ and an element $b \in B_y$ such that

$$F(\tilde{P}_1) = P_1 b.$$

Taking the product $\tilde{P} = \tilde{P}_0\tilde{P}_1$ yields a path with source \tilde{x} such that

$$\begin{aligned} F(\tilde{P}) &= F(\tilde{P}_0)F(\tilde{P}_1) \\ &\equiv (a_0e_1\beta(\tilde{e}_1)^{-1})(\beta(\tilde{e}_1)a_1e_2 \cdots e_n a_n b) \\ &\equiv Pb. \end{aligned}$$

□

THEOREM 5.5 (Unique path-lifting property). *Let P be a path in (Δ, B) from x to y such that $B_y = 1$, and let \tilde{x} be a vertex in Γ with $f(\tilde{x}) = x$. Then, there exists a unique path \tilde{P} in (Γ, A) with source vertex \tilde{x} such that $F(\tilde{P}) = P$.*

5.2. Homotopy lifting

LEMMA 5.6. *Let P and Q be paths in (Γ, A) such that $s(P) = s(Q)$ and $F(P) \searrow F(Q)$. Then, $P \searrow Q$.*

PROOF. To begin, suppose that $F(P)$ contains a round-trip e_1be_2 (i.e., $e_1 = \bar{e}_2$ and $b \in B_{e_2}$) and that $F(Q)$ is obtained from $F(P)$ by replacing e_1be_2 with b^{e_2} . Now in P there must be a subpath of the form $\tilde{e}_1a\tilde{e}_2$ where $f(\tilde{e}_1) = e_1$, $f(\tilde{e}_2) = e_2$, $\beta(\tilde{e}_1)^{-1}f_{\tilde{x}}(a)\alpha(\tilde{e}_2) = b$, and $\tilde{x} = s(\tilde{e}_2)$.

Since $b \in B_{e_2}$ and $e_2 = \bar{e}_1$,

$$(f_{\tilde{x}}(a)\alpha(\tilde{e}_2)B_{e_2}, e_2) = (\beta(\tilde{e}_1)bB_{e_2}, e_2) = (\beta(\tilde{e}_1)B_{\bar{e}_1}, \bar{e}_1)$$

as elements of $L_{f(\tilde{x})}$. Thus,

$$F_{\tilde{x}}(aA_{\tilde{e}_2}, \tilde{e}_2) = F_{\tilde{x}}(A_{\bar{e}_1}, \bar{e}_1).$$

However, F is a covering map. Hence, $\bar{e}_1 = \tilde{e}_2$ and $a \in A_{\tilde{e}_2}$. Thus, we have that $\tilde{e}_1a\tilde{e}_2$ is a round-trip in P .

Now, let P' be the path obtained from P by replacing $\tilde{e}_1 a \tilde{e}_2$ by $a^{\tilde{e}_2}$. We need to show that $P' = Q$. To this end, note that if $\tilde{y} = t(\tilde{e}_2)$, then we see that

$$f_{\tilde{y}}(a^{\tilde{e}_2})^{\beta(\tilde{e}_2)} = f_{\tilde{x}}(a)^{\alpha(\tilde{e}_2)f(\tilde{e}_2)} = f_{\tilde{x}}(a)^{\alpha(\tilde{e}_2)e_2} = b^{e_2}$$

since

$$f_{\tilde{x}}(a)^{\alpha(\tilde{e}_2)} = \alpha(\tilde{e}_2)^{-1} f_{\tilde{x}}(a) \alpha(\tilde{e}_2) = \beta(\tilde{e}_1)^{-1} f_{\tilde{x}}(a) \alpha(\tilde{e}_2) = b.$$

It follows that $F(P') = F(Q)$. Hence, by Lemma 5.3, we have that $P' = Q$. Therefore, $P \searrow Q$. □

The theorem below is an immediate consequence of Lemma 5.6.

THEOREM 5.7 (Homotopy-lifting property). *Let P and Q be paths in (Γ, A) with the same source vertex. If $F(P) \simeq F(Q)$, then $P \simeq Q$; in particular P and Q have the same target vertex.*

Combining this theorem with Theorem 5.5 we have

COROLLARY 5.8. *Let P and Q be paths in (Δ, B) from x to y such that $B_y = 1$, and let \tilde{P} and \tilde{Q} be the unique lifts of P and Q starting at \tilde{x} , a vertex in Γ with the property that $f(\tilde{x}) = x$. Then, if $P \simeq Q$, we have that $\tilde{P} \simeq \tilde{Q}$; in particular \tilde{P} and \tilde{Q} also have the same target vertex.*

5.3. Conjugation of maps and general lifting criterion

For our analogue for graphs of groups of the general lifting criterion in the topological theory of covering spaces, we need the following notion of equivalence of maps of graphs of groups, called conjugacy.

Let $G: (\Sigma, C) \rightarrow (\Gamma, A)$ be a map of graphs of groups with underlying map of graphs $g: \Sigma \rightarrow \Gamma$. For each $x \in V(\Sigma)$, choose an element $a_x \in A_{g(x)}$. Then by the

conjugate of G by the a_x 's we will mean the map G' with the same underlying map of graphs $g' = g$, defined as follows: for each $x \in V(\Sigma)$, define the homomorphism $g'_x: C_x \rightarrow A_{g(x)}$ by $g'_x(c) = g_x(c)^{a_x}$; and for each $e \in E(\Sigma)$, define its edge label by $\alpha'(e) = a_{s(e)}^{-1}\alpha(e)$. It is easy to see that the conditions of Lemma 3.3 hold for the g'_x 's and $\alpha'(e)$'s, and hence a map $G': (\Sigma, C) \rightarrow (\Gamma, A)$ is induced, called a *conjugate* of G .

Conjugation is obviously an equivalence relation on the set of maps from (Σ, C) to (Γ, A) . And as we next show, homomorphisms of fundamental groups induced by conjugate maps are the same up to conjugation by an element of the base vertex group.

LEMMA 5.9. *If $G, G': (\Sigma, C) \rightarrow (\Gamma, A)$ are conjugate maps of graphs of groups and $v \in V(\Sigma)$, then the induced homomorphisms of fundamental groups $G, G': \pi(\Sigma, C, v) \rightarrow \pi(\Gamma, A, g(v))$ are also conjugates. More precisely, there exists $a \in A_{g(v)} \subseteq \pi(\Gamma, A, g(v))$ such that $G'([P]) = G([P])^a$ for all $[P] \in \pi(\Sigma, C, v)$.*

PROOF. Suppose G' is the result of conjugating G by elements $a_x \in A_{g(x)}$. We show that the result already holds at the path level. Let $P \equiv c_0 e_1 c_1 \cdots e_n c_n$ be a path in (Σ, C) , and write $x_i = t(e_i) = s(e_{i+1})$. Then $G'(P) \in \mathcal{P}(\Gamma, A)$ is represented by

$$\begin{aligned} G'(P) &\equiv g'_{x_0}(c_0)\alpha'(e_1)g(e_1)\beta'(e_1)^{-1}g'_{x_1}(c_1)\cdots\alpha'(e_n)g(e_n)\beta'(e_n)^{-1}g'_{x_n}(c_n) \\ &= g_{x_0}(c_0)^{a_{x_0}}a_{x_0}^{-1}\alpha(e_1)g(e_1)\beta(e_1)^{-1}a_{x_1}g_{x_1}(c_1)^{a_{x_1}}\cdots g_{x_n}(c_n)^{a_{x_n}} \\ &= a_{x_0}^{-1}g_{x_0}(c_0)\alpha(e_1)g(e_1)\beta(e_1)^{-1}g_{x_1}(c_1)\cdots\alpha(e_n)g(e_n)\beta(e_n)^{-1}g_{x_n}(c_n)a_{x_n} \\ &= a_{x_0}^{-1}G(P)a_{x_n} \end{aligned}$$

If P is a loop based at v , then $x_0 = x_n = v$. We see that the result holds with $a = a_v$. □

We can now state our general lifting criterion for coverings of graphs of groups:

$$\begin{array}{ccc}
& & (\Gamma, A), u \\
& \nearrow \tilde{G} & \downarrow F \\
(\Sigma, C), v & \xrightarrow{G} & (\Delta, B)
\end{array}$$

THEOREM 5.10 (General lifting). *Suppose $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a covering map, $G: (\Sigma, C) \rightarrow (\Delta, B)$ is a map of graphs of groups, where Σ is a connected graph, and let u, v be vertices of Γ, Σ such that $f(u) = g(v)$. Then there exists a map $\tilde{G}: (\Sigma, C) \rightarrow (\Gamma, A)$ such that $\tilde{g}(v) = u$ and $F\tilde{G}$ is conjugate to G if and only if $[G\pi(\Sigma, C, v)]^{b_0} \subseteq F\pi(\Gamma, A, u)$ for some $b_0 \in B_{f(u)} = B_{g(v)}$; and if \tilde{G} exists, it is unique up to conjugacy.*

The proof is deferred to Chapter 6.

5.4. Equivalence of coverings

First we define isomorphisms of graphs of groups in the obvious way. We call a map of graphs of groups $M: (\Gamma_1, A_1) \rightarrow (\Gamma_2, A_2)$ an *isomorphism* if there exists a map of graphs of groups $N: (\Gamma_2, A_2) \rightarrow (\Gamma_1, A_1)$ such that both compositions MN and NM are identity maps; such a map N is called an *inverse* of M . If M has an inverse, then by a familiar argument, it is unique—we denote it by M^{-1} . Note also that in this situation the induced map $M: \pi(\Gamma_1, A_1, x) \rightarrow \pi(\Gamma_2, A_2, m(x))$ is an isomorphism of groups, with inverse $M^{-1}: \pi(\Gamma_2, A_2, m(x)) \rightarrow \pi(\Gamma_1, A_1, x)$, for any vertex x in Γ_1 .

Using Lemma 3.3, it is not hard to show that a map of graphs of groups $M: (\Gamma_1, A_1) \rightarrow (\Gamma_2, A_2)$ is an isomorphism if and only if three conditions hold:

- the underlying map $m: \Gamma_1 \rightarrow \Gamma_2$ is an isomorphism of graphs;
- for each $x \in V(\Gamma_1)$, the map $m_x: (A_1)_x \rightarrow (A_2)_{m(x)}$ is an isomorphism of groups;

- for each $e \in E(\Gamma_1)$, the mapping $c \mapsto m_x(c)^{\alpha_M(e)}$ is an isomorphism of groups $(A_1)_e \rightarrow (A_2)_{m(e)}$.

The three conditions are obviously necessary for M to be an isomorphism. To see that they are sufficient, note that the first two conditions allow us to define a map of graphs of groups $N: (\Gamma_2, A_2) \rightarrow (\Gamma_1, A_1)$ by applying Lemma 3.3 to the data:

- (1) $n = m^{-1}: \Gamma_2 \rightarrow \Gamma_1$;
- (2) $n_x = m_{x'}^{-1}$ for all $x \in V(\Gamma_2)$; and
- (3) $\alpha_N(e) = m_{x'}^{-1}(\alpha_M(e')^{-1})$ for all $e \in E(\Gamma_2)$,

where $x' = n(x)$ and $e' = n(e)$. Also, note below that $c' = n_x(c)$ where $c \in (A_2)_e$. We desire to show that $N = M^{-1}$. To do so, we need to prove the following:

- (i) N is a map of graphs of groups, and
- (ii) $NM = Id: (\Gamma_1, A_1) \rightarrow (\Gamma_1, A_1)$ and $MN = Id: (\Gamma_2, A_2) \rightarrow (\Gamma_2, A_2)$, the respective identity maps.

With regard to (i), to apply Lemma 3.3, we need to first show that

$$n_x(c)^{\alpha_N(e)} \in (A_1)_{e'}$$

for all $c \in (A_2)_e$. To this end, we have that

$$\begin{aligned} n_x(c)^{\alpha_N(e)} &= m_{x'}^{-1}(c)^{m_{x'}^{-1}(\alpha_M(e')^{-1})} \\ &= m_{x'}^{-1}(\alpha_M(e')c\alpha_M(e')^{-1}) \\ &= m_{x'}^{-1}(\alpha_M(e')\alpha_M(e')^{-1}m_{x'}(c')\alpha_M(e')\alpha_M(e')^{-1}) \\ &= c' \in (A_1)_{e'} \end{aligned}$$

since the fact that the mapping $c \mapsto m_x(c)^{\alpha_M(e)}$ is an isomorphism of groups $(A_1)_e \rightarrow (A_2)_{m(e)}$ implies that there exists a $c'_1 \in (A_1)_{e'}$ such that $c = \alpha_M(e')^{-1}m_{x'}(c'_1)\alpha_M(e')$.

To conclude (i), we now need to show that

$$[n_x(c)^{\alpha_N(e)}]^{n(e)} = n_y(c^e)^{\beta_N(e)}.$$

To this end, we see that

$$\begin{aligned} [n_x(c)^{\alpha_N(e)}]^{n(e)} &= [m_{x'}^{-1}(\alpha_M(e'))m_{x'}^{-1}(c)m_{x'}^{-1}(\alpha_M(e')^{-1})]^{e'} \\ &= [m_{x'}^{-1}(\alpha_M(e'))m_{x'}^{-1}(\alpha_M(e')^{-1}m_{x'}(c'_1)\alpha_M(e'))m_{x'}^{-1}(\alpha_M(e')^{-1})]^{e'} \\ &= [m_{x'}^{-1}(m_{x'}(c'_1))]^{e'} \\ &= c_1^{e'}. \end{aligned}$$

But, since $m_{x'}(c')^{\alpha_M(e')e} = m_{y'}(c'^{e'})^{\beta_M(e')}$, we know that

$$\begin{aligned} c_1^{e'} &= m_{y'}^{-1}(\beta_M(e'))[m_{x'}(c'_1)^{\alpha_M(e')e}] \beta_M(e')^{-1} \\ &= m_{y'}^{-1}(\beta_M(e'))m_{y'}^{-1}([\alpha_M(e')^{-1}m_{x'}(c'_1)\alpha_M(e')]^e)m_{y'}^{-1}(\beta_M(e')^{-1}) \\ &= \beta_N(e)^{-1}n_y(c^e)\beta_N(e) \\ &= n_y(c^e)^{\beta_N(e)}, \end{aligned}$$

thus proving (i).

As for (ii), we have that for all $a \in (A_1)_x$ and any $x \in V(\Gamma_1)$

$$NM(a) = n_{x'}(m_x(a)) = m_x^{-1}(m_x(a)) = a,$$

the identity element in $(A_1)_x$. A similar calculation will yield that $MN(a') = a'$ for all $a' \in (A_2)_{x'}$ and any $x' \in V(\Gamma_2)$. As for the respective edges, if $e \in E(\Gamma_1)$, then

$$\begin{aligned}
N[M(e)] &= N[\alpha_M(e)m(e)\beta_M(e)^{-1}] \\
&= n_{x'}(\alpha_M(e))\alpha_N(m(e))n(m(e))\beta_N(m(e))n_{y'}(\beta_M(e)^{-1}) \\
&= n_{x'}(\alpha_M(e))\alpha_N(e')n(e')\beta_N(e')n_{y'}(\beta_M(e)^{-1}) \\
&= m_x^{-1}(\alpha_M(e))m_x^{-1}(\alpha_M(e)^{-1})em_y^{-1}(\beta_M(e)^{-1})m_y^{-1}(\beta_M(e)) \\
&= e.
\end{aligned}$$

Again, a similar calculation also yields that $M[N(e')] = e'$ for all $e' \in E(\Gamma_2)$. Thus, since we have shown that NM and MN are the respective identity maps on the generators, we have that $NM = \text{Id}$ and $MN = \text{Id}$, proving (ii) above.

DEFINITION 5.11 (Equivalent coverings). Two coverings $F_i: (\Gamma_i, A_i) \rightarrow (\Delta, B)$, $i = 1, 2$, of a graph of groups (Δ, B) are *equivalent* if there exists an isomorphism of graphs of groups $M: (\Gamma_1, A_1) \rightarrow (\Gamma_2, A_2)$ such that F_2M and F_1 are conjugate maps.

$$\begin{array}{ccc}
(\Gamma_1, A_1) & \overset{M}{\dashrightarrow} & (\Gamma_2, A_2) \\
& \searrow^{F_1} & \swarrow_{F_2} \\
& & (\Delta, B)
\end{array}$$

This defines an equivalence relation on the set of coverings of (Δ, B) . It is symmetric since if F_2M is conjugate to F_1 , say by elements $b_x \in B_{f_1(x)}$ for $x \in V(\Gamma_1)$, then F_1M^{-1} is conjugate to F_2 by the elements $b_{m(x)} = b_x^{-1} \in B_{f_2(m(x))} = B_{f_1(x)}$. In fact, this relation projects to an equivalence relation on the set of conjugacy classes of coverings of (Δ, B) . The following routine exercises justify this claim:

Let $F, F' : (\Gamma, A) \rightarrow (\Delta, B)$ and $G, G' : (\Sigma, C) \rightarrow (\Gamma, A)$ be pairs of conjugate maps of graphs of groups. Then

- (1) the compositions $FG, F'G' : (\Sigma, C) \rightarrow (\Delta, B)$ are also conjugate. Hence there is a well-defined composition of conjugacy classes given by $[F][G] = [FG]$.
- (2) If G is an isomorphism, then G' is also an isomorphism.
- (3) If F is a covering map, then F' is also a covering map.
- (4) Let $F_i : (\Gamma_i, A_i) \rightarrow (\Delta, B)$ be a covering map, $i = 1, 2$. Then F_1 and F_2 are equivalent coverings if and only if there exists an isomorphism $M : (\Gamma_1, A_1) \rightarrow (\Gamma_2, A_2)$ such that $[F_2][M] = [F_1]$.

To see (1), first note that it can be easily verified that FG is defined as follows:

- (i) $fg : \Sigma \rightarrow \Delta$ is the map of graphs;
- (ii) $f_{g(x)}g_x : C_x \rightarrow B_{fg(x)}$ is the vertex homomorphism for all $x \in V(\Sigma)$; and
- (iii) $\alpha_{FG}(e) = f_{g(x)}(\alpha_G(e))\alpha_F(g(e))$ for all $e \in E(\Sigma)$.

FG as defined above yields the following for all $c \in C_x$ where $x \in V(\Sigma)$ and for all $e \in E(\Sigma)$ such that $s(e) = x$ and $t(e) = y$:

- $FG(c) = f_{g(x)}g_x(c)$, and
- $FG(e) = f_{g(x)}(\alpha_G(e))\alpha_F(g(e))fg(e)\beta_F(g(e))^{-1}f_{g(y)}(\beta_G(e^{-1}))$.

Now, we need elements $b_{fg(x)} \in B_{fg(x)}$ such that $F'G'(c) = FG(c)^{b_{fg(x)}}$ and $\alpha_{F'G'}(e) = b_{fg(x)}^{-1}\alpha_{FG}(e)$. We claim that $b_{fg(x)} = f_{g(x)}(a_x)b_{g(x)}$. In order to see this, first we note

that

$$\begin{aligned}
F'G'(c) &= f'_{g(x)}(g'_x(c)) \\
&= f_{g(x)}(g_x(c)^{a_x})^{b_{g(x)}} \\
&= b_{g(x)}^{-1} f_{g(x)}(a_x^{-1}) f_{g(x)}(g_x(c)) f_{g(x)}(a_x) b_{g(x)} \\
&= f_{g(x)}(g_x(c))^{f_{g(x)}(a_x) b_{g(x)}} \\
&= FG(c)^{b_{f_{g(x)}}}.
\end{aligned}$$

In addition,

$$\begin{aligned}
F'G'(e) &= F'(\alpha_{G'}(e)g'(e)\beta_{G'}(e)^{-1}) \\
&= F'(a_x^{-1}\alpha_G(e)g(e)\beta_G(e)^{-1}a_y) \\
&= f'_{g(x)}(a_x^{-1}\alpha_G(e))\alpha_{F'}(g(e))f'g(e)(\beta_{F'}(g(e))^{-1})f'_{g(y)}(\beta_G(e)^{-1}a_y) \\
&= f_{g(x)}(a_x^{-1}\alpha_G(e))^{b_{g(x)}}b_{g(x)}^{-1}\alpha_F(g(e))fg(e)\beta_F(g(e))^{-1}b_{g(y)}f_{g(y)}(\beta_G(e)^{-1}a_y)^{b_{g(y)}} \\
&= b_{g(x)}^{-1}f_{g(x)}(a_x^{-1})f_{g(x)}(\alpha_G(e))\alpha_F(g(e))fg(e)\beta_F(g(e))^{-1}f_{g(y)}(\beta_G(e)^{-1})f_{g(y)}(a_y)b_{g(y)} \\
&= b_{g(x)}^{-1}f_{g(x)}(a_x^{-1})\alpha_{FG}(e)fg(e)\beta_{FG}(e)^{-1}f_{g(y)}(a_y)b_{g(y)} \\
&= b_{f_{g(x)}}^{-1}\alpha_{FG}(e)fg(e)\beta_{FG}(e)^{-1}b_{f_{g(y)}} \\
&= \alpha_{F'G'}(e)f'g'(e)\beta_{F'G'}(e)^{-1}
\end{aligned}$$

Therefore, $F'G'$ is conjugate to FG whenever F' is conjugate to F and G' is conjugate to G .

As for (2), the proof is an easy application of the discussion at the beginning of this chapter. Note that since G is an isomorphism of graphs of groups we have that $g' = g$ is an isomorphism of graphs and $g'_x = ad(a_x)g_x$ is an isomorphism of groups with $g'_x^{-1}(a) = g_x^{-1}(a^{a_x^{-1}})$ for all $a \in A_{x'}$ and $x' \in V(\Gamma)$ where $g(x) = x'$. Lastly, for

all $c \in C_e$, where $e \in E(\Sigma)$, the map $c \mapsto g'_x(c)^{\alpha_{G'(e)}} = g_x(c)^{\alpha_{G(e)}}$ is an isomorphism $C_e \rightarrow A_{e'}$ with inverse $a \mapsto g_x^{-1}(a^{\alpha_{G(e)^{-1}}})$ for all $a \in A_{e'}$ where $g(e) = e'$. Thus, G'^{-1} exists and is defined as follows:

- (i) $g'^{-1} = g^{-1}: \Gamma \rightarrow \Sigma$ is the map of graphs;
- (ii) $g'_x^{-1}(a) = g_x^{-1}(a^{a_x^{-1}})$ for all $a \in A_{x'}$ where $g(x) = x'$; and
- (iii) $\alpha_{G'^{-1}}(e') = g'_x^{-1}(\alpha_{G'}(e)^{-1}) = g_x^{-1}(a_x \alpha_G(e)^{-1})$ for all $e' \in E(\Gamma)$ where $g(e) = e'$.

With regard to (3), we need to show that

- (i) $f'_x: A_x \rightarrow B_{f'(x)} = B_{f(x)}$ is injective, and
- (ii) $F'_x: L_x(\Gamma, A) \rightarrow L_{f'(x)} = L_{f(x)}$ is bijective.

In order to see (i), let $a_1, a_2 \in A_x$ for some $x \in V(\Gamma)$ such that $f'_x(a_1) = f'_x(a_2)$. Then, by the definition of f'_x we have that $f_x(a_1)^{b_x} = f_x(a_2)^{b_x}$ for some $b_x \in B_{f'(x)} = B_{f(x)}$. But, this implies that $f_x(a_1) = f_x(a_2)$ and we immediately have that $a_1 = a_2$ since F is a covering.

As for the injectivity of the map induced on the links in (ii), suppose that $(a_1 A_{e_1}, e_1), (a_2 A_{e_2}, e_2) \in L_x(\Gamma, A)$ such that $F'_x(a_1 A_{e_1}) = F'_x(a_2 A_{e_2})$. Then, we have that

$$(f'_x(a_1)\alpha'(e_1)B_{f'(e_1)}, f'(e_1)) = (f'_x(a_2)\alpha'(e_2)B_{f'(e_2)}, f'(e_2))$$

which implies that

$$(f_x(a_1)^{b_x} b_x^{-1} \alpha(e_1) B_{f(e_1)}, f(e_1)) = (f_x(a_2)^{b_x} b_x^{-1} \alpha(e_2) B_{f(e_2)}, f(e_2))$$

by the definitions of f' , f'_x , and α' . The above step yields that $f(e_1) = f(e_2)$ and

$$b_x^{-1} f_x(a_1) \alpha(e_1) B_{f(e_1)} = b_x^{-1} f_x(a_2) \alpha(e_2) B_{f(e_2)}.$$

Thus, we have that

$$f_x(a_1)\alpha(e_1)B_{f(e_1)} = f_x(a_2)\alpha(e_2)B_{f(e_2)}$$

yielding the fact that $F_x(a_1A_{e_1}, e_1) = F_x(a_2A_{e_2}, e_1)$. Hence, the injectivity of F_x implies that $(a_1A_{e_1}, e_1) = (a_2A_{e_2}, e_2)$ and we have that F'_x is indeed injective.

As for the surjectivity of F'_x , let $(bB_e, e) \in L_{f'(x)} = L_{f(x)}$. Then, $(b_x bB_e, e) \in L_{f'(x)} = L_{f(x)}$ as well. Now, since F_x is surjective, there exists $(aA_{\tilde{e}}, \tilde{e}) \in L_x$ such that $F_x(aA_{\tilde{e}}, \tilde{e}) = (b_x bB_e, e)$. Thus, we have that

$$(f_x(a)\alpha(\tilde{e})B_{f(\tilde{e})}, f(\tilde{e})) = (b_x bB_e, e)$$

which implies that

$$(b_x f'_x(a)\alpha'(\tilde{e})B_e, e) = (b_x bB_e, e).$$

From this equality, we see that $b_x f'_x(a)\alpha'(\tilde{e})B_e = b_x bB_e$ yielding that $f'_x(a)\alpha'(\tilde{e})B_e = bB_e$. Thus, $(aA_{\tilde{e}}, \tilde{e}) \in L_x$ such that $F'_x(aA_{\tilde{e}}, \tilde{e}) = (bB_e, e)$ and we have that F_x is surjective. The proof of (4) is left to the reader.

THEOREM 5.12. *Two coverings $F_i: (\Gamma_i, A_i) \rightarrow (\Delta, B)$ with Γ_i connected, $i = 1, 2$, are equivalent if and only if, for some pair of vertices $x_1 \in V(\Gamma_1)$ and $x_2 \in V(\Gamma_2)$ such that $f_1(x_1) = f_2(x_2)$, we have that $[F_1 \pi(\Gamma_1, A_1, x_1)]^{b_0} = F_2 \pi(\Gamma_2, A_2, x_2)$ for some element $b_0 \in B_{f_i(x_i)}$.*

PROOF. Suppose F_1, F_2 are equivalent coverings and let $M: (\Gamma_1, A_1) \rightarrow (\Gamma_2, A_2)$ be an isomorphism such that $F_2 M$ and F_1 are conjugate maps. Choose any vertex x_1 of Γ_1 and let $x_2 = m(x_1)$. Then by Lemma 5.9, $[F_1 \pi(\Gamma_1, A_1, x_1)]^{b_0} = (F_2 M) \pi(\Gamma_1, A_1, x_1)$ for some $b_0 \in B_{f_1(x_1)}$. However $(F_2 M) \pi(\Gamma_1, A_1, x_1) = F_2 \pi(\Gamma_2, A_2, x_2)$ since M is an isomorphism.

Conversely, assume that the condition holds for vertices $v_i \in V(\Gamma_i)$ and $b_0 \in B_{f_i(x_i)}$. Then by Theorem 5.10, there exists a map of graph of groups $M_1: (\Gamma_1, A_1) \rightarrow (\Gamma_2, A_2)$ such that $m_1(x_1) = x_2$ and F_2M_1 is conjugate to F_1 . Similarly, there exists a map of graph of groups $M_2: (\Gamma_2, A_2) \rightarrow (\Gamma_1, A_1)$ such that $m_2(x_2) = x_1$ and F_1M_2 is conjugate to F_2 . We claim that M_1 (likewise, M_2) is an isomorphism.

The composition M_2M_1 is such that $F_1(M_2M_1)$ is conjugate to F_1 since, by the exercise (1) above, $(F_1M_2)M_1$ is conjugate to F_2M_1 which is conjugate to F_1 ; and $(m_2m_1)(x_1) = x_1$. However the identity map Id on (Γ_1, A_1) also has this property. So, by the uniqueness part of Theorem 5.10, M_2M_1 and Id are conjugate maps; whence M_2M_1 is an isomorphism by the exercise (2) above. Therefore M_1 has a left inverse, namely $(M_2M_1)^{-1}M_2$.

Similarly, it follows that M_1M_2 is an isomorphism and we see that $M_2(M_1M_2)^{-1}$ is a right inverse of M_1 . Then, by a standard argument, $(M_2M_1)^{-1}M_2 = M_2(M_1M_2)^{-1} = M_1^{-1}$ and M_1 is an isomorphism. Hence F_1 and F_2 are equivalent covering maps. \square

5.5. Existence of coverings

There is a correspondence between connected coverings of a graph of groups (Δ, B) and subgroups of the fundamental group $\pi(\Delta, B, y)$, for any vertex y of Δ , which is completely analogous to the situation for coverings of topological spaces.

THEOREM 5.13 (Existence of coverings). *If $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a covering of graphs of groups and x is any vertex of Γ with $f(x) = y$, then the induced homomorphism*

$$F: \pi(\Gamma, A, x) \rightarrow \pi(\Delta, B, y)$$

is a monomorphism. Furthermore, if Δ is connected and H is any subgroup of $\pi(\Delta, B, y)$, then there exists a covering $F: (\Gamma, A) \rightarrow (\Delta, B)$ where Γ is connected,

and a vertex x of Γ such that $f(x) = y$ and $F\pi(\Gamma, A, x) = H$; and any two such coverings are equivalent.

PROOF. The first part is an immediate consequence of Theorem 5.7. A proof of the second part is given in Section 8.2. The third part is an immediate consequence of Theorem 5.12. □

CHAPTER 6

THE \sharp -CONSTRUCTION AND PROOF OF GENERAL LIFTING

First we introduce a convenient construction (that we call the “sharp-construction” or \sharp -construction) and see that it leads to an alternative way of describing conjugate maps.

6.1. The \sharp -construction

Given a graph of groups (Γ, A) , we form a graph of groups (Γ^\sharp, A) with an embedded copy of (Γ, A) as follows. For each $x \in V(\Gamma)$, let I_x be a non-empty index set. Then $V(\Gamma^\sharp)$ consists of the vertices of Γ together with a vertex x_i^\sharp for each $x \in V(\Gamma)$ and each $i \in I_x$. The edge set $E(\Gamma^\sharp)$ consists of the edges of Γ (with the same sources and targets) together with an edge denoted $[x, x_i^\sharp]$ from x to x_i^\sharp (and its inverse), for each $x \in V(\Gamma)$ and each $i \in I_x$. Denote the set of new vertices x_i^\sharp by $V^\sharp(\Gamma)$ and the set of new edges $[x, x_i^\sharp]^{\pm 1}$ by $E^\sharp(\Gamma)$ so that $V(\Gamma^\sharp) = V(\Gamma) \cup V^\sharp(\Gamma)$ and $E(\Gamma^\sharp) = E(\Gamma) \cup E^\sharp(\Gamma)$, disjoint unions. We require that the groups and monomorphisms associated to Γ^\sharp agree with those assigned by A on Γ , and that the trivial group be assigned to each new vertex x_i^\sharp and edge $[x, x_i^\sharp]$. (Since we are extending the definition of A in this trivial way, we use the notation (Γ^\sharp, A) and do not use A^\sharp .) We call (Γ^\sharp, A) a *trivial expansion* of (Γ, A) .

There is an obvious “projection” Ψ from $\mathcal{P}(\Gamma^\sharp, A)$ onto $\mathcal{P}(\Gamma, A)$ that fixes every path in $\mathcal{P}(\Gamma, A)$ and whose underlying “projection” $\psi: \Gamma^\sharp \rightarrow \Gamma$ fixes Γ and collapses

each new edge $[x, x_i^\#]$ to its source vertex x .

$$\begin{array}{ccc} \mathcal{P}(\Gamma^\#, A) & \xrightarrow{\Psi} & \mathcal{P}(\Gamma, A) \\ \downarrow & & \downarrow \\ \Gamma^\# & \xrightarrow{\psi} & \Gamma \end{array}$$

It should be noted that ψ is not a map of graphs (as it maps new edges to vertices), and thus Ψ is not a map of graphs of groups. However, it does map homotopic paths to homotopic paths, and thus induces a homomorphism $\Psi: \pi(\Gamma^\#, A) \rightarrow \pi(\Gamma, A)$. Furthermore, each section ϕ of the vertex mapping $\psi: V^\#(\Gamma) \rightarrow V(\Gamma)$ determines a splitting $\Phi: \pi(\Gamma, A) \rightarrow \pi(\Gamma^\#, A)$ of Ψ given by $\Phi([P]) = [x, \phi(x)]^{-1}[P][y, \phi(y)]$, where P is a path in (Γ, A) with source x and target y .

Now let $G: (\Sigma, C) \rightarrow (\Gamma, A)$ be a map of graphs of groups and let $(\Sigma^\#, C), (\Gamma^\#, A)$ be trivial expansions of $(\Sigma, C), (\Gamma, A)$. Then G can be extended to a map of graphs of groups $G^\#: (\Sigma^\#, C) \rightarrow (\Gamma^\#, A)$ in precisely the following ways: Extend the underlying map of graphs $g: \Sigma \rightarrow \Gamma$ to a map $g^\#: \Sigma^\# \rightarrow \Gamma^\#$ by choosing for each new edge $[x, x_i^\#] \in E^\#(\Sigma)$ a new edge $g^\#([x, x_i^\#]) \in E^\#(\Gamma)$ with source $g(x)$. This completely determines the map $g^\#$. Then to define $G^\#$ with underlying map of graphs $g^\#$ and extending G , it is enough to define edge labels for the new edges of $\Sigma^\#$. Since all new vertex and edge groups are trivial, for each $[x, x_i^\#] \in E^\#(\Sigma)$ we can choose an arbitrary element $\alpha^\#([x, x_i^\#]) \in A_{g(x)} = A_{g^\#(x)}$ as edge label. Of course, $\beta^\#([x, x_i^\#]) = 1$ as $A_{g^\#(x_i^\#)} = 1$.

The following simple observation says that each extension of G to a map $G^\#: (\Sigma^\#, C) \rightarrow (\Gamma^\#, A)$ corresponds to a conjugate of the map G .

LEMMA 6.1. *With the setup above, G is conjugate to a map G' if and only if there exists an extension of G to $G^\#: (\Sigma^\#, C) \rightarrow (\Gamma^\#, A)$ such that $G' = \Psi G^\# \Phi$, where $\Psi: \pi(\Gamma^\#, A) \rightarrow \pi(\Gamma, A)$ is the natural projection and Φ is a splitting of the natural projection $\pi(\Sigma^\#, C) \rightarrow \pi(\Sigma, C)$.*

PROOF. Let $G^\sharp: (\Sigma^\sharp, C) \rightarrow (\Gamma^\sharp, A)$ be an extension of G and let G' be the map of graphs of groups determined by the composition

$$\pi(\Sigma, C) \xrightarrow{\Phi} \pi(\Sigma^\sharp, C) \xrightarrow{G^\sharp} \pi(\Gamma^\sharp, A) \xrightarrow{\Psi} \pi(\Gamma, A).$$

For each $x \in V(\Sigma)$, let $x^\sharp = \phi(x)$ and write $a_x = \alpha^\sharp([x, x^\sharp])$. Then for each path P in (Σ, A) , $\Phi([P]) = [x, x^\sharp]^{-1}[P][y, y^\sharp]$ where $x = s(P)$ and $y = t(P)$. In particular, if $c \in C_x$ is a path of trivial length, then $\Phi(c) = [x, x^\sharp]^{-1}c[x, x^\sharp]$ and $G^\sharp(\Phi(c)) = [g(x), g^\sharp(x^\sharp)]^{-1}a_x^{-1}g_x(c)a_x[g(x), g^\sharp(x^\sharp)]$ and $\Psi(G^\sharp(\Phi(c))) = a_x^{-1}g_x(c)a_x$. It follows that the homomorphism $g'_x: C_x \rightarrow A_{g(x)}$ is given by $g'_x(c) = g_x(c)^{a_x}$.

Similarly, if $e \in E(\Sigma)$ say with $s(e) = x$ and $t(e) = y$, then $\Phi(e) = [x, x^\sharp]^{-1}e[y, y^\sharp]$ and

$$G^\sharp(\Phi(e)) = [g(x), g^\sharp(x^\sharp)]^{-1}a_x^{-1}\alpha(e)g(e)\beta(e)^{-1}a_y[g(y), g^\sharp(y^\sharp)]$$

and hence, $\Psi(G^\sharp(\Phi(e))) = a_x^{-1}\alpha(e)g(e)\beta(e)^{-1}a_y$. It follows that edge labels for G' are given by $\alpha'(e) = a_x^{-1}\alpha(e)$. It follows that G' is the conjugate of G by the a_x 's.

Conversely, assume that G' is the map obtained by conjugating G by elements $a_x \in A_{g(x)}$, $x \in V(\Sigma)$. As noted above, we can construct an extension G^\sharp of G such that $\alpha^\sharp([x, x^\sharp]) = a_x$ for each $x \in V(\Sigma)$; let $g^\sharp: \Sigma^\sharp \rightarrow \Gamma^\sharp$ be any extension of g and let $\alpha^\sharp([x, x_i^\sharp]) \in A_{g(x)}$ be arbitrarily chosen for all other new edges $[x, x_i^\sharp] \in E^\sharp(\Sigma)$. Then the first part of the proof shows that $\Psi G^\sharp \Phi = G'$, as required. \square

Note that the image of the homomorphism $\Phi: \pi(\Sigma, C) \rightarrow \pi(\Sigma^\sharp, C)$ consists only of homotopy classes of paths joining vertices of $V^\sharp(\Sigma)$ (i.e., new vertices). Denote the subcategory of all paths in $\mathcal{P}(\Sigma^\sharp, C)$ with sources and targets in $V^\sharp(\Sigma)$ by $\mathcal{P}^\sharp(\Sigma^\sharp, C)$. If $G^\sharp: (\Sigma^\sharp, C) \rightarrow (\Gamma^\sharp, A)$ is an extension of $G: (\Sigma, C) \rightarrow (\Gamma, A)$, then the induced mapping $G^\sharp: \mathcal{P}(\Sigma^\sharp, C) \rightarrow \mathcal{P}(\Gamma^\sharp, A)$ obviously restricts to a mapping of $\mathcal{P}^\sharp(\Sigma^\sharp, C)$ to $\mathcal{P}^\sharp(\Gamma^\sharp, A)$; we denote this restriction mapping by $\mathcal{P}^\sharp(G^\sharp)$.

The following is another useful way to characterize conjugate maps of graphs of groups.

LEMMA 6.2. *Two maps of graphs of groups $G_1, G_2: (\Sigma, C) \rightarrow (\Gamma, A)$ are conjugates of one another if and only if there exist extensions $G_1^\#, G_2^\#: (\Sigma^\#, C) \rightarrow (\Gamma^\#, A)$ of G_1, G_2 such that $\mathcal{P}^\#(G_1^\#) = \mathcal{P}^\#(G_2^\#)$.*

PROOF. Assume that G_2 is the conjugate of G_1 by elements $a_x \in A_{g(x)}$, $x \in V(\Sigma)$, where $g = g_1 = g_2$ is the underlying map of graphs of G_1 and G_2 . Let $G_1^\#: (\Sigma^\#, C) \rightarrow (\Gamma^\#, A)$ be any extension of G_1 . Then define an extension $G_2^\#: (\Sigma^\#, C) \rightarrow (\Gamma^\#, A)$ of G_2 , with the same underlying map of graphs $g^\#$, by letting $\alpha_2^\#([x, x_i^\#]) = a_x^{-1} \alpha_1^\#([x, x_i^\#])$ for each new edge $[x, x_i^\#] \in E^\#(\Sigma)$. To show that $\mathcal{P}^\#(G_1^\#) = \mathcal{P}^\#(G_2^\#)$, let P be a path in (Γ, A) and let $[x, x_i^\#], [y, y_j^\#] \in E^\#(\Sigma)$, where $x = s(P)$ and $y = t(P)$. Write $P^\# = [x, x_i^\#]^{-1} P [y, y_j^\#] \in \mathcal{P}^\#(\Sigma^\#, C)$. Then

$$\begin{aligned} G_2^\#(P^\#) &= g^\#([x, x_i^\#])^{-1} \alpha_2^\#([x, x_i^\#])^{-1} G_2(P) \alpha_2^\#([y, y_j^\#]) g^\#([y, y_j^\#]) \\ &= g^\#([x, x_i^\#])^{-1} \alpha_1^\#([x, x_i^\#])^{-1} a_x G_2(P) a_x^{-1} \alpha_1^\#([y, y_j^\#]) g^\#([y, y_j^\#]) \\ &= g^\#([x, x_i^\#])^{-1} \alpha_1^\#([x, x_i^\#])^{-1} G_1(P) \alpha_1^\#([y, y_j^\#]) g^\#([y, y_j^\#]) \\ &= G_1^\#(P^\#) \end{aligned}$$

as $G_2(P) = a_x^{-1} G_1(P) a_x$ by the proof of Lemma 5.9. Since $\mathcal{P}^\#(\Sigma^\#, C)$ is generated by paths of the form $P^\#$, it follows that $\mathcal{P}^\#(G_1^\#) = \mathcal{P}^\#(G_2^\#)$.

Conversely, assume that $G_1^\#, G_2^\#$ are extensions of G_1, G_2 such that $\mathcal{P}^\#(G_1^\#) = \mathcal{P}^\#(G_2^\#)$. For each $x \in V(\Sigma)$, choose a vertex $x^\# \in V^\#(\Sigma)$ adjacent to x . For $c \in C_x$, write $c^\# = [x, x^\#]^{-1} c [x, x^\#]$. Then $c^\# \in \mathcal{P}^\#(\Sigma^\#, C)$ and

$$G_i^\#(c^\#) = g_i^\#([x, x^\#])^{-1} \alpha_i^\#([x, x^\#])^{-1} (g_i)_x(c) \alpha_i^\#([x, x^\#]) g_i^\#([x, x^\#]).$$

Thus $\alpha_1^\sharp([x, x^\sharp])^{-1}(g_1)_x(c)\alpha_1^\sharp([x, x^\sharp]) = \alpha_2^\sharp([x, x^\sharp])^{-1}(g_2)_x(c)\alpha_2^\sharp([x, x^\sharp])$ so that $(g_2)_x(c) = (g_1)_x(c)^{a_x}$, where $a_x = \alpha_1^\sharp([x, x^\sharp])\alpha_2^\sharp([x, x^\sharp])^{-1}$.

Similarly, for $e \in E(\Sigma)$, consider the path $e^\sharp = [x, x^\sharp]^{-1}e[y, y^\sharp] \in \mathcal{P}^\sharp(\Gamma^\sharp, C)$ where $x = s(e)$ and $y = t(e)$. Then

$$G_i^\sharp(e^\sharp) = g_i^\sharp([x, x^\sharp])^{-1}\alpha_i^\sharp([x, x^\sharp])^{-1}\alpha_i(e)g(e)\beta_i(e)^{-1}\alpha_i^\sharp([y, y^\sharp])g_i^\sharp([y, y^\sharp]).$$

It follows that

$$\begin{aligned} \alpha_1^\sharp([x, x^\sharp])^{-1}\alpha_1(e)g(e)\beta_1(e)^{-1}\alpha_1^\sharp([y, y^\sharp]) \\ \equiv \alpha_2^\sharp([x, x^\sharp])^{-1}\alpha_2(e)g(e)\beta_2(e)^{-1}\alpha_2^\sharp([y, y^\sharp]) \end{aligned}$$

and hence, the edge labels on e can be adjusted so that

$$\alpha_1^\sharp([x, x^\sharp])^{-1}\alpha_1(e) = \alpha_2^\sharp([x, x^\sharp])^{-1}\alpha_2(e).$$

That is, we may assume that $\alpha_2(e) = a_x^{-1}\alpha_1(e)$ for all $e \in E(\Sigma)$. Thus G_1, G_2 are conjugate maps. \square

The final observation we wish to make here is that the \sharp -construction is compatible with covering maps, in the following sense.

LEMMA 6.3. *If $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a covering map of graphs of groups, and if (Δ^\sharp, B) is a trivial expansion of (Δ, B) , then there exists a trivial expansion (Γ^\sharp, A) of (Γ, A) and an extension of F which is a covering map $F^\sharp: (\Gamma^\sharp, A) \rightarrow (\Delta^\sharp, B)$.*

PROOF. We consider only the simplest case in which associated to each vertex x of Δ , there is only a single new vertex x^\sharp and thus also a single new edge $[x, x^\sharp]$ (and its inverse) in Δ^\sharp . The general case is similar.

Construct (Γ^\sharp, A) as follows. For each $x \in V(\Gamma)$, choose a transversal T_x for the right cosets of the subgroup $f_x(A_x)$ in $B_{f(x)}$; so $B_{f(x)} = f_x(A_x)T_x$ and every $b \in B_{f(x)}$

is uniquely of the form $b = f_x(a)b'$, where $a \in A_x$ and $b' \in T_x$. We assume, as we may, that $1 = 1_{B_{f(x)}} \in T_x$ for all $x \in V(\Gamma)$. Let (Γ^\sharp, A) be the associated trivial expansion given by the \sharp -construction. That is, for each $x \in V(\Gamma)$ and each $b \in T_x$, there is a new vertex x_b^\sharp and a new edge $[x, x_b^\sharp]$ (and its inverse) adjoined to Γ , to which A associates the trivial group.

It remains to define an extension $F^\sharp: (\Gamma^\sharp, A) \rightarrow (\Delta^\sharp, B)$ of F which is also a covering map. First note that there is only one extension $f^\sharp: \Gamma^\sharp \rightarrow \Delta^\sharp$ of the underlying map of graphs, namely the map satisfying $f^\sharp([x, x_b^\sharp]) = [f(x), f(x)^\sharp]$. Then let F^\sharp be the extension of F determined by the edge labels $\alpha^\sharp([x, x_b^\sharp]) = b$ for each new edge $[x, x_b^\sharp]$ in Γ^\sharp . We claim that F^\sharp is a local bijection.

Note that for $x \in V(\Gamma)$, it is enough to show that $F_x^\sharp: L_x(\Gamma^\sharp, A) \rightarrow L_{f(x)}(\Delta^\sharp, B)$ is a bijection on the new parts of these links, since we already know that F_x , and hence F_x^\sharp , is a bijection on the old part. The new elements of $L_x(\Gamma^\sharp, A)$ are of the form $(aA_{[x, x_b^\sharp]}, [x, x_b^\sharp]) = (a, [x, x_b^\sharp])$, for $a \in A_x$, $b \in T_x$, and $F_x^\sharp(a, [x, x_b^\sharp]) = (f_x(a)\alpha^\sharp([x, x_b^\sharp]), f^\sharp([x, x_b^\sharp])) = (f_x(a)b, [f(x), f(x)^\sharp])$.

Suppose $a_1, a_2 \in A_x$ and $b_1, b_2 \in T_x$ are such that $F_x^\sharp(a_1, [x, x_{b_1}^\sharp]) = F_x^\sharp(a_2, [x, x_{b_2}^\sharp])$. Then $f_x(a_1)b_1 = f_x(a_2)b_2$. However, T_x is a transversal for $f_x(A_x)$ in $B_{f(x)}$, so $b_1 = b_2$ and $f_x(a_1) = f_x(a_2)$. Moreover, f_x is injective since F is a covering map. It follows that $(a_1, [x, x_{b_1}^\sharp]) = (a_2, [x, x_{b_2}^\sharp])$ and hence, F_x^\sharp is injective.

Now let $(b, [f(x), f(x)^\sharp])$ be a new element of $L_{f(x)}(\Delta^\sharp, B)$. Since $b \in B_{f(x)}$, it is (uniquely) a product $b = f_x(a)b'$, where $a \in f_x(A_x)$ and $b' \in T_x$. Then $(a, [x, x_{b'}^\sharp]) \in L_x(\Gamma^\sharp, A)$ and $F_x^\sharp(a, [x, x_{b'}^\sharp]) = (b, [f(x), f(x)^\sharp])$. Hence F_x^\sharp is surjective.

We have shown that F^\sharp is locally bijective at each old vertex $x \in V(\Gamma)$. At each new vertex $x^\sharp \in V(\Gamma^\sharp)$, $L_{x^\sharp}(\Gamma^\sharp, A)$ consists of a single element and clearly F^\sharp is also locally bijective at these vertices. \square

6.2. Proof of general lifting

We turn now to the proof of Theorem 5.10. The proof is long and consists of lots of delicate verifications. In order to make it more manageable, we have broken it down into a number of small steps.

6.2.1. Necessity of the lifting criterion. Suppose that \tilde{G} exists. Then by Lemma 5.9, there exists $b_0 \in B_{f(u)}$ such that $[G\pi(\Sigma, C, v)]^{b_0} = (F\tilde{G})\pi(\Sigma, C, v) \subseteq F\pi(\Gamma, A, u)$; and we see that the condition is necessary for the existence of \tilde{G} . For the remainder of the proof, assume conversely that this condition holds. By replacing G by a conjugate map, we may assume that $b_0 = 1$ and hence, we will assume that

$$G\pi(\Sigma, C, v) \subseteq F\pi(\Gamma, A, u).$$

6.2.2. Form a trivial expansion. Let (Σ^\sharp, C) , (Δ^\sharp, B) be the trivial expansions of (Σ, C) , (Δ, B) with a single new vertex x^\sharp adjacent to each old vertex x . Apply Lemma 6.3 to get a covering map $F^\sharp: (\Gamma^\sharp, A) \rightarrow (\Delta^\sharp, B)$ which is an extension of F . Let $g^\sharp: \Sigma^\sharp \rightarrow \Delta^\sharp$ be the unique extension of the underlying map of graphs $g: \Sigma \rightarrow \Delta$ such that $g^\sharp([x, x^\sharp]) = [g(x), g(x)^\sharp]$ for each $x \in V(\Sigma)$. Then let $G^\sharp: (\Sigma^\sharp, C) \rightarrow (\Delta^\sharp, B)$ be the the trivial extension of G with underlying map of graphs g^\sharp , i.e., the extension with all new edge labels trivial. Note that $g^\sharp(v^\sharp) = g(v)^\sharp = f(u)^\sharp = f^\sharp(u_1^\sharp)$, where the subscript $1 = 1_{B_{f(u)}} \in T_{f(u)}$ by our convention on the transversal. Also note that every loop in (Σ^\sharp, C) at v^\sharp is homotopic to a loop of the form $[v, v^\sharp]^{-1}P[v, v^\sharp]$, where P is a loop in (Σ, C) based at v , and $G^\sharp([v, v^\sharp]^{-1}P[v, v^\sharp]) = [g(v), g(v)^\sharp]^{-1}G(P)[g(v), g(v)^\sharp]$. Similarly, every loop in (Γ^\sharp, A) at u_1^\sharp is homotopic to a loop of the form $[u, u_1^\sharp]^{-1}Q[u, u_1^\sharp]$, where Q is a loop in (Γ, A) based at u , and $F^\sharp([u, u_1^\sharp]^{-1}Q[u, u_1^\sharp]) = [f(u), f(u)^\sharp]^{-1}F(Q)[f(u), f(u)^\sharp]$. It follows that

$$G\pi(\Sigma, C, v) \subseteq F\pi(\Gamma, A, u) \iff G^\sharp\pi(\Sigma^\sharp, C, v^\sharp) \subseteq F^\sharp\pi(\Gamma^\sharp, A, u_1^\sharp).$$

6.2.3. If P and Q are two paths in $\mathcal{P}^\sharp(\Sigma^\sharp, C)$ from the base vertex v^\sharp to a vertex x^\sharp , and if $\widetilde{G^\sharp(P)}$, $\widetilde{G^\sharp(Q)}$ are the unique lifts of $G^\sharp(P)$, $G^\sharp(Q)$ starting at u_1^\sharp , then $\widetilde{G^\sharp(P)}$ and $\widetilde{G^\sharp(Q)}$ have the same target vertex. Thus, denoting this common target vertex by $\widetilde{g^\sharp}(x^\sharp)$, we get a well-defined mapping $\widetilde{g^\sharp}: V^\sharp(\Sigma) \rightarrow V^\sharp(\Gamma)$ of new vertices such that $\widetilde{g^\sharp}(v^\sharp) = u_1^\sharp$ and $f^\sharp \widetilde{g^\sharp} = g^\sharp$.

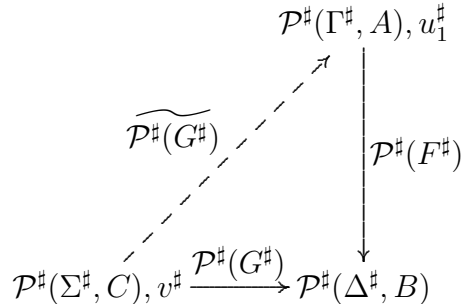
Since $\mathcal{P}^\sharp(F^\sharp): \mathcal{P}^\sharp(\Gamma^\sharp, A) \rightarrow \mathcal{P}^\sharp(\Delta^\sharp, B)$ has the unique path-lifting property and homotopy lifting property by Theorems 5.5 and 5.7, we can proceed analogously as in the topological theory of covering spaces. Let y^\sharp be the target of the path $\widetilde{G^\sharp(P)}$ and let \widetilde{R} be the unique lift of $G^\sharp(Q)^{-1}$ with source vertex y^\sharp . Then $\widetilde{G^\sharp(P)}\widetilde{R}$ is a path in (Γ^\sharp, A) and

$$F^\sharp(\widetilde{G^\sharp(P)}\widetilde{R}) = G^\sharp(P)G^\sharp(Q)^{-1} = G^\sharp(PQ^{-1}).$$

By our assumption that $G^\sharp \pi(\Sigma^\sharp, C, v^\sharp) \subseteq F^\sharp \pi(\Gamma^\sharp, A, u_1^\sharp)$, there exists a loop \widetilde{S} in (Γ^\sharp, A) based at u_1^\sharp such that $F^\sharp(\widetilde{S}) \simeq G^\sharp(PQ^{-1})$. By the homotopy lifting property, $\widetilde{G^\sharp(P)}\widetilde{R} \simeq \widetilde{S}$. Since \widetilde{S} is a loop, likewise $\widetilde{G^\sharp(P)}\widetilde{R}$ must be a loop based at u_1^\sharp . We see that \widetilde{R}^{-1} is a path in (Γ^\sharp, A) starting at u_1^\sharp and $F^\sharp(\widetilde{R}^{-1}) = F^\sharp(\widetilde{R})^{-1} = G^\sharp(Q)$. So by the unique path-lifting property, $\widetilde{R}^{-1} = \widetilde{G^\sharp(Q)}$. Therefore the target of $\widetilde{G^\sharp(Q)}$ is equal to $s(\widetilde{R}) = y^\sharp$, i.e., the same as the target of $\widetilde{G^\sharp(P)}$.

The mapping $x^\sharp \mapsto y^\sharp$ defined in the previous paragraph thus gives a well-defined map $\widetilde{g^\sharp}: V^\sharp(\Sigma) \rightarrow V^\sharp(\Gamma)$ on new vertices.

6.2.4. General lifting for $\mathcal{P}^\sharp(F^\sharp): \mathcal{P}^\sharp(\Gamma^\sharp, A) \rightarrow \mathcal{P}^\sharp(\Delta^\sharp, B)$.



There exists a unique homomorphism $\widetilde{\mathcal{P}^\#(G^\#)}: \mathcal{P}^\#(\Sigma^\#, C) \rightarrow \mathcal{P}^\#(\Gamma^\#, A)$ whose underlying map of vertices takes $v^\#$ to $u_1^\#$ and $\mathcal{P}^\#(F^\#)\widetilde{\mathcal{P}^\#(G^\#)} = \mathcal{P}^\#(G^\#)$. Furthermore, $\widetilde{\mathcal{P}^\#(G^\#)}$ preserves homotopy classes of paths and thus induces a homomorphism $\widetilde{\mathcal{P}^\#(G^\#)}: \pi^\#(\Sigma^\#, C) \rightarrow \pi^\#(\Gamma^\#, A)$, where $\pi^\#(\Sigma^\#, C)$ is the sub-groupoid of $\pi(\Sigma^\#, C)$ consisting of all homotopy classes of paths with sources and targets in $V^\#(\Sigma)$ and similarly for $\pi^\#(\Gamma^\#, A)$.

To construct this lifted homomorphism (i.e., functor), first observe that the mapping $x^\# \mapsto y^\#$ defined in Step 3 gives a well-defined map $\widetilde{g}^\#: V^\#(\Sigma) \rightarrow V^\#(\Gamma)$ on new vertices (the map of objects of the small categories). Note that $f^\#(\widetilde{g}^\#(x^\#)) = g^\#(x^\#)$ for all $x^\# \in V^\#(\Sigma)$.

Then for each path $P \in \mathcal{P}^\#(\Sigma^\#, C)$ from $x_1^\#$ to $x_2^\#$, define $\widetilde{\mathcal{P}^\#(G^\#)}(P) = \widetilde{G^\#(P)}$, the unique lift of $G^\#(P)$ in $\mathcal{P}^\#(\Gamma^\#, A)$ with source vertex $\widetilde{g}^\#(x_1^\#)$ (given by Theorem 5.5). To see that the target of $\widetilde{\mathcal{P}^\#(G^\#)}(P)$ is $\widetilde{g}^\#(x_2^\#)$, note that Σ being connected implies that $\Sigma^\#$ is also connected. So there is a path Q in $(\Sigma^\#, C)$ from $v^\#$ to $x_1^\#$. By Step 3, the unique lift $\widetilde{G^\#(Q)}$ of $G^\#(Q)$ with source $u_1^\# = \widetilde{g}^\#(v^\#)$ has target $\widetilde{g}^\#(x_1^\#)$. Hence $\widetilde{G^\#(Q)}\widetilde{G^\#(P)}$ is the unique lift of $G^\#(QP)$ starting at $u_1^\#$; so its target, which is also the target of $\widetilde{\mathcal{P}^\#(G^\#)}(P)$, is $\widetilde{g}^\#(x_2^\#)$ by Step 3. Therefore $\widetilde{\mathcal{P}^\#(G^\#)}(P)$ is a path in $(\Gamma^\#, A)$ from $\widetilde{g}^\#(x_1^\#)$ to $\widetilde{g}^\#(x_2^\#)$ (giving the required mapping of morphisms of the small categories).

To see that $\widetilde{\mathcal{P}^\#(G^\#)}$ is a homomorphism, let P be a path from $x_1^\#$ to $x_2^\#$ and let Q be a path from $x_2^\#$ to $x_3^\#$. Then $\widetilde{\mathcal{P}^\#(G^\#)}(P) \cdot \widetilde{\mathcal{P}^\#(G^\#)}(Q)$ and $\widetilde{\mathcal{P}^\#(G^\#)}(PQ)$ are both lifts of $G^\#(PQ)$ starting at $\widetilde{g}^\#(x_1^\#)$. Thus, by Lemma 5.3, $\widetilde{\mathcal{P}^\#(G^\#)}(P) \cdot \widetilde{\mathcal{P}^\#(G^\#)}(Q) = \widetilde{\mathcal{P}^\#(G^\#)}(PQ)$ and $\widetilde{\mathcal{P}^\#(G^\#)}$ is a homomorphism (i.e., functor of small categories).

The uniqueness of $\widetilde{\mathcal{P}^\#(G^\#)}$ also follows easily from Lemma 5.3. And our final claim that $\widetilde{\mathcal{P}^\#(G^\#)}$ preserves homotopy classes of paths follows immediately from Theorem 5.7.

6.2.5. Existence of \tilde{G} . Consider the composition of homomorphisms

$$\pi(\Sigma, C) \xrightarrow{\Phi} \pi^\sharp(\Sigma^\sharp, C) \xrightarrow{\widetilde{\mathcal{P}^\sharp(G^\sharp)}} \pi^\sharp(\Gamma^\sharp, A) \xrightarrow{\Psi} \pi(\Gamma, A)$$

where Ψ is the natural projection, Φ is the canonical splitting map of the natural projection $\pi(\Sigma^\sharp, C) \rightarrow \pi(\Sigma, C)$, and $\widetilde{\mathcal{P}^\sharp(G^\sharp)}$ is the homomorphism constructed in Step 4. Note that this composite homomorphism maps $V(\Sigma, C)$, $E(\Sigma, C)$ to $V(\Gamma, A)$, $E(\Gamma, A)$ and thus is a map of graphs of groups; denote it by $\tilde{G}: (\Sigma, C) \rightarrow (\Gamma, A)$, $\tilde{G} = \Psi \widetilde{\mathcal{P}^\sharp(G^\sharp)} \Phi$. We claim that \tilde{G} is a lift of G (up to conjugation) with $\tilde{g}(v) = u$.

First note that the underlying map of graphs is given on vertices by $\psi \tilde{g}^\sharp \phi: V(\Sigma) \rightarrow V(\Gamma)$. So $\tilde{g}(v) = \psi(\tilde{g}^\sharp(\phi(v))) = \psi(\tilde{g}^\sharp(v^\sharp)) = \psi(u_1^\sharp) = u$. Note also that the underlying map of graphs $\tilde{g}: \Sigma \rightarrow \Gamma$ has a unique extension $\tilde{g}^\sharp: \Sigma^\sharp \rightarrow \Gamma^\sharp$ such that $\tilde{g}^\sharp(x^\sharp) = \tilde{g}^\sharp(x^\sharp)$ for each new vertex $x^\sharp \in V^\sharp(\Sigma)$, where $\tilde{g}^\sharp: V^\sharp(\Sigma) \rightarrow V^\sharp(\Gamma)$ is the underlying (object) mapping of $\widetilde{\mathcal{P}^\sharp(G^\sharp)}$ constructed in Step 3.

Next we let $(\tilde{G})^\sharp: (\Sigma^\sharp, C) \rightarrow (\Gamma^\sharp, A)$ be the trivial extension of G with underlying map of graphs \tilde{g}^\sharp ; and verify that $(\tilde{G})^\sharp \Phi = \widetilde{\mathcal{P}^\sharp(G^\sharp)} \Phi$. To see this, let P be a path in (Σ, C) . Then simply note that $\Phi([P]) = [P^\sharp]$, where $P^\sharp = [x, x^\sharp]^{-1} P[y, y^\sharp]$ for $x = s(P)$ and $y = t(P)$, and that

$$\begin{aligned} (\tilde{G})^\sharp(P^\sharp) &= \tilde{g}^\sharp([x, x^\sharp]^{-1} \tilde{G}(P) \tilde{g}^\sharp([y, y^\sharp])) \\ &\simeq \tilde{g}^\sharp([x, x^\sharp]^{-1} \Psi \left(\widetilde{\mathcal{P}^\sharp(G^\sharp)}(P^\sharp) \right) \tilde{g}^\sharp([y, y^\sharp])) \\ &= \widetilde{\mathcal{P}^\sharp(G^\sharp)}(P^\sharp). \end{aligned}$$

From our previous observation, Step 4, and the fact that G^\sharp is a trivial extension of G , it follows that

$$\Psi \left(F^\sharp(\tilde{G})^\sharp \right) \Phi = \Psi F^\sharp \widetilde{\mathcal{P}^\sharp(G^\sharp)} \Phi = \Psi \mathcal{P}^\sharp(G^\sharp) \Phi = G.$$

However, $F^\sharp(\tilde{G})^\sharp$ is an extension of $F\tilde{G}$; so $F\tilde{G}$ and G are conjugate maps by Lemma 6.1.

Suppose $H: (\Sigma, C) \rightarrow (\Gamma, A)$ is a map of graphs of groups and let $G = FH$. Then for each extension $G^\sharp: (\Sigma^\sharp, C) \rightarrow (\Delta^\sharp, B)$ of G , there exists an extension $H^\sharp: (\Sigma^\sharp, C) \rightarrow (\Gamma^\sharp, A)$ of H such that $F^\sharp H^\sharp = G^\sharp$.

$$\begin{array}{ccc}
 & & (\Gamma^\sharp, A) \\
 & \nearrow H^\sharp & \downarrow F^\sharp \\
 (\Sigma^\sharp, C) & \xrightarrow{G^\sharp} & (\Delta^\sharp, B)
 \end{array}$$

Recall that to define the extension H^\sharp it is enough to specify an extension $h^\sharp: \Sigma^\sharp \rightarrow \Gamma^\sharp$ of the underlying map of graphs h and to specify an element $\alpha_H^\sharp([x, x^\sharp]) \in A_{h(x)}$ for each $x \in V(\Sigma)$. We proceed as follows: for each $x \in V(\Sigma)$, write the edge label $\alpha_G^\sharp([x, x^\sharp]) \in B_{g(x)}$ as its unique product $\alpha_G^\sharp([x, x^\sharp]) = f_{h(x)}(a)b$, where $a \in A_{h(x)}$ and $b \in T_{h(x)}$. Then define

$$h^\sharp(x^\sharp) = h(x)_b^\sharp, \quad h^\sharp([x, x^\sharp]) = [h(x), h(x)_b^\sharp], \quad \text{and} \quad \alpha_H^\sharp([x, x^\sharp]) = a.$$

Then we see that $f^\sharp h^\sharp = g^\sharp$, the unique extension of the underlying map of graphs g of G , and that for each $x \in V(\Sigma)$,

$$\begin{aligned}
 F^\sharp(H^\sharp([x, x^\sharp])) &= F^\sharp(a[h(x), h(x)_b^\sharp]) \\
 &= f_{h(x)}(a)\alpha_F^\sharp([h(x), h(x)_b^\sharp])[g(x), g(x)^\sharp] \\
 &= f_{h(x)}(a)b[g(x), g(x)^\sharp] \\
 &= \alpha_G^\sharp([x, x^\sharp])[g(x), g(x)^\sharp] = G^\sharp([x, x^\sharp])
 \end{aligned}$$

Both being extensions of G , it follows that $F^\sharp H^\sharp = G^\sharp$, as required.

6.2.6. Uniqueness of \tilde{G} . Let $\tilde{G}_1, \tilde{G}_2: (\Sigma, C) \rightarrow (\Gamma, A)$ be maps of graphs of groups and let $G_i = F\tilde{G}_i$, $i = 1, 2$. If G_1, G_2 are conjugate maps, then \tilde{G}_1, \tilde{G}_2 are also conjugate maps.

$$\begin{array}{ccc}
 & & (\Gamma, A) \\
 & \nearrow^{\tilde{G}_1, \tilde{G}_2} & \downarrow F \\
 (\Sigma, C) & \xrightarrow{G_1, G_2} & (\Delta, B)
 \end{array}$$

First use Lemma 6.2 to get extensions G_1^\sharp, G_2^\sharp of G_1, G_2 such that $\mathcal{P}^\sharp(G_1^\sharp) = \mathcal{P}^\sharp(G_2^\sharp)$. Then use Step 5 to get extensions $\tilde{G}_1^\sharp, \tilde{G}_2^\sharp$ of \tilde{G}_1, \tilde{G}_2 such that $F^\sharp \tilde{G}_i^\sharp = G_i^\sharp$ for $i = 1, 2$. By the uniqueness part of Step 4, it must be that $\mathcal{P}^\sharp(\tilde{G}_1^\sharp) = \mathcal{P}^\sharp(\tilde{G}_2^\sharp)$. Thus

$$\Psi \tilde{G}_1^\sharp \Phi = \Psi \left(\mathcal{P}^\sharp(\tilde{G}_1^\sharp) \right) \Phi = \Psi \left(\mathcal{P}^\sharp(\tilde{G}_2^\sharp) \right) \Phi = \Psi \tilde{G}_2^\sharp \Phi.$$

However, by Lemma 6.1, $\Psi \tilde{G}_i^\sharp \Phi$ is conjugate to \tilde{G}_i and hence \tilde{G}_1 and \tilde{G}_2 are conjugate to one another. This completes the proof of Theorem 5.10.

We list some consequences of the theorem and its proof that will be useful later. In what follows, for $i = 1, 2$, assume that $F_i: (\Gamma_i, A_i) \rightarrow (\Delta, B)$ is a covering map and let $F_i^\sharp: (\Gamma_i^\sharp, A_i) \rightarrow (\Delta^\sharp, B)$ be a covering map which is an extension of F_i to trivial expansions, where Δ^\sharp has only a single new vertex x^\sharp adjacent to each old vertex $x \in V(\Delta)$.

COROLLARY 6.4. *Suppose Γ_1, Γ_2 are connected and let v_1^\sharp, v_2^\sharp be new vertices of $\Gamma_1^\sharp, \Gamma_2^\sharp$ such that $f_1^\sharp(v_1^\sharp) = f_2^\sharp(v_2^\sharp)$. Then there exists a homomorphism $H: \mathcal{P}^\sharp(\Gamma_1^\sharp, A_1) \rightarrow \mathcal{P}^\sharp(\Gamma_2^\sharp, A_2)$ such that $h(v_1^\sharp) = v_2^\sharp$ and $F_2^\sharp H = F_1^\sharp$ if and only if $F_1^\sharp \pi(\Gamma_1^\sharp, A_1, v_1^\sharp) \subseteq F_2^\sharp \pi(\Gamma_2^\sharp, A_2, v_2^\sharp)$; and if H exists, it is unique.*

$$\begin{array}{ccc}
 \mathcal{P}^\#(\Gamma_1^\#, A_1), v_1^\# & \overset{H}{\dashrightarrow} & \mathcal{P}^\#(\Gamma_2^\#, A_2), v_2^\# \\
 \downarrow F_1^\# & & \downarrow F_2^\# \\
 & & (\Delta^\#, B)
 \end{array}$$

COROLLARY 6.5. *If $G: (\Gamma_1, A_1) \rightarrow (\Gamma_2, A_2)$ is a map of graphs of groups such that $F_2G = F_1$, then there exists an extension $G^\#: (\Gamma_1^\#, A_1) \rightarrow (\Gamma_2^\#, A_2)$ of G such that $F_2^\#G^\# = F_1^\#$.*

CHAPTER 7

COVERING TRANSFORMATIONS AND REGULAR COVERINGS

In order to obtain comparable results to that in the theory of topological covering spaces, we incorporate the \sharp -construction in our definition of covering transformations. As before, given a covering map $F: (\Gamma, A) \rightarrow (\Delta, B)$ of graphs of groups, let (Δ^\sharp, B) be the trivial expansion with a single new vertex x^\sharp adjacent to each vertex $x \in V(\Delta)$. Then use Lemma 6.3, to construct a trivial expansion (Γ^\sharp, A) and covering map $F^\sharp: (\Gamma^\sharp, A) \rightarrow (\Delta^\sharp, B)$ extending F .

DEFINITION 7.1. A *covering transformation* of $F: (\Gamma, A) \rightarrow (\Delta, B)$ is an automorphism $M: \mathcal{P}^\sharp(\Gamma^\sharp, A) \rightarrow \mathcal{P}^\sharp(\Gamma^\sharp, A)$ such that $F^\sharp M = F^\sharp$.

$$\begin{array}{ccc}
 \mathcal{P}^\sharp(\Gamma^\sharp, A) & \overset{M}{\dashrightarrow} & \mathcal{P}^\sharp(\Gamma^\sharp, A) \\
 \searrow^{F^\sharp} & & \swarrow_{F^\sharp} \\
 & & (\Delta^\sharp, B)
 \end{array}$$

THEOREM 7.2. *Suppose Γ is connected and let $v_1^\sharp, v_2^\sharp \in f^{\sharp-1}(x^\sharp)$, where x^\sharp is a new vertex in $V^\sharp(\Delta)$. Then there exists a covering transformation $M: \mathcal{P}^\sharp(\Gamma^\sharp, A) \rightarrow \mathcal{P}^\sharp(\Gamma^\sharp, A)$ such that $m(v_1^\sharp) = v_2^\sharp$ if and only if $F^\sharp \pi(\Gamma^\sharp, A, v_1^\sharp) = F^\sharp \pi(\Gamma^\sharp, A, v_2^\sharp)$; and if M exists, it is unique.*

PROOF. This follows immediately from Corollary 6.4 by a standard argument; see, for example, [4, Corollary 5.6.5]. \square

Using the unique path lifting and homotopy lifting properties, we can make the fiber $f^{\sharp-1}(x^{\sharp})$ over any new vertex $x^{\sharp} \in V^{\sharp}(\Delta)$ into a right $\pi(\Delta^{\sharp}, B, x^{\sharp})$ -space in the following standard way. For each $v_1^{\sharp} \in f^{\sharp-1}(x^{\sharp})$ and $[P] \in \pi(\Delta^{\sharp}, B, x^{\sharp})$, let \tilde{P} be the unique lift of P with source v_1^{\sharp} given by Theorem 5.5. Since the target of P is also equal to x^{\sharp} , the target of \tilde{P} is some vertex $v_2^{\sharp} \in f^{\sharp-1}(x^{\sharp})$. Furthermore, the vertex v_2^{\sharp} does not depend on the representative path taken in the homotopy class $[P]$ by Theorem 5.7. Hence, it makes sense to define $v_1^{\sharp} \cdot [P] = v_2^{\sharp}$. It is easily verified that this defines a right group action of $\pi(\Delta^{\sharp}, B, x^{\sharp})$ on the fiber $f^{\sharp-1}(x^{\sharp})$; and by the same arguments as for topological covering spaces we see that the $\pi(\Delta^{\sharp}, B, x^{\sharp})$ -space $f^{\sharp-1}(x^{\sharp})$ satisfies the following properties:

- (1) The stabilizer of $v^{\sharp} \in f^{\sharp-1}(x^{\sharp})$ is the subgroup $F^{\sharp} \pi(\Gamma^{\sharp}, A, v^{\sharp})$.
- (2) Covering transformations act as $\pi(\Delta^{\sharp}, B, x^{\sharp})$ -space automorphisms on $f^{\sharp-1}(x^{\sharp})$.
- (3) If Γ is connected, then $\pi(\Delta^{\sharp}, B, x^{\sharp})$ acts transitively on $f^{\sharp-1}(x^{\sharp})$.

We define regular coverings of graphs of groups in the same way as for topological covering spaces: A covering $F: (\Gamma, A) \rightarrow (\Delta, B)$ of graphs of groups, where Δ and Γ are connected graphs, is called *regular* if $F \pi(\Gamma, A, v)$ is a normal subgroup of $\pi(\Delta, B, f(v))$. By the same argument as for coverings of path connected topological spaces, this condition is independent of the choice of $v \in V(\Gamma)$.

LEMMA 7.3. *If Δ and Γ are connected, then the following conditions are equivalent:*

- (1) $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a regular covering;
- (2) $F^{\sharp}: (\Gamma^{\sharp}, A) \rightarrow (\Delta^{\sharp}, B)$ is a regular covering;

(3) *the group of covering transformations acts transitively on the fiber $f^{\sharp-1}(x^\sharp)$ for each $x^\sharp \in V^\sharp(\Delta)$.*

PROOF. (1) \iff (2): Let $v \in V(\Gamma)$ and note that every loop P^\sharp in (Δ^\sharp, B) based at $f(v)$ is homotopic to a loop P in (Δ, B) based at $f(v)$. Thus

$$\pi(\Delta, B, f(v)) \cong \pi(\Delta^\sharp, B, f(v)).$$

Under this isomorphism, $F(\pi(\Gamma, A, v))$ corresponds to $F^\sharp(\pi(\Gamma^\sharp, A, v))$. Hence,

$$F(\pi(\Gamma, A, v)) \triangleleft \pi(\Delta, B, f(v)) \iff F^\sharp(\pi(\Gamma^\sharp, A, v)) \triangleleft \pi(\Delta^\sharp, B, f(v)).$$

Therefore, (1) \iff (2).

(2) \iff (3): Let $v^\sharp \in f^{\sharp-1}(x^\sharp)$ and $[P] \in \pi(\Delta^\sharp, B, x^\sharp)$. Then it is easy to see that $F^\sharp \pi(\Gamma^\sharp, A, v^\sharp \cdot [P]) = [F^\sharp \pi(\Gamma^\sharp, A, v^\sharp)]^{[P]}$. So, by Theorem 7.2, there exists a covering transformation $M: \mathcal{P}^\sharp(\Gamma^\sharp, A) \rightarrow \mathcal{P}^\sharp(\Gamma^\sharp, A)$ such that $m(v^\sharp) = v^\sharp \cdot [P]$ if and only if

$$[F^\sharp \pi(\Gamma^\sharp, A, v^\sharp)]^{[P]} = F^\sharp \pi(\Gamma^\sharp, A, v^\sharp).$$

The equivalence of (2) and (3) now follows since $\pi(\Delta^\sharp, B, x^\sharp)$ acts transitively on $f^{\sharp-1}(x^\sharp)$, as Γ is connected. \square

THEOREM 7.4. *If $F: (\Gamma, A) \rightarrow (\Delta, B)$ is a regular covering of connected graphs of groups, then the group of covering transformations is isomorphic to the quotient group $\pi(\Delta, B, f(v))/F \pi(\Gamma, A, v)$ for any vertex v of Γ .*

PROOF. Let $x = f(v)$. For each $[P] \in \pi(\Delta^\sharp, B, f(v)^\sharp)$, observe that by Lemma 7.3 (3), there is a covering transformation $M_{[P]}$ such that $m_{[P]}(v^\sharp) = v^\sharp \cdot [P]$; and $M_{[P]}$ is

unique by Theorem 7.2. If $[P], [Q] \in \pi(\Delta^\#, B, x^\#)$, then by properties listed above

$$\begin{aligned} (m_{[P]}m_{[Q]})(v^\#) &= m_{[P]}(v^\# \cdot [Q]) = m_{[P]}(v^\#) \cdot [Q] \\ &= (v^\# \cdot [P]) \cdot [Q] = v^\# \cdot ([P][Q]) = m_{[P][Q]}(v^\#). \end{aligned}$$

Thus $M_{[P]}M_{[Q]} = M_{[P][Q]}$ by the uniqueness part of Theorem 7.2. That is, $[P] \mapsto M_{[P]}$ is a homomorphism from $\pi(\Delta^\#, B, x^\#)$ to the group of covering transformations. This homomorphism is onto since $\pi(\Delta^\#, B, x^\#)$ acts transitively on $f^{\#-1}(x^\#)$ and its kernel is the stabilizer subgroup of $v^\#$ which is $F^\# \pi(\Gamma^\#, A, v^\#)$. Hence the group of covering transformations is isomorphic to $\pi(\Delta^\#, B, x^\#)/F^\# \pi(\Gamma^\#, A, v^\#)$. This latter quotient group is isomorphic to $\pi(\Delta, B, x)/F \pi(\Gamma, A, v)$ by a standard translation of base-point isomorphism, and the result is established. \square

CHAPTER 8

A GENERALIZED BASS-SERRE THEORY

8.1. Covering maps associated to group actions

We begin by giving a method of constructing a graph of groups and covering map associated to a given group action (by graph automorphisms) of a group G on a graph Γ (acting on the left). We always assume, without further mention, that all group actions are without edge inversions; that is, for all $g \in G$ and all $e \in E(\Gamma)$, $g \cdot e \neq \bar{e}$. Then there is a unique way of making the quotient Γ/G into a graph such that the quotient map $g: \Gamma \rightarrow \Gamma/G$ is a map of graphs. It is often more convenient for our purposes to also make G act on the right of Γ using the equivalent right action defined by

$$x \cdot g = g^{-1} \cdot x$$

for all x in Γ and $g \in G$. With this convention, we will sometimes write group elements on the left and other times on the right, whichever is more suitable at the time.

Choose a section σ of the orbit map $g: \Gamma \rightarrow \Gamma/G$ with the property that for all $e \in E(\Gamma/G)$,

$$s(\sigma(e)) = \sigma(s(e)).$$

(In many treatments of Bass-Serre theory, the choice of section is required to correspond to a connected fundamental domain, which is done by first lifting a maximal tree. However this is unnecessary, so we do not go to that length.) Notice that for each $e \in E(\Gamma/G)$, the edges $\overline{\sigma(e)}$ and $\sigma(\bar{e})$ are in the same G -orbit. Thus, we can

choose an element $\lambda(e) \in G$ such that

$$\overline{\sigma(e)} \cdot \lambda(e) = \sigma(\bar{e}).$$

We require, as we may, that $\lambda(\bar{e}) = \lambda(e)^{-1}$.

Using $\sigma: \Gamma/G \rightarrow \Gamma$ and $\lambda: E(\Gamma/G) \rightarrow G$, we construct a graph of groups $(\Gamma/G, G_\sigma)$ with underlying graph Γ/G as follows. For each y in Γ/G , let

$$(G_\sigma)_y = G_{\sigma(y)} \text{ (the stabilizer subgroup).}$$

Since G acts without inversions, for each $e \in E(\Gamma/G)$, we see that $(G_\sigma)_e = G_{\sigma(e)} \subseteq G_{\sigma(s(e))} = (G_\sigma)_{s(e)}$; and we define the monomorphism $(G_\sigma)_e \rightarrow (G_\sigma)_{t(e)}$ via $a \mapsto a^e$ where

$$a^e = a^{\lambda(e)} = \lambda(e)^{-1} a \lambda(e).$$

Note that

$$G_{\sigma(e)}^e = G_{\sigma(e)}^{\lambda(e)} = G_{\sigma(e) \cdot \lambda(e)} = G_{\overline{\sigma(e)} \cdot \lambda(e)} = G_{\sigma(\bar{e})} \subseteq G_{\sigma(s(\bar{e}))} = G_{\sigma(t(e))}$$

and hence the image of $a \mapsto a^e$ is indeed in $G_{\sigma(t(e))}$.

Now let H be a subgroup of G . Then the orbit map $g: \Gamma \rightarrow \Gamma/G$ factors through the orbit map $h: \Gamma \rightarrow \Gamma/H$ via a unique map of graphs $f: \Gamma/H \rightarrow \Gamma/G$.

$$\begin{array}{ccc} & \Gamma & \\ & \swarrow h & \searrow g \\ \Gamma/H & & \Gamma/G \\ & \xleftarrow{f} & \end{array}$$

Restricting to the action of H on Γ , choose a section τ of the orbit map $h: \Gamma \rightarrow \Gamma/H$ and elements $\mu(e) \in H$, for all $e \in E(\Gamma/H)$, such that

$$\mu(\bar{e}) = \mu(e)^{-1} \quad \text{and} \quad \overline{\tau(e)} \cdot \mu(e) = \tau(\bar{e}).$$

Then construct a graph of groups $(\Gamma/H, H_\tau)$ from this data, as above.

THEOREM 8.1. *There exists a commutative diagram of covering maps*

$$\begin{array}{ccc}
 & \Gamma & \\
 H \swarrow & & \searrow G \\
 (\Gamma/H, H_\tau) & \xrightarrow{F} & (\Gamma/G, G_\sigma)
 \end{array}$$

such that the underlying maps of graphs of F , G , H are f , g , h ; and when Γ is connected, $F: (\Gamma/H, H_\tau) \rightarrow (\Gamma/G, G_\sigma)$ is a regular covering if and only if H is a normal subgroup of G , in which case the group of covering transformations is isomorphic to G/H .

Note that we are denoting the covering map from Γ to its quotient graph of groups modulo G also by the symbol G . There should be no confusion caused by doing this as it will always be clear from context which meaning of G is intended, and similarly for H .

PROOF. The proof is given in several parts.

8.1.1. Construction of the covering map F . We define a map of graphs of groups $F: (\Gamma/H, H_\tau) \rightarrow (\Gamma/G, G_\sigma)$ with underlying map of graphs $f: \Gamma/H \rightarrow \Gamma/G$ as follows. For $x \in V(\Gamma/H)$, $\tau(x)$ and $\sigma(f(x))$ are in the same G -orbit of vertices of Γ . So, there exists $b_x \in G$ such that

$$\tau(x) \cdot b_x = \sigma(f(x)).$$

We do the same thing for the edges of Γ/H , but in a way that is compatible with λ and μ , by first choosing an orientation $E^+(\Gamma/H)$ of the graph Γ/H . Then for each $e \in E^+(\Gamma/H)$, choose $b_e \in G$ such that

$$\tau(e) \cdot b_e = \sigma(f(e))$$

and let $b_{\bar{e}} = \mu(e)^{-1}b_e\lambda(f(e))$. Note that

$$\begin{aligned}
\tau(\bar{e}) \cdot b_{\bar{e}} &= \tau(\bar{e}) \cdot \mu(e)^{-1}b_e\lambda(f(e)) \\
&= \overline{\tau(e)} \cdot b_e\lambda(f(e)) \\
&= (\overline{\tau(e) \cdot b_e}) \cdot \lambda(f(e)) \\
&= \overline{\sigma(f(e))} \cdot \lambda(f(e)) \\
&= \sigma(\overline{f(e)}) = \sigma(f(\bar{e}))
\end{aligned}$$

and that $b_e = \mu(e)b_{\bar{e}}\lambda(f(e))^{-1} = \mu(\bar{e})^{-1}b_{\bar{e}}\lambda(f(\bar{e}))$.

To summarize, for every $z \in \Gamma/H$ (vertex or edge), we have chosen an element $b_z \in G$ such that the following two properties hold:

- (1) $\tau(z) \cdot b_z = \sigma(f(z))$;
- (2) if $z \in E(\Gamma/H)$, then $b_{\bar{z}} = \mu(z)^{-1}b_z\lambda(f(z))$.

We use these choices of group elements (that translate between the two choices of sections) to define the vertex homomorphisms and edge labels that determine F .

For $x \in V(\Gamma/H)$, define the homomorphism $f_x: H_{\tau(x)} \rightarrow G_{\sigma(f(x))}$ by $f_x(a) = a^{b_x}$.

Note that

$$f_x(H_{\tau(x)}) = H_{\tau(x)}^{b_x} = H_{\tau(x) \cdot b_x} = H_{\sigma(f(x))} \subseteq G_{\sigma(f(x))},$$

as required; and also note that f_x is injective.

Define edge labels $\alpha_F = \alpha: E(\Gamma/H) \rightarrow V(\Gamma/G, G_\sigma)$ by $\alpha(e) = b_x^{-1}b_e$, where $x = s(e)$. Note that $\beta(e) = \alpha(\bar{e}) = b_y^{-1}b_{\bar{e}} = b_y^{-1}\mu(e)^{-1}b_e\lambda(f(e))$, where $y = t(e)$.

Then for all $e \in E(\Gamma/H)$ with $s(e) = x$, $t(e) = y$, and all $c \in H_{\tau(e)}$, we check that

$$\begin{aligned}
\sigma(f(e)) \cdot f_x(c)^{\alpha(e)} &= \sigma(f(e)) \cdot c^{b_x \alpha(e)} \\
&= \sigma(f(e)) \cdot b_e^{-1} c b_e \\
&= \tau(e) \cdot c b_e \\
&= \tau(e) \cdot b_e = \sigma(f(e)),
\end{aligned}$$

yielding that $f_x(c)^{\alpha(e)} \in G_{\sigma(f(e))}$ and thus, one of the required conditions of Lemma 3.3 holds. To verify the other condition, observe further that

$$\begin{aligned}
f_x(c)^{\alpha(e)} f(e) &= c^{b_x \alpha(e) \lambda(f(e))} \\
&= c^{b_e \lambda(f(e))} \\
&= c^{\mu(e) b_y \beta(e)} \\
&= (c^e)^{b_y \beta(e)} = f_y(c^e)^{\beta(e)}.
\end{aligned}$$

Thus, applying Lemma 3.3, we obtain a map of graphs of groups $F: (\Gamma/H, H_\tau) \rightarrow (\Gamma/G, G_\sigma)$. We claim that F is a covering map.

For $x \in V(\Gamma/H)$, define

$$\tau_x: L_x(\Gamma/H, H_\tau) \rightarrow L_{\tau(x)}(\Gamma)$$

by $\tau_x(aH_{\tau(e)}, e) = \tau(e) \cdot a^{-1}$. Since $s(e) = x$, $a \in H_{\tau(x)}$, and $s(\tau(e)) = \tau(s(e)) = \tau(x)$, it follows that

$$s(\tau(e) \cdot a^{-1}) = s(\tau(e)) \cdot a^{-1} = \tau(x) \cdot a^{-1} = \tau(x),$$

and hence $\tau(e) \cdot a^{-1} \in L_{\tau(x)}(\Gamma)$. The reader will note here that τ_x is a bijection as well. Similarly, for $y \in V(\Gamma/G)$, define

$$\sigma_y: L_y(\Gamma/G, G_\sigma) \rightarrow L_{\sigma(y)}(\Gamma)$$

by $\sigma_y(bG_{\sigma(e)}, e) = \sigma(e) \cdot b^{-1}$. The map σ_y is also a bijection.

Observe that for $(aH_{\tau(e)}, e) \in L_x(\Gamma/H, H_\tau)$ and $y = f(x)$, we have that

$$\begin{aligned} \sigma_y(F_x(aH_{\tau(e)}, e)) &= \sigma_y(f_x(a)\alpha(e)G_{\sigma(f(e))}, f(e)) \\ &= \sigma(f(e)) \cdot [f_x(a)\alpha(e)]^{-1} \\ &= \sigma(f(e)) \cdot \alpha(e)^{-1}f_x(a^{-1}) \\ &= \sigma(f(e)) \cdot \alpha(e)^{-1}b_x^{-1}a^{-1}b_x \\ &= \sigma(f(e)) \cdot b_e^{-1}a^{-1}b_x \\ &= \tau(e) \cdot a^{-1}b_x \\ &= \tau_x(aH_{\tau(e)}, e) \cdot b_x. \end{aligned}$$

Thus, we have a commutative diagram:

$$\begin{array}{ccc} L_{\tau(x)}(\Gamma) & \xrightarrow{\cdot b_x} & L_{\sigma(y)}(\Gamma) \\ \tau_x \uparrow & & \uparrow \sigma_y \\ L_x(\Gamma/H, H_\tau) & \xrightarrow{F_x} & L_y(\Gamma/G, G_\sigma) \end{array}$$

It follows that F_x is a composition of bijections, and hence, is itself a bijection. Therefore, F is a covering map.

8.1.2. Construction of the covering maps H and G . These are constructed as special cases of the construction used for F , which is similar to the construction of Bass [1]. We construct H first, applying our construction to $h: \Gamma \rightarrow \Gamma/H$ using $\text{id}: \Gamma \rightarrow \Gamma$ and $\tau: \Gamma/H \rightarrow \Gamma$ for the choices of sections. For each vertex or edge

$z \in \Gamma$, we choose an elements $a_z \in H$ (translating between the sections) such that the following properties hold:

- (1) $z \cdot a_z = \tau(h(z))$;
- (2) if $z \in E(\Gamma)$, then $a_{\bar{z}} = a_z \mu(h(z))$.

Then applying our construction to this case gives us a covering map $H: \Gamma \rightarrow (\Gamma/H, H_\tau)$ with underlying map of graphs $h: \Gamma \rightarrow \Gamma/H$.

Now for each $z \in \Gamma$, let $b_z = a_z b_{h(z)} \in G$, where $a_z, b_{h(z)}$ are the choices of translating elements used in defining H, F . Note that for all $z \in \Gamma$,

- (1) $z \cdot b_z = \sigma(g(z))$; and
- (2) if $z \in E(\Gamma)$, then $b_{\bar{z}} = b_z \lambda(g(z))$.

For (1), we have that

$$\begin{aligned}
 z \cdot b_z &= z \cdot a_z b_{h(z)} \\
 &= (z \cdot a_z) \cdot b_{h(z)} \\
 &= \tau(h(z)) \cdot b_{h(z)} \\
 &= \sigma(f(h(z))) \\
 &= \sigma(g(z)).
 \end{aligned}$$

With regard to (2), we need to show that

$$\bar{z} \cdot b_{\bar{z}} = \sigma(g(\bar{z})).$$

To this end,

$$\begin{aligned}
\bar{z} \cdot b_{\bar{z}} &= \bar{z} \cdot b_z \lambda(g(z)) \\
&= \overline{z \cdot b_z} \cdot \lambda(g(z)) \\
&= \overline{\sigma(g(z))} \cdot \lambda(g(z)) \\
&= \sigma(\overline{g(z)}) \\
&= \sigma(g(\bar{z})).
\end{aligned}$$

Hence, these elements satisfy the required properties of translating elements for constructing a covering map $G: \Gamma \rightarrow (\Gamma/G, G_\sigma)$ with underlying map of graphs $g: \Gamma \rightarrow \Gamma/G$ (where the sections are $\text{id}: \Gamma \rightarrow \Gamma$ and $\sigma: \Gamma/G \rightarrow \Gamma$). Furthermore, by our choice of translating elements such that $b_z = a_z b_{h(z)}$, for each $z \in \Gamma$, it follows that $FH = G$, as desired.

8.1.3. Expanding the group action using the \sharp -construction. We show that our \sharp -construction is compatible with the action of G on Γ . First form the trivial expansion of $(\Gamma/G, G_\sigma)$ with a single new vertex y^\sharp adjacent to each old vertex y of Γ/G . Then form Γ^\sharp so that the fiber over each new vertex $y^\sharp \in V^\sharp(\Gamma/G)$ is a copy of the group G by taking

$$V^\sharp(\Gamma) = \{y_b^\sharp \mid y \in V(\Gamma/G) \text{ and } b \in G\}$$

and take $[b\sigma(y), y_b^\sharp]$ as the new edge attaching each new vertex y_b^\sharp . Observe that the action of G on Γ extends to an action on Γ^\sharp in an obvious way (by left multiplication): for all $b_1, b_2 \in G$ and $y \in V(\Gamma/G)$,

$$b_1 \cdot y_{b_2}^\sharp = y_{b_1 b_2}^\sharp \quad \text{and} \quad b_1 \cdot [b_2 \sigma(y), y_{b_2}^\sharp] = [b_1 b_2 \sigma(y), y_{b_1 b_2}^\sharp]$$

Also observe that $\Gamma^\sharp/G = (\Gamma/G)^\sharp$ and that we have a commutative diagram of graphs and maps:

$$\begin{array}{ccc} & \Gamma^\sharp & \\ h^\sharp \swarrow & & \searrow g^\sharp \\ \Gamma^\sharp/H & \xrightarrow{f^\sharp} & \Gamma^\sharp/G \end{array}$$

and the quotient maps $h^\sharp, g^\sharp, f^\sharp$ are extensions of h, g, f .

Next we apply the constructions in 8.1.1 and 8.1.2 to this expanded situation. By taking some care in how we extend the sections τ, σ to the trivial expansions $\Gamma^\sharp/H, \Gamma^\sharp/G$ and in how we choose the translating elements between these sections, we can ensure that the covering maps constructed agree with the earlier construction given in Lemma 6.3. To begin, choose a right transversal T_x for the subgroup $f_x(H_{\tau(x)})$ of $G_{\sigma(f(x))}$ for each $x \in V(\Gamma/H)$. As always, we assume that the identity element of $G_{\sigma(f(x))}$ is in T_x . The following observation is key.

LEMMA 8.2. *Let $b \in G$ and $y \in V(\Gamma/G)$. Then there exists a unique triple of elements $a' \in H, x \in V(\Gamma/H), b' \in T_x$ such that $f(x) = y$ and*

$$b = a'b_x b'.$$

PROOF. Existence: Let $x = h(b\sigma(y))$. Then $f(x) = g(b\sigma(y)) = y$ and $\tau(x)$ is in the same H -orbit as $b\sigma(y)$. So there exists $a \in H$ such that $a\tau(x) = b\sigma(y)$. Thus $b\sigma(f(x)) = a\tau(x) = ab_x\sigma(f(x))$ and we see that $b = ab_x b_0$ for some $b_0 \in G_{\sigma(f(x))}$. Write $b_0 = f_x(a_0)b'$, where $a_0 \in H_{\tau(x)}$ and $b' \in T_x$. Then $b = ab_x f_x(a_0)b' = aa_0 b_x b' = a'b_x b'$, where $a' = aa_0 \in H, x \in V(\Gamma/H)$, and $b' \in T_x$, as required.

Uniqueness: Suppose $a'' \in H, x' \in V(\Gamma/H), b'' \in T_{x'}$ is another such triple. Note that $b\sigma(y) = a'b_x b'\sigma(f(x)) = a'b_x\sigma(f(x)) = a'\tau(x)$ and likewise that $b\sigma(y) = a''\tau(x')$. So $x = h(a'\tau(x)) = h(b\sigma(y)) = h(a''\tau(x')) = x'$ and $a'\tau(x) = a''\tau(x)$. Thus $(a'')^{-1}a' \in$

$H_{\tau(x)}$. Now $a'b_xb' = b = a''b_xb''$ and hence

$$b'' = b_x^{-1}a_0b_xb' = f_x(a_0)b'$$

where $a_0 = (a'')^{-1}a' \in H_{\tau(x)}$ and $b', b'' \in T_x$. However, T_x is a right transversal for $f_x(H_{\tau(x)}) \leq G_{\sigma(f(x))}$ and f_x is injective. So $b' = b''$ and $a_0 = 1$; whence also $a' = a''$ and uniqueness follows. \square

It follows immediately from the lemma that every H -orbit of vertices of Γ^\sharp contains a unique vertex of the form $f(x)^\sharp_{b_xb'}$, where $x \in V(\Gamma/H)$ and $b' \in T_x$; we identify this orbit with $x^\sharp_{b'}$ and define $\tau(x^\sharp_{b'}) = f(x)^\sharp_{b_xb'}$. Similarly, each H -orbit of edges (directed from old vertices to new vertices) contains a unique edge of the form $[b_x\sigma(f(x)), f(x)^\sharp_{b_xb'}] = [\tau(x), f(x)^\sharp_{b_xb'}]$, where $x \in V(\Gamma/H)$ and $b' \in T_x$; identify this orbit with $[x, x^\sharp_{b'}]$ and define $\tau([x, x^\sharp_{b'}]^{\pm 1}) = [\tau(x), f(x)^\sharp_{b_xb'}]^{\pm 1}$. In this way, we have identified Γ^\sharp/H with $(\Gamma/H)^\sharp$ and extended τ to a section of the orbit map $h^\sharp: \Gamma^\sharp \rightarrow \Gamma^\sharp/H$. Since $\tau([x, x^\sharp_{b'}]^{-1}) = \tau([x, x^\sharp_{b'}])^{-1}$, we can take $\mu([x, x^\sharp_{b'}]) = 1$ on new edges.

Extend σ to new vertices and edges by $\sigma(y^\sharp) = y_1^\sharp$ and $\sigma([y, y^\sharp]^{\pm 1}) = [\sigma(y), y_1^\sharp]^{\pm 1}$, where subscripts $1 = 1_G$. Since $\sigma([y, y^\sharp]^{-1}) = \sigma([y, y^\sharp])^{-1}$, we can take $\lambda([y, y^\sharp]) = 1$ on new edges.

Since G acts freely on the new vertices and edges of Γ^\sharp , the trivial group is assigned by G_σ, H_τ to each new vertex and new edge. It follows that $(\Gamma^\sharp/G, G_\sigma), (\Gamma^\sharp/H, H_\tau)$ are trivial expansions of $(\Gamma/G, G_\sigma), (\Gamma/H, H_\tau)$. To define $F^\sharp: (\Gamma^\sharp/G, G_\sigma) \rightarrow (\Gamma^\sharp/H, H_\tau)$, it remains to choose the translating elements between the sections τ and σ for each new vertex $x^\sharp_{b'}$ and new edge $[x, x^\sharp_{b'}]$ of Γ^\sharp/H . For this, we take $b(x^\sharp_{b'}) = b([x, x^\sharp_{b'}]^{\pm 1}) = b_xb' \in G$. Note that $\tau(x^\sharp_{b'}) \cdot b(x^\sharp_{b'}) = \sigma(f^\sharp(x^\sharp_{b'}))$ and $\tau([x, x^\sharp_{b'}]^{\pm 1}) \cdot b([x, x^\sharp_{b'}]^{\pm 1}) = \sigma(f^\sharp([x, x^\sharp_{b'}]^{\pm 1}))$. Hence, by the construction in 8.1.1, we get a covering map F^\sharp

extending F . Since each $\alpha_F^\sharp([x, x_{b'}^\sharp]) = b_x^{-1}b([x, x_{b'}^\sharp]) = b_x^{-1}(b_x b') = b'$, it follows that F^\sharp is the same extension of F given by the construction of Lemma 6.3.

Finally, given $b \in G$ and $y \in V(\Gamma/G)$, apply the lemma to get unique $a' \in H$, $x \in V(\Gamma/H)$, $b' \in T_x$ such that $f(x) = y$ and $b = a'b_x b'$. Then define the translating elements $a(y_b^\sharp) = a([b\sigma(y), y_b^\sharp]^{\pm 1}) = a' \in H$ and $b(y_b^\sharp) = b([b\sigma(y), y_b^\sharp]^{\pm 1}) = b \in G$. Then the reader can easily verify that $z \cdot a_z = \tau(h^\sharp(z))$ and $z \cdot b_z = \sigma(g^\sharp(z))$ also hold for all new vertices and edges of Γ^\sharp . Hence, by the construction in 8.1.2, we get covering maps H^\sharp, G^\sharp extending H, G . Furthermore, we have chosen the translating elements such that $a_z b_{f(z)} = b_z$ for each vertex or edge z of Γ^\sharp . Therefore $F^\sharp H^\sharp = G^\sharp$.

$$\begin{array}{ccc}
 & \Gamma^\sharp & \\
 H^\sharp \swarrow & & \searrow G^\sharp \\
 (\Gamma^\sharp/H, H_\tau) & \xrightarrow{F^\sharp} & (\Gamma^\sharp/G, G_\sigma)
 \end{array}$$

8.1.4. Homomorphism induced by λ and the action of $\pi(\Gamma^\sharp/G, G_\sigma, y^\sharp)$ on the fiber $g^{\sharp-1}(y^\sharp)$. The function $\lambda: E(\Gamma^\sharp/G) \rightarrow G$ together with the inclusions $G_{\sigma(y)} \hookrightarrow G$, $y \in \Gamma^\sharp/G$, preserve the defining relations $\bar{e}ce = c^e$, $e \in E(\Gamma^\sharp/G)$, $c \in G_{\sigma(e)}$, and thus determine a homomorphism of groupoids, which we also denote by $\lambda: \pi(\Gamma^\sharp/G, G_\sigma) \rightarrow G$. For any vertex $y \in V(\Gamma^\sharp/G)$, let $\lambda_y: \pi(\Gamma^\sharp/G, G_\sigma, y) \rightarrow G$ denote the homomorphism of groups obtained by the restriction of λ .

Note that the fiber of the map g^\sharp over any new vertex $y^\sharp \in V^\sharp(\Gamma/G)$ is of the form

$$g^{\sharp-1}(y^\sharp) = \{y_b^\sharp \mid b \in G\}$$

which can be viewed as a copy of the group G . We make the following observations:

- (1) The action of $\pi(\Gamma^\sharp/G, G_\sigma, y^\sharp)$ on the fiber $g^{\sharp-1}(y^\sharp)$ is given by the composition of λ_{y^\sharp} followed by right multiplication in G ; that is, for each

$[P] \in \pi(\Gamma^\sharp/G, G_\sigma, y^\sharp)$ and $y_b^\sharp \in g^{\sharp-1}(y^\sharp)$, we have that $y_b^\sharp \cdot [P] = y_{b'}^\sharp$, where $b' = b\lambda_{y^\sharp}([P])$.

- (2) The homomorphism $\lambda_y: \pi(\Gamma/G, G_\sigma, y) \rightarrow G$ is onto and Γ/G is connected if and only if Γ is connected.

To see (1), let P be a path in $(\Gamma^\sharp/G, G_\sigma)$ whose underlying path in Γ^\sharp/G is a loop at y^\sharp and let $b \in G$. Then, by Theorem 5.5, there is a unique lift of P to a path \tilde{P} in Γ^\sharp with source y_b^\sharp . Assume that $\tilde{P} = e_1e_2 \cdots e_n$ and let $v_i = t(e_i) = s(e_{i+1})$; and note that $v_0 = y_b^\sharp$ and $v_n = y_{b'}^\sharp$ for some $b' \in G$. Then $y_b^\sharp \cdot [P] = y_{b'}^\sharp$ and

$$P = G^\sharp(\tilde{P}) \equiv b_0g(e_1)b_1g(e_2)b_2 \cdots g(e_n)b_n$$

where $b_0 = \alpha_G(e_1) = b_{v_0}^{-1}b_{e_1} = b^{-1}b_{e_1}$ as $v_0 = y_b^\sharp$; $b_n = \beta(e_n)^{-1} = \lambda(g(e_n))^{-1}b_{e_n}^{-1}b_{v_n} = \lambda(g(e_n))^{-1}b_{e_n}^{-1}b'$ as $v_n = y_{b'}^\sharp$; and for $1 \leq i \leq n-1$, $b_i = \beta_G(e_i)^{-1}\alpha_G(e_{i+1}) = \lambda(g(e_i))^{-1}b_{e_i}^{-1}b_{e_{i+1}}$ since $\alpha_G(e_{i+1}) = b_{v_i}^{-1}b_{e_{i+1}}$ and $\beta_G(e_i) = b_{v_i}^{-1}b_{e_i}\lambda(g(e_i))$. Thus, we see that

$$\lambda_{y^\sharp}([P]) = b_0\lambda(g(e_1))b_1\lambda(g(e_2))b_2 \cdots \lambda(g(e_n))b_n = b^{-1}b'.$$

Hence $b' = b\lambda_{y^\sharp}([P])$ and (1) is established.

Next observe that λ_y is onto iff $\lambda_{y^\sharp}: \pi(\Gamma^\sharp/G, G_\sigma, y) \rightarrow G$ is onto; Γ is connected iff Γ^\sharp is connected; and Γ/G is connected iff Γ^\sharp/G is connected. Thus to prove (2), it suffices to prove the corresponding statement for the graph Γ^\sharp and vertex y^\sharp .

To this end, assume first that λ_{y^\sharp} is an onto map and that Γ^\sharp/G is connected. Then it follows immediately from (1) that the fiber $g^{\sharp-1}(y^\sharp)$ lies in a single component of Γ^\sharp . Also note that if P is a path in $(\Gamma^\sharp/G, G_\sigma)$ from some new vertex z^\sharp to y^\sharp , then there is a lift of P joining each vertex in $g^{\sharp-1}(z^\sharp)$ to a vertex in $g^{\sharp-1}(y^\sharp)$. Since Γ^\sharp/G is connected, it follows that all new vertices of Γ^\sharp lie in the same component as $g^{\sharp-1}(y^\sharp)$. Since every old vertex is joined to a new vertex by an edge, we see that Γ^\sharp is connected.

Conversely, assume that Γ^\sharp is connected. Then for each $b \in G$, there is a path p in Γ^\sharp from y_1^\sharp to y_b^\sharp . Since $g^\sharp(y_1^\sharp) = g^\sharp(y_b^\sharp) = y^\sharp$, we see that $G^\sharp([p]) \in \pi(\Gamma^\sharp/G, G_\sigma, y^\sharp)$ and $y_1^\sharp \cdot G^\sharp([p]) = y_b^\sharp$. Thus, by (1), it follows that $b = 1 \cdot \lambda_{y^\sharp}(G^\sharp([p]))$. Hence λ_{y^\sharp} is onto; and certainly the quotient graph Γ^\sharp/G is connected.

8.1.5. $F \pi(\Gamma/H, H_\tau, x) = \lambda_{f(x)}^{-1}(H^{b_x})$. First, for any $b' \in T_x$, we show that $F^\sharp \pi(\Gamma^\sharp/H, H_\tau, x_{b'}^\sharp) = \lambda_{f(x)^\sharp}^{-1}(H^{b_x b'})$. Let $[P] \in \pi(\Gamma^\sharp/G, G_\sigma, f(x)^\sharp)$. Since $x_{b'}^\sharp$ is in the H -orbit of the vertex $f(x)_{b_x b'}^\sharp$ in Γ^\sharp , it follows that $[P] \in F^\sharp \pi(\Gamma^\sharp/H, H_\tau, x_{b'}^\sharp)$ if and only if $f(x)_{b_x b'}^\sharp$ and $f(x)_{b_x b'}^\sharp \cdot [P]$ are in the same H -orbit. By 8.1.4(1), this is the case if and only if there exists $a \in H$ such that $ab_x b' = b_x b' \lambda_{f(x)^\sharp}([P])$; or equivalently $\lambda_{f(x)^\sharp}([P]) \in H^{b_x b'}$, and our assertion follows.

Now consider the commutative diagram

$$\begin{array}{ccccc} \pi(\Gamma^\sharp/H, H_\tau, x_1^\sharp) & \xrightarrow{F^\sharp} & \pi(\Gamma^\sharp/G, G_\sigma, f(x)^\sharp) & \xrightarrow{\lambda_{f(x)^\sharp}} & G \\ \text{ad}[x, x_1^\sharp] \uparrow & & \text{ad}[f(x), f(x)^\sharp] \uparrow & & \uparrow = \\ \pi(\Gamma/H, H_\tau, x) & \xrightarrow{F} & \pi(\Gamma/G, G_\sigma, f(x)) & \xrightarrow{\lambda_{f(x)}} & G \end{array}$$

where we have taken $b' = 1 \in T_x$. The vertical maps are isomorphisms and we have shown that the image of F^\sharp is $\lambda_{f(x)^\sharp}^{-1}(H^{b_x})$. It follows that the image of F is $\lambda_{f(x)}^{-1}(H^{b_x})$, as required.

8.1.6. If Γ is connected, then $F: (\Gamma/H, H_\tau) \rightarrow (\Gamma/G, G_\sigma)$ is a regular covering if and only if H is a normal subgroup of G ; and in this case the group of covering transformations is isomorphic to G/H and the following sequence is exact for any $x \in V(\Gamma/H)$.

$$1 \rightarrow \pi(\Gamma/H, H_\tau, x) \xrightarrow{F} \pi(\Gamma/G, G_\sigma, f(x)) \xrightarrow{\bar{\lambda}_{f(x)}} G/H \rightarrow 1$$

Here $\bar{\lambda}_{f(x)}$ is the composition of $\lambda_{f(x)}$ and the quotient map $G \rightarrow G/H$.

This follows immediately from Theorem 5.12, 8.1.5, 8.1.4(2), and Theorem 7.4. This completes the proof of Theorem 8.1. \square

One of the main theorems of Bass-Serre Theory follows from the case when H is the trivial subgroup of G .

COROLLARY 8.3 (Main Theorem of Bass-Serre Theory). *Let G be a group acting on a connected graph Γ . Then the sequence*

$$1 \rightarrow \pi(\Gamma, x) \xrightarrow{G} \pi(\Gamma/G, G_\sigma, f(x)) \xrightarrow{\lambda_{f(x)}} G \rightarrow 1$$

is exact for any vertex x of Γ . In particular, Γ is a tree if and only if $\lambda_{f(x)}$ is an isomorphism.

REMARK 8.4 (Uniqueness of the associated covering). For $i = 1, 2$, let σ_i be a section of the orbit map $g: \Gamma \rightarrow \Gamma/G$ and let $\lambda_i(e) \in G$, for all $e \in E(\Gamma/G)$, be group elements satisfying:

$$\lambda_i(\bar{e}) = \lambda_i(e)^{-1} \quad \text{and} \quad \overline{\sigma_i(e)} \cdot \lambda_i(e) = \sigma_i(\bar{e});$$

likewise, let τ_i be a section of the orbit map $h: \Gamma \rightarrow \Gamma/H$ and let $\mu_i(e) \in H$, for all $e \in E(\Gamma/H)$, be group elements satisfying:

$$\mu_i(\bar{e}) = \mu_i(e)^{-1} \quad \text{and} \quad \overline{\tau_i(e)} \cdot \mu_i(e) = \tau_i(\bar{e}).$$

Let $F_i: (\Gamma/H, H_{\tau_i}) \rightarrow (\Gamma/G, G_{\sigma_i})$ be a covering map obtained by choosing translating elements between the sections τ_i and σ_i using the construction of 8.1.1. Then there exists a commutative diagram

$$\begin{array}{ccc} (\Gamma/H, H_{\tau_1}) & \xrightarrow{M} & (\Gamma/H, H_{\tau_2}) \\ F_1 \downarrow & & \downarrow F_2 \\ (\Gamma/G, G_{\sigma_1}) & \xrightarrow{N} & (\Gamma/G, G_{\sigma_2}) \end{array}$$

where M and N are isomorphisms of graphs of groups.

8.2. Proof of the existence of coverings

The proof is long and involved, and thus we will again break it up into steps as we did before with the general lifting proof.

8.2.1. Define $\lambda: \pi(\Delta, B) \rightarrow \pi(\Delta, B, y_0)$. Let (Δ, B) be a graph of groups where Δ is connected. Now, we fix a vertex $y_0 \in V(\Delta)$ with the goal of defining a map

$$\lambda: \pi(\Delta, B) \rightarrow G$$

where $G = \pi(\Delta, B, y_0)$. To begin, choose a maximal tree T in Δ , and for each $y \in V(\Delta)$, define

$$\begin{aligned} \lambda: B_y &\rightarrow G \text{ by} \\ \lambda(b) &= p_y^{-1} b p_y, \end{aligned}$$

where p_y is the unique reduced path in T from y to y_0 . Now, for each $e \in E(\Delta)$, define

$$\lambda(e) = p_{y_1}^{-1} e p_{y_2},$$

where $y_1 = s(e)$ and $y_2 = t(e)$. Then for each $e \in E(\Delta)$ and $c \in B_e$ we have that

$$\begin{aligned} \lambda(\bar{e})\lambda(c)\lambda(e) &= (p_{y_2}^{-1} \bar{e} p_{y_1})(p_{y_1}^{-1} c p_{y_1})(p_{y_1}^{-1} e p_{y_2}) \\ &= p_{y_2}^{-1} (\bar{e} c e) p_{y_2} \\ &= p_{y_2}^{-1} c^e p_{y_2} \\ &= \lambda(c^e). \end{aligned}$$

Similarly, for all $y \in V(\Delta)$ and $b, b' \in B_y$,

$$\lambda(b)\lambda(b') = (p_y^{-1} b p_y)(p_y^{-1} b' p_y) = p_y^{-1} b b' p_y = \lambda(bb').$$

Thus, a groupoid homomorphism, $\lambda: \pi(\Delta, B) \rightarrow G$ is induced where $\lambda_{y_0}: \pi(\Delta, B, y_0) \rightarrow G$ is the identity map.

8.2.2. Construct the graph $\Delta \times G$. Let $\Delta \times G$ be the graph with

$$V(\Delta \times G) = V(\Delta) \times G \text{ and } E(\Delta \times G) = E(\Delta) \times G.$$

Furthermore, for $(e, g) \in E(\Delta \times G)$, define

$$s(e, g) = (s(e), g); t(e, g) = (t(e), g\lambda(e)); \text{ and } \overline{(e, g)} = (\bar{e}, g\lambda(e)).$$

In order to see the last portion of the statement above, note that

$$\begin{aligned} s(\overline{(e, g)}) &= s(\bar{e}, g\lambda(e)) = (s(\bar{e}), g\lambda(e)) = (t(e), g\lambda(e)) = t(e, g), \text{ and} \\ t(\overline{(e, g)}) &= t(\bar{e}, g\lambda(e)) = (t(\bar{e}), g) = (s(e), g) = s(e, g). \end{aligned}$$

Note also that G acts on $\Delta \times G$ (on the left) by $b' \cdot (z, b) = (z, b'b)$ for all $z \in \Delta$ and $b, b' \in G$.

8.2.3. Define a quotient graph $\Gamma = \Delta \times_B G$. To begin, we say that $(z_1, b_1) \sim (z_2, b_2)$ if $z_1 = z_2$ and $b_1\lambda(b) = b_2$ for some $b \in B_{z_i}$, where $i = 1, 2$. This equivalence respects the G -action. To see this, first note that $b' \cdot (z_1, b_1) = (z_1, b'b_1)$ and $b' \cdot (z_2, b_2) = (z_2, b'b_2)$. Thus, if $(z_1, b_1) \sim (z_2, b_2)$, then $z_1 = z_2 = z$, say, and $b_1\lambda(b) = b_2$ for some $b \in B_z$. Hence, $b'b_1\lambda(b) = b'b_2$ yielding that $(z_1, b'b_1) \sim (z_2, b'b_2)$. So, the action of G projects to an action on Γ , with

$$b' \cdot [z, b] = [z, b'b].$$

Note then that $\Gamma/G = \Delta$ via the identification $G([z, b]) = z$.

Now, define a section σ of the orbit map $g: \Gamma \rightarrow \Delta$ by $\sigma(z) = [z, 1]$, where $1 \in G$. Thus, we have that $G_{\sigma(z)} = \{b \in G \mid b \cdot [z, 1] = [z, 1]\}$, and

$$\begin{aligned} b \cdot [z, 1] = [z, 1] &\Leftrightarrow (z, b) \sim (z, 1) \\ &\Leftrightarrow 1 \cdot \lambda(b') = b, \end{aligned}$$

for some $b' \in B_z$. But, this is true if and only if $b \in \lambda(B_z)$. Hence,

$$G_{\sigma(z)} = \lambda(B_z) \leq G.$$

Identifying B_z with its image $\lambda(B_z) = G_{\sigma(z)}$ we see that

$$(\Gamma/G, G_\sigma) = (\Delta, B).$$

8.2.4. Γ is a tree (the Bass-Serre tree). Since Δ is connected and λ_{y_0} is an isomorphism (hence a surjection), by 8.1.4(2), Γ is connected. Hence, Γ is a tree by Corollary 8.3.

8.2.5. Define an appropriate covering $F: (\Gamma/H, H_\tau) \rightarrow (\Gamma/G, G_\sigma) = (\Delta, B)$, where $H \leq \pi(\Delta, B, y_0) = G$. To this end, let H be any subgroup of G . We desire to define the above covering in such a way that $F\pi(\Gamma/H, H_\tau, x_0) = H$ for appropriate choice of section τ of the orbit map $h: \Gamma \rightarrow \Gamma/H$ and vertex $x_0 \in V(\Gamma/H)$ where $f(x_0) = y_0$. To begin, choose $x_0 = h([y_0, 1])$. Then, if $h: \Gamma \rightarrow \Gamma/H$ is the orbit map, we have that

$$f(x_0) = (fh)[y_0, 1] = g[y_0, 1] = y_0.$$

Now, we choose τ such that $\tau(x_0) = [y_0, 1] = \sigma(y_0)$ and take $b_{x_0} = 1 \in G$. Then,

$$\tau(x_0) \cdot b_{x_0} = \sigma(f(x_0)).$$

Hence, by 8.1.5,

$$\begin{aligned} F\pi(\Gamma/H, H_\tau, x_0) &= \lambda_{f(x_0)}^{-1}(H^{b_{x_0}}) \\ &= \lambda_{y_0}^{-1}(H) \\ &= H, \end{aligned}$$

since λ_{y_0} is the identity.

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