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Rotationally invariant approximation to charge-monopole scattering

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A semiclassical approximation derived directly from the Feynman path integral is employed in the study of electric-charge-magnetic-monopole scattering. We show that this approximation, unlike perturbation theory, is consistent with rotational invariance. The semiclassical cross section is explicitly evaluated. It differs from the classical differential cross section for sufficiently large scattering angles due to the interference between the several classical trajectories contributing to the scattering at such angles. It is found that when the scattering angle is not too near the backward direction the semiclassical cross section approaches the classical limit rather slowly as the Dirac quantization number becomes large, or equally as $\hbar \rightarrow 0$ with the product of electric and magnetic charges held fixed.

I. INTRODUCTION

In this paper we demonstrate how to apply a semiclassical approximation derived from the Feynman path integral¹ to the problem of electric-charge-magnetic-monopole scattering. A semiclassical treatment of this system is already contained in the noteworthy study of semiclassical methods of Ford and Wheeler.² Their approach is based on a partial-wave expansion of the scattering amplitude with JWKB phase shifts. Our method by contrast does not rely on separating the Schrödinger equation into partial waves and may therefore enjoy a wider range of application. Further, because it is based on the Feynman path integral, this method seems better adapted to field-theoretic generalization than the Ford-Wheeler approach based on partial waves, in particular since the partial-wave series is divergent for this system.

In studying the problem of charge-monopole scattering it appears that semiclassical methods offer the only known consistent approximation schemes. An exact treatment reveals a partial-wave expansion which is formally divergent (although summable).³ On the other hand, perturbation theory is beset with serious difficulties. The Lagrangian has an explicit dependence on an external direction (which we denote as \hat{n})^{4,5} and, as a consequence,

the scattering amplitude computed to any finite order in perturbation theory violates rotational invariance. Further, the parameter characterizing the strength of the interaction is not small due to the Dirac quantization condition^{4,5} $eg/4\pi = n\hbar$, where n is an integer or half-integer. Here e and g are the electric and magnetic charges.

The semiclassical approximation does not contain these difficulties. The method of Ford and Wheeler ensures rotational invariance from the outset, while we are able to show directly that the approximation we adopt is consistent with rotational invariance as well. Further, we show that our final approximation to the scattering amplitude is independent of the initial choice of \hat{n} when the Dirac quantization condition is imposed.

Because more than one classical trajectory contributes to the scattering process for a range of scattering angles, the semiclassical approximation yields a nonclassical cross section for charge-monopole scattering. These trajectories interfere in the semiclassical approach. The interference terms are of the form $a \exp[\pm ib(eg/\hbar)]$, where a and b are independent of \hbar . Thus if eg is held fixed as $\hbar \rightarrow 0$, the interference terms vanish in the distribution sense and the usual classical cross section is recovered. Such a limit corresponds to letting $|n| \rightarrow \infty$ in the relation $eg/4\pi = n\hbar$. We shall

see that the approach to this limit is rather slow when scattering is not too near the backward direction. Actually, however, for monopole fields which emerge from gauge theories,⁶ making n large seems inappropriate since n is a topological quantum number characterizing the solutions. A more natural limit in this context would be to hold n fixed as $\hbar \rightarrow 0$. Then the exponentials become independent of \hbar and the quantum corrections become appreciable relative to the classical cross section near the limit. Thus, it seems that purely classical considerations applied to Yang-Mills monopoles may not always be accurate.

Another important feature of charge-monopole scattering is the presence of infinitely many "rainbow" angles, angles at which the number of contributing classical trajectories changes. The classical differential cross section diverges at these angles. Also present is a limit point at $\theta = \pi$ of these rainbow angles (the backward "glory"). Because of these phenomena, one might expect the exact scattering amplitude to possess interesting analytic structure in $\cos\theta$. The analyticity of the exact amplitude is investigated in the following paper.⁷

Our semiclassical approximation is arrived at by performing stationary-phase approximations on the path integral. Ford and Wheeler's answer is found by taking a large-angular-momentum and large- n limit of the partial-wave series.³ Pechukas¹ has shown that both methods yield the same semiclassical cross section when applied to central potentials. Here we find the same result in application to the charge-monopole system (away from rainbows and the backward glory).⁸ Such a result is remarkable considering the drastic differences between the two approaches.

In Sec. II we summarize the results from the classical theory of charge-monopole scattering and write down the form of its quantum corrections in the semiclassical approximation. The rotational properties of the semiclassical approximation are discussed in detail in Sec. III. Section IV describes the explicit computation of the semiclassical cross section. Concluding remarks are made in Sec. V. Appendix A gives a brief derivation of the semiclassical scattering amplitude. Appendix B contains a detailed derivation of the JWKB phases for charge-monopole scattering. The corresponding analysis for central potentials is also indicated.

II. CLASSICAL AND SEMICLASSICAL CHARGE-MONOPOLE SYSTEM

We review the classical theory of a charged particle in a magnetic monopole field in part A and

consider the quantum corrections to classical scattering in part B.

A. Classical scattering

The Lagrangian describing the motion of a particle of charge e and mass m in the field of a magnetic monopole is given by

$$L(\vec{q}, \dot{\vec{q}}) = \frac{1}{2} m \dot{\vec{q}}^2 + e A_i(\hat{n}, \vec{q}) \dot{q}_i, \quad (2.1)$$

where⁹

$$A_i(\hat{n}, \vec{q}) = \frac{g}{4\pi} \frac{(\hat{n} \times \vec{q})_i (\hat{n} \cdot \vec{q})}{q [q^2 - (\hat{n} \cdot \vec{q})^2]}. \quad (2.2)$$

Here \hat{n} is a unit vector characterizing the vector potential and $q = |\vec{q}|$.¹⁰ The curl of (2.2) yields the Coulomb-type magnetic field:

$$\vec{\nabla} \times \vec{A} = -\frac{g}{4\pi} \frac{\vec{q}}{q^3}. \quad (2.3)$$

The equation of motion which follows from the Lagrangian is

$$\ddot{q}_i = \alpha \frac{(\dot{\vec{q}} \times \vec{q})_i}{q^3}, \quad (2.4)$$

where

$$\alpha = -\frac{eg}{4\pi m}. \quad (2.5)$$

From (2.4) one can deduce the following constants of motion:

(a) The angular momentum $\vec{J} = m\vec{C}$, where

$$\vec{C} = \vec{q} \times \dot{\vec{q}} - \alpha \hat{q}. \quad (2.6)$$

The constancy of \vec{C} follows from taking the cross product of the equation of motion $\ddot{\vec{q}} = \alpha(\dot{\vec{q}} \times \vec{q})/q$ with \vec{q} and noting that $\hat{q} \cdot \dot{\vec{q}} = 0$.

(b) The magnitude of the orbital angular momentum $m |\vec{q} \times \dot{\vec{q}}|$. This follows from $C^2 = |\vec{q} \times \dot{\vec{q}}|^2 + \alpha^2$.

(c) The magnitude of the velocity $v = q$. This is true since the energy is simply $\frac{1}{2} m \dot{\vec{q}}^2$.

It is well known that the solutions of (2.4) are trajectories which are confined to the surface of a cone with cone axis $-\epsilon(\alpha)\hat{C}$ and opening angle 2ψ , where $\cos\psi = |\alpha|/C$. If we choose the cone axis to be the third direction, the solutions are given by¹¹

$$\vec{q} = q(\sin\psi \cos\varphi, \sin\psi \sin\varphi, \cos\psi), \quad (2.7a)$$

$$q^2 = v^2 t^2 + s^2, \quad (2.7b)$$

$$\tan\psi = \frac{sv}{|\alpha|}, \quad (2.7c)$$

$$\varphi = -\frac{\epsilon(\alpha)}{\sin\psi} \tan^{-1}\left(\frac{vt}{s}\right), \quad (2.7d)$$

where s is the impact parameter.

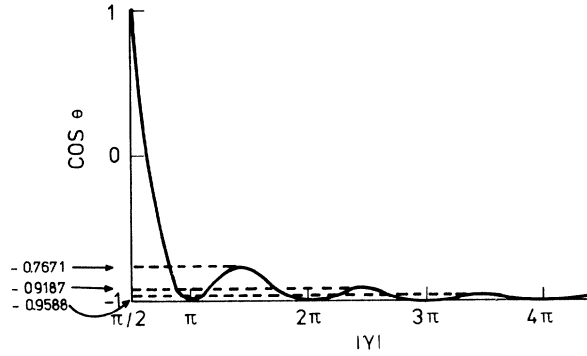


FIG. 1. Cosine of the scattering angle θ versus $|\gamma| = (\pi/2)[1 + (\alpha/vs)^2]^{1/2}$. The figure shows the presence of multiple classical trajectories which contribute to the same θ when θ is sufficiently large.

For a given s , the scattering angle θ is given by

$$\cos \frac{\theta}{2} = \frac{\pi}{2\gamma} \sin \gamma, \quad (2.8)$$

where

$$\gamma = \frac{\pi \epsilon(\alpha)}{2 \sin \psi} = \epsilon(\alpha) \frac{\pi}{2} \left[1 + \left(\frac{\alpha}{sv} \right)^2 \right]^{1/2}. \quad (2.9)$$

We have plotted $\cos \theta$ as a function of $|\gamma|$ in Fig. 1. The figure shows that more than one trajectory contributes to the classical cross section at certain scattering angles, and that the number $N(\theta)$ of such contributing trajectories changes discontinuously with θ . The rainbow angles are located at positions where $\tan \gamma = \gamma$. The first three are indicated in Fig. 1. Note that these angles accumulate at $\theta = \pi$.

The classical differential cross section σ_c for a scattering angle θ is given by

$$\sigma_c = \sum_{\nu}^{N(\theta)} |\sigma_c(\gamma_{\nu})|, \quad (2.10)$$

$$\sigma_c(\gamma_{\nu}) = \left[s \frac{\partial s}{\partial \cos \theta} \right]_{\gamma = \gamma_{\nu}},$$

where the γ_{ν} 's are the roots of (2.8) for a given $\cos \theta$. $\sigma_c(\gamma)$ can be computed from (2.8) and (2.9):

$$f(\vec{p}'', \vec{p}') = \lim_{t_f \rightarrow \infty, t_i \rightarrow -\infty} \lim_{q'' \rightarrow \infty} q'' \int d^3 q' \langle \vec{q}'' | U(t_f - t_i) | \vec{q}' \rangle \exp \left[\frac{i}{\hbar} (\vec{p}' \cdot \vec{q}' - \vec{p}'' \cdot \vec{q}'' + \frac{p'^2}{2m} (t_f - t_i)) \right]. \quad (2.15)$$

The functional integral representation for the propagator is⁹

$$\langle \vec{q}'' | U(t_f - t_i) | \vec{q}' \rangle = \int Dq \exp \left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(\vec{q}, \dot{\vec{q}}) \right). \quad (2.16)$$

The semiclassical propagator is obtained from

$$\sigma_c(\gamma) = \left(\frac{2\alpha}{\pi^2 v} \right)^2 \frac{\gamma^4}{[(2\gamma/\pi)^2 - 1]^2 \sin \gamma \cos \gamma (\tan \gamma - \gamma)}. \quad (2.11)$$

Note that it is singular at the rainbow angles as well as at $\theta = \pi$.

It is useful to know the asymptotic form of \vec{q} for large times. If we write

$$\vec{q}(t) = \vec{x}(t) + \vec{v}(t)t, \quad (2.12a)$$

where $\vec{v}(t)$ is the velocity, it follows from (2.7) that

$$\vec{x}(t) \rightarrow \epsilon(\alpha)s(\sin \gamma, \pm \cos \gamma, 0) \text{ as } t \rightarrow \pm \infty, \quad (2.12b)$$

$$\vec{v}(t) \rightarrow v(\pm \sin \psi \cos \gamma, -\sin \psi \sin \gamma, \pm \cos \psi) \text{ as } t \rightarrow \pm \infty. \quad (2.12c)$$

Note that

$$\vec{x}(t) \cdot \vec{v}(t) \rightarrow 0 \text{ as } t \rightarrow \pm \infty. \quad (2.12d)$$

In what follows, we often denote the variables at the initial and final times under consideration by single and double primes.

B. Quantum corrections

In passing to the quantum case, we wish to describe the system in terms of scattering states. If the momentum of the incident particle is \vec{p}' , the scattering wave function is

$$\langle \vec{q}'' | \psi^* \rangle = \lim_{t_f \rightarrow +\infty, t_i \rightarrow -\infty} \langle \vec{q}'' | U(t_f - t_i) | \vec{p}' \rangle \times \exp \left(\frac{i}{\hbar} \frac{p'^2}{2m} (t_f - t_i) \right), \quad (2.13)$$

where $U(t_f - t_i)$ is the evolution operator for the system. The amplitude for scattering to a final momentum state $|\vec{p}''\rangle$ is

$$f(\vec{p}'', \vec{p}') = \lim_{q'' \rightarrow \infty} q'' \exp \left(-\frac{i}{\hbar} \vec{p}'' \cdot \vec{q}'' \right) \langle \vec{q}'' | \psi^* \rangle, \quad (2.14)$$

where $\vec{p}'' = p' \hat{q}''$, and we consider for simplicity nonforward scattering. Substituting (2.13) and inserting a complete set of states $|\vec{q}'\rangle$, we can write

(2.16) by a well-known stationary-phase approximation on the path integral.^{1,12} One finds,

$$\langle \vec{q}'' | U(t_f - t_i) | \vec{q}' \rangle = \sum_{\nu} \delta_{\nu}(2\pi i \hbar)^{-3/2} |S_{\vec{q}', \vec{q}''}^{(\nu)}|^{1/2} \times \exp \left(\frac{i}{\hbar} S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i) \right). \quad (2.17)$$

Here $|S^{(\nu)}|$ is the absolute value of $\det(\partial^2 S^{(\nu)} / \partial q_i' \partial q_j')$ and $S^{(\nu)} \equiv S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i)$ is the Hamilton's principal function given by

$$S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i) = \int_{t_i}^{t_f} dt L(\vec{q}, \dot{\vec{q}}). \quad (2.18)$$

The integration is along the ν th classical path connecting the initial position \vec{q}' with the final position \vec{q}'' in a time $t_f - t_i$. The δ_ν 's are JWKB

phase factors.

The semiclassical scattering amplitude can be obtained from (2.15) and (2.17) (cf. Ref. 1 or Appendix A). We find

$$f(p'', p') = \sum_{\nu}^{\delta_\nu} \epsilon_\nu |[\sigma_C(\gamma_\nu)]^{1/2}| \exp\left(\frac{i}{\hbar} B^{(\nu)}\right), \quad (2.19)$$

where

$$B^{(\nu)}(\vec{p}'', \vec{p}') = \lim_{t_f \rightarrow \infty, t_i \rightarrow -\infty} \lim_{q'' \rightarrow \infty} [S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i) + E(t_f - t_i) + \vec{p}' \cdot \vec{q}' - \vec{p}'' \cdot \vec{q}''] \quad (2.20)$$

Here $E = p'^2/2m$, and \vec{q}' is defined in terms of \vec{p}' through

$$\frac{\partial S^{(\nu)}}{\partial q_i'} = -p_i'. \quad (2.21)$$

The JWKB phase factor ϵ_ν will be computed in Appendix B.

The function $B^{(\nu)}$ simplifies considerably for charge-monopole scattering if we use (2.1), (2.12) and the constancy of v along the trajectory. We find

$$B^{(\nu)} = e \int_{C_\nu} \vec{A} \cdot d\vec{q}, \quad (2.22)$$

where the integration is over the entire trajectory C_ν . Unlike the phase of the Coulomb scattering amplitude, we shall see in Sec. IV that $B^{(\nu)}$ is finite.³

We note from (2.10) and (2.19) that the quantum differential cross section σ_Q differs from σ_C by the interference terms

$$\sum_{\nu \neq \rho} \epsilon_\nu^* \epsilon_\rho |[\sigma_C(\gamma_\nu) \sigma_C(\gamma_\rho)]^{1/2}| \exp\left(-i \frac{B^{(\nu)} - B^{(\rho)}}{\hbar}\right). \quad (2.23)$$

We may point out that our approximation breaks down near the rainbow angles and $\theta = \pi$. This is due to the fact that near these angles the stationary points in the integral in (2.15) are not well separated. The approximation can be corrected by a known procedure near these angles (see Refs. 1 and 2). We have not carried out this procedure, as it is quite involved in our formulation.

III. ROTATIONAL PROPERTIES OF THE EXACT AND SEMICLASSICAL AMPLITUDES

This section is divided into three parts. In part A, we give a preliminary discussion of the rotational properties of some important classical variables, while in part B, the rotational proper-

ties of the exact and semiclassical amplitudes are discussed and shown to be mutually consistent. Finally in part C, we show that the semiclassical differential cross section is independent of \hat{n} .

A. Classical considerations

We begin by studying the transformation of the vector potential under rotations of \hat{n} . By (2.3), $\vec{\nabla} \times \vec{A}$ is unchanged when \hat{n} is rotated except when \vec{q} is proportional to \hat{n} or $R\hat{n}$. These singular configurations need not concern us for the moment. Thus,

$$\vec{A}(R\hat{n}, \vec{q}) - \vec{A}(\hat{n}, \vec{q}) = \vec{\nabla} \Lambda(R, \hat{n}, \vec{q}) \quad (3.1)$$

for any rotation R . \vec{A} falls off as q^{-1} as $q \rightarrow \infty$. Thus, if $\vec{q}(t)$ is a classical trajectory, (3.1) shows that

$$[\partial_t \Lambda]_{\vec{q}=\vec{q}(t)} \rightarrow 0 \text{ as } |t| \rightarrow \infty. \quad (3.2)$$

Also note that since $g^{-1}A$ has the dimension of inverse length, Λ depends only on the unit vector \hat{q} :

$$\Lambda = \Lambda(\mathbf{R}, \hat{n}, \hat{q}). \quad (3.3)$$

[Λ may be geometrically determined from the solid angle swept out by \hat{n} under the rotation R , as seen by the observer at \hat{q} .¹³ Equation (3.3) states that, in fact, it depends only on the direction of the observer.] As $t \rightarrow \pm\infty$ along a classical trajectory, $\hat{q}(t) \rightarrow \pm \hat{v}(\pm\infty)$ from (2.12a). Further, the canonical momentum

$$\vec{p} = m \dot{\vec{q}} + e \vec{A}(\hat{n}, \vec{q}) \quad (3.4)$$

approaches $m \vec{v}(\pm\infty)$ in these limits. Thus,

$$\Lambda \rightarrow \Lambda(\mathbf{R}, \hat{n}, \pm \hat{v}(\pm\infty)) \text{ as } t \rightarrow \pm\infty. \quad (3.5)$$

Next let us consider the transformation of \vec{p} under a rotation R of \vec{q} and \vec{q} . By (3.4) and (3.1),

$$p_i - p_i^R = R_{ij} p_j + e R_{ij} \partial_j \Lambda(\vec{R}, \hat{n}, \hat{q}). \quad (3.6)$$

Here \vec{R} is the transpose of R , and we have used the identity $A_i(R\hat{n}, R\vec{q}) = R_{ij} A_j(\hat{n}, \vec{q})$. Thus

$$p_i^R \rightarrow R_{ij} p_j \text{ as } t \rightarrow \pm\infty \quad (3.7)$$

by (3.2)

Finally, we consider the properties of $S^{(\nu)}$ and $B^{(\nu)}$. If $\vec{q}(t)$ is a classical trajectory with $\vec{q}(t_i) = \vec{q}'$ and $\vec{q}(t_f) = \vec{q}''$, then $R\vec{q}(t)$ is the classical trajectory with the boundary conditions rotated by R . Thus

$$\begin{aligned} S^{(\nu)}(R\vec{q}'', R\vec{q}', t_f - t_i) &= \int_{t_i}^{t_f} dt L(R\vec{q}, R\dot{\vec{q}}) \\ &= S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i) \\ &\quad + e[\Lambda(\vec{R}, \hat{n}, \hat{q}'') - \Lambda(\vec{R}, \hat{n}, \hat{q}')]. \end{aligned} \quad (3.8)$$

Similarly, it follows from (2.22) that

$$\begin{aligned} B^{(\nu)}(R\vec{p}'', R\vec{p}') &= B^{(\nu)}(\vec{p}'', \vec{p}') \\ &\quad + e[\Lambda(\vec{R}, \hat{n}, \hat{p}'') - \Lambda(\vec{R}, \hat{n}, -\hat{p}')], \end{aligned} \quad (3.9)$$

where we have used (3.5).

B. Exact and semiclassical transformation laws

Here we first find the transformation under rotations of the exact quantum-mechanical propagation function and scattering amplitude. Then we will show that the corresponding semiclassical approximations transform in an identical way.

Consider first the exact propagation function. Using the rotational invariance of the measure, $Dq = D(Rq)$, and the identity

$$\begin{aligned} \int_{t_i}^{t_f} dt L(R\vec{q}, R\dot{\vec{q}}) &= \int_{t_i}^{t_f} dt L(\vec{q}, \dot{\vec{q}}) \\ &\quad + e[\Lambda(\vec{R}, \hat{n}, \hat{q}'') - \Lambda(\vec{R}, \hat{n}, \hat{q}')], \end{aligned} \quad (3.10)$$

it readily follows from (2.16) that

$$\begin{aligned} \langle R\vec{q}'' | U(t_f - t_i) | R\vec{q}' \rangle &= \exp\left(\frac{i}{\hbar} e\Lambda(\vec{R}, \hat{n}, \hat{q}'')\right) \\ &\quad \times \langle \vec{q}'' | U(t_f - t_i) | \vec{q}' \rangle \\ &\quad \times \exp\left(-\frac{i}{\hbar} e\Lambda(\vec{R}, \hat{n}, \hat{q}')\right). \end{aligned} \quad (3.11)$$

The transformation property of the scattering amplitude can be obtained from (3.11), (2.15), the identity $d^3q' = d^3(Rq')$ and the change of variables from \vec{q}' to $\vec{x}' = \vec{q}' - (\vec{p}'/m)t_i$.¹⁴ We find

$$\begin{aligned} f(R\vec{p}'', R\vec{p}') &= \exp\left(\frac{i}{\hbar} \Lambda(\vec{R}, \hat{n}, \hat{p}'')\right) f(\vec{p}'', \vec{p}') \\ &\quad \times \exp\left(-\frac{i}{\hbar} \Lambda(\vec{R}, \hat{n}, -\hat{p}')\right). \end{aligned} \quad (3.12)$$

The semiclassical propagator is given by (2.17). By examining the transformation law for $S^{(\nu)}$ [Eq. (3.8)], we see that the semiclassical answer will have the correct transformation law, if $|S_{\vec{q}'\vec{q}''}^{(\nu)}|$ is invariant under rotations. To prove the latter, we differentiate (3.8) on \vec{q}'' and \vec{q}' and find

$$\begin{aligned} \frac{\partial^2 S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i)}{\partial q_i'' \partial q_j'} &= \frac{\partial^2 S^{(\nu)}(R\vec{q}'', R\vec{q}', t_f - t_i)}{\partial q_i'' \partial q_j'} \\ &= \tilde{R}_{ik} \frac{\partial^2 S^{(\nu)}(\vec{\xi}, \vec{\eta}, t_f - t_i)}{\partial \xi_k \partial \eta_l} \Bigg|_{\substack{\vec{\xi}=R\vec{q}'' \\ \vec{\eta}=R\vec{q}'}} R_{lj}. \end{aligned} \quad (3.13)$$

The result then follows by taking the determinant and using $\det R = 1$. Similarly, the proof for the semiclassical scattering amplitude follows from (2.19) and (3.9) and the rotational invariance of $\sigma_c(\gamma_\nu)$.

The rotational invariance of the semiclassical differential cross section is evident from the structure of the interference terms (2.23) and (3.9). This suggests that the interference terms are independent of \hat{n} . We give a geometrical proof of this fact in the next part.

It may be noted that the preceding proof is easily generalized to show the consistency of the semiclassical approximation with rotational invariance to all orders of \hbar . To show this for the propagator, we write (2.16) as

$$\begin{aligned} \langle \vec{q}'' | U(t_f - t_i) | \vec{q}' \rangle &= \sum_\nu \exp\left(\frac{i}{\hbar} S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i)\right) \\ &\quad \times \left[\int Dq \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(\vec{q}, \dot{\vec{q}}) \right. \right. \\ &\quad \left. \left. - \frac{i}{\hbar} S^{(\nu)}(\vec{q}'', \vec{q}', t_f - t_i) \right) \right]. \end{aligned} \quad (2.16')$$

The higher orders in \hbar for the ν th term in (2.16') are generated by expanding the (arbitrary) trajectory in the action in a power series about the ν th classical trajectory and keeping suitable terms. But by (3.8) and (3.10), the expression in the square brackets is invariant under the rotation $\vec{q}' \rightarrow R\vec{q}'$, $\vec{q}'' \rightarrow R\vec{q}''$. Therefore, so are the contributions of each term of the power series to the square bracket. The required result then follows from (3.8) and (3.10). A similar proof can also be constructed for scattering amplitudes.

C. Invariance of σ_Q under rotations of \hat{n}

The following discussion is similar to that of Schwinger.⁵ Let us begin by simplifying the expression for $B^{(\nu)}$ given by (2.22). The contour of integration C_ν can be deformed to C'_ν without chang-

ing the value of $B^{(\nu)}$ provided both contours have the same end points and the following condition is satisfied:

$$\int_S d\vec{S} \cdot (\vec{\nabla} \times \vec{A}) = 0. \quad (3.14)$$

Here S is the surface enclosed by C_ν and C'_ν .

Substituting (2.3), (3.14) becomes¹⁵

$$0 = -\frac{g}{4\pi} \int_S d\vec{S} \cdot \frac{\vec{q}}{q^3} + \begin{cases} \pm \frac{1}{2}g & \text{if a ray along} \\ & \pm \hat{n} \text{ intersects } S, \\ 0 & \text{otherwise} \end{cases} \quad (3.15)$$

where the sign ambiguity in the last term is due to the uncertainty in the relative orientation of $d\vec{S}$ and \hat{n} . We will consider only such deformations where C'_ν is obtained by continuously changing C_ν along the surface of the cone defined by the trajectory holding the end points \vec{q}' and \vec{q}'' of the trajectory fixed. Here we exclude the vertex point from the definition of the cone's surface. It is evident that these deformations will satisfy (3.15) since a surface element $d\vec{S}$ on the cone has no component in the \hat{q} direction. Thus

$$B^{(\nu)} = e \int_{C'_\nu} \vec{A} \cdot d\vec{q}. \quad (3.16)$$

Furthermore, let us restrict the deformed path C'_ν to an arc segment of a circle on the cone. The polar angle φ of the arc segment goes from γ to $-\gamma$. [Cf. (2.12c). Note that $\hat{q}' = -\hat{\beta}'$, $\hat{q}'' = \hat{\beta}''$.] Let us choose $\gamma > 0$, that is, $eg < 0$ for definiteness. [Cf. (2.9) and (2.5).] Then, for $\gamma \geq \pi$, (3.16) will contain closed-loop integrals, so we can write for any γ ,

$$B^{(\nu)} = e \int_{\zeta_\nu} \vec{A} \cdot d\vec{q} + k_\nu \Phi_\nu, \quad (3.17)$$

where k_ν is the winding number ($=0, 1, 2, \dots$) (which we define as the number of times the trajectory C_ν crosses itself), ζ_ν runs from γ to $-\gamma + 2k_\nu\pi$, and

$$\Phi_\nu = e \oint \vec{A} \cdot d\vec{q}. \quad (3.18)$$

Here the integral is from π to $-\pi$. From Gauss's law and (3.15),

$$\Phi_\nu = \frac{eg}{4\pi} \Omega_\nu + \begin{cases} \pm \frac{1}{2}eg & \text{if } \hat{n} \text{ is inside the cone} \\ 0 & \text{if } \hat{n} \text{ is outside the cone,} \end{cases} \quad (3.19)$$

where Ω_ν is the solid angle of the cone.

Now let us consider the interference from two trajectories ν and ρ . The trajectories will trace

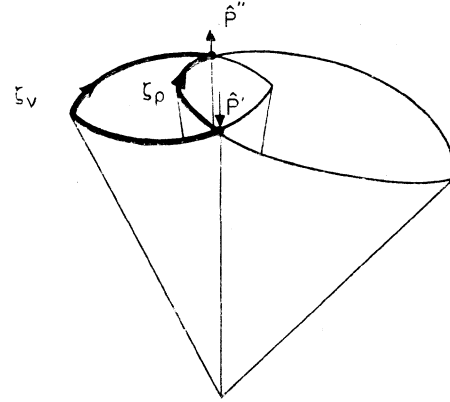


FIG. 2. The contours ζ_ν and ζ_ρ on the ν th and ρ th cones which contribute to the same scattering angle (see Sec. III C).

out two different cones which have common vertex points and asymptotic directions $\hat{\beta}'$ and $\hat{\beta}''$. We wish to examine the difference $B^{(\nu)} - B^{(\rho)}$. From (3.17)

$$B^{(\nu)} - B^{(\rho)} = e \int_{\zeta_\nu} \vec{A} \cdot d\vec{q} - e \int_{\zeta_\rho} \vec{A} \cdot d\vec{q} + k_\nu \Phi_\nu - k_\rho \Phi_\rho, \quad (3.20)$$

where ζ_ν and ζ_ρ are arcs of circles from \vec{q}' to \vec{q}'' on the ν th and ρ th cone respectively as illustrated in Fig. 2. Since $\zeta_\nu - \zeta_\rho$ is a closed contour, we can replace the first two terms in (3.20) by a flux integral $\Phi_{\nu\rho}$. Using Gauss's law,

$$\Phi_{\nu\rho} = e \int_{S_{\nu\rho}} d\vec{S} \cdot (\vec{\nabla} \times \vec{A}), \quad (3.21)$$

where $S_{\nu\rho}$ is the surface enclosed by ζ_ν and ζ_ρ . Once again applying (3.15),

$$\Phi_{\nu\rho} = \frac{eg}{4\pi} \Omega_{\nu\rho} + \begin{cases} \pm \frac{1}{2}eg & \text{if } \hat{n} \text{ intersects } S_{\nu\rho} \\ 0 & \text{if } \hat{n} \text{ does not} \\ & \text{intersect } S_{\nu\rho} \end{cases} \quad (3.22)$$

where $\Omega_{\nu\rho}$ is the solid angle subtended by $S_{\nu\rho}$ at the vertex. Thus

$$\frac{1}{\hbar} (B^{(\nu)} - B^{(\rho)}) = \frac{1}{\hbar} (\Phi_{\nu\rho} + k_\nu \Phi_\nu - k_\rho \Phi_\rho). \quad (3.23)$$

Let us consider what happens to (3.23) if we rotate \hat{n} . Obviously the Φ 's will not change if we restrict \hat{n} from passing through either one of the cone surfaces, so neither will $(1/\hbar)(B^{(\nu)} - B^{(\rho)})$. If on the other hand we allow \hat{n} to pass through a surface, the right-hand side of (3.23) picks up an additional multiple of $\pm eg/2\hbar$, which from the Dirac quantization condition equals $2\pi n$ ($n=0, \pm 1, \pm 2, \dots$).⁹ But this added term does not contribute in the inter-

ference term (2.23). Thus, the semiclassical differential cross section is unaffected by the position of \hat{n} .

IV. COMPUTATION OF $B^{(\nu)}$

If \hat{n} is not a tangent to the cone, Eq. (3.16) gives $B^{(\nu)}$ with C'_ν restricted to an arc segment of a circle on the cone from γ to $-\gamma$. Here \hat{n} is yet to be specified. We have found it convenient to choose¹⁶

$$\hat{n} = \frac{\vec{p}' \times \vec{p}''}{|\vec{p}' \times \vec{p}''|} . \quad (4.1)$$

To evaluate $B^{(\nu)}$ with this choice of \hat{n} , we proceed as follows. Let R_0 be the rotation which brings \hat{n} to the cone axis, which we can choose to be the third axis:

$$R_0 \hat{n} = \hat{n}_0 = (0, 0, 1) . \quad (4.2)$$

Then (3.16) and (3.1) show that

$$B^{(\nu)} = e \int_{-\gamma}^{\gamma} d\varphi q \sin\psi A_\varphi(\hat{n}_0, \vec{q}) - e[\Lambda(R_0, \hat{n}, \hat{p}'') - \Lambda(R_0, \hat{n}, -\hat{p}')] , \quad (4.3)$$

where A_φ is the component of \vec{A} in the direction $\hat{\varphi} = (-\sin\varphi, \cos\varphi, 0)$. In deriving (4.3), we have used the fact that $\hat{q}(t) \rightarrow -\hat{p}'$ as $t \rightarrow -\infty$, and $-\hat{p}''$ as $t \rightarrow +\infty$. The first term in (4.3) is trivial to integrate using the explicit form of A [Eq. (2.2)]. One finds

$$e \int_{-\gamma}^{\gamma} d\varphi q \sin\psi A_\varphi(\hat{n}_0, \vec{q}) = -\frac{e\mathcal{G}}{2\pi} \gamma \cos\psi \quad (4.4)$$

It remains to compute Λ 's. According to Zwanziger,¹⁷ for an arbitrary rotation R ,

$$\Lambda(R, \hat{n}, \hat{m}) = \frac{\mathcal{G}}{4\pi} \omega(R, \hat{n}, \hat{m}) , \quad (4.5)$$

where

$$\cos\omega(R, \hat{n}, \hat{m}) = \frac{(R\hat{n} \times \hat{m})}{|R\hat{n} \times \hat{m}|} \cdot \frac{(\hat{n} \times \hat{m})}{|\hat{n} \times \hat{m}|} \quad (4.6a)$$

and

$$\sin\omega(R, \hat{n}, \hat{m}) = \frac{(R\hat{n} \times \hat{m})}{|R\hat{n} \times \hat{m}|} \cdot \frac{[\hat{m} \times (\hat{n} \times \hat{m})]}{|\hat{n} \times \hat{m}|} . \quad (4.6b)$$

(Zwanziger's calculations are in four dimensions. However, they are readily adapted to three dimensions.)

Using (4.2) and the expressions for \vec{p}' and \vec{p}'' given by (2.12c), one finds¹⁸

$$\omega(R_0, \hat{n}, \hat{p}'') = -\omega(R_0, \hat{n}, -\hat{p}') \quad (4.7)$$

and

$$\cos\omega(R_0, \hat{n}, -\hat{p}') = \frac{\pi}{|\sin\theta|} \frac{\cos\gamma \sin\gamma}{\gamma} , \quad (4.8a)$$

$$\sin\omega(R_0, \hat{n}, -\hat{p}') = \frac{\sin^2\gamma}{|\sin\theta|} \frac{\pi}{\gamma} \left[1 - \left(\frac{\pi}{2\gamma} \right)^2 \right]^{1/2} . \quad (4.8b)$$

The explicit form of R_0 is not necessary for the calculations. Finally,

$$B^{(\nu)} = -\frac{e\mathcal{G}}{2\pi} [\gamma \cos\psi - \omega(R_0, \hat{n}, -\hat{p}')] . \quad (4.9)$$

We have verified this answer using geometrical considerations based on the fact that Λ is proportional to a certain solid angle (as mentioned earlier) (cf. Ref. 12). We will not reproduce these calculations here.

To complete the evaluation of the scattering amplitude

$$f(\vec{p}'', \vec{p}') = \sum_{\nu}^{N(\theta)} \epsilon_{\nu} [|\sigma_c(\gamma_{\nu})|^{1/2} \exp\left(\frac{i}{\hbar} B^{(\nu)}\right)] , \quad (2.19)$$

it remains to specify the phase factor ϵ_{ν} . Let $\gamma_1, \gamma_2, \dots, \gamma_{N(\theta)}$ denote the γ 's with increasing modulus which contribute to the sum, with associated phase factors $\epsilon_1, \epsilon_2, \dots, \epsilon_{N(\theta)}$. We show in Appendix B that

$$\epsilon_{\nu} = \exp\left(-\frac{i\pi\nu}{2}\right) . \quad (4.10)$$

The semiclassical cross section found here is identical to that found by Schwinger *et al.*³ [their Eqs. (3.60) and (3.61)¹⁹], which was derived using the methods of Ford and Wheeler.

V. CONCLUDING REMARKS

From the expressions for the differential cross-section [cf. equations (2.10), (2.11), (2.23), (4.9), and (4.10)], one sees that $G_A \equiv (\pi^2 v / 2\alpha)^2 \sigma_A$ is independent of v for a fixed value of γ (i.e., $\cos\theta$) for both $A=C$ and $A=Q$. G_Q has been plotted in Figs. 3(a)–3(d) for $n=1, 3, 20, 50$. G_C is plotted in Fig. 3(e). For the same n , we have plotted the ratio of the interference terms to the classical cross section $R \equiv (G_Q - G_C)/G_C$ in Figs. 4(a)–4(d). (The differential cross sections are invariant under $n \rightarrow -n$ so that negative n need not be considered. Also in the figures we emphasize the region where multiple trajectories contribute. The classical and quantum cross sections are of course equal in the region of one contributing trajectory.) From the figures we see that even for n as large as 50, the corrections to the classical cross section can be appreciable at some angles. However, the interference terms oscillate more rapidly with increasing n . Thus, when averaged

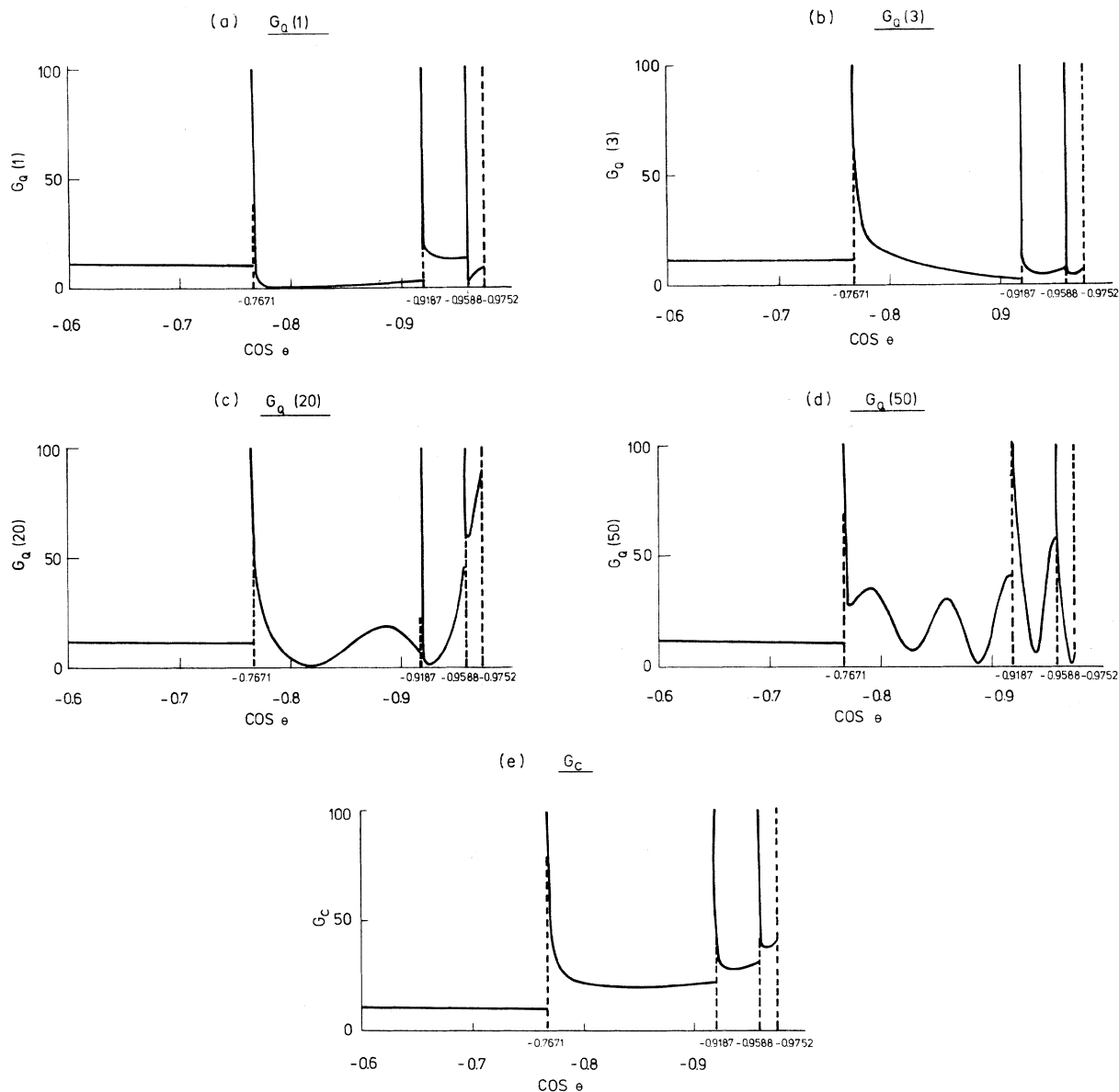


FIG. 3. (a)–(e) The normalized classical and semiclassical differential cross sections G_C and $G_Q(n)$ versus the cosine of the scattering angle θ . The normalization is given by $G_A = (\pi^2 v / 2\alpha)^2 \sigma_A$ for both $A = Q$ and C . $G_Q(n)$ is plotted for $n = 1, 3, 20, 50$.

over small angular intervals, the interference terms will tend to zero as $n \rightarrow \infty$, and the classical answer will be recovered. But it is noteworthy that the classical limit is approached quite slowly away from the backward direction. Figure 3(d) shows that even for $n = 50$, the period of oscillation in the region of three contributing trajectories ($-0.7671 > \cos\theta > -0.9187$) is about 7° . The oscillations are quite rapid in the region with nine or more contributing trajectories ($\cos\theta < -0.9752$) for any $n \neq 0$. We have not plotted this region in Figs. 3 and 4.

We point out once again that our approximation breaks down near the rainbow angles and near $\theta = \pi$. For a discussion of these regions see Ford and Wheeler.³ Even upon excluding these regions our answer for f exhibits a very nonanalytic nature. The analytic continuation of f from an interval where N trajectories contribute to one where N' trajectories contribute ($N \neq N'$) does not coincide with the actual value of f in the second region. Thus, the semiclassical scattering amplitude is only piecewise analytic. It is interesting to investigate whether any such nonanalytic feature

persists in the exact amplitude. We have investigated this question in the following paper⁷ and find the answer to be negative. Even so, however, we have found some curious analytic structure of the exact amplitude. By finding an integral representation for the formally divergent exact partial-wave expansion of f , we find that a branch-point singularity exists at $\cos\theta=1$ on the physical sheet. The analytic continuation of f to unphysical sheets exhibits additional logarithmic singularities at $\cos\theta=-1$. By contrast, when a well-known over-all phase is removed, the Coulomb scattering amplitude has only a simple pole in the forward direction.

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APPENDIX A

Here we indicate the derivation of (2.19). Substituting (2.17) in (2.15), and making a stationary-phase approximation on the integral, we find

$$f(\vec{p}'', \vec{p}') = \sum_{\nu}^{\mathcal{N}(\theta)} \epsilon_{\nu} \left(\lim q'' \frac{|S_{\vec{q}'\vec{q}'}^{(\nu)}|^{1/2}}{|S_{\vec{q}'\vec{q}'}^{(\nu)}|^{1/2}} \right) \exp\left(\frac{i}{\hbar} B^{(\nu)}\right), \quad (\text{A1})$$

where \lim indicates the usual limits on q'' , t_f , and t_i , and \vec{q}' is given by (2.21). Using well-known results of classical mechanics,²⁰ one can show that

$$\frac{|S_{\vec{q}'\vec{q}'}^{(\nu)}|^{1/2}}{|S_{\vec{q}'\vec{q}'}^{(\nu)}|^{1/2}} = \left| \frac{\partial \vec{q}''(\vec{q}', \vec{p}', t_f - t_i)}{\partial \vec{q}'} \right|^{-1/2}. \quad (\text{A2})$$

The notation $\vec{q}''(\vec{q}', \vec{p}', t)$ indicates the position along a classical trajectory at time t which at time zero had position \vec{q}' and momentum \vec{p}' . The right-hand side is to be evaluated along the ν th classical tra-

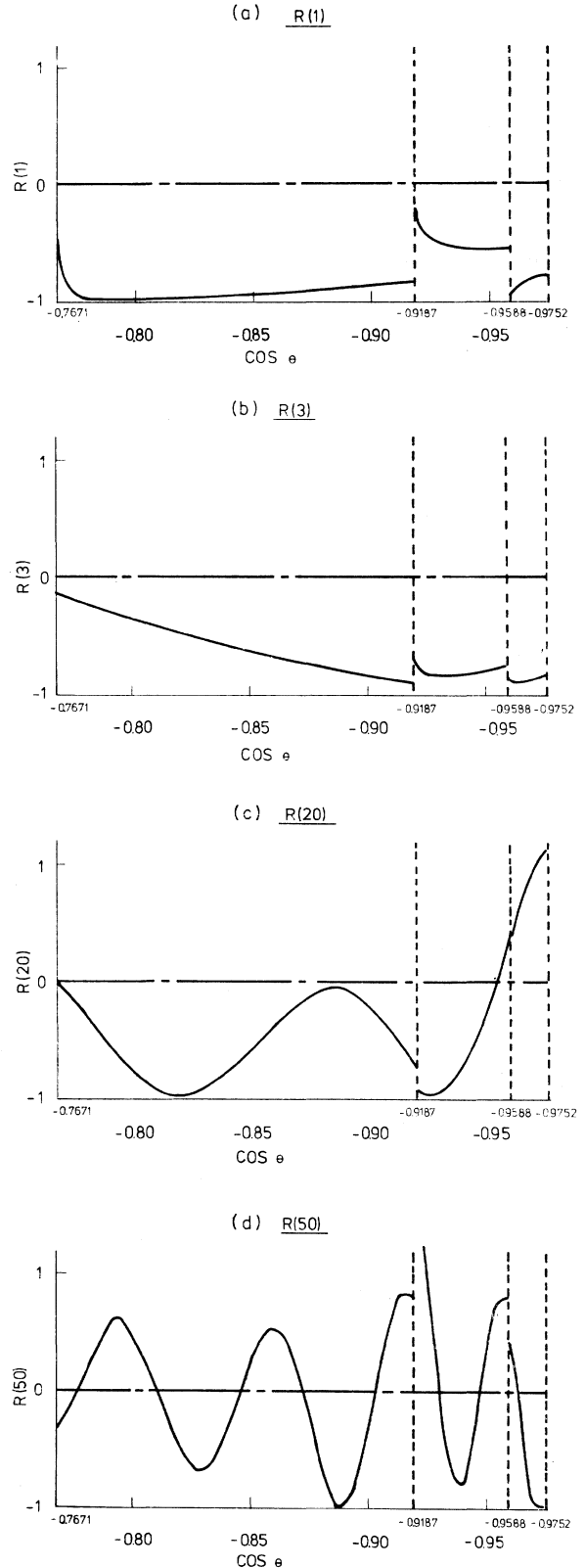


FIG. 4. (a)–(d) Plots of the ratio $R(n) = [G_Q(n) - G_C] / G_C$ for $n = 1, 3, 20, 50$.

jectory. Finally, Pechukas shows that¹⁰

$$\lim q'' \left| \frac{\partial \vec{q}''(\vec{q}', \vec{p}', t_f - t_i)}{\partial \vec{q}'} \right|^{-1/2} = |[\sigma_c(\gamma_\nu)]^{1/2}| \quad (\text{A3})$$

from which (2.19) follows.

APPENDIX B

Here we discuss how the JWKB phases ϵ_ν were determined to be $\exp(-i\pi\nu/2)$ in (4.10). According to a well-known result,¹ the integer ν is the number of zeros of the determinant

$$\Delta(\gamma, t) = \det \left(\frac{\partial q_i''(\vec{q}', \vec{p}', t)}{\partial q_j'} \right) \quad (\text{B1})$$

when $\gamma = \gamma_\nu$ and the time t varies from $-\infty$ to $+\infty$. Here $\vec{q}''(\vec{q}', \vec{p}', t)$ is the position on the trajectory at time t , which at time $-\infty$ had position \vec{q}' and momentum \vec{p}' . Also a zero of order n is to be counted as n zeros. Since in our problem $\vec{p}' = m\vec{v}'$, we can replace the initial momentum by the initial velocity in (B1).

As a preliminary to the analysis, we prove the following two identities:

$$v_i'' = \frac{\partial q_i''}{\partial q_j'} v_j', \quad (\text{B2})$$

$$(\vec{q}'' \times \vec{v}')_i = \frac{\partial q_i''}{\partial q_j'} (\vec{q}' \times \vec{v}')_j. \quad (\text{B3})$$

Here \vec{v}'' is the velocity at time t . The proof of (B2) follows from time-translation invariance. Let ξ be the value of a variable on a trajectory at time t and ξ^T its value at time $t+T$. Then one has the obvious identity $\vec{q}''(\vec{q}', \vec{v}', t) = \vec{q}''(\vec{q}^T, \vec{v}^T, t)$. At large negative times the particle is free so that $\vec{q}^T = \vec{q}' + \vec{v}'T$, $\vec{v}^T = \vec{v}'$. Using this fact and comparing linear terms in T , one gets (B2).

Equation (B3) is a consequence of rotational invariance of the equation of motion. If R is an arbitrary rotation, $R\vec{q}''(\vec{q}', \vec{v}', t) = \vec{q}''(R\vec{q}', R\vec{v}', t)$. Choosing R to be an infinitesimal rotation around \vec{v}' , that is, $R_{ij} \approx \delta_{ij} + \delta\epsilon_{ijk}v'_k$, one finds (B3).

Let us now rewrite Δ in a convenient form. If \hat{m}^α and \hat{n}^α ($\alpha = 1, 2, 3$) are two orthonormal right-handed coordinate systems, then $\Delta = \det N$, where

$$N_{\alpha\beta} = n_i^\alpha \frac{\partial q_i''}{\partial q_j'} m_j^\beta.$$

If we now make the choice

$$\hat{m}^1 = \vec{v}', \quad \hat{m}^2 = \frac{\vec{q}' \times \vec{v}'}{|\vec{q}' \times \vec{v}'|}, \quad \hat{m}^3 = \hat{m}^1 \times \hat{m}^2, \quad (\text{B4})$$

$$\hat{n}^1 = \hat{v}'', \quad \hat{n}^2 = \hat{n}^3 \times \hat{n}^1, \quad \hat{n}^3 = \frac{\vec{v}'' \times (\vec{q}'' \times \vec{v}')}{|\vec{v}'' \times (\vec{q}'' \times \vec{v}')|}, \quad (\text{B5})$$

and use (B2) and (B3), we find

$$\Delta = \frac{v'' \hat{n}^2(\vec{q}'' \times \vec{v}')}{v' |\vec{q}' \times \vec{v}'|} \frac{\partial \vec{q}''}{\partial s} \cdot [\vec{v}'' \times (\vec{q}'' \times \vec{v}')] \frac{1}{|\vec{v}'' \times (\vec{q}'' \times \vec{v}')|}. \quad (\text{B6})$$

Here s is the impact parameter $\hat{m}^3 \cdot \vec{q}'$ and the differentiation with respect to s is carried out holding $\hat{m}^a \cdot \vec{q}'$ ($a = 1, 2$), \vec{v}' , and t fixed. (We sometimes do not use the identity $v'' = v'$ in order to facilitate our later discussion of central potentials.) If we substitute

$$\hat{n}^2 \cdot (\vec{q}'' \times \vec{v}') = \frac{1}{v''} |\vec{v}'' \times (\vec{q}'' \times \vec{v}')|, \quad (\text{B7})$$

Δ simplifies to

$$\Delta = \frac{\lambda}{v' |\vec{q}' \times \vec{v}'|}, \quad (\text{B8})$$

where

$$\lambda = \frac{\partial \vec{q}''}{\partial s} \cdot [\vec{v}'' \times (\vec{q}'' \times \vec{v}')]. \quad (\text{B9})$$

We exclude from our considerations zero-energy and/or zero-impact-parameter trajectories. So for us it suffices to consider the zeros of λ .

Let us first observe some elementary properties of λ . (1) From the expressions (2.7) for the trajectories, one sees that λ is a differentiable function of s and t for $0 < s < \infty$ and $-\infty < t < +\infty$. (2) The following asymptotic properties of λ can be derived from its definition:

$$\lambda \rightarrow |\vec{q}' \times \vec{v}'| v' \quad \text{as } t \rightarrow -\infty, \quad (\text{B10})$$

$$\lambda \rightarrow -v'^4 t^2 \frac{s}{\sigma_c(\gamma)} \quad \text{as } t \rightarrow +\infty, \quad (\text{B11})$$

where $\sigma_c(\gamma) = s \partial s / \partial \cos \theta$. The proof of (B10) follows by noticing that \vec{q}'' , $\vec{v}'' - \vec{q}'$, \vec{v}' , and $\partial q_i'' / \partial s = (\partial q_i'' / \partial q_j) m_j^3 - m_i^3$ as $t \rightarrow -\infty$. To show (B11), we use

$$\vec{q}'' \sim \vec{v}'' t = v' \hat{v}'' t \quad \text{as } t \rightarrow +\infty. \quad (\text{B12})$$

(Here, because of energy conservation, the equality $v'' = v'$ as $t \rightarrow +\infty$ is also valid for central potentials.) Thus

$$\frac{\partial \vec{q}''}{\partial s} \sim v' t \frac{\partial \hat{v}''}{\partial s}, \quad (\text{B13})$$

$$\vec{v}'' \times (\vec{q}'' \times \vec{v}') \sim t [\vec{v}'' (\vec{v}'' \cdot \vec{v}') - \vec{v}' v'^2].$$

Noticing also that since \hat{v}'' is a unit vector, we have $\vec{v}'' \cdot \partial \hat{v}'' / \partial s = 0$, (B9) simplifies to

$$\lambda \sim -v'^4 t^2 \frac{\partial (\hat{v}'' \cdot \hat{v}')}{\partial s}, \quad (\text{B14})$$

which is the same as (B11).

We can divide the zeros of λ into three classes

for charge-monopole scattering:

$$(A) \quad \vec{q}'' = \rho \vec{v}' ,$$

$$(B) \quad \vec{q}'' \times \vec{v}' = \kappa \vec{v}'' , \quad \kappa \neq 0 ,$$

$$(C) \quad \frac{\partial \vec{q}''}{\partial s} = a \vec{v}'' + b \vec{q}'' \times \vec{v}' \quad \text{and} \quad \vec{v}'' \times (\vec{q}'' \times \vec{v}') \neq 0 .$$

The configurations (A) and (B) have a simple physical interpretation. For convenience let us introduce the angle $\chi = \varphi - \gamma$. It has the significance of the change in φ as time increases from $-\infty$ to t . Along the trajectory, $|\chi|$ increases monotonically from 0 to $2|\gamma|$.²¹ Now case (A) occurs once for each complete revolution of the trajectory about the cone axis, that is whenever $|\chi| = 2n\pi$ ($n=1, 2, \dots$). Case (B) occurs only on certain backward scattering trajectories ($\cos\theta = -1$). On such trajectories, it occurs at the point of closest approach to the monopole (the origin). More precisely, it occurs for $|\gamma| = (2n-1)\pi$ ($n=1, 2, \dots$) and $t=0$ or equivalently $|\chi| = |\gamma|$ [cf. (2.7d) and (2.9)]. To show this, note from (B) that $\vec{q}'' \cdot \vec{v}'' = 0$ which means $t=0$. An explicit calculation of (B) at $t=0$ shows $|\gamma| = (2n-1)\pi$.

On the other hand, the location of case (C) zeros seems to have no obvious interpretation.

In the following, we will show that the zeros of λ are simple in all three cases. In addition, we will extract useful information on the nature of these zeros, in particular, on the sign of λ at these zeros. The latter will be especially important for the analysis of case (C).

Case (A)

It is convenient to rewrite λ in terms of $(\partial \vec{q}''/\partial s)_x$ rather than $(\partial \vec{q}''/\partial s)_t \equiv \partial \vec{q}''/\partial s$. For this note that

$$\left(\frac{\partial \vec{q}''}{\partial s} \right)_t = \left(\frac{\partial \vec{q}''}{\partial s} \right)_x + \left(\frac{\partial \vec{q}''}{\partial \chi} \right)_s \left(\frac{\partial \chi}{\partial s} \right)_t , \quad (B15)$$

$$\left(\frac{\partial \vec{q}''}{\partial t} \right)_s = \left(\frac{\partial \vec{q}''}{\partial \chi} \right)_s \left(\frac{\partial \chi}{\partial t} \right)_s . \quad (B16)$$

Since $(\partial \chi/\partial t)_s$ is never zero, $(\partial \vec{q}''/\partial \chi)_s$ is parallel or antiparallel to the velocity \vec{v}'' . Hence, (B9) gives

$$\lambda = \left(\frac{\partial \vec{q}''}{\partial s} \right)_x \cdot [\vec{v}'' \times (\vec{q}'' \times \vec{v}')] . \quad (B17)$$

Also, $\vec{q}'' = \rho \vec{v}'$ at $|\chi| = 2n\pi$ gives

$$\left(\frac{\partial \vec{q}''}{\partial s} \right)_x = \left(\frac{\partial \rho}{\partial s} \right)_x \vec{v}' \quad \text{at} \quad |\chi| = 2n\pi . \quad (B18)$$

To examine if the case (A) zero is simple, let us compute $\dot{\lambda}$. Using (B17) and (B18), we find

$$\dot{\lambda} = - \left(\frac{\partial \rho}{\partial s} \right)_{|\chi|=2n\pi} |\vec{v}'' \times \vec{v}'|^2 \quad \text{at} \quad |\chi| = 2n\pi . \quad (B19)$$

Here if $|\vec{v}'' \times \vec{v}'|$ is zero, then \vec{v}'' is proportional to $\vec{v}' = (1/\rho)\vec{q}''$. Thus, the orbital angular momentum and hence s vanishes. Since we exclude this case, $\dot{\lambda} = 0$ implies

$$\left(\frac{\partial \rho}{\partial s} \right)_{2n\pi} = 0 \quad (B20)$$

or

$$\rho \left(\frac{\partial \rho}{\partial s} \right)_x v^2 = \frac{1}{2} \left(\frac{\partial q''^2}{\partial s} \right)_x = 0 \quad \text{at} \quad |\chi| = 2n\pi . \quad (B21)$$

The form of q''^2 in terms of $|\chi|$ is, from (2.7),

$$q''^2 = s^2 \csc^2 \left(\frac{\pi |\chi|}{2 |\gamma|} \right) \quad (B22)$$

Thus,

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial q''^2}{\partial s} \right)_{|\chi|} &= \frac{q''^2}{s} \left\{ 1 - \left[1 - \left(\frac{\pi}{2\gamma} \right)^2 \right] \frac{\pi |\chi|}{2 |\gamma|} \right. \\ &\quad \left. \times \cot \left(\frac{\pi |\chi|}{2 |\gamma|} \right) \right\} , \\ &\equiv \frac{q''^2}{s} \Sigma \end{aligned} \quad (B23)$$

where we have used (2.9). The expression Σ at $|\chi| = 2n\pi$ can be written in two different ways in terms of the angle $y = \pi^2 n / |\gamma|$:

$$\Sigma = (1 - y \cot y) + \left(\frac{y}{2\pi n} \right)^2 y \cot y \quad (B24)$$

$$= 1 - y \left[1 - \left(\frac{y}{2\pi n} \right)^2 \right] \cot y . \quad (B25)$$

We want to show that Σ cannot vanish for the allowed range of y . Since $|\chi|$ is equal to $2n\pi$ and it is bounded by $2|\gamma|$, we have $|\gamma| \geq n\pi$. Thus, the range of y is $0 < y \leq \pi$. For $0 < y \leq \pi/2$, the second term in (B24) is positive, while the first term is also positive due to the well-known inequality $y \cot y < 1$ for $0 < y < \pi/2$. Thus, $\Sigma > 0$ for this range. On the other hand, for $\pi/2 < y \leq \pi$, $\cot y < 0$ and so (B25) shows that $\Sigma > 0$. This completes the proof that λ has only simple zeros in case (A).

Useful information on the sign of $\dot{\lambda}$ at case (A) zeros can be inferred from the preceding analysis. By (B19), (B21), and (B23) we have

$$\dot{\lambda} = \left[- \frac{\Sigma}{\rho} \right] \frac{q''^2}{s} \frac{|\vec{v}'' \times \vec{v}'|^2}{v'^2} \quad \text{at} \quad |\chi| = 2n\pi . \quad (B26)$$

It has been show that $\Sigma > 0$. Also $\rho < 0$ since the incoming velocity \vec{v}' is directed towards the origin and $\vec{q}'' = \rho \vec{v}'$. Thus, $\dot{\lambda} > 0$ at case A zeros.

Case (B)

Let us compute $\dot{\lambda}$ at $t=0$ and $|\gamma| = (2n-1)\pi$. From (B9) and $\vec{q}'' \times \vec{v}' = \kappa \vec{v}''$ ($\kappa \neq 0$),

$\sigma_C(\gamma)$ as $t \rightarrow +\infty$ [so long as we are away from the infinities of $\sigma_C(\gamma)$]. Note that by (2.11), $\text{sgn}[1/\sigma_C(\gamma)] = \text{sgn}[\sin 2\gamma \times (\tan \gamma - \gamma)]$.

We will successively discuss the following regions of γ : (a) $\pi/2 < |\gamma| < \pi$, (b) $|\gamma| = \pi$, (c) $\pi < |\gamma| < |\bar{\gamma}_1|$, (d) $|\gamma| = |\bar{\gamma}_1|$, (e) $|\bar{\gamma}_1| < |\gamma| < 2\pi, \dots$. The $\bar{\gamma}_\nu$'s are the roots of $\tan \gamma = \gamma$ arranged in order of increasing modulus.

(a) In Fig. 5(a), we plot λ/q^{n-2} versus $|\chi|$. The crosses indicate the points where λ cannot vanish, while the first plus, for example, signifies the fact that if λ has a [case (C)] zero for $0 < |\chi| < |\gamma|$, then $\dot{\lambda} > 0$ at that zero. [Cf. case (C) in table.] Similarly, the minus indicates that if λ has a zero for $|\gamma| < |\chi| < \pi$, then $\dot{\lambda} < 0$ at that zero.

The function λ/q^{n-2} starts out with a positive slope in $|\chi|$, and here approaches a negative limit as $|\chi| \rightarrow 2|\gamma|$ since $\sigma_C(\gamma) > 0$ by (2.11). Thus λ/q^{n-2} has an odd number of zeros in the intermediate region. From the constraints indicated on the graph, we then see that there can only be one zero, and it has to be located in the interval $|\gamma| < |\chi| < \pi$. This conclusion is indicated by the dashed line.

(b) A similar plot is made in 5(b). The dots indicate points where λ/q^{n-2} must vanish. The zero at $|\chi| = |\gamma| = \pi$ is a case (B) zero while the zero at $|\chi| = 2\pi$ follows from $\sigma_C(\pi) = \infty$. The dashed curve can be inferred from Fig. 5(a) using continuity in $|\gamma|$.

(c) In this region, $\sigma_C(\gamma) < 0$ so that λ now has an even number of zeros. The dashed curve in Fig. 5(c) follows either from the constraints on the zeros (including $\dot{\lambda} > 0$ at $|\chi| = 2\pi$ from the table) or from continuity in γ applied to Fig. 5(b). Thus, there are two zeros in this region.

(d), (e) The dashed curves of Figs. 5(d) and 5(e) are inferred from similar considerations. Here we find two zeros for (d) and three for (e) in the interval $0 < |\chi| < 2|\gamma|$.

(f), (g), ... Continuing the arguments in this fashion, we find that the ν th branch γ_ν in γ which contributes to a given $\cos \theta$ (cf. Fig. 1) is associated with precisely ν zeros. Thus Eq. (4.10) follows.

We may remark that the signs of $\dot{\lambda}$ at the case (A) and (B) zeros inferred from the dashed curves are in agreement with the table. This provides a check on our arguments.

We finally indicate how the number of zeros of $\Delta = \det(\partial q_i''/\partial q_j')$ can be found for central potentials using the preceding techniques. The reasoning is essentially that of Pechukas.¹ Equations (B2) and (B3) are still valid so that (B8) and (B9) are unchanged. Equations (B10) and (B11) are also true with γ replaced by s . It follows that there are zeros at (A) $v'' = 0$, that is at turning points of zero impact parameter trajectories, (B) $\vec{q}'' \times \vec{v}'' = 0$, that is, at every half

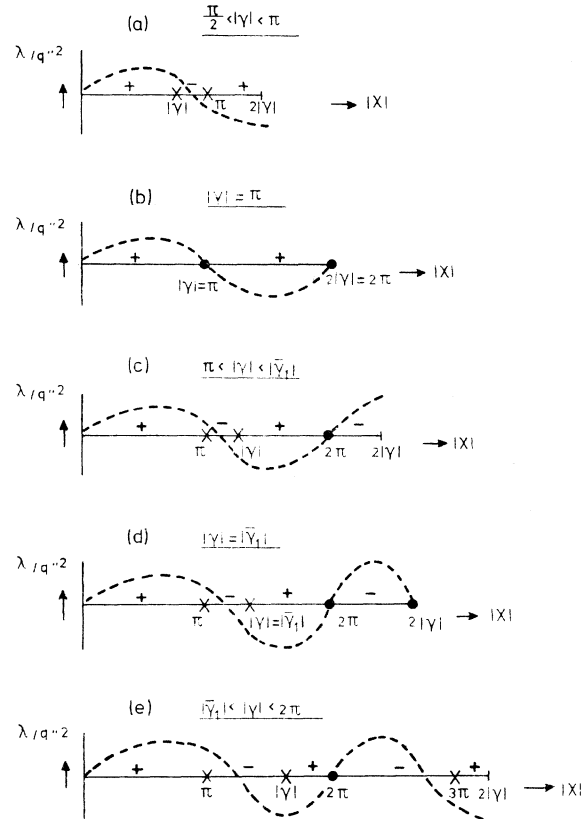


FIG. 5. (a)–(e) A schematic plot of λ/q^{n-2} versus $|\chi|$ for various values of $|\gamma|$. The figures are used to infer the number of zeros of $\Delta = \det(\partial q_i''/\partial q_j')$. The dots indicate the points where λ/q^{n-2} must have a zero. The crosses indicate the points where λ/q^{n-2} cannot have a zero. The pluses or minuses indicate the sign of $\dot{\lambda}$ at a zero of λ/q^{n-2} in the corresponding (open) interval of $|\chi|$, provided λ/q^{n-2} has a zero in that interval (see Appendix B).

circuit of the particle around the potential, (C) $\partial \vec{q}''/\partial s = a\vec{v}'' + b\vec{q}'' \times \vec{v}''$ with $\vec{v}'' \times (\vec{q}'' \times \vec{v}'') \neq 0$. (The configuration $\vec{q}'' \times \vec{v}'' = \kappa \vec{v}''$, $\kappa \neq 0$ is impossible since the trajectory is confined to a plane.) Case (C) can be analyzed by computing $\dot{\lambda}$. Then at a zero, (B44) holds where $\vec{C} = \vec{q}'' \times \vec{v}'' = \vec{q}'' \times \vec{v}''$ and ξ is the sign of $\vec{C} \cdot (\vec{q}'' \times \vec{v}'')$. Thus, ξ is positive for $t \rightarrow -\infty$ and flips sign after every half circuit. Further, $\vec{q}'' \cdot \vec{v}'' < 0$ before the closest distance of approach of the particle to the potential center and > 0 afterwards. Thus, as in the charge-monopole case, (B44) fixes the sign of $\dot{\lambda}$ at a zero depending on where the zero is located. On repeating the arguments which follow (B47), it is then found that there is no case (C) zero at all if $(-1)^N \sigma_C(s) < 0$ (where N is the number of half circuits), while there is precisely one case (C) zero if $(-1)^N \sigma_C(s) > 0$. Further, this zero is located in the region $\vec{q}'' \cdot \vec{v}'' > 0$.

$$\dot{\lambda} = \frac{\partial \vec{q}''}{\partial s} \cdot [\dot{\vec{v}}'' \times (\vec{q}'' \times \vec{v}') + \vec{v}'' \times (\vec{v}'' \times \vec{v}')] \text{ at } t=0. \quad (\text{B27})$$

Using $\vec{q}'' \cdot \dot{\vec{v}}'' = 0$ (which follows from the equation of motion), $\vec{v}'' \cdot \vec{v}' = 0$ (which follows from $\vec{q}'' \times \vec{v}' = \kappa \vec{v}''$) and (2.7b), $\dot{\lambda}$ simplifies to

$$\dot{\lambda} = s \dot{\vec{v}}'' \cdot \vec{v}' - v'^2 \vec{v}'' \cdot \frac{\partial \vec{q}''}{\partial s} \text{ at } t=0. \quad (\text{B28})$$

The first term can be evaluated from the equation of motion, the following expressions for \vec{q}'' and \vec{v}'' at $t=0$:

$$\vec{q}'' = s(\sin\psi, 0, \cos\psi), \quad (\text{B29})$$

$$\vec{v}'' = -\epsilon(\alpha)(0, v', 0), \quad (\text{B30})$$

and the form of \vec{v}' given by (2.12c). We find

$$s \dot{\vec{v}}'' \cdot \vec{v}' = -2v'^3 \cos^2\psi \text{ at } t=0, \quad (\text{B31})$$

where we have substituted $|\gamma| = (2n-1)\pi$. Similarly, from (B29) and (2.12c),

$$-v'^2 \vec{v}' \cdot \frac{\partial \vec{q}''}{\partial s} = v'^3 (\cos 2\psi - \frac{1}{2} \sin^2 2\psi) \text{ at } t=0. \quad (\text{B32})$$

Therefore, at $t=0$,

$$\dot{\lambda} = -v'^3 (1 + \frac{1}{2} \sin^2 2\psi), \quad (\text{B33})$$

$$\dot{\lambda} < 0 \text{ for } s \neq 0. \quad (\text{B34})$$

Thus, the case (B) zeros are also simple.

Case (C)

Let us assume that a case (C) zero occurs at time t_0 . Then $(\partial \vec{q}'' / \partial s)_{t_0} = a \vec{v}'' + b (\vec{q}'' \times \vec{v}')$. Next define a differential operator D ,

$$D = \frac{\partial}{\partial s} - [av'_j + b(\vec{q}'' \times \vec{v}')_j] \frac{\partial}{\partial q'_j} \quad (\text{B35})$$

The coefficients of D are independent of the variable time on the trajectory since a and b are evaluated at t_0 . The advantage of D is that by (B2) and (B3),

$$D \vec{q}'' = 0 \text{ at } t_0. \quad (\text{B36})$$

Further, because of the same equations, λ can be written for any time as

$$\lambda = Dq''_i \cdot [\vec{v}'' \times (\vec{q}'' \times \vec{v}')_i]. \quad (\text{B37})$$

Thus,

$$\dot{\lambda} = Dv''_i \cdot [\vec{v}'' \times (\vec{q}'' \times \vec{v}')_i] \text{ at } t_0. \quad (\text{B38})$$

We want to rewrite $\dot{\lambda}$ in a form that exhibits its positivity properties. To facilitate this, we derive some identities from energy and angular momentum conservation. The former gives

$$\vec{v}'' \cdot D \vec{v}'' = 0. \quad (\text{B39})$$

The latter can be written in the form

$$\vec{C} = \vec{q}'' \times \vec{v}'' - \alpha \hat{q}'' = \vec{q}' \times \vec{v}' + \alpha \hat{v}', \quad (\text{B40})$$

where we have used $\hat{q}' = -\hat{v}'$. Thus, at t_0 ,

$$D \vec{C} = \vec{q}'' \times D \vec{v}'' = D \vec{q}' \times \vec{v}'. \quad (\text{B41})$$

We now solve (B41) for $D \vec{v}''$ for the purpose of substitution in (B38). Using (B39),

$$\vec{v}'' \times D \vec{C} = -D \vec{v}'' (\vec{q}'' \cdot \vec{v}'') \text{ at } t_0. \quad (\text{B42})$$

Further, (B41) shows that $D \vec{C}$ is orthogonal to \vec{q}'' and \vec{v}' and, hence, can be written in the form

$$D \vec{C} = \xi |D \vec{C}| \frac{\vec{q}'' \times \vec{v}'}{|\vec{q}'' \times \vec{v}'|} \text{ at } t_0, \quad (\text{B43})$$

where $\xi = +1$ or -1 . Substituting (B42) and (B43) into (B38), we find

$$\dot{\lambda} = -\frac{\xi}{\vec{q}'' \cdot \vec{v}''} \frac{|D \vec{C}|}{|\vec{q}'' \times \vec{v}'|} |\vec{v}'' \times (\vec{q}'' \times \vec{v}')|^2 \text{ at } t_0. \quad (\text{B44})$$

From the definition of a case (C) zero, the last factor is nonzero. Thus, $\dot{\lambda}$ can vanish only if $D \vec{C} = 0$. Since by (B40), (B41), and the definition of D ,

$$\begin{aligned} \vec{C} \cdot D \vec{C} &= (\vec{q}' \times \vec{v}') \cdot (D \vec{q}' \times \vec{v}') \\ &= v' |\vec{q}' \times \vec{v}'| > 0 \text{ for } s \neq 0, \end{aligned} \quad (\text{B45})$$

we conclude that the case (C) zeros are also simple.

Let us now investigate the sign of $\dot{\lambda}$ at t_0 . For this note that $\xi = \text{sgn}[\vec{C} \cdot (\vec{q}'' \times \vec{v}')]]$ by (B43) and (B45). A simple calculation using (2.7) and (2.12c) gives

$$\vec{C} \cdot (\vec{q}'' \times \vec{v}') = Cq'' v' \sin^2\psi \sin|\chi|. \quad (\text{B46})$$

Since $\vec{q}'' \cdot \vec{v}'' < 0$ for $t < 0$ ($|\chi| < |\gamma|$) and $\vec{q}'' \cdot \vec{v}'' > 0$ for $t > 0$ ($|\chi| > |\gamma|$), it follows that

$$\text{sgn} \dot{\lambda} = \text{sgn}[\epsilon(|\gamma| - |\chi|) \sin|\chi|] \text{ at } t_0. \quad (\text{B47})$$

Note that a case (C) zero cannot occur at the following locations: (1) $\vec{q}'' \cdot \vec{v}'' = 0$ ($|\chi| = |\gamma|$). Otherwise (B42) and (B43) will give $\vec{v}'' \times (\vec{q}'' \times \vec{v}') = 0$ which is forbidden in case (C). (2) $\sin|\chi| = 0$. Otherwise from (B46) and (B43), $\vec{C} \cdot D \vec{C} = 0$ which contradicts (B45).

We are ready to determine the number of zeros of λ on a trajectory with a given γ . The following table summarizes the results we have already obtained:

Case	$\lambda(\chi) = 0$	$\text{sgn}(\dot{\lambda})$ at $\lambda = 0$
(A)	$ \chi = 2n\pi$	+
(B)	$ \chi = \gamma = (2n-1)\pi$	-
(C)	$ \chi \neq \gamma , \chi \neq n\pi$	$\text{sgn}[\epsilon(\gamma - \chi) \sin \chi]$

Here $n = 1, 2, \dots$. Further (B10) and (B11) show that $\text{sgn}(\lambda) = +1$ for $t \rightarrow -\infty$ and $\text{sgn}(\lambda) = \text{sgn}[1/$

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- ¹⁹In passing from our solution to theirs, note the difference in the definition of θ in the two papers; namely, (2.8) of our paper as opposed to (3.58) of Schwinger *et al.*
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