# A CLASSIFYING FAMILY OF SPACES FOR
# THE COHOMOLOGY OF PROFINITE GROUPS

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ABSTRACT

In the study of homological algebra, one useful tool for studying the cohomology of a discrete group is that group's classifying space. In some sense, the classifying space captures both the group itself and a description of its cohomology for any action of the group on any coefficient module. While some constructions for a classifying space also apply to topological groups, the relationship of the resulting space to the group's cohomology is unclear.

Profinite groups are a special case of topological groups, determined entirely as the limits of inverse systems of finite, discrete groups. The goal of this work is to construct for profinite groups as close an analog as possible to the classifying space of a discrete group. In particular, we are interested in the construction for a finite, discrete group's classifying space achieved by first producing the nerve of the group as a category and then taking its geometric realization to obtain a space with isomorphic cohomology groups. We proceed by extending each step of this process to apply to a profinite group using inverse limits, followed by correcting for a lack of continuity (in the sense of compatibility with inverse limits) in singular cohomology by applying alternative cohomology theories to the resulting sequence of spaces. The end result has a promising isomorphism to the cohomology of the group, with the possibility of a further isomorphism.

## DEDICATION

To Christ, my lord and savior, and my church family which I was adopted into through Him. To my grandfathers, Vernon Lee and Billie Putman.

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I would like to thank my advisor, Dr. Corson, first and foremost; the contents of this paper, from initial concept to the essential proofs which closed the remaining gaps, would not exist without him.

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CONTENTS

# CHAPTER 1

# BACKGROUND

In this chapter, we will collect a number of definitions and basic results which are needed later. An understanding of basic topology and set, group and category theory is assumed. A natural starting point is to describe categorical limits and colimits, as our main object of study is profinite groups, which are both defined in terms of limits and whose study also involves the dual construction, colimits.

## 1.1 LIMITS AND COLIMITS

Given a category $\mathscr{C}$, we wish to define limits, also called *inverse* or *projective limits*, and colimits, also called *direct limits*, within $\mathscr{C}$. First, $(I, \preceq)$ is a *partially ordered set* (sometimes abbreviated poset) if the following conditions are met for every $i, j, k \in I$:

a. $i \preceq i$

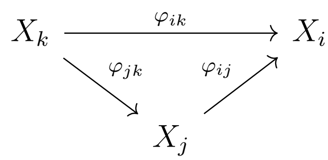
b. $i \preceq j$ and $j \preceq k$ imply $i \preceq k$

c. $i \preceq j$ and $j \preceq i$ imply $i = j$

Furthermore, such a partially ordered set is called *directed* if it also satisfies the following:

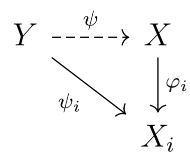
d. For every pair $i, j$, there exists $k \in I$ with both $i, j \preceq k$

An inverse or projective system in the category $\mathscr{C}$, then, is a collection $\{X_i : i \in I\}$ of objects of $\mathscr{C}$ indexed by such a directed partially ordered set, along with a collection of morphisms $\varphi_{ij} : X_j \to X_i$ for every pair $i \preceq j$ such that $\varphi_{ii}$ is the identity on $X_i$ and diagrams of the form below commute whenever they are defined:



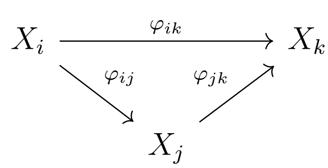

Given such an inverse or projective system along with another object $Y$ of $\mathscr{C}$, a collection

of morphisms $\psi_i : Y \to X_i$ is called *compatible* if $\varphi_{ij}\psi_j = \psi_i$ whenever $i \preceq j$. The limit $X$ of an inverse system is an object $X$ of $\mathscr{C}$ satisfying the following universal property:

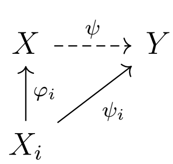



For every such $Y$ and compatible collection of morphisms, there exists a unique morphism $\psi : Y \to X$ such that $\varphi_i\psi = \psi_i$ for every $i \in I$.

A direct or inductive system in $\mathscr{C}$ is the dual of an inverse or projective system, namely a collection $\{X_i : i \in I\}$ where $(I, \preceq)$ is again a directed partially ordered set, but with morphisms $\varphi_{ij} : X_i \to X_j$ whenever $i \preceq j$ instead, such that $\varphi_{ii}$ is the identity on $X_i$ and which make diagrams of the following form commute whenever they are defined:




A dual notion of compatible morphisms applies to a direct system, where a collection of morphisms $\psi_i : X_i \to Y$ is called compatible if $\psi_j\varphi_{ij} = \psi_i$ whenever $i \preceq j$. Then the colimit or direct limit $X$ of such a system satisfies the following universal property:



For every Y with a collection of compatible morphisms, there exists a unique morphism $\psi : X \to Y$ with $\psi\varphi_i = \psi_i$ for every $i \in I$.

In the category of topological groups, the morphisms are continuous homomorphisms between groups. In topological groups or spaces, the limit of an inverse system $\{X_i\}$ may be constructed as the subgroup or the subspace respectively of $\prod_{i \in I} X_i$, with its topology inherited as a subspace of the product topology, containing only the tuples $(x_i)$ such that $\varphi_{ij}(x_j) = x_i$ whenever $i \preceq j$. The limit of a system of finite, discrete groups or spaces is called a *profinite* group or space; topologically, profinite groups and spaces are compact, Hausdorff, and either finite or uncountably infinite in cardinality. If an inverse system has every map $\varphi_{ij} : X_j \to X_i$ surjective, this is called a *surjective system*; in the case of finite, discrete groups or spaces, this also makes the projection maps $\varphi_i : X \to X_i$ surjective. Dually, the colimit of a direct system of groups may be defined as the quotient of the direct sum of those groups by the subgroup generated by all elements of the form $x_i - \varphi_{ij}(x_i)$. The same kind of constructions also apply to topological rings and modules, with the appropriate continuous morphisms for each category.

The following results about inverse and direct limits will be needed later:

If $(I, \preceq)$ is a directed partially ordered set and $J \subset I$ with $(J, \preceq)$ also a directed partially ordered set (for the same comparator $\preceq$), $J$ is *cofinal in* $I$ if for every $i \in I$ there exists some $j \in J$ with $i \preceq j$. Note that "is cofinal in" is clearly a transitive relation, that is, if $K \subset J \subset I$ such that $K$ is cofinal in $J$ and $J$ is cofinal in $I$, then $K$ is also cofinal in $I$. For an inverse or direct system $\{X_i, \varphi_{ij}\}_{i \in I}$ indexed by $I$, if $J$ is cofinal in $I$ then $\{X_i, \varphi_{ij}\}_{i \in J}$ is also an inverse or direct system, called a *cofinal subsytem* of $\{X_i, \varphi_{ij}\}_{i \in I}$.

Lemma 1.1. *If* $\{X_i, \varphi_{ij}\}_{i\in I}$ *is an inverse (respectively, direct) system of topological groups, modules or rings and* $\{X_i, \varphi_{ij}\}_{i\in J}$ *is a cofinal subsytem, then* $\varprojlim_{i\in I} X_i \cong \varprojlim_{j\in J} X_j$ *(respectively,* $\varinjlim_{i\in I} X_i \cong \varinjlim_{j\in J} X_j$*).*

Proof. For an inverse system, let $\varphi_k : \varprojlim_{i\in I} X_i \to X_k$ and $\varphi'_k : \varprojlim_{j\in J} X_j \to X_k$ be the canonical projections. Define a map $\overline{\varphi}_k : \varprojlim_{j\in J} X_j \to X_k$ by choosing $k' \in J$ with $k \preceq k'$ and letting $\overline{\varphi}_k := \varphi_{kk'}\varphi'_{k'}$. Note that this definition does not depend on the choice of $k'$ since if $k \preceq k''$ also, there exists a $k''' \in J$ with $k', k'' \preceq k'''$ and $\varphi_{kk'}\varphi'_{k'} = \varphi_{kk'}\varphi_{k'k'''}\varphi'_{k'''} = \varphi_{kk'''}\varphi'_{k'''} = \varphi_{kk''}\varphi_{k''k'''}\varphi'_{k'''} = \varphi_{kk''}\varphi'_{k''}$. Hence $\overline{\varphi}_k$ are compatible, so they induce a map $\overline{\varphi} : \varprojlim_{j\in J} X_j \to \varprojlim_{i\in I} X_i$ such that $\varphi_k\overline{\varphi} = \overline{\varphi}_k$ for every $k \in I$. For any $(x_{k'}) \in \varprojlim_{j\in J} X_j$, if $\overline{\varphi}(x_{k'}) = (y_k)$ then $y_{k'} = x_{k'}$ for every $k' \in J$, and $\overline{\varphi}$ is injective since this is the only element of $\varprojlim_{j\in J} X_j$ with this property. Also, if $(y_k) \in \varprojlim_{i\in I} X_i$, let $(x_{k'}) \in \varprojlim_{j\in J} X_j$ satisfy $x_{k'} = y_{k'}$ for every $k' \in J$; then clearly $\overline{\varphi}(x_{k'}) = (y_k)$ since any $k \notin J$ has some $k'$ with $k \preceq k'$ and $y_k = \varphi_{kk'}(x_{k'})$. Hence $\overline{\varphi}$ is surjective also, and so gives an isomorphism.

For a direct system with canonical morphisms $\varphi_k : X_k \to \varinjlim_{i\in I} X_i$ and $\varphi'_k : X_k \to \varinjlim_{j\in J} X_j$, define for any $k \in I$ with $k \preceq k' \in J$ the map $\overline{\varphi}_k : X_k \to \varinjlim_{j\in J} X_j$ as the composition $\varphi'_{k'}\varphi_{kk'}$. These maps also don't depend on the choice of $k'$ by an equation dual to the inverse system case, so they are compatible with the direct system and induce a map $\overline{\varphi} : \varinjlim_{i\in I} X_i \to \varinjlim_{j\in J} X_j$ with $\overline{\varphi}\varphi_k = \overline{\varphi}_k$ for every $k \in I$. If $(x_k) \in \varinjlim_{i\in I} X_i$ and $\overline{\varphi}(x_k) = (y_{k'})$, then $x_{k'} = y_{k'}$ for every $k' \in J$, and if $k \notin J$, there exists some $k' \in J$ so that $k \preceq k'$, making $y_{k'} = \varphi_{kk'}(x_k)$. Then $\overline{\varphi}$ is injective since $\overline{\varphi}(x_k) = (y_{k'}) = 1$ only if $\varphi_{kk'}(x_k) = 1$ for every $k$, which makes $(x_k) = 1$ also. If $(y_{k'}) \in \varinjlim_{j\in I} X_j$, then any $(x_k) \in \varinjlim_{i\in I} X_i$ with $x_{k'} = y_{k'}$ for every $k' \in J$ has $\overline{\varphi}(x_k) = (y_{k'})$, so $\overline{\varphi}$ is also surjective, and so gives an isomorphism. □

Lemma 1.2. *Let* $\{X_i, \varphi_{ij}\}$ *be an inverse system of compact Hausdorff nonempty topological spaces* $X_i$ *over the directed set* $I$*. Then* $\varprojlim_{i\in I} X_i$ *is nonempty; in particular the inverse limit of an inverse system of nonempty finite sets is nonempty.*

PROOF. For each $j \in I$, define a subset $Y_j$ of $\prod X_i$ to consist of those $(x_k)$ with $\varphi_{kj}(x_j) = x_k$ whenever $k \preceq j$. Each such $Y_j$ is a closed nonempty subset of $\prod X_i$. If $j \preceq j'$, then $Y_j \supseteq Y_{j'}$, so the collection of subsets $\{Y_j : j \in I\}$ has the finite intersection property (any intersection of finitely many $Y_j$ is nonempty) since $I$ is directed. Then, since $\prod X_i$ is compact, $\bigcap Y_j$ is nonempty. However, $\varprojlim_{i \in I} X_i = \bigcap_{j \in I} Y_j$, so the inverse limit is also nonempty. □

LEMMA 1.3. *Let $\{A_i, \varphi_{ij}\}$ be a direct system of abelian groups over a directed poset $I$, $A = \varinjlim A_i$ its direct limit and $\varphi_i : A_i \to A$ the canonical homomorphisms. Then:*

- **(a):** *$A = \bigcup_{i \in I} \varphi_i(A_i)$;*
- **(b):** *Let $x_i \in A_i$ with $\varphi_i(x_i) = 0$; then there exists some $k \succeq i$ with $\varphi_{ik}(x_i) = 0$;*
- **(c):** *If $\varphi_{ik}$ is an injection for each $k \succeq i$, then $\varphi_i$ is an injection;*
- **(d):** *If $\varphi_{ik}$ is surjective for each $k \succeq i$, then $\varphi_i$ is a surjection.*

PROOF. (a) follows by construction of the direct limit. For (b), $\varphi_i(x) = 0$ means there exists some $j, k \in I$ so that $k \succeq i, j$ and $\varphi_{ik}(x_i) = \varphi_{jk}(0)$, but clearly $\varphi_{jk}(0) = 0$ in $A_k$, as needed. (c) follows from (b) since $\varphi_i(x_i) = 0$ implies $\varphi_{ik}(x_i) = 0$ for some $k \succeq i$. But $\varphi_{ik}$ is an injection, so this is only possible if $x_i = 0$. For (d), let $a \in A$; then $a = \varphi_j(y_j)$ where $y_j \in A_j$ for some $j \in I$. Choose $k \succeq i, j$; since $\varphi_{ik}$ is surjective, there exists $x_i \in A_i$ with $\varphi_{ik}(x_i) = \varphi_{jk}(y_j)$, which means $\varphi_i(x_i) = \varphi_j(y_j) = a$. □

## 1.2 SIMPLICIAL SETS AND SPACES

Many of the constructions necessary to our goal will involve the application of simplicial sets or spaces, which are defined in this section. Our initial definitions are modeled after Friedman [4], with some additions.

A *totally ordered set* is a partially ordered set $(I, \leq)$ which has the additional property that for every $i, j \in I$, either $i \leq j$, $j \leq i$, or both; in other words, every pair of elements of $I$ is comparable. When two elements $i, j$ in a totally ordered set have $i \neq j$ and $i \leq j$ or

$i \geq j$ respectively, we will write this as $i < j$ or $i > j$. The *simplex category* $\Delta$ has finite totally ordered sets as its objects and order-preserving maps between those sets as its morphisms. Since order-preserving bijections are the isomorphisms of $\Delta$, finite totally ordered sets are unique up to them, so we may unambiguously call the set with $n+1$ elements $[n]$, labelling these elements as $0 < 1 < 2 < ... < n-1 < n$. Furthermore, any morphism of $\Delta$ may be written as a composition of the injective order-preserving maps $\delta_i : [n] \to [n+1]$ and the surjective order-preserving maps $\sigma_i : [n] \to [n-1]$, which are defined as follows:

$$\delta_i(j) = \begin{cases} j, & j < i \\ j+1 & j \geq i \end{cases} ; \ \sigma_i(j) = \begin{cases} j, & j \leq i \\ j-1, & j > i \end{cases}$$

Note that, by definition, compositions of these maps satisfy the following identities:

(1) $\delta_i \delta_j = \delta_j \delta_{i-1}$ whenever $i > j$

(2) $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i$ whenever $i < j$

(3) $\sigma_i \delta_j = \begin{cases} \delta_{j-1} \sigma_i & \text{if } i < j-1 \\ \text{id} & \text{if } i = j \text{ or } i = j-1 \\ \delta_j \sigma_{i-1} & \text{if } i > j \end{cases}$

A *simplicial set* $X$ is a contravariant functor from $\Delta$ into the category **Set**. A more concrete description of $X$ is that it is a collection of sets $X[0], X[1], X[2], ...$ with an $X[n]$, called the *$n$-simplices* of $X$, defined for every $n \in \mathbb{N}$, and which are equipped with the following maps:

a. For each map $\delta_i : [n-1] \to [n]$, there is a map $X(\delta_i) = d_i : X[n] \to X[n-1]$, the $i$th face map on $X[n]$.

b. For each map $\sigma_i : [n+1] \to [n]$, there is a map $X(\sigma_i) = s_i : X[n] \to X[n+1]$, the $i$th degeneracy map on $X[n]$.

These maps satisfy the following identities, which are dual to the identities for $\delta_i$ and $\sigma_i$ above:

(1) $d_i d_j = d_{j-1} d_i$ whenever $i < j$

(2) $s_i s_j = s_j s_{i-1}$ whenever $i > j$

(3) $$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$

If an $n$-simplex $x \in X[n]$ has some $i \in [n-1]$ and $y \in X[n-1]$ so that $s_i(y) = x$, the simplex $x$ will be called a *degenerate simplex*. For an $n$-simplex $x \in X[n]$, any other simplex $y$ which is an image of $x$ under a composition of face maps (so that $y = d_{i_1} d_{i_2} ... d_{i_k}(x)$ for some $i_1, ..., i_k \in \mathbb{N}$) will be called a *face of* $x$. A morphism $X \to Y$ between two simplicial sets is just a collection of set maps $X[n] \to Y[n]$ which commute with the face and degeneracy maps; we will denote the category of simplicial sets as **sSet**. Furthermore, if every $X[n]$ is finite when viewed as a set this will be called a simplicial finite set, and the full subcategory of **sSet** with the simplicial finite sets as its objects is **sFSet**, the category of simplicial finite sets. The standard $n$-simplex $\Delta^n$ is a simplicial set defined as the functor $\text{Hom}_{\Delta}(\bullet, [n])$; it has only one nondegenerate $n$-simplex, no nondegenerate $k$-simplices for $k > n$, and for all $m < n$ the nondegenerate $m$-simplices are faces of that $n$-simplex.

For a simplicial set $X$ and $n \in \mathbb{N}$, the *$n$-truncation of* $X$, denoted $X^{\leq n}$, is the collection $X[0], X[1], ..., X[n]$, that is, the $i$-simplices of $X$ for every $i \leq n$, along with all of the face and degeneracy maps which have these simplices as their domains or ranges. From another perspective, let $\Delta_{\leq n}$ be the full subcategory of $\Delta$ on the objects $[0], [1], ..., [n]$; then $X^{\leq n}$ is just the restriction of $X$ as a functor to $\Delta_{\leq n}$. For each $n \in \mathbb{N}$, we may view the $n$-truncated simplicial sets as a category, denoted by $\mathbf{sSet}_n$. We may also take the $n$-truncation of a simplicial finite set, in which case the forgetful functor will take the result to a finite set. Analagously to the case of all simplicial sets, the category of $n$-truncated simplicial finite sets will be denoted $\mathbf{sFSet}_n$.

For any of the categories **sSet**, **sFSet**, $\mathbf{sSet}_n$, or $\mathbf{sFSet}_n$, we will define the geometric realization functor $|\cdot|$ into the category of compactly-generated Hausdorff topological spaces. First, the geometric realization of a standard $n$-simplex is $|\Delta^n| = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : 0 \leq x_i \leq 1, \sum x_i = 1\}$, which is precisely the standard topological $n$-simplex. The maps $\delta_i$ and $\sigma_i$ induce maps $\delta_i : |\Delta^n| \to |\Delta^{n+1}|$ and $\sigma_i : |\Delta^n| \to |\Delta^{n-1}|$ which are inclusion as the $i$th face of the higher-dimensional simplex and collapsing onto the $i$th face, which is a lower-dimensional simplex, respectively. Then the geometric realization of any other simplicial set $X$ is $|X| = \left( \coprod_{n=0}^{\infty} (X[n] \times |\Delta^n|) \right) / \sim$ where $\sim$ is the equivalence relation generated by the degeneracy and face maps on $X$ in the following manner: $(x, \delta_i(p)) \sim (d_i(x), p)$ for all $x \in X[n+1], p \in |\Delta^n|$ and $(x, \sigma_i(p)) \sim (s_i(x), p)$ for $x \in X[n-1], p \in |\Delta^n|$. This extends in an obvious way to $n$-truncated simplicial sets by taking the disjoint union only up to $n$. In either case, the geometric realization has a natural description as a CW-complex with an $n$-cell $\{x\} \times |\Delta^n|$ for every nondegenerate $n$-simplex $x$ and attaching maps determined by the the equivalence relation $\sim$.

Given a topological space $X$, the *singular simplicial complex of* $X$ is the simplicial set $SX$ defined in the following manner: The $n$-simplices $SX[n]$ are all of the continuous maps $\eta : |\Delta^n| \to X$, with face and degeneracy maps given by $d_i(\eta) = \eta\delta_i$ and $s_i(\eta) = \eta\sigma_i$, where the right-hand side is the map produced by precomposition with $\delta_i : |\Delta^{n-1}| \to |\Delta^n|$ and $\sigma_i : |\Delta^{n+1}| \to |\Delta^n|$ respectively. The geometric realization $|SX|$ of this simplicial set has a natural continuous map $\epsilon : |SX| \to X$ which takes the image of each $\{\eta\} \times |\Delta^n|$ under the equivalence $\sim$ (from the definition of the realization) to the image of $\eta : |\Delta^n| \to X$. Furthermore, $\epsilon$ is a weak homotopy equivalence, and if $X$ is itself a CW-complex then $\epsilon$ is homotopic to a cellular homotopy equivalence.

A simplicial space is defined identically to a simplicial set, except that it acts as a functor into the category of topological spaces instead. This means that each collection $X[n]$ of $n$-simplices is a topological space and the face and degeneracy maps must also be

continuous. In this case the geometric realization $|X|$ is defined the same as above, with the products $X[n] \times |\Delta^n|$ taken to be the usual topological product spaces for each $n$. The category of simplicial sets is a subcategory of simplicial spaces consisting of those having the discrete topology. For simplicial spaces $X$, $Y$, the product $X \times Y$ is defined as the simplicial space whose $n$-simplices are $(X \times Y)[n] := X[n] \times Y[n]$ with the product topology, with face and degeneracy maps applied componentwise, so
$d_i(x, y) = (d_i(x), d_i(y))$ and $s_i(x, y) = (s_i(x), s_i(y))$.

## 1.3 GROUP COHOMOLOGY

Since our goal is to construct an object which represents the cohomology of a profinite group, we will define the notion of cohomology and establish a few important results about it here.

In general, a *chain complex* is a sequence of abelian groups or modules and a sequence of homomorphisms between the groups such that the image of each homomorphism is contained in the kernel of the next. A chain complex has the form $... \leftarrow C_{n-1} \overset{d_n}{\leftarrow} C_n \leftarrow ...$ where $C_n$ are the groups in question and $d_n$ the homomorphisms, called the boundary maps. The degree $n$ of the groups varies over either the integers or the natural numbers; in the latter case the chain ends on the left with $0 \leftarrow C_0 \leftarrow C_1 \leftarrow ....$ The homology groups of a chain complex are the groups $H_n(C_*) = \ker(d_n)/\mathrm{im}(d_{n+1})$. The dual of a chain complex is a *cochain complex*, a sequence of homomorphisms between abelian groups or modules of the form $... \to C^{n-1} \overset{\partial^n}{\to} C^n \to ...$ with groups $C^n$ and coboundary maps $\partial^n$, which again have $\partial^{n+1}\partial^n = 0$ for all $n$. As before, $n$ can vary over the integers or the natural numbers, and if it varies over the natural numbers the chain begins with $0 \to C^0 \to C^1 \to ....$ The cohomology groups of a cochain complex are $H^n(C^*) = \ker(\partial^{n+1})/\mathrm{im}(\partial^n)$.

A *chain map* between two chain (respectively, cochain) complexes $C^1_*, C^2_*$ (respectively, $C^*_1, C^*_2$) is a family of maps $f_n : C^1_n \to C^2_n$ (respectively, $f^n : C^n_1 \to C^n_2$) which commute with the boundary (coboundary) maps of both complexes. Given two chain maps

$f, g : C^1_* \to C^2_*$, a *chain homotopy* is a sequence of homomorphisms $h_n : C^1_n \to C^2_{n+1}$ such that $hd + dh = f - g$; analagously a cochain homotopy between maps $f, g : C^*_1 \to C^*_2$ is a sequence of homomorphisms $h^n : C^n_1 \to C^{n-1}_2$ with $\partial h + h\partial = f - g$. An augmentation (over $\mathbb{Z}$) of a chain complex is a surjective map $\varepsilon : C_0 \to \mathbb{Z}$ such that $\varepsilon d_1 : C_1 \to C_0 \to \mathbb{Z}$ is the zero map; an augmented chain complex is a chain complex along with such a map $\varepsilon$. Any chain map $\tau : C^1_* \to C^2_*$ between two augmented chain complexes (where $\varepsilon'$ is the augmentation map on $C^2_*$) is said to preserve augmentation if $\varepsilon'\tau = \varepsilon : C^1_0 \to \mathbb{Z}$. The reduced chain complex $\tilde{C}_*$ of an augmented chain complex $C_*$ is defined by $\tilde{C}_n = C_n$ for $n \neq 0$, $\tilde{C}_0 = \ker \varepsilon$, and $\tilde{d}_n = d_n$. The homology group $H_n(\tilde{C}_*)$ is called the reduced homology group of $C$ and is also denoted by $\tilde{H}_n(C_*)$.

Two chain complexes $C^1_*, C^2_*$ (with or without augmentation) are called *chain equivalent* if there exist two chain maps $f : C^1_* \to C^2_*$, $g : C^2_* \to C^1_*$ such that $fg$ is chain homotopic to the identity on $C^2_*$ and $gf$ is chain homotopic to the identity on $C^1_*$; in this case the maps $f$ and $g$ are each called a *chain equivalence*. In particular, when two chain complexes are chain equivalent, their homology groups are isomorphic, and similarly any two cochain complexes which are chain equivalent (which is defined essentially the same way as for chain complexes) have isomorphic cohomology groups. The 0 complex, which has 0 at all degrees, has natural maps into and out of any chain (or cochain) complex; a sequence of morphisms $t_n : C_n \to C_{n+1}$ (or $t^n : C^n \to C^{n+1}$) such that $td + dt = \mathrm{id} = \mathrm{id} + 0$ (or $\partial t + t\partial = \mathrm{id}$) gives a chain homotopy between the identity and the 0 map, which may be viewed as the composition $C_* \to 0 \to C_*$ (respectively $C^* \to 0 \to C^*$) of these natural maps. Such a sequence is called a *contracting homotopy*, and suffices to show that the (co)homology of the complex is 0. Giving a (co)chain equivalence between a complex and 0 is called a *chain contraction*. The mapping cone of a chain or cochain map $\tau : C^1_* \to C^2_*$ (respectively $\tau : C^*_1 \to C^*_2$) is the chain complex $C^\tau_*$ or $C^*_\tau$ respectively, which satisfies $C^\tau_n = C^1_{n-1} \oplus C^2_n$ ($C^n_\tau := C^{n+1}_1 \oplus C^n_2$, respectively) and $d^\tau_n(x_1, x_2) = (-d^1_{n-1}(x_1), \tau(x_1) + d^2_n(x_2))$ (or in the cochain case, $\partial_\tau^{n+1}(f_1, f_2) = (-\partial_1^{n+2}(f_1), \tau(f_1) + \partial_2^{n+1}(f_2)))$. A chain or cochain map is a chain equivalence if and only if its mapping cone is chain homotopic to the 0 complex.

For a topological group $G$, the action taking each discrete $G$-module $M$ to the group of $G$-invariants $M^G = \{x \in M : gx = x \text{ for every } g \in G\}$ is a functor from the category of discrete $G$-modules into the category **Ab** of discrete abelian groups. This functor is equivalent to $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \bullet)$ where $G$ is given the trivial action on $\mathbb{Z}$. This functor is left exact, but not necessarily right exact, and its derived functors are $Ext^n_{\mathbb{Z}[G]}(\mathbb{Z}, \bullet)$. We define the *cohomology groups* of $G$ with coefficients in $M$, denoted by $H^n(G, M)$, to be precisely $Ext^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$; note that $H^0(G, M) = M^G$.

A more convenient way to calculate the cohomology of a topological group $G$ with coefficients in a discrete $G$-module $A$ is with the following cochain complex. Let $C^n(G, A)$ be the group of all continuous functions $\sigma : G^{n+1} \to A$ such that
$\sigma(gg_0, ..., gg_n) = g\sigma(g_0, ..., g_n)$, and define a coboundary map
$\partial^{n+1} : C^n(G, A) \to C^{n+1}(G, A)$ by $(\partial^{n+1}\sigma)(g_0, g_1 ..., g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i\sigma(g_0, ..., \hat{g}_i, ..., g_{n+1})$
where $\hat{g}_i$ indicates omission. Then $C^*(G, A)$ is the cochain complex
$0 \to C^0(G, A) \to C^1(G, A) \to ... \to C^n(G, A) \overset{\partial^{n+1}}{\to} C^{n+1}(G, A) \to ...$, and the cohomology of $C^*(G, A)$ is precisely $H^*(G, A)$.

We will need the following results regarding the cohomology groups of a profinite group, adapted from Ribes and Zalesskii [10]:

If $\varphi : G \to H$ is a continuous homomorphism of profinite groups, $A$ is a discrete module of $G$ and $B$ is a discrete module of $H$, and $f : B \to A$ is a group homomorphism, then the maps $\varphi$ and $f$ are *compatible* if $f(\varphi(g) \cdot b) = g \cdot f(b)$ for all $g \in G$, $b \in B$. Any such pair of maps induces homomorphisms $(\varphi, f) : C^n(H, B) \to C^n(G, A)$ for all $n \geq 0$ given by
$[(\varphi, f)\sigma](g_0, ..., g_n) = f(\sigma(\varphi(g_0), ..., \varphi(g_n)))$ whenever $\sigma \in C^n(H, B)$ and each $g_i \in G$. In fact this is a map of cochain complexes, so it gives a map $(\varphi, f)^n : H^n(H, B) \to H^n(G, A)$ on the cohomology groups.

Now, let $I$ be a directed poset, $\{G_i, \varphi_{ij}, I\}$ an inverse system over $I$ of profinite groups, and $\{A_i, f_{ij}, I\}$ a direct system over $I$ of discrete abelian groups where each $A_i$ is a $G_i$-module such that for any pair $i, j \in I$ with $i \preceq j$ the maps $\varphi_{ij} : G_j \to G_i$ and $f_{ij} : A_i \to A_j$ are compatible. Then for each $n$ we naturally obtain the direct systems $\{C^n(G_i, A_i)\}_{i\in I}$ and $\{H^n(G_i, A_i)\}_{i\in I}$. If $G = \varprojlim_{i\in I} G_i$ and $A = \varinjlim_{i\in I} A_i$ with $\varphi_i : G \to G_i$ and $f_i : A_i \to A$ the morphisms given by the universal property of each limit, then $A$ can be considered a $G$-module as follows: Given $a \in A$ and $x \in G$, there exists some $i \in I$ and $a_i \in A_i$ with $f_i(a_i) = a$, so define $xa := f_i((\varphi_i(x))a_i)$. This is a well-defined, continuous action of $G$ on $A$.

Lemma 1.4. *Under the above assumptions, $C^n(G, A) \cong \varinjlim_{i\in I} C^n(G_i, A_i)$ for all $n \in \mathbb{N}$, and these isomorphisms commute with the operators $\partial$ in the following way: for each $i \in I$ the diagram*

$$\begin{array}{ccc} \varinjlim_{i\in I} C^n(G_i, A_i) & \xrightarrow{\partial^{n+1}} & \varinjlim_{i\in I} C^{n+1}(G_i, A_i) \\ \downarrow \cong & & \downarrow \cong \\ C^n(G, A) & \xrightarrow{\partial^{n+1}} & C^{n+1}(G, A) \end{array}$$

*commutes.*

Proof. Fix $n$. For each $i \in I$ define $\Psi_{ni} : C^n(G_i, A_i) \to C^n(G, A)$ as follows: Given $\sigma_i \in C^n(G_i, A_i)$, put $\Psi_{ni}(\sigma_i) = f_i \sigma_i \varphi_i$. Then the maps $\Psi_{ni}$ are compatible with the morphisms $\zeta_{ij} : C^n(G_i, A_i) \to C^n(G_j, A_j)$ of the direct system which exist whenever $i \preceq j$, so they induce morphisms $\Psi_n : \varinjlim_{i\in I} C^n(G_i, A_i) \to C^n(G, A)$. Then the maps $\Psi_n$ commute with $\partial$, and it remains to show that each $\Psi_n$ is an isomorphism.

$\Psi_n$ **is injective:** Suppose that $\sigma \in \varinjlim_{i\in I} C^n(G_i, A_i)$ and $\Psi_n(\sigma) = 0$. Let $k \in I$ and $\sigma_k \in C^n(G_k, A_k)$ be such that $\zeta_k(\sigma_k) = \sigma$ (where $\zeta_k : C^n(G_k, A_k) \to \varinjlim_{i\in I} C^n(G_i, A_i)$ is the morphism given by the universal property of the direct limit). For any $i \succeq k$, let

$\sigma_i = \zeta_{ki}(\sigma_k)$; then $0 = \Psi_n(\sigma) = \Psi_{ni}(\sigma_i) = f_i\sigma_i\varphi_i$. Also define $X_i := \{(x_{i,0}, ..., x_{i,n}) \in G_i^{n+1} : \sigma_i(x_i) \neq 0\}$. We claim that for some $i \succeq k$, $X_i = \emptyset$, that is, $\sigma_i = 0$, in which case $\sigma = 0$ and $\Psi_n$ is injective. Since $\sigma_i$ is continuous, $G_i^{n+1}$ is compact and $A_i$ is discrete, $\sigma_i$ can only take on a finite number of values, so $X_i$ is closed in $G_i^{n+1}$, therefore also compact. However, $i \succeq j \succeq k$ implies that $f_{ij}(X_i) \subseteq X_j$, since if $(x_{i,0}, ..., x_{i,n}) \in X_i$, then $0 \neq \sigma_i(x_{i,0}, ..., x_{i,n}) = (f_{ji}\sigma_j\varphi_{ji})(x_{i,0}, ..., x_{i,n})$ so that $\sigma_j(\varphi_{ji}(x_{i,0}, ..., x_{i,n})) \neq 0$ as well, so $\varphi_{ji}(x_{i,0}, ..., x_{i,n}) \in X_j$. Therefore, $\{X_i, \varphi_{ji} : i, j \succeq k\}$ is an inverse system of compact spaces. If $(x_0, ..., x_n) \in \varprojlim_{i \succeq k} X_i \subseteq G^{n+1}$, then $\Psi_n(\sigma)(x) = \Psi_{ni}(\sigma_i)(x_0, ..., x_n) = (f_i\sigma_i\varphi_i)(x_0, ..., x_n) = (f_i\sigma_i)(x_{i,0}, ..., x_{i,n})$. Since $\sigma_i(x_{i,0}, ..., x_{i,n}) \neq 0$ if $i \succeq k$, it follows from Lemma 1.3 that $\Psi_n(\sigma)(x_0, ..., x_n) \neq 0$. But $\Psi_n(x_0, ..., x_n) = 0$ by assumption, so $\varprojlim_{i \succeq k} X_i = \emptyset$, and by Lemma 1.2 there must exist some $i \in I$ with $X_i = \emptyset$, as needed.

$\Psi_n$ **is surjective:** If $\sigma \in C^n(G, A)$, we claim that for some $i \in I$ there exists $\sigma_i \in C^n(G_i, A_i)$ with $\sigma = \Psi_n(\sigma_i) = f_i\sigma_i\varphi_i$. Since $G$ is compact and $A$ is discrete, $\sigma(G^{n+1})$ is finite. Thus, there exists $j_0 \in I$ such that for every $j \succeq j_0$ there is some $G_j$-submodule $B_j$ of $A_j$ for which the restriction of $f_j$ maps $B_j$ isomorphically onto $\sigma(G^{n+1})$. Since $\sigma(G^{n+1})$ is finite, the subspace $\ker(\sigma) = \{(g_0, ..., g_n) : \sigma(g_0, ..., g_n) = 0\}$ is open in $G^{n+1}$. Thus (replacing $j_0$ with a larger index if needed) there exists an open subspace $U_{j_0}$ of $G_{j_0}^{n+1}$ so that $U := \varphi_{j_0}^{-1}(U_{j_0}) \subset \ker(\sigma)$. For $j \succeq j_0$, let $U_j := \varphi_{j_0 j}^{-1}(U_{j_0})$; then $G^{n+1}/U = \varprojlim_{j \succeq j_0} G_j^{n+1}/U_j$ (as topological spaces). Since $G^{n+1}/U$ is finite and each $G^{n+1}/U \to G_j^{n+1}/U_j$ is a surjection, there exists some $i \succeq j_0$ so that the projection $G^{n+1}/U \to G_i^{n+1}/U_i$ is the identity map. Let $\overline{\sigma} : G^{n+1}/U \to A$ be the map induced by $\sigma$. Then there is a unique continuous map $\overline{\sigma}_i : G_i^{n+1}/U_i \to B_i$ so that the diagram below commutes:

$$\begin{array}{ccccc} G^{n+1}/U & \xrightarrow{\overline{\sigma}} & \sigma(G^{n+1}) & \hookrightarrow & A \\ \downarrow \cong & & \uparrow & & \\ G_i^{n+1}/U_i & \xrightarrow{\overline{\sigma}_i} & B_i & & \end{array}$$

Let $\sigma_i$ be the composition $G_i^{n+1} \to G_i^{n+1}/U_i \xrightarrow{\overline{\sigma}_i} B_i \hookrightarrow A_i$; this $\sigma_i$ satisfies $\Psi_n(\sigma_i) = \sigma$, as claimed. Therefore, $\Psi_n$ is also surjective, so it is an isomorphism for each $n$. $\square$

COROLLARY 1.5. *For each $n \geq 0$, $H^n(G, A) \cong \varinjlim_{i \in I} H^n(G_i, A_i)$.*

PROOF. Since $\varinjlim$ is an exact functor on abelian groups, $\varinjlim_{i\in I} H^n(G_i, A_i) \cong H^n\left(\varinjlim_{i\in I} C^*(G_i, A_i)\right)$ where the cochain complexes $C^*(G_i, A_i)$ form a direct system under the maps $g_{ij} = (\varphi_{ij}, f_{ij}) : C^n(G_i, A_i) \to C^n(G_j, A_j)$ given by $g_{ij}(\sigma_i) = f_{ij}\sigma_i\varphi_{ij}$ whenever $\sigma_i \in C^n(G_i, A_i)$ and $j \succeq i$. The maps $g_{ij}$ then determine a map of cochain complexes $C^*(G_i, A_i) \to C^*(G_j, A_j)$ since they commute with the coboundary operators $\partial^n$. Thus, to get an isomorphism $H^n(G, A) \cong \varinjlim_{i\in I} H^n(G_i, A_i)$ it suffices to prove the existence of isomorphisms $C^n(G, A) \cong \varinjlim_{i\in I} C^n(G_i, A_i)$ which commute with the coboundary maps $\partial^n$, which we have done in Lemma 1.4. $\square$

If $G$ is a profinite group, $A$ is a discrete $G$-module, and $K$ is a closed normal subgroup of $G$ with the projection $\varphi_K : G \to G/K$, then $A^K$ is a $G/K$ module in a natural way: $\varphi_K(x) \cdot a = x \cdot a$ for every $x \in G$, $a \in A^K$. Furthermore, $\varphi_K$ and the inclusion $f_K : A^K \to A$ are clearly compatible, so they induce maps $\mathrm{Inf} = \mathrm{Inf}_G^{G/K} : H^n(G/K, A^K) \to H^n(G, A)$, called *inflations*. In this way, we obtain an inverse system $\{G/K : K \text{ open and normal in } G\}$ and a corresponding direct system $\{A^K\}$ satisfying the hypotheses of Lemma 1.4 and Corollary 1.5, with the additional property that $G = \varprojlim G/K$ and $A = \varinjlim A^K$. Thus:

COROLLARY 1.6. *For each $n \geq 0$, $H^n(G, A) \cong \varinjlim_{i\in I} H^n(G/K, A^K)$ with Inf: $H^n(G/K, A^K) \to H^n(G, A)$ as the maps given by the universal property of the direct limit. Furthermore, this isomorphism also holds if the limit on the right is taken over some cofinal family of open normal subgroups of $G$ instead.*

## 1.4 COHOMOLOGY THEORIES ON TOPOLOGICAL SPACES

A *cohomology theory* on topological spaces assigns to each topological space $X$ a sequence of cohomology groups $H^n(X)$ for each $n \in \mathbb{N}$. Some cohomology theories on topological spaces have a coefficient group $A$, an abelian group (or module); in this case the groups will be denoted be $H^n(X, A)$. Analagously, a homology theory assigns to each topological space $X$ a sequence of homology groups $H_n(X)$, possibly with coefficients in $A$, denoted by $H_n(X, A)$. For a given topological space $X$ and abelian discrete group $A$, we will be interested in the following homology and cohomology theories. The results below are standard, and largely adapted from Hatcher [7] or Spanier [11].

**Singular Cohomology:**

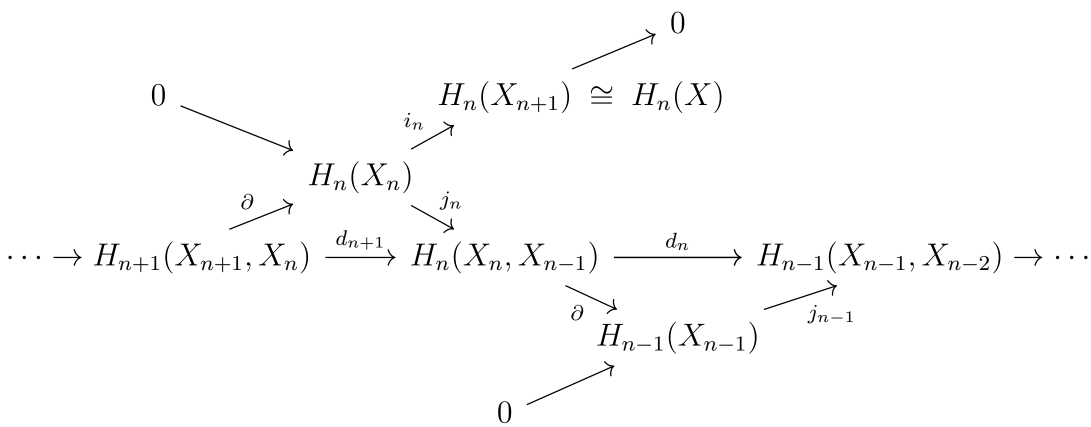
Let $C_n(X)$ be the free abelian group on the basis $SX[n]$, the set of all singular $n$-simplices on $X$. If $\sigma : |\Delta^n| \to X$ is a singular $n$-simplex and $(v_0, ..., v_n)$ are the vertices of $|\Delta^n|$, then $\delta_i : |\Delta^{n-1}| \to |\Delta^n|$ is the inclusion such that $\delta_i(v_j) = \begin{cases} v_j & j < i \\ v_{j+1} & j > i \end{cases}$. Denote by $\sigma^{(i)}$ the composition $|\Delta^{n-1}| \xrightarrow{\delta_i} |\Delta^n| \xrightarrow{\sigma} X$, which is a singular $(n-1)$-simplex. Then, with the boundary operator $d_n\sigma := \sum_{i=0}^{n} (-1)^i \sigma^{(i)}$, form the singular chain complex $0 \leftarrow C_0(X) \leftarrow C_1(X) \leftarrow ... \leftarrow C_{n-1}(X) \xleftarrow{d_n} C_n(X) \leftarrow ....$ The singular homology groups on $X$, $H_n(X)$, are the homology groups of this complex, with reduced homology groups $\tilde{H}_n(X)$. Then let $C^n(X, A) = \mathrm{Hom}(C_n(X), A)$ with the coboundary maps $\partial^n : C^{n-1}(X, A) \to C^n(X, A)$ defined such that $(\partial^n f)(\sigma) = f(d_n\sigma)$. This forms the cochain complex $0 \to C^0(X, A) \to C^1(X, A) \to ... \to C^{n-1}(X, A) \xrightarrow{\partial^n} C^n(X, A) \to ...$, and the singular cohomology of $X$ with coefficients in $A$, denoted by $H^*(X, A)$, is the cohomology of this cochain complex.

**Cellular Cohomology:**

Given a subspace $Y \subset X$, we have a short exact sequence of chain complexes $0 \to C_*(Y) \overset{i_*}{\to} C_*(X) \overset{j_*}{\to} C_*(X)/C_*(Y) \to 0$; that is, for each $n$ there is a short exact sequence $0 \to C_n(Y) \overset{i_n}{\to} C_n(X) \overset{j_n}{\to} C_n(X)/C_n(Y) \to 0$ where the maps involved commute with the boundary map on the two chain complexes $C_*(Y)$, $C_*(X)$. Here each $i_n$ is induced by the inclusion $Y \to X$, and each $j_n$ by the inclusion $(X, \emptyset) \to (X, Y)$. The boundary map of $C_*(X)$ leaves $C_*(Y)$ invariant, so we obtain a boundary map $d'$ on the quotient. Let $C_n(X, Y) := C_n(X)/C_n(Y)$; this gives a chain complex $... \leftarrow C_{n-1}(X, Y) \overset{d'_n}{\leftarrow} C_n(X, Y) \leftarrow ....$ The homology groups of this chain complex are $H_n(X, Y)$, the relative homology groups of the pair of spaces $(X, Y)$. Furthermore, the short exact sequence above gives rise to the long exact sequence:

$$0 \leftarrow H_0(X, Y) \leftarrow ... \leftarrow H_{n-1}(X) \overset{i_{n-1}}{\leftarrow} H_{n-1}(Y) \overset{\partial}{\leftarrow} H_n(X, Y) \overset{j_n}{\leftarrow} H_n(X) \overset{i_n}{\leftarrow} H_n(Y) \leftarrow ...$$

If $X$ is a CW-complex and its $n$-skeleta are denoted $X_n$, then $H_k(X_n, X_{n-1}) = 0$ for $k \neq n$, and if $k = n$ it is free abelian with a basis in one-to-one correspondence with the $n$-cells of $X$. Also, $H_k(X_n) = 0$ for all $k > n$, and the map $i_k : H_k(X_n) \to H_k(X)$ induced by inclusion is an isomorphism for $k < n$ and surjective for $k = n$. Using this, we define $C_n^{\text{cell}}(X) := H_n(X_n, X_{n-1})$, and define boundary maps by fitting the long exact sequences given by the inclusions $X_{n-1} \subset X_n$ into the following commutative diagram:



Specifically, $d_n$ is defined to be the composition $j_{n-1}\partial$. Then the *cellular homology of $X$* is the homology of the chain complex

$... \leftarrow H_{n-1}(X_{n-1}, X_{n-2}) \leftarrow H_n(X_n, X_{n-1}) \leftarrow H_{n+1}(X_{n+1}, X_n) \leftarrow ...$ with these boundary maps.

If $X$ is a simplicial set and $x \in X[n]$ is nondegenerate, denote by $[x]$ the $n$-cell of $|X|$ given as the image of $\{x\} \times |\Delta^n|$ under the equivalence relation $\sim$. Then:

Lemma 1.7. *For any nondegenerate $x \in X[n]$ where $n > 0$,*

$$d_n[x] = \sum_{0 \leq i \leq n, d_i(x) \text{ nondegenerate}} (-1)^i [d_i(x)]$$

Proof. For each $x \in X[n]$, the basis element $[x]$ in $C_n^{\text{cell}}(|X|)$ is represented by the singular $n$-simplex $\sigma_x \colon |\Delta^n| \to |X|_n$ given by $\sigma_x(t) = (x, t)$. So $d_n([x])$ is represented by the singular $n$-chain $d\sigma_x = \sum_{i=0}^{n} (-1)^i \sigma_x^{(i)}$. But $\sigma_x^{(i)} \colon |\Delta^{n-1}| \to |X|_{n-1}$ is given by $\sigma_x^{(i)}(t) = \sigma_x(\delta_i(t)) = (x, \delta_i(t)) = (d_i(x), t)$. Hence $d_i(\sigma_x)$ represents the basis element $[d_i(x)]$ in $C_{n-1}^{\text{cell}}(|X|)$ if $d_i(x)$ is nondengenerate, and is zero otherwise. This gives the desired formula. ☐

The cellular cochain complex of a topological space $X$ with coefficients in $A$ is the dual of the cellular chain complex, given by $C^n_{\text{cell}}(X, A) := \text{Hom}\,(H_n(X_n, X_{n-1}), A)$ with coboundary maps $\partial^n$ given by precomposition with $d_n$ as defined above; then the cellular cohomology groups of $X$ with coefficients in $A$, $H^n_{\text{cell}}(X, A)$, are the cohomology groups of this complex.

**Alexander-Spanier Cohomology:**

Let $\Phi^n(X, A)$ be the module of all functions $f : X^{n+1} \to A$ with addition and scalar multiplication defined pointwise, such that if $x_0, x_1, ..., x_n \in X$, then $f(x_0, ..., x_n) \in A$, and if $f_1, f_2 \in \Phi^n(X, A)$ and $z \in \mathbb{Z}$, then $(zf_1)(x_0, ..., x_n) = z(f_1(x_0, ..., x_n))$ and $(f_1 + f_2)(x_0, ..., x_n) = f_1(x_0, ..., x_n) + f_2(x_0, ..., x_n)$. Define a coboundary operator $\partial : C^{n-1}(X, A) \to C^n(X, A)$ by the formula $(\partial f)(x_0, ..., x_n) = \sum_{i=0}^{n} (-1)^i f(x_0, ..., \hat{x}_i, ..., x_n)$ where, as usual, $\hat{x}_i$ indicates omission; this makes $\Phi^*(X, A)$ a cochain complex. An element $f \in \Phi^n(X, A)$ is called *locally zero* if there is a covering $\mathscr{U}$ of $X$ by open sets such that $f$ is zero on any $(n+1)$-tuple of $X$ which lies in an element of $\mathscr{U}$. Note that if $f$ is locally zero for a given $\mathscr{U}$ and $\mathscr{U}^{n+1} = \bigcup_{U \in \mathscr{U}} U^{n+1} \subset X^{n+1}$, then $f$ is zero on $\mathscr{U}^{n+1}$. The subset of $\Phi^n(X, A)$ consisting of locally zero functions is a subgroup, $\Phi^n_0(X, A)$, and if $f$ is zero on $\mathscr{U}^{n+1}$, then $\partial f$ is also zero on $\mathscr{U}^{n+2}$, which means $\Phi^*_0(X, A)$ with the same coboundary map is a cochain subcomplex of $\Phi^*(X, A)$. Define $\overline{C}^*(X, A)$ to be the quotient cochain complex $\Phi^*(X, A)/\Phi^*_0(X, A)$ (that is, each group $\overline{C}^n(X, A) := \Phi^n(X, A)/\Phi^n_0(X, A)$). Then the *Alexander-Spanier cohomology* of $X$ with coefficients in $A$, $\overline{H}^*(X, A)$, is the cohomology of this quotient complex.

For a given open cover $\mathscr{U}$ of $X$, let $C_*(\mathscr{U})$ be the chain complex $... \leftarrow C_{n-1}(\mathscr{U}) \overset{d_n}{\leftarrow} C_n(\mathscr{U}) \leftarrow ...$ where $C_n(\mathscr{U})$ is the free abelian group generated by ordered tuples $(x_0, ..., x_n)$ of points in $X$ such that there exists some $U \in \mathscr{U}$ such that $x_0, ..., x_n \in U$, with the boundary operator $d_n(x_0, ..., x_n) = \sum_{i=0}^{n} (-1)^i (x_0, ..., \hat{x}_i, ..., x_n)$. Then let $C^*(\mathscr{U}, A)$ be the cochain complex which is dual to $C_*(\mathscr{U})$; specifically, $C^n(\mathscr{U}, A) := \text{Hom}(C_n(\mathscr{U}), A)$ with coboundary maps $\partial^n$ defined by precomposition with $d_n$. If $\mathscr{V}$ is a refinement of $\mathscr{U}$, the map taking any $f \in C^n(\mathscr{U}, A)$ and restricting to only those tuples $(x_0, ..., x_n)$ such that $x_0, ..., x_n \in V$ for some $V \in \mathscr{V}$ (which we will call "restriction to tuples in $\mathscr{V}$") is a cochain map $C^*(\mathscr{U}, A) \to C^*(\mathscr{V}, A)$. Furthermore, given two open coverings $\mathscr{U}, \mathscr{V}$ which are not necessarily related by refinement, let $\mathscr{W} = \mathscr{U} \cap \mathscr{V} := \{U \cap V : U \in \mathscr{U}, V \in \mathscr{V}\}$; then $\mathscr{W}$ is a refinement of both $\mathscr{U}$ and $\mathscr{V}$. This gives a directed partial order on all open coverings of $X$ by refinement, hence a direct system $\{C^*(\mathscr{U}, A)\}$ of the cochain complexes indexed by the open covers with maps given by restriction. This system has the colimit $\varinjlim C^*(\mathscr{U}, A)$, itself a cochain complex; we wish to show that this complex is isomorphic to the Alexander-Spanier cochain complex $\overline{C}^*(X, A)$. If $f \in \Phi^n(X, A)$ and $\mathscr{U}$ is any open covering of $X$, then the restriction of $f$ to tuples in $\mathscr{U}$ is an element of $C^n(\mathscr{U}, A)$, and for every $n$ we may compose this restriction with the canonical map $C^n(\mathscr{U}, A) \to \varinjlim C^n(\mathscr{U}, A)$ to obtain a cochain map $\lambda : \Phi^*(X, A) \to \varinjlim C^*(\mathscr{U}, A)$.

LEMMA 1.8. *The cochain map $\lambda$ is surjective with kernel equal to $\Phi_0^*(X, A)$.*

PROOF. To see that $\lambda$ is surjective, let $u \in \varinjlim C^n(\mathscr{U}, A)$. Then for some cover $\mathscr{V}$, there exists $v \in C^n(\mathscr{V}, A)$ so that the canonical morphism $C^n(\mathscr{V}, A) \to \varinjlim C^n(\mathscr{U}, A)$ maps $v$ to $u$. Define $f_v \in \Phi(X, A)$ as follows:

$$f_v(x_0, ..., x_n) = \begin{cases} v(x_0, ..., x_n) & \text{if } x_0, ..., x_n \in V \text{ for some } V \in \mathscr{V} \\ 0 & \text{otherwise} \end{cases}$$

By definition, the restriction of $f_v$ to tuples in $\mathscr{V}$ is precisely $v$, which means $\lambda(f_v) = u$.

Next, an element $f \in \Phi^n(X, A)$ is in the kernel of $\lambda$ if and only if there exists some open cover $\mathscr{U}$ such that the restriction of $f$ to tuples in $\mathscr{U}$ is 0. Thus $\lambda f = 0$ if and only if there is an open cover $\mathscr{U}$ so that $f$ is zero on $\mathscr{U}^{n+1}$. By the definition of $\Phi_0^n(X, A)$, this means $f \in \Phi_0^n(X, A)$. $\square$

Corollary 1.9. *For the Alexander-Spanier cohomology of a space $X$ with coefficients in $A$, there is a canonical isomorphism $\overline{H}^*(X, A) \cong \varinjlim H^n(\mathscr{U}, A)$ where $H^n(\mathscr{U}, A)$ is the cohomology of $C^n(\mathscr{U}, A)$.*

## 1.5 SHEAF THEORY AND ČECH COHOMOLOGY

We wish to define one more important cohomology theory on topological spaces which will prove especially useful to our goal. However, the definition and study of this form of cohomology requires a small detour to establish some basic definitions and results about presheaves and sheaves. This section is largely adapted from information in Spanier [11], with some help from Stacks [1].

Let $X$ be a topological space. A *presheaf* $\Gamma$ is a contravariant functor from the category whose objects are open subsets $U$ of $X$ and whose morphisms are inclusion maps $i : U \to V$ to some other concrete category $\mathscr{C}$, such as sets, vector spaces, or groups, which assigns to each inclusion map $i$ a restriction map $\Gamma(i) : \Gamma(V) \to \Gamma(U)$ such that for any composition $i \circ j$, $\Gamma(i \circ j) = \Gamma(j) \circ \Gamma(i)$, and the identity $\mathrm{id}_U : U \to U$ satisfies $\Gamma(\mathrm{id}_U) = \mathrm{id}_{\Gamma(U)}$. For $\gamma \in \Gamma(U)$, if $V \subset U$ with the inclusion map $i : V \to U$, then take $\gamma|V$ to mean $\Gamma(i)(\gamma)$, which is an element of $\Gamma(V)$. If $\mathscr{U}$ is a collection of open subsets of $X$, a compatible $\mathscr{U}$ family of $\Gamma$ is an indexed family $\{\gamma_U \in \Gamma(U)\}_{U \in \mathscr{U}}$ such that $\gamma_U|U \cap V = \gamma_V|U \cap V$ for every pair $U, V \in \mathscr{U}$.

A presheaf is called a *sheaf* if it also satisfies the following conditions:

- **(G) The Gluing condition:** Given a collection $\mathscr{U}$ of open subsets of $X$ with $V = \bigcup_{U \in \mathscr{U}} U$ and a compatible $\mathscr{U}$ family $\{\gamma_U\}_{U \in \mathscr{U}}$, there is an element $\gamma \in \Gamma(V)$ such that $\gamma|U = \gamma_U$ for all $U \in \mathscr{U}$.
- **(M) The Monopresheaf condition:** Given a collection $\mathscr{U}$ of open subsets of $X$ with $V = \bigcup_{U \in \mathscr{U}} U$, for any two elements $\gamma_1, \gamma_2 \in \Gamma(V)$, if $\gamma_1|U = \gamma_2|U$ for all $U \in \mathscr{U}$, then $\gamma_1 = \gamma_2$. Equivalently for presheaves of modules or abelian groups, if for some $\gamma \in \Gamma(V)$, $\gamma|U = 0$ for all $U \in \mathscr{U}$, then $\gamma = 0$.

For each presheaf $\Gamma$ of modules we obtain another presheaf $\hat{\Gamma}$, called its completion, whose elements are compatible families of $\Gamma$. Given a collection $\mathscr{U}$ of open sets, let $\Gamma(\mathscr{U})$ be the collection of compatible $\mathscr{U}$ families of $\Gamma$. If $\mathscr{V}$ is another collection of open sets which refines $\mathscr{U}$, there is a homomorphism $\Gamma(\mathscr{U}) \to \Gamma(\mathscr{V})$ which assigns to each compatible $\mathscr{U}$ family $\{\gamma_U\}$ the compatible $\mathscr{V}$ family $\{\gamma_V\}$ such that if $V \in \mathscr{V}$ is contained in $U \in \mathscr{U}$, then $\gamma_V = \gamma_U|V$; this is uniquely defined since $\{\gamma_U\}$ is a compatible family. For a fixed open set $W$, let $\mathscr{U}$ vary over the family of open coverings of $W$; then the collection $\{\Gamma(\mathscr{U})\}$ is a direct system of modules, so define $\hat{\Gamma}(W) := \varinjlim\{\Gamma(\mathscr{U})\}$. If $W' \subset W$ and $\mathscr{U}$ is an open covering of $W$, then $\mathscr{U}' := \{U \cap W' : U \in \mathscr{U}\}$ is an open covering of $W'$ which refines $\mathscr{U}$, giving a homomorphism $\Gamma(\mathscr{U}) \to \Gamma(\mathscr{U}')$. By passing to limits, this gives a homomorphism $\hat{\Gamma}(W) \to \hat{\Gamma}(W')$. Furthermore, there is a natural homomorphism $\alpha : \Gamma \to \hat{\Gamma}$ which assigns to each $\gamma \in \Gamma(V)$ the element of $\hat{\Gamma}(V)$ represented by the compatible $\mathscr{V}$ family $\{\gamma\}$, where $\mathscr{V}$ is just $\{V\}$. This presheaf $\hat{\Gamma}$ is called the *completion of* $\Gamma$, and depends only on the values $\Gamma(U)$ for small open subsets $U$ of $X$.

LEMMA 1.10. *A presheaf* $\Gamma$ *of modules is a sheaf if and only if* $\alpha : \Gamma \to \hat{\Gamma}$ *is an isomorphism. In particular, the completion* $\hat{\Gamma}$ *is a sheaf.*

PROOF. The condition **(M)** is satisfied if and only if $\alpha$ is injective: for $\gamma \in \Gamma(V)$, $\alpha(\gamma) = 0$ means that the element of $\hat{\Gamma}(V)$ represented by the compatible $\mathscr{V}$ family $\{\gamma\}$ maps to 0 in the limit $\varinjlim\{\Gamma(\mathscr{U})\}$ over all compatible $\mathscr{U}$ families running over open covers $\mathscr{U}$ of $V$. This occurs if and only if there exists such a $\mathscr{U}$ with $\gamma|U = 0$ for all $U \in \mathscr{U}$, and condition **(M)** is fulfilled if and only if $\gamma = 0$ whenever this is the case.

If the condition **(G)** is satisfied, then for every open cover $\mathscr{U}$ of an open set $V$ and every compatible $\mathscr{U}$ family $\{\gamma_U\}_{U\in\mathscr{U}}$ there exists an element $\gamma \in \Gamma(V)$ such that $\gamma|U = \gamma_U$ for all $U \in \mathscr{U}$, which means that $\alpha(\gamma)$ is precisely the element of $\hat{\Gamma}(V)$ representing the family $\{\gamma_U\}_{U\in\mathscr{U}}$. Hence $\alpha$ is surjective in this case. If $\alpha$ is an isomorphism, then the condition **(G)** is satisfied since any family $\{\gamma_U\}_{U\in\mathscr{U}}$ has an element $\eta \in \hat{\Gamma}(V)$ representing it such that $\gamma = \alpha^{-1}(\eta)$ satisfies $\gamma|U = \gamma_U$ for every $U \in \mathscr{U}$. □

Any map $\tau : \Gamma_1 \to \Gamma_2$ between two presheaves of modules on $X$ induces a map $\hat{\tau} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ between their completions, defined as follows: Let $[\gamma] \in \hat{\Gamma}_1(V)$ be represented by the compatible $\mathscr{U}$ family $\{\gamma_U\}_{U\in\mathscr{U}}$ where $\mathscr{U}$ covers $V$; then $\hat{\tau}([\gamma])$ is represented by the compatible $\mathscr{U}$ family $\{\tau(\gamma_U)\}_{U\in\mathscr{U}}$. This induced map makes the following diagram commute:

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\tau} & \Gamma_2 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \hat{\Gamma}_1 & \xrightarrow{\hat{\tau}} & \hat{\Gamma}_2 \end{array}$$

If $\mathcal{B}$ is a basis for the topology on $X$, we may view $\mathcal{B}$ as a category with inclusion maps as its morphisms, similarly to the collection of all open sets, and define a *presheaf on* $\mathcal{B}$ to be a contravariant functor $\Gamma_{\mathcal{B}}$ from $\mathcal{B}$ to another concrete category $\mathscr{C}$. Then we similarly define the *completion of* $\Gamma_{\mathcal{B}}$ to be a sheaf $\hat{\Gamma}_{\mathcal{B}}$ on $X$ (that is, a functor from *all* of the open subsets of $X$ to $\mathscr{C}$) as follows: For any collection $\mathscr{U}$ of elements of $\mathcal{B}$, let $\Gamma_{\mathcal{B}}(\mathscr{U})$ be the collection of compatible $\mathscr{U}$ families of $\Gamma_{\mathcal{B}}$, and for a fixed open set $W$ of $X$ let $\mathscr{U}$ vary over the family of open coverings of $W$ by elements of $\mathcal{B}$. This gives a direct system $\{\Gamma_{\mathcal{B}}(\mathscr{U}) : \mathscr{U} \subset \mathcal{B}\}$ as before, so we may define $\hat{\Gamma}_{\mathcal{B}}(W) := \varinjlim\{\Gamma_{\mathcal{B}}(\mathscr{U}) : \mathscr{U} \subset \mathcal{B}\}$. Note that if $\mathcal{B}$ is all of the open sets of $X$, this definition of the completion of $\bar{\Gamma}$ is identical to the one above. Additionally, we may view $\hat{\Gamma}_{\mathcal{B}}$ as a presheaf on $\mathcal{B}$ by restricting attention to the elements of $\mathcal{B}$, and then define a map (of presheaves on $\mathcal{B}$) $\alpha : \Gamma_{\mathcal{B}}(U) \to \hat{\Gamma}_{\mathcal{B}}(U)$ for every $U \in \mathcal{B}$ identically to before. Furthermore, suppose that $\Gamma_{\mathcal{B}}$ is a presheaf on a basis $\mathcal{B}$ and $\Gamma$ is a presheaf on $X$ such that the restriction of $\Gamma$ to $\mathcal{B}$ is identical to $\Gamma_{\mathcal{B}}$; that is, $\Gamma(U) = \Gamma_{\mathcal{B}}(U)$ for every $U \in \mathcal{B}$ and the diagram below commutes whenever $V \subset U$ are both in $\mathcal{B}$:

$$\begin{array}{ccc} \Gamma(U) & \xleftrightarrow{=} & \Gamma_{\mathcal{B}}(U) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ \Gamma(V) & \xleftrightarrow{=} & \Gamma_{\mathcal{B}}(V) \end{array}$$

Then the completion $\hat{\Gamma}$ of $\Gamma$ is identical to $\hat{\Gamma}_{\mathcal{B}}$ since every open cover $\mathscr{U}$ of an open subset $V$ of $X$ has a refinement $\mathscr{U}'$ contained in $\mathcal{B}$, so open covers contained in $\mathcal{B}$ form a cofinal family for the direct system $\{\Gamma(\mathscr{U})\}$ which determines $\hat{\Gamma}(U)$. This also means that if $\Gamma_1$, $\Gamma_2$ are any two presheaves which both restrict to the same presheaf $\Gamma_{\mathcal{B}}$ on a basis $\mathcal{B}$ (in the sense described above), then $\hat{\Gamma}_1 = \hat{\Gamma}_{\mathcal{B}} = \hat{\Gamma}_2$.

For a presheaf $\Gamma$ on the space $X$ and an element $x \in X$, the collection of open subsets $U$ of $X$ which contain $x$ may be indexed by inclusion. With this ordering, the restriction maps $\Gamma(U) \to \Gamma(V)$ whenever $x \in V \subset U$ form a direct system. Define the *stalk of* $\Gamma$ *at* $x$ to be $\Gamma_x := \varinjlim\{\Gamma(U) : x \in U\}$. If $x \in U$ and $\gamma \in \Gamma(U)$, let the image of $\gamma$ under the canonical map $\Gamma(U) \to \Gamma_x$ be denoted $\gamma_x$ and called the *germ* of $\gamma$ at $x$. For any open subset $U$ of $X$ and element $x \in U$, each $\gamma \in \Gamma(U)$ may be viewed as a map $\gamma : U \to \bigsqcup_{x \in U} \Gamma_x$ by taking $\gamma(x) := \gamma_x$.

Example 1.11. If $A$ is an abelian group, the *constant presheaf* $A_X$ has $A_X(U) = A$ for every nonempty open subset $U$ of $X$ and $A_X(\emptyset) = 0$, with every restriction map either the identity $A \to A$ if the subset is nonempty or the unique map to 0 if the subset is empty. For each $x \in X$, the system $\{A_X(U) : x \in U\}$ is just the constant system for $A$, so the stalk $(A_X)_x$ at $x$ is precisely $A$. Furthermore, for every $U$ with $x \in U$, each $\gamma \in A_X(U)$ is just some $a \in A$, so its germ $\gamma_x = \gamma(x)$ is $a$.

The *constant sheaf* $\hat{A}_X$ is the completion of $A_X$. If $\mathscr{U}$ is a collection of open sets such that $x \in U$ for every $U \in \mathscr{U}$, then any compatible $\mathscr{U}$ family $\{\gamma_U\}_{U \in \mathscr{U}}$ must have some $a \in A$ with $\gamma_U(x) = a$ for every $U \in \mathscr{U}$; hence $(\hat{A}_X)_x$ is also precisely $A$, as before. Hence, for an open subset $U$ of $X$ each $\gamma \in \hat{A}_X(U)$ is a map $\gamma : U \to \bigsqcup_{x \in U} A$, but since $\gamma$ assigns each $x$ to just one element of $A$, this may be viewed instead as a map $U \to A$.

If $\varphi : X \to Y$ is a continuous map of topological spaces, and $\Gamma_X$ is a presheaf on $X$, then we obtain a presheaf $\varphi\Gamma_X$ on $Y$, called the *pushforward of* $\Gamma_X$, defined by the formula $\varphi\Gamma_X(U) := \Gamma_X(\varphi^{-1}(U))$. If $V \subset U$, then $\varphi^{-1}(V) \subset \varphi^{-1}(U)$, and the restriction map of the pushforward is the map which makes the following diagram commute:

$$\begin{array}{ccc} \varphi\Gamma_X(U) & \xleftrightarrow{=} & \Gamma_X(\varphi^{-1}(U)) \\ r\downarrow & & \downarrow r \\ \varphi\Gamma_X(V) & \xleftrightarrow{=} & \Gamma_X(\varphi^{-1}(V)) \end{array}$$

If $\hat{\Gamma}_X$ is the completion of $\Gamma_X$ and $\widehat{\varphi\Gamma}_X$ is the completion of the pushforward of $\Gamma_X$, then

$\varphi\hat{\Gamma}_X$ is also a sheaf, and we obtain a natural injective presheaf map $\widehat{\varphi\Gamma}_X \to \varphi\hat{\Gamma}_X$ as follows: For any collection $\mathscr{U}$ of open subsets of $Y$, $\varphi^{-1}(\mathscr{U})$ is a collection of open subsets of $X$; in particular, if $\mathscr{U}$ covers an open subset $V$ of $Y$ then $\varphi^{-1}(\mathscr{U})$ covers the open subset $\varphi^{-1}(V)$ of $X$. Thus, any compatible $\mathscr{U}$ family of $\varphi\Gamma_X$ corresponds to a compatible $\varphi^{-1}(\mathscr{U})$ family of $\Gamma_X$ under the identification $\varphi\Gamma_X(U) = \Gamma_X(\varphi^{-1}(U))$ for each $U \in \mathscr{U}$. This gives maps $\varphi\Gamma_X(\mathscr{U}) \to \Gamma_X(\varphi^{-1}(\mathscr{U}))$ which commute with the maps $\varphi\Gamma_X(\mathscr{U}) \to \varphi\Gamma_X(\mathscr{V})$ and $\Gamma_X(\varphi^{-1}(\mathscr{U})) \to \Gamma_X(\varphi^{-1}(\mathscr{V}))$ given by refinement, so if $V$ is an open subset of $Y$, they determine compatible maps between the system $\{\varphi\Gamma_X(\mathscr{U})\}$ running over all covers of $V$ in $Y$ and $\{\Gamma_X(\mathscr{W})\}$ running over all covers of $\varphi^{-1}(V)$ in $X$. Note that the latter system generally has more covers, in the sense that $\varphi^{-1}(\mathscr{U})$ is a cover of $\varphi^{-1}(V)$ for each cover $\mathscr{U}$ of $V$, but not necessarily *every* cover of $\varphi^{-1}(V)$ in $X$ is of that form. This determines a map $\widehat{\varphi\Gamma}_X(V) = \varinjlim\{\varphi\Gamma_X(\mathscr{U})\} \to \varinjlim\{\Gamma_X(\mathscr{W})\} = \hat{\Gamma}_X(\varphi^{-1}(V)) = \varphi\hat{\Gamma}_X(V)$ which is clearly injective but not necessarily surjective. These maps clearly commute with the restriction maps, so they give an injective presheaf map as claimed.

**Čech Cohomology:**

Let $\mathcal{B}$ be a basis of $X$, $\Gamma$ a presheaf of abelian groups on $\mathcal{B}$, and $\mathscr{U}$ an open cover of $X$ consisting of elements of $\mathcal{B}$. For $n \geq 0$ define $\check{C}^n(\mathscr{U}, \Gamma)$ to be the module of functions $f$ which assign to any ordered $(n+1)$-tuple $U_0, U_1, ..., U_n$ of elements of $\mathscr{U}$ an element $f(U_0, U_1, ..., U_n) \in \Gamma(U_0 \cap U_1 \cap ... \cap U_n)$. The coboundary $\partial : \check{C}^n(\mathscr{U}, \Gamma) \to \check{C}^{n+1}(\mathscr{U}, \Gamma)$ is defined by $\partial f(U_0, ..., U_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(U_0, ..., \hat{U}_i, ..., U_{n+1})|_{(U_0 \cap ... \cap U_{n+1})}$ where, as usual, $\hat{U}_i$ denotes omission of $U_i$. Since $\partial\partial = 0$, this makes $\check{C}^*(\mathscr{U}, \Gamma)$ a cochain complex, with cohomology groups $\check{H}^*(\mathscr{U}, \Gamma)$.

Let $\mathscr{V} \subset \mathcal{B}$ be a refinement of $\mathscr{U}$ and $\lambda : \mathscr{V} \to \mathscr{U}$ a function such that $V \subset \lambda(V)$ for all $V \in \mathscr{V}$. This gives a cochain map $\lambda^* : \check{C}^*(\mathscr{U}, \Gamma) \to \check{C}^*(\mathscr{V}, \Gamma)$ defined as $\lambda^* f(V_0, ..., V_n) = f(\lambda(V_0), ..., \lambda(V_n))|_{(V_0 \cap ... \cap V_n)}$. If $\mu : \mathscr{V} \to \mathscr{U}$ is another such function, a cochain homotopy $D : \check{C}^n(\mathscr{U}, \Gamma) \to \check{C}^{n-1}(\mathscr{V}, \Gamma)$ between $\lambda^*$ and $\mu^*$ is defined by $Df(V_0, ..., V_{n-1}) = \sum_{j=0}^{n-1} (-1)^j f(\lambda(V_0), ..., \lambda(V_j), \mu(V_j), ..., \mu(V_{n-1}))|_{(V_0 \cap ... \cap V_{n-1})}$. Thus, there is a well-defined homomorphism $\lambda^* : \check{H}^*(\mathscr{U}, \Gamma) \to \check{H}^*(\mathscr{V}, \Gamma)$ with $\lambda^*[f] = [\lambda^* f]$ which is independent of the choice of such $\lambda$. These maps $\lambda^*$ determined by refinement make $\{\check{H}^*(\mathscr{U}, \Gamma) : \mathscr{U} \subset \mathcal{B}$ is an open cover of $X\}$ a direct system, and the *Čech cohomology* of $X$ with coefficients in $\Gamma$ is defined to be $\check{H}^*(X, \Gamma) := \varinjlim_{\mathscr{U} \subset \mathcal{B}} \check{H}^*(\mathscr{U}, \Gamma)$. Furthermore, if $A$ is an abelian group, then the *Čech cohomology of $X$ with coefficients in* $A$ is defined as follows: Take $\mathcal{B}$ to consist of *all* open subsets of $X$ and $A_X$ to be the constant presheaf as described in Example 1.11; then $\check{H}^*(X, A) := \check{H}^*(X, A_X)$.

Let $\mathcal{B}$ be *all* open subsets of $X$, so that $\Gamma$ is a presheaf on $X$. If an open cover $\mathscr{U}$ of $X$ is $\mathscr{U} = \{U_x : x \in X\}$ where $x \in U_x$ for every $x \in X$, call this cover *indexed by* $X$. If $\mathscr{V}$ is any open cover of $X$ and $\mathscr{U}$ is a cover indexed by $X$, we may choose for each $x$ some $V_x \in \mathscr{V}$ with $x \in V_x$ and then define a common refinement $\mathscr{W} = \{U_x \cap V_x : x \in X\}$ of both $\mathscr{U}$ and $\mathscr{V}$ which is also indexed by $X$. Thus, the collection of open covers indexed by $X$ is cofinal in the collection of open covers as a directed partially ordered set indexing

$\{\check{H}^*(\mathscr{U},\Gamma)\}$, which means $\check{H}^*(X,\Gamma) \cong \varinjlim_{\mathscr{U} \text{ indexed by } X} \check{H}^*(\mathscr{U},\Gamma)$. Furthermore, if $\mathscr{V}$ is a refinement of $\mathscr{U}$ and both are indexed by $X$ with $V_x \subset U_x$ for every $x \in X$, this gives a map $\gamma : \mathscr{V} \to \mathscr{U}$ defined by $\gamma(V_x) = U_x$ for each $x \in X$, which then induces a canonical cochain map $\gamma^* : \check{C}^*(\mathscr{U},\Gamma) \to \check{C}(\mathscr{V},\Gamma)$. We may use this to define a cochain complex $\check{C}^*(X,\Gamma) := \varinjlim_{\mathscr{U} \text{ indexed by } X} \check{C}^*(\mathscr{U},\Gamma)$. Since the direct limit functor is exact, it commutes with cohomology, so $\varinjlim_{\mathscr{U} \text{ indexed by } X} \check{H}^*(\mathscr{U},\Gamma)$ is isomorphic to the cohomology of this cochain complex, that is, $\check{H}^*(X,\Gamma)$ as defined above is isomorphic to the cohomology groups of $\check{C}^*(X,\Gamma)$.

LEMMA 1.12. *For a basis $\mathcal{B}$ of $X$, there is a covariant functor from the category of short exact sequences of presheaves on $\mathcal{B}$ to the category of exact sequences which assigns to any short exact sequence $0 \to \Gamma' \to \Gamma \to \Gamma'' \to 0$ of presheaves on $\mathcal{B}$ an exact sequence*
$... \to \check{H}^n(X,\Gamma') \to \check{H}^n(X,\Gamma) \to \check{H}^n(X,\Gamma'') \to \check{H}^{n+1}(X,\Gamma') \to ...$

PROOF. For any open cover $\mathscr{U} \subset \mathcal{B}$ there is a short exact sequence of cochain complexes $0 \to \check{C}^*(\mathscr{U},\Gamma') \to \check{C}^*(\mathscr{U},\Gamma) \to \check{C}^*(\mathscr{U},\Gamma'') \to 0$. This gives an exact cohomology sequence for $\mathscr{U}$, and the result follows from passing this to the direct limits defining the Čech cohomology groups. □

A presheaf of modules $\Gamma$ on $\mathcal{B}$ is *locally zero* if, for any $\gamma \in \Gamma(V)$ there exists an open cover $\mathscr{U} \subset \mathcal{B}$ of $V$ with $\gamma|U = 0$ for all $U \in \mathscr{U}$; this is equivalent to the completion $\hat{\Gamma}$ being the zero presheaf (which satisfies $\Gamma(U) = 0$ for all $U$) and to the condition that for every $x \in X$, the stalk $\Gamma_x = 0$. A homomorphism $\tau : \Gamma_1 \to \Gamma_2$ between presheaves on $X$ is called *locally injective* if its kernel is locally 0, and a *local isomorphism* if both its kernel and its cokernel are locally 0. An open cover on a topological space $X$ is *locally finite* if for every $x \in X$ there exists an open subset $U$ of $X$ with $x \in U$ which has nonempty intersection with only finitely many elements of the cover; a space $X$ is called *paracompact* if every open cover admits a refinement which is locally finite. In particular, compact spaces are also paracompact. For an open cover $\mathscr{U}$, let $\mathscr{U}^* := \{U^*\}_{U\in\mathscr{U}}$ where $U^* = \cup\{U' \in \mathscr{U} : U' \cap U \neq \emptyset\}$. Another open cover $\mathscr{V}$ is a *star refinement* of $\mathscr{U}$ if $\mathscr{V}^*$ is a refinement of $\mathscr{U}$.

LEMMA 1.13. *If $X$ is a paracompact Hausdorff space with basis $\mathcal{B}$ and $\Gamma$ is a locally zero presheaf on $\mathcal{B}$, then $\check{H}^*(X, \Gamma) = 0$.*

PROOF. Let $\mathscr{U} \subset \mathcal{B}$ be a locally finite open covering of $X$ and $f$ an $n$-cochain of $\check{C}^*(\mathscr{U}, \Gamma)$. Let $\mathscr{W}$ be a locally finite open star refinement of $\mathscr{U}$. For each $x \in X$, since $\Gamma$ is locally zero, there is an open neighborhood $V_x \in \mathcal{B}$ contained in some element of $\mathscr{W}$ such that if $x \in U_0 \cap ... \cap U_n$ with $U_0, ..., U_n \in \mathscr{U}$, then $V_x \subset U_0 \cap ... \cap U_n$ and $f(U_0, ..., U_n)|V_x = 0$; this is only a finite number of conditions since $\mathscr{U}$ is locally finite. Let $\mathscr{V} := \{V_x\}_{x\in X}$ and define $\lambda : \mathscr{V} \to \mathscr{U}$ so that for each $x \in X$ there is some $W_x \in \mathscr{W}$ with $V_x \subset W_x \subset W_x^* \subset \lambda(V_x)$. Then, if $V_{x_0} \cap ... \cap V_{x_n} \neq \emptyset$, it follows that $V_{x_0} \subset W_{x_j}^*$ for each $j$ so that $V_{x_0} \subset \lambda(V_{x_j})$ for each $j$. Therefore, $f(\lambda(V_{x_0}), ..., \lambda(V_{x_n}))|V_{x_0} = 0$, so $\lambda^* f = 0$ in $C^*(\mathscr{V}, \Gamma)$. Thus, $\check{H}^n(X, \Gamma) = 0$ for all $n$. □

COROLLARY 1.14. *If $X$ is a paracompact Hausdorff space with basis $\mathcal{B}$ and $\tau : \Gamma_1 \to \Gamma_2$ is a local isomorphism of presheaves on $\mathcal{B}$, then the induced map $\tau_* : \check{H}^*(X, \Gamma_1) \to \check{H}^*(X, \Gamma_2)$ is an isomorphism.*

This gives rise to one important property of the completion of a presheaf:

COROLLARY 1.15. *If $X$ is a paracompact Hausdorff space with basis $\mathcal{B}$ and $\Gamma$ is a presheaf on $\mathcal{B}$, then the natural homomorphism $\alpha : \Gamma \to \hat{\Gamma}$ induces a cohomological isomorphism $\alpha_* : \check{H}^*(X, \Gamma) \to \check{H}^*(X, \hat{\Gamma})$.*

PROOF. By Corollary 1.14, it suffices to show that $\alpha$ is a local isomorphism. Let $\gamma \in [\ker(\alpha)]\,(V)$; then $\gamma \in \Gamma(V)$ and there exists an open covering $\mathscr{U} \subset \mathcal{B}$ of $V$ such that $\gamma|U = 0$ for all $U \in \mathscr{U}$. Thus $\ker(\alpha)$ is locally zero.

On the other hand, let $\gamma' \in [\mathrm{coker}(\alpha)](V)$. Then there exists an open covering $\mathscr{U} \subset \mathcal{B}$ of $V$ and a compatible $\mathscr{U}$ family $\{\gamma_U\}_{U\in\mathscr{U}}$ representing $\gamma'$. For each $U \in \mathscr{U}$, $\gamma'|U$ is represented by $\gamma_U \in \alpha(\Gamma(U))$, so $\gamma'|U = 0$, and $\mathrm{coker}(\alpha)$ is also locally zero. $\square$

Now, let $\Gamma$ be a presheaf on $X$, $\mathcal{B}$ a basis of $X$, and $\Gamma_{\mathcal{B}}$ a presheaf on $\mathcal{B}$. Since every open cover $\mathscr{U}$ of $X$ has a refinement $\mathscr{U}' \subset \mathcal{B}$, covers contained in $\mathcal{B}$ are cofinal in the collection of all open covers of $X$. Hence, if $\Gamma$ is a presheaf on $X$ whose restriction to $\mathcal{B}$ is identical to $\Gamma_{\mathcal{B}}$ (in the sense given before), then:

$$\check{H}^*(X,\Gamma) = \varinjlim \check{H}^*(\mathscr{U},\Gamma) = \varinjlim_{\mathscr{U}\subset\mathcal{B}} \check{H}^*(\mathscr{U},\Gamma) = \varinjlim_{\mathscr{U}\subset\mathcal{B}} \check{H}^*(\mathscr{U},\Gamma_{\mathcal{B}}) = \check{H}^*(X,\Gamma_{\mathcal{B}})$$

Thus, if $\Gamma$ is a presheaf on $X$, its Čech cohomology is identical to that of any presheaf on any basis $\mathcal{B}$ to which $\Gamma$ restricts identically. In this sense, Čech cohomology does not really depend on the choice of a basis $\mathcal{B}$, and unless otherwise specified will be assumed to use the basis consisting of *all* open subsets of $X$. Finally, if $X$ is a paracompact Hausdorff space, then by Corollary 1.15 the maps $\alpha : \Gamma \to \hat{\Gamma}$ on presheaves of $X$ and $\alpha : \Gamma_{\mathcal{B}} \to \hat{\Gamma}_{\mathcal{B}}$ on presheaves of $\mathcal{B}$ induce the following commutative diagram in Čech cohomology, whose rows are isomorphisms:

$$\begin{array}{ccc} \check{H}^*(X,\Gamma_{\mathcal{B}}) & \xrightarrow{\alpha^*} & \check{H}^*(X,\hat{\Gamma}_{\mathcal{B}}) \\ {\scriptstyle =}\updownarrow & & \updownarrow{\scriptstyle =} \\ \check{H}^*(X,\Gamma) & \xrightarrow{\alpha^*} & \check{H}^*(X,\hat{\Gamma}) \end{array}$$

A similar commutative diagram which removes the term in the lower-left, but has $\hat{\Gamma}_{\mathcal{B}}$ as a sheaf on $X$ in the lower-right, shows that $\check{H}^*(X,\Gamma_{\mathcal{B}}) \cong \check{H}^*(X,\hat{\Gamma}_{\mathcal{B}})$ by a natural isomorphism which is induced by $\alpha : \Gamma_{\mathcal{B}} \to \hat{\Gamma}_{\mathcal{B}}$, where the left-hand side treats $\Gamma_{\mathcal{B}}$ as a presheaf on $\mathcal{B}$, but the right-hand side treats $\hat{\Gamma}_{\mathcal{B}}$ as a *sheaf on* $X$; that is, the cohomology is computed using the basis consisting of *every* open subset of $X$.

# CHAPTER 2

# FUNCTORS ON SIMPLICIAL SETS

## 2.1 FUNCTOR EXTENSION TO INVERSE LIMITS

Our ultimate goal involves extending a well-established construction on finite groups, which uses simplicial sets, to profinite groups. To this end, we will establish here that taking inverse limits of $n$-truncated simplicial finite sets commutes cleanly with certain kinds of functors.

Analagous to the situation of topological groups, the limit of an inverse system of $n$-truncated simplicial finite sets will be called an $n$-truncated simplicial profinite set. By adding such inverse limits for every system in $\mathbf{sFSet}_n$, we extend the category to $\widehat{\mathbf{sFSet}}_n$, the $n$-truncated simplicial profinite sets. Note that an $n$-truncated simplicial profinite set is really an $n$-truncated simplicial space for which the space of $m$-simplices for each $m \leq n$ is compact and Hausdorff, and the face and degeneracy maps between them are continuous.

If $X$ is a simplicial space or an $m$-truncated simplicial space (including elements of $\widehat{\mathbf{sFSet}}_m$), define an *equivalence relation on* $X$ to be a simplicial subset $R$ of $X \times X$ such that $R[n] \subseteq X[n] \times X[n]$ is an equivalence relation on $X[n]$ when viewed as a space. Similarly, the quotient $X/R$ of $X$ by an equivalence relation $R$ is given by $(X/R)[n] = X[n]/R[n]$ for each $n$. Given a morphism $f : X \to Y$ in $\widehat{\mathbf{sFSet}}_m$, let the *kernel of* $f$ be the simplicial subspace $K \subseteq X \times X$ such that $K[n] := \{(a, b) : f_n(a) = f_n(b)\}$ (where $f_n : X[n] \to Y[n]$ is the $n$th degree map of $f$). Note that $K[n]$ is clearly an equivalence relation on $X[n]$ for each $n$, so $K$ is also an equivalence relation on $X$. Next, let $\{R_i, i \in I\}$ be the set of all open equivalence relations on $X \in \widehat{\mathbf{sFSet}}_m$ (that is, those which are open as subsets of $X \times X$) with the indexing set $I$ ordered so that $i \leq j$ if and only if $R_i \supseteq R_j$. Then the *canonical projective system for* $X$ is the inverse system $\{X/R_i, p_{ij}\}$ where $p_{ij} : X/R_j \to X/R_i$ is the natural projection, which exists whenever $R_i \supseteq R_j$. By definition, this system is surjective, its constituent simplicial sets are finite, and $\varprojlim X/R_i = X$. We will prove the following claim regarding $\widehat{\mathbf{sFSet}}_m$:

THEOREM 2.1. *Let $\mathscr{C}$ be a category with inverse limits. Given a covariant functor $F : sFSet_m \to \mathscr{C}$, there exists an extension $\hat{F} : \widehat{\boldsymbol{sFSet}}_m \to \mathscr{C}$ such that $\varprojlim F(X_i) = \hat{F}\left(\varprojlim X_i\right)$ for any surjective inverse system $\{X_i, \varphi_{ij}\}$ in $\boldsymbol{sFSet}_m$. Additionally, for any surjective inverse system in $\widehat{\boldsymbol{sFSet}}_m$, $\varprojlim \hat{F}(X_i) = \hat{F}\left(\varprojlim X_i\right)$.*

Given a functor $F : \mathbf{sFSet}_m \to \mathscr{C}$ and a surjective inverse system $\{X_i, \varphi_{ij}\}$, let $X = \varprojlim X_i$ and let $\{X/R_i, p_{ij}\}$ be the canonical projective system for $X$. Then apply the functor $F$ to this inverse system to get $\{F(X/R_i), F(p_{ij})\}$, an inverse system in $\mathscr{C}$. As $\mathscr{C}$ has inverse limits, we may then define $\hat{F}(X)$ to be the limit of this system. For the projection maps $p_i : X \to X/R_i$ define $\hat{F}(p_i)$ to be the projection map $\hat{F}(X) \to F(X/R_i)$ which is given for $\hat{F}(X)$ as an inverse limit.

In case $X$ is finite, $\hat{F}$ may be chosen to coincide with $F$ since the system $\{X/R_i, p_{ij}\}$ may be indexed by a finite set $I$ and has a maximal element in the diagonal of $X \times X$. Thus $\hat{F}(X) = F(X)$ is the limit of the finite inverse system $\{F(X/R_i), F(p_{ij})\}$.

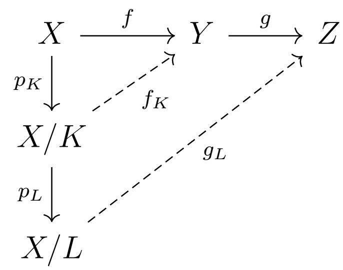
If $f : X \to Y$ is a map in $\widehat{\mathbf{sFSet}}_m$ and $Y$ is an $m$-truncated simplicial finite set, then the kernel $K$ of $f$ is an open equivalence relation on $X$, which means the map $p_K : X \to X/K$ is a projection map for $X$ as a limit of the canonical projective system $\{X/R_i, p_{ij}\}$. Furthermore, the map $f_K : X/K \to Y$ is a map in $\mathrm{sFSet}_m$, so we may define $\hat{F}(f) := F(f_K)\hat{F}(p_K)$ in this situation. If in addition $g : Y \to Z$ is a map in $\mathbf{sFSet}_m$, then $\hat{F}(h)$ for the map $h = gf$ with kernel $L$ has the similar definition $\hat{F}(h) := F(h_L)\hat{F}(p_L)$.

Lemma 2.2. *If $X = \varprojlim X_i$ is an inverse limit of a surjective system $\boldsymbol{sFSet}_m$, then for any map $f : X \to Y$ to a $m$-truncated simplicial finite set, there exists an $i$ and a function $f_i$ such that $f = f_i p_i$ where $p_i : X \to X_i$ is the projection map.*

Proof. If $X$ is finite, then there exists some $i$ so that $X = X_i$ and we are done. If not, let $K$ be the kernel of $f$ and for any $i$ let $L_i = ker(p_i)$ so that $X_i = X/L_i$ . Then for every $0 \leq n \leq m$, $L_i[n]$ is the preimage of the diagonal $D_i \subseteq X_i[n] \times X_i[n]$ under the projection map $X \times X \to X_i \times X_i$ ; in particular it is a closed subset of $X[n] \times X[n]$. Also, $K[n]$ is an open subset of $X[n] \times X[n]$ as the preimage under the projection map of the diagonal in $X/K \times X/K$, which is finite and discrete. Thus $L_i[n] \setminus K[n]$ is closed in $X[n] \times X[n]$ for every $i$. Now consider $\bigcap_{i \in I} (L_i[n] \setminus K[n]) = \left( \bigcap_{i \in I} L_i[n] \right) \setminus K[n] = D \setminus K[n]$ (where $D$ is the diagonal in $X \times X$), which is clearly empty. Therefore, since $X[n] \times X[n]$ is compact, there exists some finite set $i_1, ..., i_l$ so that $\bigcap_{k=1}^{l} (L_{i_k}[n] \setminus K[n]) = \emptyset$. Let $i_n \succeq i_k$ for $k = 1, 2, ..., l$; then $L_{i_n} \subseteq L_{i_k}$ for all such $k$ and in particular $L_{i_n}[n] \subseteq L_{i_k}[n]$ , so $\bigcap_{k=1}^{l} (L_{i_k}[n] \setminus K[n])$ is just $L_{i_n}[n] \setminus K[n]$. Since $L_{i_n}[n] \setminus K[n] = \emptyset$, $L_{i_n}[n] \subseteq K$. Then choose $i \succeq i_0, i_1, ..., i_n$ so that $L_i[n] \subseteq L_{i_n}[n] \subseteq K[n]$ for all $n$; thus $L_i \subseteq K$. Then there exists a natural projection $q : X/L_i \to X/K$ which commutes with $p_K$ and $p_i$, and we may define $f_i := f_K q$ so that $f = f_K p_K = f_K q p_i = f_i p_i$, as desired. □

Lemma 2.3. *As defined above, $\hat{F}(gf) = \hat{F}(h) = F(g)\hat{F}(f)$.*

Proof. By definition, $\hat{F}(h) = F(h_L)\hat{F}(p_L)$. Also note that $L \supseteq K$ so that the index of $L$ in the canonical projective system is less than or equal to that of $K$, which gives a map $p_{KL} : X/L \to X/K$ with $p_L = p_{KL} p_K$. This gives the following commutative diagram:




Therefore, $F(h_L)\hat{F}(p_L) = F(h_L)\hat{F}(p_{KL}p_K) = F(h_L)F(p_{KL})\hat{F}(p_K)$. Now, the first two maps are between simplicial finite sets, so

$F(h_L)F(p_{KL}) = F(h_L p_{KL}) = F(gf_K) = F(g)F(f_K)$ since F is a functor on $\mathbf{sFSet}_m$. Thus $\hat{F}(gf) = F(g)F(f_K)\hat{F}(p_K)$, which is $F(g)\hat{F}(f)$ by definition. □

Next, let $f : X \to Y$ be a map in $\widehat{\mathbf{sFSet}}_m$, where $\{X/R_i, p_{ik}\}$ is the canonical projective system for $X$ indexed by $I$ and $\{Y/Q_j, q_{j\ell}\}$ is the canonical projective system for $Y$ indexed by $J$. Then, denoting by $q_j$ the projection map $Y \to Y/Q_j$, note that for every $j$ $\hat{F}(q_{\ell j}q_j f) = F(q_{\ell j})\hat{F}(q_j f)$ by Lemma 2.3 since $Y/Q_j$ and $Y/Q_l$ are simplicial finite sets. Note also that the family of maps $\hat{F}(q_j f)$ are compatible with the projections $F(q_{j\ell})$ in the inverse system $\{F(Y/Q_j), F(q_{j\ell})\}$. Thus by the universal property of inverse limits, there exists a unique map $X \to Y$ so that $\hat{F}(q_j)\hat{F}(f) = \hat{F}(q_j f)$ for all $j$ in $J$; define this to be $\hat{F}(f)$ in this situation.

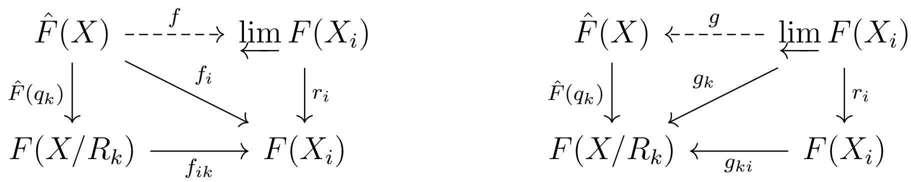
Lemma 2.4. *If $f : X \to Y$ and $g : Y \to Z$ are maps in $\widehat{\boldsymbol{sFSet}}_m$, then $\hat{F}(gf) = \hat{F}(g)\hat{F}(f)$. In particular, $\hat{F}$ is a functor from $\widehat{\boldsymbol{sFSet}}_m$ to $\mathscr{C}$.*

Proof. Let $r_k : Z \to Z/S_k$ be the projections to the finite simplicial sets forming the canonical projective system for $Z$. Then, as defined above, $\hat{F}(g) : F(Y) \to F(Z)$ and $\hat{F}(gf) : F(X) \to F(Z)$ are the unique maps such that $\hat{F}(r_k)\hat{F}(g) = \hat{F}(r_k g)$ and $\hat{F}(r_k)\hat{F}(gf) = \hat{F}(r_k gf)$ for every $k$. Then let $K = \ker(r_k g)$, let $q_K : Y \to Y/K$ be the natural quotient map, and let $g_K : Y/K \to Z/S_k$ be the unique map so that $r_k g = g_K q_K$ . By Lemma 2.3, $\hat{F}(r_k G) = \hat{F}(g_K q_K) = F(g_K)\hat{F}(q_K)$. Therefore,

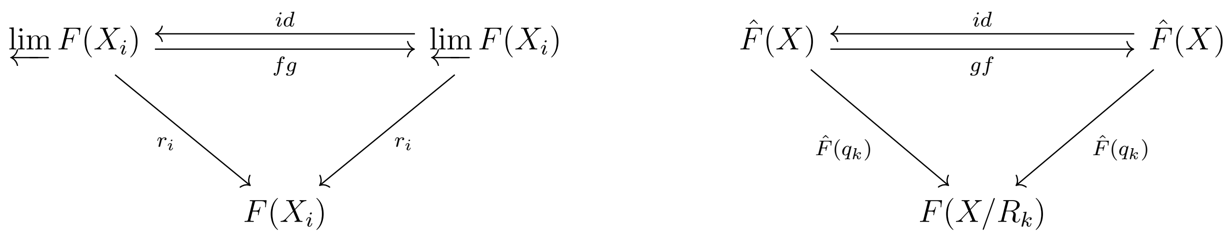
$\hat{F}(r_k)\hat{F}(g)\hat{F}(f) = \hat{F}(r_k g)\hat{F}(f) = F(g_K)\hat{F}(q_K)\hat{F}(f) = F(g_K)\hat{F}(q_K f)$. By Lemma 2.3 again, $F(g_K)\hat{F}(q_K f) = \hat{F}(g_K q_K f) = \hat{F}(r_k g f)$. But then, $\hat{F}(r_k)\hat{F}(g)\hat{F}(f) = \hat{F}(r_k g f)$ for every $k$, which is the definition of the unique map $\hat{F}(gf)$. Thus $\hat{F}(g)\hat{F}(f) = \hat{F}(gf)$. □

LEMMA 2.5. *If* $\{X_i, p_{ij}\}$ *is a surjective inverse system in* $\boldsymbol{sFSet}_m$*, then* $\hat{F}\left(\varprojlim X_i\right) = \varprojlim F(X_i)$ *where the right hand side indicates a limit in* $\mathscr{C}$ *of the inverse system* $\{F(X_i), F(p_{ij})\}$.

PROOF. First let $X := \varprojlim X_i$, and write the canonical projective system for $X$ as $\{X/R_k, q_{kl}\}$, so that by definition $\hat{F}(\varprojlim X_i) = \hat{F}(X) = \varprojlim F(X/R_k)$. For any $i$, $X_i$ is a $m$-truncated finite simplicial set and $p_i : X \to X_i$ is a projection, so there exists some $k$ so that the kernel $K_i$ of $p_i$ is the equivalence relation $R_k$; in particular we have the identity map $X/R_k \to X_i$; we will denote this map by $f_{ik}$. On the other hand, for any $k$ there exists an $i$ so that the kernel $K_i$ is contained in $R_k$, which means $K_i = R_l$ with $R_l \succeq R_k$. Hence there exists a projection $q_{kl} : X/R_l \to X/R_k$, and $X_i \to X/R_l$ is the identity map; define $g_{ki} : X_i \to X/R_k$ to be the composition of these two maps. The maps $f_{ik}$ and $g_{ki}$ are maps of $m$-truncated simplicial finite sets, so we may apply the functor $F$ to them to obtain maps $F(f_{ik}) : F(X/R_k) \to F(X_i)$ and $F(g_{ki}) : F(X_i) \to F(X/R_k)$ respectively. Let $r_i : \varprojlim F(X_i) \to F(X_i)$ be the projection maps defined for this limit in $\mathscr{C}$; then for each $i$ define $f_i : \hat{F}(X) \to F(X_i)$ to be the composition $F(f_{ik})\hat{F}(q_k) = \hat{F}(f_{ik}q_k)$, and for each $k$ define $g_k : \varprojlim F(X_i) \to F(X/R_k)$ to be the composition $F(g_{ki})r_i$. Observe that the maps $f_i$ and $g_k$ are compatible with the respective inverse systems, so by the universal property of inverse limits in $\mathscr{C}$ they respectively give maps $f : \hat{F}(X) \to \varprojlim F(X_i)$, $g : \varprojlim F(X_i) \to \hat{F}(X)$ which commute with all relevant projection maps. This gives the following pair of commutative diagrams:




Using these, we will show that the following diagrams also commute:




Since the identity map is the unique one which can be placed where $fg$ (respectively, $gf$) is in order for the left (respectively, right) diagram to commute, this will suffice to show that $f$ and $g$ are inverses, so that $f$ is an isomorphism in $\mathscr{C}$.

In the left diagram, obviously the identity commutes with $r_i$. Next, $r_i fg = F(f_{ik})\hat{F}(q_k)g$ where $k$ is, as in the definition of $f_{ki}$, chosen so that $X/R_k = X_i$. Then $F(f_{ik})\hat{F}(q_k)g = F(f_{ik})F(g_{ki})r_i = F(f_{ik}g_{ki})r_i$ where $g_{ki}$ chooses $R_l = R_k$ since clearly $K_i = R_k$ here; therefore, this map is precisely $r_i$. In the right diagram, the identity similarly commutes with $\hat{F}(q_k)$, and $\hat{F}(q_k)gf = F(g_{ki})r_i f$ where $g_{ki}$ is the composition $X_i \to X/R_\ell \to X/R_k$ for some $\ell$, so $F(g_{ki})r_i f = F(g_{ki})F(f_{i\ell})\hat{F}(q_\ell) = \hat{F}(g_{ki}f_{i\ell}q_\ell)$. But then, $g_{ki}f_{i\ell}q_\ell$ is the following sequence of maps: $X \overset{q_l}{\to} X/R_\ell \to X_i \to X/R_\ell \overset{q_{k\ell}}{\to} X/R_k$ where the second and third arrows are just identities, so $g_{ki}f_{i\ell}q_\ell = q_{k\ell}q_\ell = q_k$, and thus $\hat{F}(q_k)gf = \hat{F}(q_k)$, as desired. □

Lemma 2.6. *If $\{X_i, p_{ij}\}$ is a surjective inverse system in $\widehat{\boldsymbol{sFSet}}_m$, then* $\hat{F}\left(\varprojlim X_i\right) = \varprojlim \hat{F}(X_i)$.

Proof. For each $i$, let $\{X_i/R\}$ denote the canonical projective system for $X_i$. Then by definition, $\hat{F}(X_i) = \varprojlim F(X_i/R)$. Now define an inverse system as follows: Index by ordered pairs $(i, R)$ with $i \in I$ and $R$ an equivalence relation on $X_i$; let $(i, R) \succeq (j, S)$ if both $i \succeq j$ and $R \subseteq \ker(p_{ji})$. This gives a projection $q_{jS,iR} : X_i/R \to X_j/S$ whenever $(i, R) \succeq (j, S)$. Then the inverse system is $\{X_i/R, q_{iR,jS}\}$, and similarly this creates an inverse system $\{F(X_i/R), F(q_{iR,jS})\}$ in $\mathscr{C}$. By construction, $\varprojlim X_i/R = \varprojlim X_i = X$ and $\varprojlim F(X_i/R) = \varprojlim \hat{F}(X_i)$, but also by Lemma 2.5 $\varprojlim F(X_i/R) = \hat{F}(X)$, so $\varprojlim \hat{F}(X_i) = \hat{F}(X)$. □

This completes the proof of Theorem 2.1. In particular, this theorem applies to the geometric realization functor, so that if $X = \varprojlim X_i$ is the inverse limit of a surjective system of $n$-truncated finite simplicial sets, we have the topological space $\widehat{|X|} = \widehat{|\varprojlim X_i|} = \varprojlim |X_i|$. Furthermore, we claim $\widehat{|X|} = |X|$ where the right-hand side denotes the geometric realization of $X$ as a simplicial space. To see this, first note $|X| := \left(\coprod_{m=0}^{n} (X[m] \times |\Delta^m|)\right) / \sim$, but for each $m$, $X[m] \times |\Delta^m| = \left(\varprojlim X_i[m]\right) \times |\Delta^m| = \varprojlim (X_i[m] \times |\Delta^m|)$ since inverse limits commute with direct products. Then, clearly $\coprod_{m=0}^{n} \varprojlim (X_i[m] \times |\Delta^m|) = \varprojlim \coprod_{m=0}^{n} (X_i[m] \times |\Delta^m|)$ for the finite disjoint union, and it remains only to see that the equivalence relation $\sim$ on this space is determined by the relations $\sim_i$ of $|X_i| = \left(\coprod_{m=0}^{n} (X_i[m] \times |\Delta^m|)\right) / \sim_i$ for each $i$. This, too, is clear, since $(x, p) \sim (y, q)$ if and only if there exists some $i$ with $\varphi_i(x) = x_i, \varphi_i(y) = y_i$, and $(x_i, p) \sim_i (y_i, q)$. Thus:

$$|X| = \left(\varprojlim \coprod_{m=0}^{n} (X_i[m] \times |\Delta^m|)\right) / \sim = \varprojlim \left(\left(\coprod_{m=0}^{n} (X_i[m] \times |\Delta^m|)\right) / \sim_i\right) = \varprojlim |X_i|.$$

This is equal to $\widehat{|X|}$. From now on, $\widehat{|X|}$ will be denoted $|X|$ in this situation since there is no ambiguity in the meaning of this notation.

## 2.2 GEOMETRIC REALIZATION

We defined the geometric realization of a simplicial set $X$ as $|X| = \left( \coprod_{n=0}^{\infty} (X[n] \times |\Delta^n|) \right) / \sim$ for a particular equivalence relation $\sim$. A more precise description of the effect of this relation will be needed; specifically, we will show that the geometric realization can be built upward, in the sense that the realization of an $n$-truncated simplicial set does not depend on the $m$-simplices for any value $m > n$. For any order-preserving map $\alpha : [n] \to [m]$ of finite totally ordered sets, we will denote the corresponding map on $X$ as $\alpha^* : X[m] \to X[n]$. When viewing $X$ as a functor from $\Delta$, $\alpha^*$ is just $X(\alpha)$. In particular, $\delta_i^* = d_i$ and $\sigma_i^* = s_i$ are the usual face and degeneracy maps for $X$ as a simplicial set, respectively. We wish to prove:

THEOREM 2.7. *Let $\sim$ be the equivalence relation above, and suppose that $(x, u) \sim (y, v)$ for some $x, y \in X[n]$. Then there exist $p_0 \le p_1 \le n$ and $z \in X[p_1]$, together with three order-preserving maps $\alpha : [p_0] \to [n]$, $\beta : [p_0] \to [p_1]$, and $\gamma : [n] \to [p_1]$ where $\alpha, \beta$ are injective, $\gamma$ is surjective, and $\alpha^*(x) = \beta^*(z)$, $\gamma^*(z) = y$. Furthermore, there exists $t \in [p_0]$ so that $\alpha(t) = u$ and $\beta(t) = \gamma(v)$. Also, if $y$ is a nondegenerate simplex, then $p_1 = n$ and $\gamma$ is the identity map.*

Proving this theorem will require a few lemmas.

LEMMA 2.8. *Let $\alpha : [e] \to [c]$ be a composition of injective and surjective order-preserving maps, so that $\alpha^* : X[c] \to X[e]$ is a composition of face and degeneracy maps. Then either $\alpha$ is the identity map, or $\alpha = \delta_{i_1}\delta_{i_2}...\delta_{i_n}\sigma_{j_1}\sigma_{j_2}...\sigma_{j_m}$ for some $n, m$, that is, $\alpha$ may be written as a surjective map followed by an injective map; this means $\alpha^* = s_{j_m}s_{j_{m-1}}...s_{j_1}d_{i_n}d_{i_{n-1}}...d_{i_1}$. Furthermore, this rearrangement does not change the effect of $\alpha$ when it is extended to a map $|\Delta^e| \to |\Delta^c|$.*

PROOF. If $\alpha$ is not the identity map, then $\alpha = \beta_1\beta_2...\beta_p$ where each $\beta_k$ is either $\delta_i$ or $\sigma_i$ for some $i$. In particular, this expression contains at least one surjective map or at least one injective map. If $\alpha$ is not already in the desired form, there must be at least one of each kind of map, so that $\alpha = \beta_1...\beta_\ell\sigma_k\delta_{i_1}...\delta_{i_n}\sigma_{j_1}...\sigma_{j_m}$ with $m, \ell$ possibly equal to 0 but $n \geq 1$. Note that $\sigma_k$ is the first surjective map not in the desired order, with injective maps before it. Then, by the identities on order-preserving maps:

$$\sigma_k\delta_{i_1} = \begin{cases} \delta_{i_1-1}\sigma_k & \text{if } k < i_1 - 1 \\ \text{id} & \text{if } k = i_1 \text{ or } i_1 - 1 \\ \delta_{i_1}\sigma_{k-1} & \text{if } k > i_1 \end{cases}$$

If $k = i_1$ or $i_1 - 1$ we may delete both terms, and otherwise we may write $\sigma_k\delta_{i_1} = \sigma_{k'}\delta_{i'_1}$ so that $\alpha = \beta_1...\beta_\ell\delta_{i'_1}\sigma_{k'}...\delta_{i_n}\sigma_{j_1}...\sigma_{j_m}$. This process may be repeated with each following term until either the surjective map is annihilated into an identity map, or has been moved to the right of $\delta_{i_n}$ so that $\alpha = \beta_1...\beta_\ell\delta_{i'_1}...\delta_{i'_n}\sigma_{j'_1}...\sigma_{j_{m'}}$ where $m'$ is either $m$ or $m+1$ (and $j'_1$ is the index resulting from shifting $\sigma_k$ all the way to the right in the latter case) and possibly one of the $\delta_{i_q}$ removed. At this point either $\alpha$ has the desired form or there exist some $k'', n', \ell'$ so that $\alpha = \beta_1...\beta_{\ell'}\sigma_{k''}\delta_{i_1}...\delta_{i_{n'}}\sigma_{j'_1}...\sigma_{j_{m'}}$ again. In the latter case, $\sigma_{k''}$ may be shifted to the right or annihilated the same way $\sigma_k$ was, and this process can continue to be repeated until $\alpha$ has the desired form. Finally, note that the extension of surjective and injective order-preserving maps to maps on $|\Delta^e|$ retain the three identities defined for such maps, so rearranging $\alpha$ this way keeps its effect on $|\Delta^e|$ the same. □

Lemma 2.9 (Degeneracy Uniqueness). *Suppose that $x \in X[p]$, $y \in X[q]$ are nondegenerate simplices and that there exist surjective maps $\alpha : [n] \to [p]$, $\beta : [n] \to [q]$ such that $\alpha^*(x) = \beta^*(y)$. Then $p = q$, $x = y$, and $\alpha = \beta$.*

Proof. Without loss of generality, we may assume $p \geq q$. Define $\gamma : [p] \to [n]$ by choosing $\gamma(i) \in \alpha^{-1}(i)$ for each $i \in [p]$; note that $\gamma$ is injective. Then $\alpha\gamma = \text{id}$ by definition. Therefore, $x = (\alpha\gamma)^*x = \gamma^*\alpha^*x = \gamma^*\beta^*y = (\beta\gamma)^*y$. Since $\beta\gamma$ is an injective map followed by a surjective map, by Lemma 2.8 we may rewrite it as a surjective map $\phi : [p] \to [m]$ for some $m$ followed by an injective map $\psi : [m] \to [q]$ so that the following diagram commutes:

$$\begin{array}{ccc} [p] & \xrightarrow{\beta\gamma} & [q] \\ {\scriptstyle\phi}\big\downarrow & \nearrow{\scriptstyle\psi} & \\ [m] & & \end{array}$$

Thus, $x = (\beta\gamma)^*y = (\psi\phi)^*y = \phi^*(\psi^*y)$, where either $\phi$ is the identity map or $\phi^*$ is a degeneracy map. Since $x$ is nondegenerate, $\phi$ must be the identity $[p] \to [p]$, which means that $\beta\gamma = \psi$, an injective map from $[p]$ to $[q]$. However, $p \geq q$, so this is only possible if $p = q$, which means $\psi$ is also the identity map. Now, note that the above is independent of our choice of $\gamma(i)$ for each $i$ from $\alpha^{-1}(i)$, so $\alpha\gamma = \beta\gamma = \mathrm{id}$ for every such $\gamma$. Therefore, $\alpha = \beta$, and since $\mathrm{id}^* = \mathrm{id}$, we also have $\mathrm{id}(x) = y$, so $x = y$ also. □

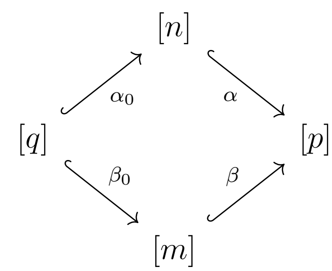
Note that by the definition of degeneracy, any degenerate $n$-simplex $x \in X[n]$ has some $i \in [n-1], y \in X[n-1]$ so that $s_i(y) = x$; if $y$ is also degenerate then there exists $j \in [n-2], z \in X[n-2]$ with $x = s_i(y) = s_j s_i(z)$. This descent can continue as long as the simplex in question is degenerate, but it must be finite since $X[0]$ contains no degenerate simplices (because there is no "$X[-1]$" to contain simplices which may map into $X[0]$ by degeneracy maps), so there must exist some simplex $w$ which is nondegenerate such that $s_{i_1}s_{i_2}...s_{i_k}(w) = x$. By degeneracy uniqueness, this $w$ is in fact the *only* non-degenerate simplex with this property, so for any degenerate simplex $x$ we may refer to *the* unique nondegenerate simplex whose image it is under a composition of degeneracy maps.

Lemma 2.10. *Let $\sigma_p : [r] \to [p]$ and $\sigma_q : [r] \to [q]$ be surjective maps such that $x \in X[p]$, $y \in X[q]$, $t \in [r]$, $u \in [q]$, and $v \in [p]$ satisfy $\sigma_p^* x = \sigma_q^* y$ and $\sigma_q(t) = u$, $\sigma_p(t) = v$. Then there exists a map $\alpha : [q] \to [p]$ so that $\alpha^* x = y$ and $\alpha(u) = v$. Furthermore, $\alpha = \beta_0\gamma$ where $\beta_0$ is injective and $\gamma$ is surjective, so we may write $\alpha^* = s_{k_1}...s_{k_m}d_{l_1}d_{l_2}...d_{l_n}$.*

PROOF. Since $\sigma_p^*(x) = \sigma_q^*(y)$ is a degenerate simplex, there exists some $m \leq p, q$ and $z \in X[m]$ which is the unique nondegenerate simplex corresponding to it. If $z = x$ or $z = y$, we may assume without loss of generality the former case. Then $\sigma_p$ may be considered a composition of $\sigma_q : [r] \to [q]$ with another surjective map $\gamma : [q] \to [p]$. Therefore, if $\beta_0 = \text{id} : [p] \to [p]$, then $\alpha = \beta_0\gamma$ has the desired properties. Otherwise, both $x$ and $y$ are degenerate simplices, say $\gamma^* z = y$ and $\beta^* z = x$ for some surjective maps $\gamma : [q] \to [m]$, $\beta : [p] \to [m]$. Then $(\gamma\sigma_q)^* z = (\beta\sigma_p)^* z$, which by degeneracy uniqueness implies that $\gamma\sigma_q = \beta\sigma_p$. Now let $\beta_0 : [m] \to [p]$ be any injective map so that $\beta_0(i) \in \beta^{-1}(i)$ for each $i$, and define $\alpha = \beta_0\gamma$. Clearly $\beta\beta_0 = \text{id}$, so $\beta_0^*(x) = \beta_0^*(\beta^* z) = (\beta\beta_0)z = z$, and thus $\alpha^* x = (\beta_0\gamma)^* x = \gamma^*\beta_0^* x = \gamma^* z = y$; furthermore, $\alpha(u) = \beta_0\gamma(u) = \beta_0\gamma\sigma_q(t) = \beta_0\beta\sigma_p(t) = \sigma_p(t) = v$. Thus $\alpha = \beta_0\gamma$ has the claimed properties in this case also. $\square$

LEMMA 2.11. *If there exists $p \geq m, n$ and injective maps $\alpha : [n] \to [p]$ and $\beta : [m] \to [p]$ so that $x \in X[n]$, $y \in X[m]$ satisfy $x = \alpha^* z$ and $y = \beta^* z$ for some $z \in X[p]$, then there exist $q \leq m, n$ and injective maps $\alpha_0 : [q] \to [n]$, $\beta_0 : [q] \to [m]$ so that $\alpha_0^* x = \beta_0^* y$. Since $\alpha_0^*, \beta_0^*$ are surjective maps, they are expressible as compositions of face maps. Furthermore, if there exist $u \in |\Delta^n|$ and $v \in |\Delta^m|$ so that $\alpha(u) = \beta(v)$, then there exists $t \in |\Delta^p|$ so that $\alpha_0(t) = u$, $\beta_0(t) = v$.*

PROOF. Consider the intersection $\alpha[n] \cap \beta[m]$, and let $q := |\alpha[n] \cap \beta[m]| + 1$. Since this intersection is a well-ordered set (as a subset of the well-ordered set $[p]$), it is isomorphic to $[q]$ by a unique order-preserving map $\gamma : \alpha[n] \cap \beta[m] \to [q]$. For each element $a \in [q]$, let $\alpha_0(a)$ be the element $b$ in $[n]$ such that $\alpha(b) = \gamma^{-1}(a)$; similarly, let $\beta_0(c)$ be the element $d$ in $[m]$ such that $\beta(d) = \gamma^{-1}(c)$. This gives the following commutative diagram of order-preserving injective maps:




Now, since the above diagram commutes,
$\alpha_0^*x = \alpha_0^*\alpha^*z = (\alpha\alpha_0)^*z = (\beta\beta_0)^*z = \beta_0^*\beta^*z = \beta_0^*y$, as claimed. Also, let $t \in |\Delta^p|$ be the element corresponding to $\alpha(u)$ under the extension of the order-preserving isomorphism. Then, by the definition of the maps, we have $\alpha_0(t) = u$, $\beta_0(t) = v$. $\square$

Proof of Theorem 2.7. First note that since the relation $\sim$ is generated by face and degeneracy maps, there must exist a chain of order-preserving maps

$$[n] \xleftarrow{\gamma_{-1}} [p_0] \xleftarrow{\gamma_0} [p_1] \xleftarrow{\gamma_1} ... \xleftarrow{\gamma_{i-1}} [p_i] \xrightarrow{\eta_i} [p_{i+1}] \xrightarrow{\eta_{i+1}} ... \xrightarrow{\eta_{j-1}} [p_j] \xleftarrow{\gamma_j} [p_{j+1}] \xleftarrow{\gamma_{j+1}} ... \to ... \xleftarrow{\gamma_{r-1}} [p_r] \xleftarrow{\gamma_r} [n]$$

where each $\gamma_i$ or $\eta_j$ is either an injection or a surjection, so that the corresponding chains $|\Delta^n| \leftarrow ... \leftarrow |\Delta^n|$ and $X[n] \to ... \to X[n]$ have sequences of elements
$u = t_{-1}, t_0, ..., t_r, t_{r+1} = v$ where $t_i \in |\Delta^{p_i}|$ and $x = x_{-1}, x_0, ..., x_r, x_{r+1} = y$ where $x_i \in X[p_i]$ so that for every map $\gamma_i : [p_{i+1}] \to [p_i]$, $\gamma_i(t_{i+1}) = t_i$ and $\gamma_i^*(x_i) = x_{i+1}$ and for every map $\eta_j : [p_j] \to [p_{j+1}]$, $\eta_j(t_j) = t_{j+1}$ and $\eta_j^*(x_{j+1}) = x_j$. Next, we will describe three moves which may be made on such a chain relating $(x, u)$ to $(y, v)$ to produce a new chain which also relates $(x, u)$ to $(y, v)$.

The first move is *condensing*: Consider the subchains **(1)**
$[p_i] \xleftarrow{\gamma_i} [p_{i+1}] \xleftarrow{\gamma_{i+1}} [p_{i+2}] \xleftarrow{\gamma_{i+2}} [p_{i+3}]$ or **(2)** $[p_j] \xrightarrow{\eta_j} [p_{j+1}] \xrightarrow{\eta_{j+1}} [p_{j+2}] \xrightarrow{\eta_{j+2}} [p_{j+3}]$. By Lemma 2.8 we may rewrite either sequence as a composition of a single surjective map first, then a single injective map. Therefore the subchain **(1)** may be replaced by a new subchain
$[p_i] \xleftarrow{\gamma_i'} [p_{i+1}'] \xleftarrow{\gamma_{i+1}'} [p_{i+2}']$ (note that $p_{i+2}' = p_{i+3}$), and **(2)** by $[p_j] \xrightarrow{\eta_j'} [p_{j+1}'] \xrightarrow{\eta_{j+1}'} [p_{j+2}']$ (similarly, $p_{j+2}' = p_{i+3}$). In each case the first map is a surjection and the second an injection, which

means $\gamma_i'^*$ and $\eta_{j+1}'^*$ are compositions of face maps, while $\gamma_{i+1}'^*$ and $\eta_j'^*$ are compositions of degeneracy maps.

The second move applies to any subchain of the form $[p_{i-1}] \overset{\gamma_{i-1}}{\leftarrow} [p_i] \overset{\eta_i}{\rightarrow} [p_{i+1}]$ where both $\gamma_{i-1}$ and $\eta_i$ are surjective. By Lemma 2.10, this subchain may be replaced by a new subchain of the form $[p_{i-1}] \overset{\gamma_{i-1}'}{\leftarrow} [p_i'] \overset{\gamma_i'}{\leftarrow} [p_{i+1}]$ where $\gamma_i'$ is injective and $\gamma_{i-1}'$ is surjective (so $\gamma_i'^*$ is a composition of degeneracy maps and $\gamma_{i-1}'^*$ is a composition of face maps).

The third move applies to any subchain of the form $[p_{i-1}] \overset{\eta_{i-1}}{\rightarrow} [p_i] \overset{\gamma_i}{\leftarrow} [p_{i+1}]$ where both $\eta_{i-1}$ and $\gamma_i$ are injective. By Lemma 2.11, this subchain may be replaced by a new subchain of the form $[p_{i-1}] \overset{\gamma_{i-1}'}{\leftarrow} [p_i'] \overset{\eta_i'}{\rightarrow} [p_{i+1}]$ where both $\gamma_{i-1}'$and $\eta_i'$ are injective (so the corresponding maps on $X$ are compositions of face maps).

By applying the first move to the original chain as many times as possible, we may obtain a new chain of exactly the form $[n] \leftarrow [p_0] \leftarrow [p_1] \rightarrow [p_2] \rightarrow [p_3] \leftarrow ... \leftarrow X[n]$; that is, maps only occur in the same direction twice in a row, with the first map of each pair surjective and the second one injective. This means every subchain of the form $[p_{i-1}] \overset{\gamma_{i-1}}{\leftarrow} [p_i] \overset{\eta_i}{\rightarrow} [p_{i+1}]$ has surjective maps on both sides of $[p_i]$ and every subchain of the form $[p_{i-1}] \overset{\eta_{i-1}}{\rightarrow} [p_i] \overset{\gamma_i}{\leftarrow} [p_{i+1}]$ has injective maps on both sides, so we may apply the second and third moves to each such subchain, followed by condensing the chain as much as possible again. Doing so repeatedly clearly collapses the entire chain into a chain of only three maps, specifically $[n] \overset{\gamma_{-1}}{\leftarrow} [p_0] \overset{\eta_0}{\rightarrow} [p_1] \overset{\gamma_1}{\leftarrow} [n]$ where both $\gamma_{-1}$ and $\eta_0$ are injective maps and $\gamma_1$ is surjective, possibly the identity map $[n] \rightarrow [n]$ (if $y$ is a nondegenerate simplex, this must be the case). Setting $\alpha = \gamma_{-1}$, $\beta = \eta_0$, $\gamma = \gamma_1$ gives these maps the claimed properties by the above descriptions of such chains. □

If $X$ is a simplicial set and $X^{\leq m}$ the $m$-truncation of $X$, then $|X^{\leq m}| = \left( \coprod_{n=0}^{m} (X[n] \times |\Delta^n|) \right) / \sim$. Using Theorem 2.7, clearly the $m$-skeleton $|X|_m$ of $|X|$ may be described in terms of only the $n$-simplices and maps between them for $n \leq m$, so it is exactly $|X^{\leq m}|$. This means we may also describe the realization of an entire simplicial set as the ascending union of the realizations of its $m$-truncations: $|X| = \bigcup_{m=0}^{\infty} |X|_m = \bigcup_{m=0}^{\infty} |X^{\leq m}|$.

## 2.3 THE DOLD-KAN CORRESPONDENCE

The Dold-Kan correspondence is a well-established theorem relating simplicial sets to chain complexes, which is necessary for several of our future results; most immediately, we will use it to establish the basic properties of the classifying space of a discrete group as it relates to the cohomology of the group itself. This section is based largely on the work in Mathew [8], Raksit [9], and Goerss and Jardine [6], with some added work to adapt the result to cohomology.

A *simplicial group* is a simplicial set $G$ with the additional properties that each $G[n]$ has the structure of a group, and that the face and degeneracy maps are group homomorphisms. Alternatively, $G$ is a functor from $\Delta$ to the category of groups. If each $G[n]$ is abelian (that is, if $G$ is a functor from $\Delta$ to the category of *abelian* groups), then it is called a simplicial abelian group.

Let $A$ be a simplicial abelian group. The *Moore complex of* $A$ is the chain complex $A_*$ such that $A_n = A[n]$, equipped with the boundary map $d(a_n) = \sum_{i=0}^{n}(-1)^i d_i(a_n)$. This forms a chain complex since for any $a_n \in A_n$:

$$dd(a_n) = \sum_{i=0}^{n-1}(-1)^i d_i \left(\sum_{j=0}^{n}(-1)^j d_j(a_n)\right) = \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^i(-1)^j d_i d_j(a_n)$$

$$= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i}(-1)^{i+j} d_i d_j(a_n) + \sum_{j=i+1}^{n}(-1)^{i+j} d_{j-1} d_i(a_n)\right)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{i+j} d_i d_j(a_n) + \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}(-1)^{i+j+1} d_j d_i(a_n)$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{i+j} d_i d_j(a_n) - \sum_{j=0}^{n-1}\sum_{i=0}^{j}(-1)^{i+j} d_j d_i(a_n) = 0$$

Furthermore, for any degenerate simplex $s_i a_{n-1}$ where $a_{n-1} \in A_{n-1}$, $d(s_i a_{n-1}) = \sum_{j=0}^{n} (-1)^j d_j s_i(a_{n-1}) = \sum_{0 \leq j \leq n, j \neq i, i+1} (-1)^j d_j s_i(a_{n-1})$ since the terms $(-1)^i d_i s_i(a_{n-1}) + (-1)^{i+1} d_{i+1} s_i(a_{n-1})$ are equal to $(-1)^i a_{n-1} + (-1)^{i+1} a_{n-1} = 0$. Then each remaining term is either equal to $s_{i-1} d_j(a_{n-1})$ or $s_i d_{j-1}(a_{n-1})$, and in either case the terms of $d(s_i a_{n-1})$ are all degenerate simplices. Thus we may define a subcomplex $DA_*$ where $DA_n$ consists of sums of degenerate $n$-simplices $s_i a_{n-1}$.

Next, we define the normalized complex $NA_*$ as follows: First, define for each $n$ the group

$$NA_n := \{a_n \in A_n : d_i(a_n) = 0 \text{ for all } i < n\}$$

Note that for any $a_n \in NA_n$, $d(a_n) = \sum_{i=0}^{n} (-1)^i d_i(a_n) = (-1)^n d_n(a_n)$ since all of the other terms are 0 by the definition of $NA_n$. Then for any $i \in \{0, ..., n-1\}$, $d_i d(a_n) = (-1)^n d_i d_n(a_n) = (-1)^n d_{n-1} d_i(a_n) = 0$, so $NA_*$ is a subcomplex of $A_*$. For each $n$, since $DA_n$ and $NA_n$ are both subgroups of $A_n$, there is a natural map $NA_n \oplus DA_n \to A_n$ determined by inclusion on each factor.

Now, for $k \geq -1$, let $N_k A_n := \begin{cases} A_n, & k = -1 \\ \bigcap_{i=0}^{k} \ker d_i, & 0 \leq k \leq n-1 \\ NA_n, & k > n-1 \end{cases}$, and let

$$D_k A_n := \begin{cases} 0, & k = -1 \\ \langle s_j(A_{n-1}) : j \leq k \rangle, & 0 \leq k \leq n-1 \\ DA_n, & k > n-1 \end{cases}.$$

Note that by definition, $N_{n-1} A_n = \bigcap_{i=0}^{n-1} \ker d_i = NA_n$ and $D_{n-1} A_n = \langle s_j(A_{n-1}) : j \leq n-1 \rangle = DA_n$. Similar to the map $NA_n \oplus DA_n \to A_n$, there is a natural map $N_k A_n \oplus D_k A_n \to A_n$ for each $k$ and $n$. Then:

LEMMA 2.12. *For each $k$, the natural map $N_k A_n \oplus D_k A_n \to A_n$ is an isomorphism. In particular, $NA_n \oplus DA_n \to A_n$ is an isomorphism.*

Proof. Clearly $A_n = A_n \oplus 0 = N_{-1}A_n \oplus D_{-1}A_n$. We will proceed by induction to obtain for each $k \geq 0$ a natural splitting $N_kA_n \oplus D_kA_n = A_n$. As the base case, note that for $k = 0$ and any $n$, $\ker d_0 \oplus \mathrm{im} s_0 = A_n$ since $s_0 : A_{n-1} \to A_n$ is a split injection with the section $d_0$.

To induct, suppose there is some $k$ so that $N_{k-1}A_n \oplus D_{k-1}A_n = A_n$ is a natural splitting for every $n$. Then whenever $j < k$, $s_ks_j = s_js_{k-1}$, so $s_k(D_{k-1}A_{n-1}) \subset D_{k-1}A_n$ and we have the exact sequence:

$$0 \to A_{n-1}/D_{k-1}A_{n-1} \xrightarrow{s_k} A_n/D_{k-1}A_n \to A_n/D_kA_n \to 0$$

Also, $d_{k+1}s_j = s_jd_k$ so that $d_{k+1}(D_{k-1}A_n) \subset D_{k-1}A_{n-1}$. This gives $d_{k+1} : A_n/D_{k-1}A_n \to A_{n-1}/D_{k-1}A_{n-1}$ as a section of $s_k$ since $d_{k+1}s_k = \mathrm{id}$; hence the above sequence is split.

Furthermore, $s_k(N_{k-1}A_{n-1}) \subset N_{k-1}A_n$ since whenever $j < k$, $d_js_k = s_{k-1}d_j$. We claim the map $(\iota + s_k) : N_kA_n \oplus N_{k-1}A_{n-1} \to N_{k-1}A_n$ (where $\iota$ is inclusion) is an isomorphism. First, if $a \in N_{k-1}A_n$, then $a - s_kd_k(a) \in N_kA_n$ since $d_k$ is a section of the split injection $s_k$, and $(\iota + s_k)(a - s_kd_k(a), d_k(a)) = a - s_kd_k(a) + s_kd_k(a) = a$. Next, $\iota$ is injective as an inclusion map, and if $b \in N_{k-1}A_{n-1}$ has $s_k(b) = 0$, then $b = d_ks_k(b) = 0$, so $(\iota + s_k)$ is also injective. This gives another split exact sequence:

$$0 \to N_{k-1}A_{n-1} \xrightarrow{s_k} N_{k-1}A_n \to N_kA_n \to 0$$

These two split exact sequences form the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{k-1}A_{n-1} & \longrightarrow & N_{k-1}A_n & \xrightarrow{a \mapsto a - s_kd_ka} & N_kA_n & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\
0 & \longrightarrow & A_{n-1}/D_{k-1}A_{n-1} & \longrightarrow & A_n/D_{k-1}A_n & \longrightarrow & A_n/D_kA_n & \longrightarrow & 0
\end{array}$$

The first two columns are isomorphisms by the inductive hypothesis, and the first square consists of natural inclusions and projections, so it commutes. For the second square, $a - s_k d_k a$ is in the same class as $a$ in $A_n/D_k A_n$, making it commute also. Therefore, the third column is also an isomorphism, as needed. □

From this Lemma, we have $NA_* \to (A/DA)_*$ as an isomorphism of chain complexes. Additionally:

COROLLARY 2.13. *For any $k \geq 0$, $D_k A_n = D_{k-1}A_n \oplus s_k(N_{k-1}A_{n-1})$. Furthermore, $DA_n = \bigoplus_{k=0}^{n-1} s_k(N_{k-1}A_{n-1})$, so that $DA_* = \bigoplus_{k=0}^{n-1} s_k(N_{k-1}A_{*-1})$ as a chain complex.*

PROOF. First, we will establish the formula $D_k A_n = D_{k-1}A_n \oplus s_k(N_{k-1}A_{n-1})$ using induction. For the base case, by definition,
$D_0 A_n = s_0(A_{n-1}) = 0 \oplus s_0(A_{n-1}) = D_{-1}A_n \oplus s_0(N_{-1}A_{n-1})$.

To induct, suppose the formula holds for $k-1$; then any $a \in D_k A_n$ satisfies $a = x + s_k(b)$ for some $x \in D_{k-1}A_n$ and $b \in A_{n-1} = N_{k-1}A_{n-1} \oplus D_{k-1}A_{n-1}$ by Lemma 2.12. Let $b_1 \in N_{k-1}A_{n-1}$, $b_2 \in D_{k-1}A_{n-1}$ be such that $b = b_1 + b_2$. Then $s_k(b_2) \in s_k(D_{k-1}A_{n-1}) \subset D_{k-1}A_n$, so $a = x + s_k(b_1) + s_k(b_2)$, which is in $D_{k-1}A_n + s_k(N_k A_{n-1})$. Hence, $D_k A_n = D_{k-1}A_n + s_k(N_{k-1}A_{n-1})$. Furthermore $D_{k-1}A_n \cap s_k(N_{k-1}A_{n-1}) \subset D_{k-1}A_n \cap N_{k-1}A_n = 0$ by Lemma 2.12. Thus $D_k A_n = D_{k-1}A_n \oplus s_k(N_{k-1}A_{n-1})$.

Finally:

$$DA_n = D_{n-1}A_n = D_{n-2}A_n \oplus s_k(N_{n-2}A_{n-1}) = \ldots = \bigoplus_{k=0}^{n-1} s_k(N_{k-1}A_{n-1})$$

giving the desired decomposition. □

Let $C_*$ be a chain complex, and define a simplicial abelian group $SC_*$ as follows: Let $SC_*[n] := \bigoplus_{\phi:[n]\twoheadrightarrow[k]} C_k$, which has one factor $C_k$ for every surjective map $\phi : [n] \twoheadrightarrow [k]$. By Lemma 2.8, the composition of any map $\alpha : [m] \to [n]$ with the surjection $\beta : [n] \twoheadrightarrow [k]$ indexing a factor $C_k$ may be rewritten as a composition $[m] \twoheadrightarrow [j] \hookrightarrow [k]$ of a surjective map

$\theta$ followed by an injective map $\iota$. Define a map $\tau_{kj} : C_k \to C_j$ by

$$\tau_{kj}(c_k) = \begin{cases} c_k, & j = k \\ (-1)^k dc_k, & j = k-1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \iota = \delta_k.$$

Composing this map with the inclusion $C_j \to SC_*[m]$ determined by the surjection $[m] \twoheadrightarrow [j]$ gives a map $C_k \to SC_*[m]$, and the direct sum of all such maps defines the morphism $SC_*(\alpha) : SC_*[n] \to SC_*[m]$. In this way we obtain the face and degeneracy maps required for $SC_*$ to be a simplicial abelian group. Now, any chain map $C_* \to C'_*$ determines a simplicial homomorphism $SC_* \to SC'_*$ by taking the sum of the maps $C_k \to C'_k$ in each degree, so $S$ acts as a functor from chain complexes to simplicial abelian groups.

For any simplicial abelian group $A$, we have $NA_k \subset A_k = A[k]$, so for any surjective map $\phi : [n] \twoheadrightarrow [k]$ the map $A(\phi) : A[k] \to A[n]$ restricts to a map $A(\phi) : NA_k \to A_n = A[n]$. Thus, for any $n$ we may define a map $\Phi_n : \bigoplus_{\phi:[n]\twoheadrightarrow[k]} NA_k \to A[n]$ as the sum of the maps $A(\phi)$. This gives a map $\Phi : S(NA_*) \to A$ of simplicial abelian groups.

THEOREM 2.14. *For any simplicial abelian group $A$, the map $\Phi : S(NA_*) \to A$ is an isomorphism.*

PROOF. It suffices to show that each $\Phi_n$ is an isomorphism. Clearly $S(NA_*)[0] = NA_0 = A_0$ since there are no degenerate simplices in $A[0]$, so $\Phi_0$ is an isomorphism.

To see that $\Phi_n$ is surjective for all $n$, we induct on $n$, using $\Phi_0$ as the base case. Suppose that $\Phi_m : S(NA_*)[m] \to A[m]$ is surjective for all $m < n$. By Lemma 2.12, $A[n] = NA_n \oplus DA_n$. Clearly $NA_n$ is in the image of the factor $NA_n$ (corresponding to the identity surjection $[n] \twoheadrightarrow [n]$) under $\Phi_n$, so it remains to see that $DA_n$ is in the image of $\Phi_n$. However, by inductive hypothesis, $A[n-1]$ is contained in the image of $\Phi_{n-1}$, and $DA_n$ is just the subgroup generated by the images of $A[n-1]$ under the degeneracy maps, so is also in the image of $\Phi_n$.

Next we check that $\Phi_n$ is injective, also by induction on $n$ with the base case $\Phi_0$. Suppose that $(a_\phi) \in \bigoplus_{\phi:[n]\twoheadrightarrow[k]} NA_k$ satisfies $\Phi_n(a_\phi) = 0$. By assumption, $\sum_{\phi:[n]\twoheadrightarrow[k]} A(\phi)(a_\phi) = 0 \in A[n]$, so suppose that some $a_\phi$ is nonzero. First note that if all of the other factors are 0, then the factor $a_{\mathrm{id}} \in NA_n$ given by the identity surjection must also be 0 by the canonical splitting $A[n] = NA_n \oplus DA_n$.

Then, to each surjection $\phi : [n] \twoheadrightarrow [k]$ with $k \neq n$, we may assign a section $\psi_\phi : [k] \hookrightarrow [n]$ so that $\psi_\phi(i) := \max\{j \in [n] : \phi(j) = i\}$. Then define a partial order on the surjective maps as follows: If for all $i \in [k]$, $\psi_{\phi_1}(i) \leq \psi_{\phi_2}(i)$, then $\phi_1 \leq \phi_2$; in particular this means that if $\phi_2\psi_{\phi_1} = \mathrm{id}_{[k]}$ then $\phi_1 \leq \phi_2$. Now suppose that $\tau$ is a maximal element of the set $\{\phi : a_\phi \neq 0\}$ with respect to this ordering. Then the factor of the element $A(\psi_\tau)(a_\phi)$ in $S(NA_*)[k]$ indexed by $\mathrm{id}_k : [k] \twoheadrightarrow [k]$ is $a_\tau$. Since $(a_\phi) \in \ker(\Phi_n(A))$, we have $A(\psi_\tau)(a_\phi) \in \ker(\Phi_k(A))$, and by inductive hypothesis this makes $A(\psi_\tau)(a_\phi) = 0$. However, $a_\tau$ is a factor of this element, so $a_\tau = 0$, contradicting the assumption that $a_\tau \neq 0$. Thus $\Phi_n$ is injective, and therefore an isomorphism. □

We are now prepared to prove the main result of this section:

THEOREM 2.15 (Dold-Kan Correspondence). *The functor $A \to NA_*$ defines an equivalence of categories between chain complexes of abelian groups and simplicial abelian groups. Furthermore, the complexes $NA_*$, $A_*$, and $(A/DA)_*$ are equivalent by a natural chain homotopy.*

PROOF. By Theorem 2.14, $S(NA_*)$ is isomophic to $A$, so for the equivalence of categories it suffices to show that for any chain complex we have $C_*$, $N(SC_*) = C_*$.

Now, $N(SC_*)[n]$ consists of elements of $\bigoplus_{\phi:[n]\twoheadrightarrow[k]} C_k$ which the face maps $d_i$, for $i < n$, take to 0; we claim this consists of only the factor $C_n$ given by the identity surjection $[n] \twoheadrightarrow [n]$. Certainly $C_n \subset N(SC_*)[n]$ since if $i < n$, then each map $\delta_i : [n-1] \to [n] \twoheadrightarrow [n]$ satisfies $SC_*(\delta_i) = d_i : C_n \to C_{n-1}$ equal to 0 by definition. On the other hand, if $k < n$ then the surjective map $\phi : [n] \twoheadrightarrow [k]$ is a composition $[n] \overset{\sigma_i}{\twoheadrightarrow} [n-1] \twoheadrightarrow [k]$ for some $i$, so every factor $C_k$ with $k < n$ is a member of $D(SC_*)[n]$, making the factor zero in the quotient $N(SC_*)[n] = SC_*[n]/D(SC_*)[n]$.

That $NA_*$ and $(A/DA)_*$ are not only chain homotopic, but in fact naturally *isomorphic* complexes, is the content of Lemma 2.12, so it will suffice to produce a chain homotopy between $A_*$ and $NA_*$. For this, we will first define a chain contraction of $DA_*$. For each $k \geq 0$, let $t_k : s_k(N_{k-1}A_{n-1}) \to s_k(N_{k-1}A_n)$ be given by $t_k(a) = \begin{cases} (-1)^k s_k, & n \geq k+1 \\ 0, & n < k+1 \end{cases}$.

Then, since $d_k s_k(a) = a = d_{k+1}s_k(a)$, each $a \in s_k(N_{k-1}A_{n-1})$ satisfies:

$$dt_k(a) = \sum_{i=0}^{n+1}(-1)^{i+k}d_i s_k(a)$$

$$= \sum_{i=0}^{k-1}(-1)^{i+k}s_{k-1}d_i(a) + (-1)^{2k}a + (-1)^{2k+1}a + \sum_{i=k+2}^{n+1}(-1)^{i+k}s_k d_{i-1}(a)$$

Since $a \in s_k(N_{k-1}A_{n-1})$, there is some $b \in N_{k-1}A_{n-1}$ with $a = s_k(b)$, so for each $i \in 0, 1, ..., k-1$, we have $s_{k-1}d_i(a) = s_{k-1}d_i s_k(b) = s_{k-1}s_{k-1}d_i(b) = 0$. Thus only the last term is left, and:

$$dt_k(a) = \sum_{i=k+2}^{n+1}(-1)^{i+k}s_k d_{i-1}(a) = \sum_{i=k+1}^{n}(-1)^{i+k+1}s_k d_i(a)$$

On the other hand,

$$t_k d(a) = \sum_{i=0}^{n}(-1)^{i+k}s_k d_i(a) = s_k d_k(a) + \sum_{i=k+1}^{n}(-1)^{i+k}s_k d_i(a)$$

Therefore,

$$dt_k(a) + t_k d(a) = s_k d_k(a) = s_k d_k s_k(b) = s_k(b) = a$$

This means $dt_k + t_k d = \mathrm{id}$, and $t_k$ is a chain contraction of the complex $s_k(N_{k-1}A_{*-1})$. Since $DA_* = \bigoplus_{k=0}^{n-1} s_k(N_{k-1}A_{*-1})$ by Corollary 2.13, the map $T := t_0 \oplus ... \oplus t_{n-1} : DA_n \to DA_{n+1}$ gives a chain contraction of $DA_*$.

Now, let $D : A_* \to DA_*$ and $N : A_* \to NA_*$ be the natural projections given by the splitting $A_* = NA_* \oplus DA_*$, and consider the composition
$TD : A_n \xrightarrow{D} DA_n \xrightarrow{T} DA_{n+1} \hookrightarrow A_{n+1}$. For $a \in A_n$,

$$d(TD)(a) + (TD)(da) = dT(Da) + Td(Da) = Da = a - Na$$

This gives a chain homotopy between the composition $A_* \xrightarrow{N} NA_* \hookrightarrow A_*$ and the identity map on $A_*$. On the other hand, $NA_* \hookrightarrow A_* \xrightarrow{N} NA_*$ is just the identity on $NA_*$, so $NA_*$ and $A_*$ are homotopy equivalent, as needed. □

If $Y$ is any simplicial set, we may obtain a simplicial abelian group $\mathbb{Z}(Y)$ by taking $\mathbb{Z}(Y)[n] := \mathbb{Z}(Y[n])$, the collection of formal sums $\sum_{y \in Y[n]} k_y y$ where $k_y \in \mathbb{Z}$ and only finitely many of the $k_y$ are nonzero. The face and degeneracy maps are determined termwise; so, $d_i\left(\sum_{y \in Y[n]} k_y y\right) := \sum_{y \in Y[n]} k_y d_i(y)$ and $s_i\left(\sum_{y \in Y[n]} k_y y\right) := \sum_{y \in Y[n]} k_y s_i(y)$. This makes $\mathbb{Z}(Y)_*$ a chain complex with $\mathbb{Z}(Y)_n = \mathbb{Z}(Y)[n]$ and boundary maps $d : \mathbb{Z}(Y)_n \to \mathbb{Z}(Y)_{n-1}$ determined by $d(y) = \sum_{i=0}^{n} (-1)^i d_i(y)$.

Now, let $A$ be an abelian group and $M(Y[*], A)$ the cochain complex defined so that $M(Y[n], A)$ is the collection of all functions $f : Y[n] \to A$ with the coboundary map $\partial f(y) := \sum_{i=0}^{n+1} (-1)^i f(d_i(y))$, and let $C^*(Y, A)$ be any subcomplex of $M(Y[*], A)$. In particular we may choose some condition which is stable under $\partial$ in the sense that if $f$ satisfies the condition, then so does $\partial f$, and make $C^*(Y, A)$ the subcomplex consisting of functions which fulfill that condition. Then (by applying the action of $\mathbb{Z}$ on $A$ as a $\mathbb{Z}$-module) $C^*(Y, A)$ may instead be viewed as a subcomplex of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(Y)_*, A)$ with the coboundary map $\partial f(y) = f(dy)$, such as functions $f : \mathbb{Z}(Y)_n \to A$ fulfilling the same condition as before for elements of $y$ with the additional property that $f(k_y y) = k_y f(y)$; this definition clearly gives the same cochain complex. Now, the split exact sequence of chain groups

$$0 \to D\mathbb{Z}(Y)_* \to \mathbb{Z}(Y)_* \to N\mathbb{Z}(Y)_* \to 0$$

given by Lemma 2.12 induces a split exact sequence of cochain groups

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(N\mathbb{Z}(Y)_*, A) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(Y)_*, A) \xrightarrow{r} \mathrm{Hom}_{\mathbb{Z}}(D\mathbb{Z}(Y)_*, A) \to 0$$

where the map $r$ is just restriction to degenerate simplices. Restricting $r$ to the subcomplex $C^*(Y, A)$ gives the short exact sequence

$$0 \to \ker(r|C^*(Y, A)) \to C^*(Y, A) \to \mathrm{im}(r|C^*(Y, A)) \to 0$$

Let $NC^*(Y, A) := \ker(r|C^*(Y, A))$ and $DC^*(Y, A) := \mathrm{im}(r|C^*(Y, A))$, and indicate the cohomology of each cochain complex by replacing $C$ with $H$. Then this gives the following long exact sequence in cohomology:

$$... \to DH^{n-1}(Y, A) \to NH^n(Y, A) \to H^n(Y, A) \to DH^n(Y, A) \to NH^{n+1}(Y, A) \to ...$$

We will use this to obtain a dual to the Dold-Kan correspondence:

Corollary 2.16. *The inclusion $NC^*(Y, A) \to C^*(Y, A)$ gives a natural isomorphism on cohomology; that is, $NH^n(Y, A) \cong H^n(Y, A)$.*

Proof. By the long exact sequence above, it suffices to show that $DH^n(Y, A) = 0$ for all $n$. Let $T$ be the chain contraction of $D\mathbb{Z}(Y)_*$ given in the proof of Theorem 2.15; then $T^\#$ given by $\left[T^\# f\right](x) := f(T(x))$ gives a contracting homotopy of $\mathrm{Hom}_{\mathbb{Z}}(D\mathbb{Z}(Y)_*, A)$ which descends to the subcomplex $DC^*(Y, A)$. Hence $DH^*(Y, A) = 0$, as needed. $\square$

If $X$ is a topological space, then by definition the singular chain complex $C_*(X) = \mathbb{Z}(SX)_*$. The complex $N\mathbb{Z}(SX)_*$ is in this case called the normalized singular chain complex, and by Corollary 2.16 its dual complex $C^*_{\mathrm{norm}}(X, A) := \mathrm{Hom}(N\mathbb{Z}(SX)_*, A)$ has the same cohomology groups as $C^*(X, A) = \mathrm{Hom}(\mathbb{Z}(SX)_*, A)$. Furthermore, if $X$ is a CW-complex, the map $\epsilon : |SX| \to X$ induces a chain homotopy equivalence between

$N\mathbb{Z}(SX)_*$ and $C_*^{\text{cell}}(X)$, which makes $H_*^{\text{cell}}(X) \cong H_*(X)$ and $H^*_{\text{cell}}(X, A) \cong H^*(X, A)$ by natural isomorphisms.

## 2.4 CLASSIFYING SPACE OF A DISCRETE GROUP

As our goal is the establishment of an analogue to the classifying space of a discrete group, it will be useful to describe that construction and its consequences first, which we are now fully prepared to do. The constructions in this section are generally well-known, but the specific proofs are original.

A category is called small if its collection of objects and its collection of morphisms may both be considered sets. For any small category $\mathscr{C}$ we may define the simplicial set $N(\mathscr{C})$, the nerve of $\mathscr{C}$, as follows: The 0-simplices $N(\mathscr{C})[0]$ are the objects of $\mathscr{C}$. For $n > 0$, the $n$-simplices are ordered $n$-tuples of morphisms which may be composed in the order of the tuple (so, in particular, $N(\mathscr{C})[1]$ is all of the morphisms of $\mathscr{C}$). The face maps on $N(\mathscr{C})[1]$ $d_0$ and $d_1$ take the morphism to its target and source respectively; for $n > 1$ the face maps are $d_0(f_1, f_2, ..., f_n) = (f_2, ..., f_n)$, $d_n(f_1, ..., f_{n-1}, f_n) = (f_1, ..., f_{n-1})$, and $d_i(f_1, ..., f_i, f_{i+1}, .., f_n) = (f_1, ..., f_i f_{i+1}, ..., f_n)$ for $0 < i < n$. The degeneracy maps act by adding the identity morphism at the $i$th position, so $s_0(f_1, ..., f_n) = (\text{id}, f_1, ..., f_n)$ and $s_i(f_1, ..., f_i, .., f_n) = (f_1, ..., f_i, \text{id}, ..., f_n)$ for $0 < i \leq n$.

A discrete group $G$ may be viewed as a category with a single object, $*$, and the elements of the group $g$ acting as morphisms $g : * \to *$, with composition defined as the product between two elements. Under this view, $G$ is clearly a small category, and the nerve $N(G)$ (which we will notate $NG$ for convenience) is defined as above. We may also view the nerve as taking $NG[n] := G^n$ (with $G^0$ the single-element set $\{*\}$) with face and degeneracy maps acting by multiplication and insertion of the identity element 1 respectively, just as above. The nondegenerate simplices of $NG$ are those tuples $(g_1, ..., g_n)$ such that $g_i \neq 1$ for all $i$.

Let $\mathcal{E}G$ be the simplicial set defined as follows: $\mathcal{E}G[n] := G^{n+1}$ with face maps $d_i(g_0, ..., g_n) := (g_0, ..., \hat{g}_i, ..., g_n)$ and degeneracy maps $s_i(g_0, ..., g_{n-1}) := (g_0, ..., g_i, g_i, ..., g_{n-1})$. The geometric realization of this simplicial set $EG := |\mathcal{E}G|$ is a CW-complex with one $n$-cell $[g_0, ..., g_n]$ for each nondegenerate simplex $(g_0, ..., g_n)$ and gluing maps corresponding to each face map; furthermore this space is contractible by a homotopy taking $x \in [g_0, ..., g_n]$ along the line segment in $[1, g_0, ..., g_n]$ from $x$ to the vertex $[1]$. Every element $g \in G$ acts on $\mathcal{E}G$ by diagonal multiplication, namely, $g(g_0, ..., g_n) = (gg_0, ..., gg_n)$, which obviously commutes with the face and degeneracy maps. In fact, this action is free since $g(g_0, ..., g_n) = (g_0, ..., g_n)$ only when $g = 1$. This further determines a free action of $G$ on $EG$ by $g[g_0, ..., g_n] = [gg_0, ..., gg_n]$, and the quotient of $EG$ by this action is a topological space $BG$ with $\pi_1(BG) \cong G$ and $EG$ as a universal covering space.

Each element $(g_0, ..., g_n)$ of $\mathcal{E}G[n]$ may be rewritten as $(g'_0, g'_0g'_1, g'_0g'_1g'_2, ..., g'_0g'_1g'_2...g'_n) = g'_0(1, g'_1, g'_1g'_2, ..., g'_1g'_2...g'_n)$ where $g'_0 = g_0$ and $g'_i = g_{i-1}^{-1}g_i$ for all $i > 0$. For elements written this way, $d_i(g_0(1, g_1, g_1g_2, ..., g_1g_2...g_n)) = g_0d_i(1, g_1, g_1g_2, ..., g_1g_2...g_n)$, which is $g_0g_1(1, g_2, g_2g_3, ..., g_2g_3...g_n)$ if $i = 0$, $g_0(1, g_1, ..., g_1g_2...g_{i-1}, g_1g_2...g_ig_{i+1}, ..., g_1...g_n)$ if $0 < i < n$, or $g_0(1, g_1, ..., g_1...g_{n-1})$ if $i = n$. Also, $s_i(g_0(1, g_1, ..., g_1...g_n)) = g_0s_i(1, g_1, ..., g_1...g_n)$, which is $g_0(1, 1, g_1, ..., g_1...g_n)$ if $i = 0$ or $g_0(1, g_1, ..., g_1...g_i, g_1...g_i1, ..., g_1...g_i1g_{i+1}...g_n)$ if $0 < i \leq n$.

LEMMA 2.17. *There exists a natural surjective map $p : \mathcal{E}G \to NG$ with the kernel $K_p$ such that $K_p[n] = \{(g_0(1, g_1, ..., g_1...g_n), g'_0(1, g_1, ..., g_1...g_n)) : g_0, g'_0 \in G\}$.*

PROOF. For each $n$, define a map of sets $p_n : \mathcal{E}G[n] \to NG[n]$ by $p_n(g_0(1, g_1, g_1g_2, ..., g_1g_2...g_n)) := (g_1, g_2, ..., g_n)$. If $p_n(g_0(1, g_1, ..., g_1...g_n)) = p_n(g'_0(1, g'_1, ..., g'_1...g'_n))$, then $(g_1, g_2, ..., g_n) = (g'_1, g'_2, ..., g'_n)$, so $g_i = g'_i$ for all $1 \leq i \leq n$. However, $g_0$ is not necessarily equal to $g'_0$, so pairs of the form

$(g_0(1, g_1, ..., g_1...g_n), g_0'(1, g_1, ..., g_1...g_n))$ are precisely the elements of the kernel of $p_n$. Clearly any $n$-simplex $(g_1, ..., g_n)$ occurs as $p_n(1(1, g_1, ..., g_1...g_n))$, so $p_n$ is also surjective as a set map.

Let $p$ be the map on $\mathcal{E}G$ determined by $p_n$ for each $n$; then it remains to see that $p$ commutes with the face and degeneracy maps. By the observations above,

$$p(d_i(g_0(1, g_1, ...., g_1...g_n))) = \begin{cases} p(g_0 g_1(1, g_2, ..., g_2...g_n)) & i = 0 \\ p(g_0(1, g_1, ..., g_1...g_{i-1}, g_1...g_i g_{i+1}, ..., g_1...g_n)) & 0 < i < n \\ p(g_0(1, g_1, ..., g_1...g_{n-1})) & i = n \end{cases}$$

$$= \begin{cases} (g_2, ..., g_n) & i = 0 \\ (g_1, ..., g_i g_{i+1}, ..., g_n) & 0 < i < n \\ (g_1, ..., g_{n-1}) & i = n \end{cases}$$

Since $d_i(p(g_0(1, g_1, ..., g_1...g_n))) = d_i(g_1, ..., g_n)$ takes exactly the same values, $p$ commutes with $d_i$. Also,

$$p(s_i(g_0(1, g_1, ..., g_1...g_n))) = \begin{cases} p(g_0(1, 1, g_1, ..., g_1...g_n)) & i = 0 \\ p(g_0(1, g_1, ..., g_1...g_i, g_1...g_i 1, ..., g_1...g_i 1 g_{i+1}...g_n)) & 0 < i \leq n \end{cases}$$

$$= \begin{cases} (1, g_1, ..., g_n) & i = 0 \\ (g_1, ..., g_i, 1, g_{i+1}, ..., g_n) & 0 < i \leq n \end{cases}$$

Again, $s_i(p(g_0(1, g_1, ..., g_1...g_n))) = s_i(g_1, ..., g_n)$ takes the same values, so $p$ commutes with $s_i$. Finally, the naturality of $p$ is trivial to verify. □

Under the geometric realization functor, the map $p$ is taken to a quotient map of spaces $|p| : EG \to |NG|$. Furthermore, since
$p(g(g_0(1, g_1, ..., g_1...g_n))) = (g_1, ..., g_n) = p(g_0(1, g_1, ..., g_1...g_n))$ for any $g \in G$,

$|p|(g[g_0(1, g_1, ..., g_1...g_n)]) = [g_1, ..., g_n]$. Hence $|p|$ acts precisely as the quotient map of $EG$ under the action of $G$. Thus, $|NG| = BG$.

Recall the definition of $C^n(G, A)$ as functions $f : G^{n+1} \to A$ such that $gf(g_0, ..., g_n) = f(gg_0, ..., gg_n)$. In fact this gives a description of $C^n(G, A)$ as a collection of functions $f : \mathcal{E}G[n] \to A$ fulfilling a condition which is stable under the coboundary map since $\partial f(g_0, ..., g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0, ..., \hat{g}_i, ..., g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(d_i(g_0, ..., g_{n+1}))$. Hence, by Corollary 2.16, the cohomology of $C^*(G, A)$ is isomorphic to that of the kernel of the map $r$ given by the restriction of each function to the degenerate elements, those $(g_0, ..., g_n)$ with $g_i = g_{i+1}$ for some $i$.

Lemma 2.18. *There exists a natural injective cochain map* $q : \ker(r) \to C^*_{cell}(EG, A)$ *whose image is the subcomplex generated by the homomorphisms* $f : C_n^{cell}(EG) \to A$ *which satisfy* $f(g[g_0, ..., g_n]) = gf([g_0, ..., g_n])$ *for all* $g \in G$.

Proof. Let $q : \ker(r) \to C^*_{\text{cell}}(EG, A)$ be given by the formula $(qf)([g_0, ..., g_n]) := f(g_0, ..., g_n)$. To see that this map is injective, note that $f \in \ker(q)$ if and only if $f(g_0, ..., g_n) = 0$ for every nondegenerate element $(g_0, ..., g_n)$ of $\mathcal{E}G[n]$. However, $f \in \ker(r)$, so it also takes every *degenerate* element to 0; hence $f = 0$. By definition, every $f \in \ker(r) \leq C^*(G, A)$ satisfies $gf(g_0, ..., g_n) = f(gg_0, ..., gg_n)$, so

$$g\left[(qf)([g_0, ..., g_n])\right] = g\left[f(g_0, ..., g_n)\right] = f(gg_0, ..., gg_n)$$

$$= (qf)([gg_0, ..., gg_n]) = (qf)](g[g_0, ..., g_n])$$

giving $q$ the desired image.

It remains to show that $q$ is a cochain map. For this, it suffices to show that for any $f \in \ker(r)$, $q\partial f = \partial qf$. By Lemma 1.7, $d[g_0, ..., g_n] = \sum_{0 \leq i \leq n, d_i(g_0,...,g_n) \text{ nondegenerate}} (-1)^i [d_i(g_0, ..., g_n)]$, which is the sum $\sum_{0 \leq i \leq n, g_{i-1} \neq g_{i+1}} (-1)^i [g_0, ..., \hat{g}_i, ..., g_n]$. Therefore, any $f \in C^n_{\text{cell}}(EG, A)$ satisfies

$(\partial f)[g_0, ..., g_{n+1}] = \sum_{0 \le i \le n+1, g_{i-1} \ne g_{i+1}} (-1)^i f[g_0, ..., \hat{g}_i, ..., g_{n+1}]$, and in particular any $f \in \ker(r)$ satisfies $(\partial q f)([g_0, ..., g_{n+1}]) = \sum_{0 \le i \le n+1, g_{i-1} \ne g_{i+1}} (-1)^i f(g_0, ..., \hat{g}_i, ..., g_{n+1})$. On the other hand, $(q \partial f)([g_0, ..., g_{n+1}]) = (\partial f)(g_0, ..., g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, ..., \hat{g}_i, ..., g_{n+1})$. However, since $f \in \ker(r)$, each term with $g_{i-1} = g_{i+1}$ for some $i$ vanishes, so this is precisely equal to the above, as needed.

Finally, let $H$ be another group such that $A$ is also an $H$-module, and $\theta : G \to H$ a group homomorphism. Let $r_G$, $r_H$ be the respective restriction maps on $C^*(G, A)$ and $C^*(H, A)$. Then $\theta$ induces maps $\theta^* : C^*(H, A) \to C^*(G, A)$ and $\theta^*_{\text{cell}} : C^*_{\text{cell}}(EH, A) \to C^*_{\text{cell}}(EG, A)$. To show naturality it suffices to prove that for any $n$, the following diagram commutes:

$$\begin{array}{ccc} \ker(r_G) & \xleftarrow{\theta^n} & \ker(r_H) \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ C^*_{\text{cell}}(EG, A) & \xleftarrow{\theta^n_{\text{cell}}} & C^*_{\text{cell}}(EH, A) \end{array}$$

Let $f \in \ker(r_H)$; we wish to show that $q\theta^n(f) = \theta^n_{\text{cell}} q(f)$. For this, observe that:

$$(q\theta^n f)([g_0, ..., g_{n+1}]) = (\theta^n f)(g_0, ..., g_{n+1}) = f(\theta(g_0), ..., \theta(g_{n+1}))$$

$$= (qf)([\theta(g_0), ..., \theta(g_{n+1})]) = (\theta^n_{\text{cell}} q f)([g_0, ..., g_{n+1}])$$

This gives the needed equality.

□

Theorem 2.19. *For any discrete group $G$ and $G$-module $A$ on which $G$ acts trivially, $H^n(BG, A) \cong H^n_{cell}(BG, A) \cong H^n(G, A)$ by natural isomorphisms.*

Proof. By Lemma 2.18, we have a natural injective map $q : \ker(r) \to C^n_{\text{cell}}(EG, A)$ whose image is the functions $f$ with $f(g[g_0, ..., g_n]) = gf([g_0, ..., g_n])$. Since $G$ acts trivially on $A$, this last term is equal to $f([g_0, ..., g_n])$. On the other hand, the natural map

$|p|^* : C^*_{\text{cell}}(BG, A) \to C^*_{\text{cell}}(EG, A)$ induced by $|p| : EG \to BG$ is also injective, and its image in $C^*_{\text{cell}}(EG, A)$ is the same subcomplex. This is because $(|p|^* f)[g_0(1, g_1, ..., g_1...g_n)] = f(|p|_*[g_0(1, g_1, ..., g_1...g_n)])$ and $|p|_*[g_0(1, g_1, ..., g_1...g_n)] = |p|_*[gg_0(1, g_1, ..., g_1...g_n)] = |p|_* g[g_0(1, g_1, ..., g_1...g_n)]$ for all $g \in G$. Therefore, $\text{im}(|p|^*) = \text{im}(q)$ with both maps injective, which gives a natural cochain equivalence (in fact, an *isomorphism*) between $C^*_{\text{cell}}(BG, A)$ and $\ker(r)$. Finally, $C^*_{\text{cell}}(BG, A)$ is naturally cochain equivalent to $C^*(BG, A)$ by the remarks following Corollary 2.16 and $C^*(G, A)$ has isomorphic cohomology to $\ker(r)$ by Corollary 2.16, so the desired natural isomorphisms follow. $\square$

# CHAPTER 3

# COHOMOLOGICAL ISOMORPHISMS

Acheiving our ultimate goal will require the chaining together of several isomorphisms between different cohomology theories of topological spaces. The basic goal of this chapter is to establish all of those isomorphisms. The first three sections of this chapter are adapted from Spanier [11], while the last section is original, but inspired by those results.

## 3.1 THE SINGULAR CHAIN COMPLEX OF AN OPEN COVER

Let $X$ be a topological space. The singular chain complex $0 \leftarrow C_0(X) \leftarrow C_1(X) \leftarrow ... \leftarrow C_{n-1}(X) \overset{d_n}{\leftarrow} C_n(X) \leftarrow ...$ may be augmented by the map $\varepsilon : C_0(X) \to \mathbb{Z}$ which takes $\varepsilon(\sigma) = 1$ for every singular 0-simplex of $X$.

A singular simplex $\sigma : |\Delta^m| \to |\Delta^n|$ is linear if $\sigma\left(\sum t_i v_i\right) = \sum t_i \sigma(v_i)$ for $\sum t_i = 1$; if $\sigma$ is linear, then $\sigma^{(i)}$ is also linear for any $i$, so the set of linear simplices in $|\Delta^n|$ forms a subcomplex $C_*^{\text{lin}}\left(|\Delta^n|\right) \subset C_*\left(|\Delta^n|\right)$. Let $b_n$ be the barycenter of $|\Delta^n|$, defined as $b_n := \sum_{i=0}^{n} \frac{1}{n+1} v_i$, and define a homomorphism $\beta_n : C_m^{\text{lin}}\left(|\Delta^n|\right) \to C_{m+1}^{\text{lin}}\left(|\Delta^n|\right)$ such that $(\beta_n\sigma)(v_{i+1}) = \sigma(v_i), (\beta_n\sigma)(v_0) = b_n$. Let $\xi_n : |\Delta^n| \to |\Delta^n|$ be the identity map, viewed as a linear $n$-simplex on $|\Delta^n|$; this map is also linear, so it is in $C_n^{\text{lin}}\left(|\Delta^n|\right)$. Finally, let $\tau : \mathbb{Z} \to C_0^{\text{lin}}\left(|\Delta^n|\right)$ be defined as $\tau(1) = (b_n)$; then $\beta_n$ is a chain homotopy between the identity on $C_*^{\text{lin}}\left(|\Delta^n|\right)$ and the composition $\tau\varepsilon$, where $\varepsilon$ is the augmentation map which takes $\varepsilon(\sigma) = 1$ for every linear 0-simplex of $|\Delta^n|$.

We wish to define an augmentation-preserving chain map $sd : C_*(X) \to C_*(X)$ and a chain homotopy $D : C_*(X) \to C_*(X)$ which are functorial in $X$, such that for any continuous map $f : X \to Y$ the following squares commute (where $C(f)$ is the induced map $C_*(X) \to C_*(Y)$):

$$\begin{array}{ccc} C_*(X) & \xrightarrow{sd} & C_*(X) \\ {\scriptstyle C(f)}\downarrow & & \downarrow{\scriptstyle C(f)} \\ C_*(Y) & \xrightarrow{sd} & C_*(Y) \end{array} \qquad\qquad \begin{array}{ccc} C_*(X) & \xrightarrow{D} & C_*(X) \\ {\scriptstyle C(f)}\downarrow & & \downarrow{\scriptstyle C(f)} \\ C_*(Y) & \xrightarrow{D} & C_*(Y) \end{array}$$

Both maps are defined on $m$-chains by induction on $m$. If $c \in C_0(X)$, then $sd(c) = c$ and $D(c) = 0$. Then, if $sd$ and $D$ are defined on $p$-chains for all $0 \leq p < m$ and $m \geq 1$, define $sd(\xi_n) := \beta_n(sd(d(\xi_n))$, $D(\xi_n) := \beta_n(sd(\xi_n) - \xi_n - Dd(\xi_n))$. For $\sigma : |\Delta^n| \to X$ let $sd(\sigma) := C(\sigma)(sd(\xi_n))$ and $D(\sigma) := C(\sigma)(D(\xi_n))$; this gives $sd$ and $D$ the needed properties.

If $X$ is a metric space and $c = \sum n_\sigma \sigma$, define $\text{mesh}(c) := \sup \{\text{diam}\, \sigma\,(|\Delta^n|)\,|n_\sigma \neq 0\}$.

Lemma 3.1. *Let $|\Delta^n|$ have a linear metric and let $c$ be a linear $m$-chain of $|\Delta^n|$. Then $mesh(sd(c)) \leq \dfrac{m}{m+1} mesh(c)$.*

Proof. We will induct on $m$, using the inductive definition of $sd$. For the base case, note that when $m = 0$, $sd(c) = c$ and $\text{mesh}(c) = 0$, and clearly $0 = 0$. Next, if $\sigma$ is a linear $m$-simplex on $|\Delta^n|$, then it will suffice to show $\text{mesh}(sd(\sigma)) \leq \dfrac{m}{m+1}\text{mesh}(\sigma)$. If $b = \sum \dfrac{1}{m+1}\sigma(v_i)$ then we claim the distance between $b$ and any point $y = \sum t_j \sigma(v_j)$ with $\sum t_j = 1$ and $t_j \geq 0$ for all $j$ is at most $\dfrac{m}{m+1}\text{mesh}(\sigma)$. First,
$||b - y|| \leq ||b - \sum t_j \sigma(v_j)|| = ||\sum t_j(b - \sigma(v_j))|| \leq \sum t_j ||b - \sigma(v_j)|| = \sum t_j ||\sigma(v_j) - b||$
and, similarly, for each $j$
$||\sigma(v_j) - b|| \leq ||\sigma(v_j) - \sum \dfrac{1}{m+1}\sigma(v_i)|| \leq \sum \dfrac{1}{m+1}||\sigma(v_j) - \sigma(v_i)||$, which means
$||b - y|| \leq \sum t_j \sum \dfrac{1}{m+1}||\sigma(v_i) - \sigma(v_j)||$. This is at most
$\sum \dfrac{1}{m+1} \sup ||\sigma(v_i) - \sigma(v_j)|| = \dfrac{m}{m+1} \sup ||\sigma(v_i) - \sigma(v_j)||$. However,
$\sup ||\sigma(v_i) - \sigma(v_j)|| \leq \text{diam}\, \sigma\,(|\Delta^n|) = \text{mesh}(\sigma)$, so $||b - y|| \leq \dfrac{m}{m+1}\text{mesh}(\sigma)$, as claimed.

Therefore, $\text{mesh}(sd(\sigma)) \leq \sup\left(\frac{m}{m+1}\text{mesh}(\sigma), \text{mesh}(sd(\partial\sigma))\right)$, and by induction $\text{mesh}(sd(\partial\sigma)) \leq \frac{m-1}{m}\text{mesh}(\partial\sigma) \leq \frac{m}{m+1}\text{mesh}(\sigma)$, completing the needed inequality. □

Let $sd^k : C_*(X) \to C_*(X)$ be the augmentation-preserving chain map defined for $k \geq 0$ by repeated application of the map $sd$ defined above: $sd^0 := \text{id}$, and $sd^k := sd(sd^{k-1})$ for all $k \geq 1$. Using Lemma 3.1, we obtain:

Corollary 3.2. *Let $|\Delta^n|$ have a linear metric and let $c \in C_m^{lin}(|\Delta^n|)$. Then $mesh(sd^k(c)) \leq \left(\frac{m}{m+1}\right)^k mesh(c)$.*

Given a collection $\mathscr{U}$ of open subsets of $X$, we define $C_*^{\text{sing}}(\mathscr{U})$ to be the subcomplex of $C_*(X)$ generated by singular $n$-simplices $\sigma : |\Delta^n| \to X$ such that $\sigma(|\Delta^n|) \subset U$ for some $U \in \mathscr{U}$. Note that if $\sigma(|\Delta^n|) \subset U$, then the image of $\sigma^{(i)} : |\Delta^{n-1}| \to X$ is a subset of $\sigma(|\Delta^n|)$, so also a subset of $U$, which means $d\sigma \in C_{n-1}^{\text{sing}}(\mathscr{U})$. If $\mathscr{U}$ is an open cover of $X$, then since $sd$ and $D$ are natural, $sd\left(C_*^{\text{sing}}(\mathscr{U})\right) \subset C_*^{\text{sing}}(\mathscr{U})$ and $D(C_*^{\text{sing}}(\mathscr{U})) \subset C_*^{\text{sing}}(\mathscr{U})$.

Lemma 3.3. *Let $\mathscr{U}$ be an open cover of $X$. For any singular $n$-simplex $\sigma$ of $X$, there exists some $k \geq 0$ such that $sd^k(\sigma) \in C_*^{sing}(\mathscr{U})$.*

Proof. Since $X = \bigcup_{U \in \mathscr{U}} U$, it follows that $|\Delta^n| = \bigcup_{U \in \mathscr{U}} \sigma^{-1}(U)$ forms an open cover of $|\Delta^n|$. Let $|\Delta^n|$ be given a linear metric and let $\lambda > 0$ be a Lebesgue number for this cover relative to this metric; that is, every set of diameter less than $\lambda$ is contained is some element in the cover. Choose $k \geq 0$ so that $\left(\frac{n}{n+1}\right)^k \text{diam}|\Delta^n| \leq \lambda$. By Corollary 3.2, $\text{mesh}(sd^k\xi_n) \leq \lambda$, so every singular simplex of $sd^k\xi_n$ maps into $\sigma^{-1}(U)$ for some $U \in \mathscr{U}$. Thus, $sd^k\sigma = C(\sigma)sd^k\xi_n$ is a chain in $C_*^{\text{sing}}(\mathscr{U})$. □

Theorem 3.4. *Let $\mathscr{U}$ be an open cover of $X$. Then the inclusion map $C_*^{sing}(\mathscr{U}) \to C_*(X)$ is a chain equivalence. Hence, $H^*(X, A) \cong H^*_{sing}(\mathscr{U}, A)$ where $H^*_{sing}(\mathscr{U}, A)$ is the cohomology given by $C^n_{sing}(\mathscr{U}, A) = Hom(C_n^{sing}(\mathscr{U}), A)$ with the same boundary maps as $H^*(X, A)$.*

PROOF. For each singular simplex $\sigma$ in $X$, let $k(\sigma)$ be the smallest nonnegative integer such that $sd^{k(\sigma)}(\sigma) \in C_*^{\text{sing}}(\mathscr{U})$; such an integer must exist by Lemma 3.3; clearly $k(\sigma) = 0$ if and only if $\sigma \in C_*^{\text{sing}}(\mathscr{U})$. Furthermore, $k\left(\sigma^{(i)}\right) \leq k(\sigma)$ for $0 \leq i \leq \deg\sigma$. Define $\overline{D} : C_*(X) \to C_*(X)$ by $\overline{D}(\sigma) = \sum_{j=0}^{k(\sigma)-1} Dsd^j(\sigma)$. Then $\overline{D}(\sigma) = 0$ if and only if $\sigma \in C_*^{\text{sing}}(\mathscr{U})$. Also,

$d\overline{D}(\sigma) = \sum sd^{j+1}(\sigma) - \sum sd^j(\sigma) - \sum Dsd^j(d\sigma) = sd^{k(\sigma)}(\sigma) - \sigma - \sum_{j=0}^{k(\sigma)-1}\sum_{i=0}^{n}(-1)^i Dsd^j\left(\sigma^{(i)}\right)$,

and $\overline{D}d(\sigma) = \sum_{i=0}^{n}(-1)^i \sum_{j=0}^{k\left(\sigma^{(i)}\right)-1} Dsd^j\left(\sigma^{(i)}\right)$. Therefore,

$\sigma + d\overline{D}(\sigma) + \overline{D}d(\sigma) = \sum_{i=0}^{n}(-1)^i \sum_{j=k\left(\sigma^{(i)}\right)}^{k(\sigma)-1} Dsd^j\left(\sigma^{(i)}\right) + sd^{k(\sigma)}(\sigma)$ is in $C_*^{\text{sing}}(\mathscr{U})$. Define $\tau : C_*(X) \to C_*^{\text{sing}}(\mathscr{U})$ by $\tau(\sigma) = \sigma + d\overline{D}(\sigma) + \overline{D}d(\sigma)$; then $\tau$ is a chain map preserving augmentation. Clearly, if $i : C_*^{\text{sing}}(\mathscr{U}) \to C_*(X)$ is the inclusion map, then $\tau \circ i = \text{id}_{C_*(\mathscr{U})}$ and $\overline{D}$ is a chain homotopy between $i \circ \tau$ and $\text{id}_{C_*(X)}$, making $i$ a chain equivalence. □

## 3.2 FINE PRESHEAVES

A presheaf on a topological space $X$ is *fine* if, for any locally finite open covering $\mathscr{U}$ of $X$, there exists an indexed family $\{e_U\}_{U\in\mathscr{U}}$ of endomorphisms of $\Gamma$ such that for every open set $V \subset X$, the following conditions hold:

- **(a):** For every $\gamma \in \Gamma(V)$, $e_U(\gamma)|(V - \overline{U}) = 0$
- **(b):** If $V$ has nonempty intersection with only finitely many elements of $\{\overline{U} : U \in \mathscr{U}\}$, then every $\gamma \in \Gamma(V)$ has $\gamma = \sum_{U\in\mathscr{U}} e_U(\gamma)$.

Condition (a) guarantees that $e_U(\gamma) = 0$ whenever $V \cap \overline{U} = \emptyset$, which makes the sum in condition (b) finite. Furthermore, if $\Gamma$ is fine then so is $\hat{\Gamma}$, since any endomorphism of $\Gamma$ with properties (a) and (b) induces an endomorphism of $\hat{\Gamma}$ with those same properties.

A *cochain complex of presheaves* $\Gamma^*$ is a functor into the category of cochain complexes, or equivalently a collection of presheaves $\Gamma^n$ along with coboundary maps $\partial^n : \Gamma^{n-1} \to \Gamma^n$ which for each open set $U$ of $X$ satisfy all of the conditions necessary for $\Gamma^*(U)$ to be a cochain complex. Let $Z^k$ and $B^{k+1}$ denote the kernel and image, respectively, of the coboundary map $\partial : \Gamma^k \to \Gamma^{k+1}$. Then the cohomology groups $H^k(\Gamma^*) = Z^k/B^k$ are presheaves of groups on $X$.

**The Singular Presheaf:**

The singular presheaf on $X$ with coefficients $A$, denoted by $C^*(\bullet, A)$, assigns to each open subset $U$ of $X$ the cochain complex $C^*(U, A)$. Now, $\gamma \in C^*(V, A)$ is in the kernel of the completion map $\alpha : C^*(\bullet, A) \to \hat{C}^*(\bullet, A)$ if and only if there exists some open cover $\mathscr{U}$ with $\gamma|U = 0$ for all $U \in \mathscr{U}$, which means $\gamma$ is zero on all of $C_*^{\text{sing}}(\mathscr{U})$. However, by Theorem 3.4, $C_*^{\text{sing}}(\mathscr{U})$ is chain equivalent to $C_*(V)$, so in this case $[\gamma] = [0]$ in $H^*(V, A)$. Next, let $\gamma' \in \hat{C}^n(V, A)$ be represented by a compatible $\mathscr{U}$ family $\{\gamma_U\}_{U \in \mathscr{U}}$ where $\mathscr{U}$ is an open cover of $V$; we wish to show that $\alpha$ is surjective by finding $\gamma \in C^n(V, A)$ such that $\alpha(\gamma) = \gamma'$. Let $\gamma$ be the map such that for any $U \in \mathscr{U}$ and any simplex $\sigma : |\Delta^n| \to U$, we have $\gamma(\sigma) := \gamma_U(\sigma)$, and for any $\sigma : |\Delta^n| \to V$ which is not in $C_*^{\text{sing}}(\mathscr{U})$, we have $\gamma(\sigma) := 0$. Clearly $\alpha(\gamma) = \gamma'$, so $\alpha$ is surjective.

Now, if $\mathscr{U}$ is an open cover of $V$ and $\mathscr{V}$ a refinement of $\mathscr{U}$, then we obtain a map $C^*_{\text{sing}}(\mathscr{U}, A) \to C^*_{\text{sing}}(\mathscr{V}, A)$ given by restriction to the simplices with image in elements of $\mathscr{V}$; that is, this map is induced by the inclusion $C_*^{\text{sing}}(\mathscr{V}) \to C_*^{\text{sing}}(\mathscr{U})$. This gives a direct system of cochain complexes $\{C^*_{\text{sing}}(\mathscr{U}, A)\}$, and for each $\mathscr{U}$ there is a canonical map $C^*_{\text{sing}}(\mathscr{U}, A) \to \hat{C}^*(V, A)$ given by assigning an element $f$ to the element $\gamma \in \hat{C}^n(V, A)$ which is represented by a compatible $\mathscr{U}$-family $\{\gamma_U\}_{U \in \mathscr{U}}$ such that $\gamma_U(\sigma) = f(\sigma)$ for all $\sigma \in C_n(U)$. These maps are clearly compatible, so they determine a map $\varinjlim C^*_{\text{sing}}(\mathscr{U}, A) \to \hat{C}^*(V, A)$. Furthermore, this is an isomorphism: Any $\gamma$ represented by a compatible $\mathscr{U}$-family $\{\gamma_U\}_{U \in \mathscr{U}}$ clearly has some $f \in C^*_{\text{sing}}(\mathscr{U}, A)$ which maps to it, so the image of $f$ in the limit does the same; on the other hand the image of any $f \in C^*_{\text{sing}}(\mathscr{U}, A)$ is 0 only if there exists some refinement $\mathscr{V}$ of $\mathscr{U}$ with $f(\sigma) = 0$ for all $\sigma \in C_*^{\text{sing}}(\mathscr{V})$, in which case $f$ is just a representative of 0 in the limit. By Theorem 3.4 each open cover $\mathscr{U}$

has a chain equivalence $i_{\mathscr{U}}^* : C^*(V,A) \to C^*_{\text{sing}}(\mathscr{U},A)$, so the cohomology groups of each complex are isomorphic. This makes the maps $H^*_{\text{sing}}(\mathscr{U},A) \to H^*_{\text{sing}}(\mathscr{V},A)$ isomorphisms also, so we obtain an isomorphism $H^*(V,A) \to \varinjlim H^*_{\text{sing}}(\mathscr{U},A)$. But $\varinjlim C^*_{\text{sing}}(\mathscr{U},A) \cong \hat{C}^*(V,A)$ and the map $H^*(C^*(V,A)) \to \varinjlim H^*_{\text{sing}}(\mathscr{U},A) \to H^*(\hat{C}^*(V,A))$ is in fact the same as the one induced by $\alpha$. Therefore $\alpha$ induces an isomorphism $H^*(V,A) \to H^*(\hat{C}^*(V,A))$. Hence, $\alpha$ induces an isomorphism $\alpha^* : H^*(C^*(\bullet,A)) \to H^*(\hat{C}^*(\bullet,A))$.

The singular presheaf is not fine, but the presheaf $C^n(\bullet,A)$ for any fixed $n$ is. If $\mathscr{U}$ is a locally finite open cover of $X$ and for each $x \in X$, $U_x$ is chosen so that $x \in U_x \in \mathscr{U}$, then define $e_U : C^n(V,A) \to C^n(V,A)$ by the following:

$$(e_U f)(\sigma) = \begin{cases} f(\sigma) & U = U_{\sigma(v_0)} \\ 0 & U \neq U_{\sigma(v_0)} \end{cases}$$

If $V' \subset V$, then the following commutative square where the vertical maps are induced by the inclusion shows that $e_U$ is an endomorphism of $C^n(\bullet,A)$:

$$\begin{array}{ccc} C^n(V,A) & \xrightarrow{e_U} & C^n(V,A) \\ \downarrow & & \downarrow \\ C^n(V',A) & \xrightarrow{e_U} & C^n(V',A) \end{array}$$

If $\sigma : |\Delta^n| \to X$ satisfies $\sigma(|\Delta^n|) \subset (V - \overline{U}) \subset (V - U)$ for some $U \in \mathscr{U}$, then $U \neq U_{\sigma(v_0)}$, so $(e_U f)(\sigma) = 0$, satisfying condition (a). For any $\sigma$ with image in an open set $V$ there is a unique $U$, namely $U_{\sigma(v_0)}$, with $(e_U f)(\sigma) \neq 0$ for all $f \in C^n(V,A)$, so whenever $V$ has nonempty intersection with only finitely many $U \in \mathscr{U}$, we have $\left(\sum e_U f\right)(\sigma) = (e_{U_{\sigma(v_0)}} f)(\sigma) = f(\sigma)$ for every $f \in C^n(V,A)$, so condition (b) is also satisfied.

**The Alexander-Spanier Presheaf:**

The Alexander-Spanier presheaf of $X$ with coefficients $A$, written $\Phi^*(\bullet, A)$, assigns to each open subset $U$ of $X$ the cochain complex $\Phi^*(U, A)$. The kernel of the completion map $\alpha : \Phi^*(\bullet, A) \to \hat{\Phi}^*(\bullet, A)$ is precisely the locally zero functions $\Phi_0^*(\bullet, A)$, so if $\alpha$ is surjective, it induces an isomorphism $\overline{C}^*(\bullet, A) \to \hat{\Phi}^*(\bullet, A)$ where $\overline{C}^*(\bullet, A) = \Phi^*(\bullet, A)/\Phi_0^*(\bullet, A)$. To see that $\alpha$ is indeed surjective, let $\gamma' \in \hat{\Phi}^n(V, A)$ be represented by a compatible $\mathscr{U}$ family $\{\gamma_U\}_{U\in\mathscr{U}}$ where $\mathscr{U}$ is an open cover of $V$. Then for each $U \in \mathscr{U}$, $\gamma_U : U^{n+q} \to G$ satisfies $\gamma_U|(U \cap W)^{n+1} = \gamma_W|(U \cap W)^{n+1}$ for all $U, W \in \mathscr{U}$. Thus, we may define a function $\gamma : V^{n+1} \to A$ to have $\gamma|_{U^{n+1}} = \gamma_U$ for all $U \in \mathscr{U}$ and $\gamma(x_0, ..., x_n) = 0$ if $x_0, ..., x_n$ are not all in some $U \in \mathscr{U}$. It then follows that $\alpha(\gamma) = \gamma'$.

Similarly to the singular presheaf, the presheaf of just $\Phi^n(\bullet, G)$ for a fixed $n$ is fine. Let $\mathscr{U}$ be a locally finite open covering of $X$, and for each $x \in X$ choose an element $U_x \in \mathscr{U}$ containing $x$, and for $f \in \Phi^n(V, A)$ define $e_U(f) \in \Phi^n(V, G)$ by the following:

$$(e_U f)(x_0, ..., x_n) := \begin{cases} f(x_0, ..., x_n) & U = U_{x_0} \\ 0 & U \neq U_{x_0} \end{cases}$$

If $V' \subset V$, then the following commutative square with vertical maps induced by inclusion shows that $e_U$ is an endomorphism of $\Phi^n(\bullet, A)$:

$$\begin{array}{ccc} \Phi^n(V, A) & \xrightarrow{e_U} & \Phi^n(V, A) \\ \downarrow & & \downarrow \\ \Phi^n(V', A) & \xrightarrow{e_U} & \Phi^n(V', A) \end{array}$$

If $(x_0, ..., x_n) \in (V^{n+1} - \overline{U}^{n+1}) \subset (V^{n+1} - U^{n+1})$, then $U_{x_0} \neq U$, so $(e_U f)(x_0, ..., x_n) = 0$, satisfying condition (a). Next, given $x_0, ..., x_n$, there is a unique $U$ (namely $U_{x_0}$) such that $(e_U f)(x_0, ..., x_n) \neq 0$, so $\left(\sum e_U f\right)(x_0, ..., x_n) = (e_{U_{x_0}} f)(x_0, ..., x_n) = f(x_0, ..., x_n)$. Thus, if $V$

has nonempty intersection with only finitely many $U \in \mathscr{U}$, then any $f \in \Phi^n(V, A)$ satisfies $\sum e_U f = f$, so condition (b) follows.

From the above, both $C^*(\bullet, A)$ and $\Phi^*(\bullet, A)$ are cochain complexes of fine presheaves, in the sense that they are cochain complexes of presheaves such that for each $n$ the presheaves $C^n(\bullet, A)$, $\Phi^n(\bullet, A)$ are fine.

Given an open cover $\mathscr{U}$ of a space $X$, a *shrinking of* $\mathscr{U}$ is an open cover $\mathscr{V}$ of $X$ in one to one correspondence with $\mathscr{U}$ such that if $U \in \mathscr{U}$ corresponds to $V_U \in \mathscr{V}$, then $\overline{V}_U \subset U$. Any locally finite open cover of a normal Hausdorff space has shrinkings, and any shrinking of a locally finite open cover is clearly also locally finite.

LEMMA 3.5. *If* $\Gamma$ *is a fine presheaf on a paracompact Hausdorff space* $X$*, then* $\check{H}^n(X, \Gamma) = 0$ *for* $n > 0$.

PROOF. Let $\mathscr{U} = \{U\}$ be a locally finite open cover of $X$ and let $\mathscr{U}' = \{U'\}$ be a shrinking of $\mathscr{U}$. Let $\{e_U\}_{U \in \mathscr{U}}$ be the endomorphisms making $\Gamma$ fine with respect to the covering $\mathscr{U}'$ indexed by the covering $\mathscr{U}$. Let $\mathscr{V} = \{V\}$ be a refinement of $\mathscr{U}$ such that each $V \in \mathscr{V}$ has nonempty intersection with only finitely many elements of $\mathscr{U}$ and for any $U \in \mathscr{U}$, either $V \subset U$ or $V \subset (X - \overline{U}')$. Let $\lambda : \mathscr{V} \to \mathscr{U}$ be a function such that $V \subset \lambda(V)$ for all $V \in \mathscr{V}$.

Since each $e_U$ is an endomorphism of $\Gamma$, $e_U$ induces a cochain map $e_U : \check{C}^*(\mathscr{U}, \Gamma) \to \check{C}^*(\mathscr{U}, \Gamma)$ such that for any $\psi \in \check{C}^n(\mathscr{U}, \Gamma)$ and $U_0, ..., U_n \in \mathscr{U}$, $(e_U\psi)(U_0, ..., U_n) = e_U(\psi(U_0, ..., U_n))$. Then $e_U$ acts similarly as a cochain map on $\check{C}^*(\mathscr{V}, \Gamma)$ and commutes with the cochain map $\lambda^* : \check{C}^*(\mathscr{U}, \Gamma) \to \check{C}^*(\mathscr{V}, \Gamma)$ induced by $\lambda$. Let $n > 0$ and $\psi \in \check{C}^n(\mathscr{U}, \Gamma)$ a cocycle. Define $\psi_U \in \check{C}^n(\mathscr{V}, \Gamma)$ by $\psi_U = e_U(\lambda^*\psi)$; then $\psi_U$ is a cocycle for each $U \in \mathscr{U}$, and if $V_0, ..., V_n \in \mathscr{V}$ then $\psi_U(V_0, ..., V_n) = 0$ except for a finite number of $U \in \mathscr{U}$. Therefore $\sum \psi_U$ exists, and $\sum \psi_U = \lambda^*(\psi)$.

Define $\psi'_U \in \check{C}^{n-1}(\mathscr{V}, \Gamma)$ by $\psi'_U(V_0, ..., V_{n-1})$

$$= \begin{cases} e_U(\psi(U, \lambda(V_0), ..., \lambda(V_{n-1}))|(V_0 \cap ... \cap V_{n-1}) & V_0 \cap ... \cap V_{n-1} \subset U \\ 0 & V_0 \cap ... \cap V_{n-1} \subset X - \overline{U}' \end{cases}$$

Then $\partial \psi'_U = \psi_U$ for all $U$, and because $\sum \psi'_U$ can be formed (since for any $(n-1)$-tuple $V_0, ..., V_{n-1}$, $\psi'_U(V_0, ..., V_{n-1}) = 0$ for all but finitely many $U \in \mathscr{U}$), $\lambda^* \psi = \sum \psi_U = \partial \left(\sum \psi'_U\right)$. Therefore $\lambda^* \psi$ is a coboundary for any $\psi \in \check{C}^n(\mathscr{U}, \Gamma)$, so $H^n(X, \Gamma) = 0$. □

Lemma 3.6. *Let $\Gamma^*$ be a cochain complex of presheaves of modules on $X$. For every $k$ there is the following exact sequence, functorial in $\Gamma^*$:*

$$0 \to \ker\left[\check{H}^0(X, B^k) \to \check{H}^1(X, Z^{k-1})\right] \to \check{H}^0(X, Z^k) \to H^k(\hat{\Gamma}^*(X)) \to 0$$

Proof. First note that $\hat{\Gamma}^k(X) = \check{H}^0(X, \Gamma^k)$. From the short exact sequence of presheaves $0 \to Z^k \to \Gamma^k \to B^{k+1} \to 0$, Lemma 1.12 gives an exact sequence $0 \to \check{H}^0(X, Z^k) \to \check{H}^0(X, \Gamma^k) \to \check{H}^0(X, B^{k+1}) \to \check{H}^1(X, Z^k)$. Since $B^{k+1} \subset \Gamma^{k+1}$, a similar exactness property gives $\check{H}^0(X, B^{k+1}) \subset H^0(X, \Gamma^{k+1})$. Using these, we obtain $\check{H}^0(X, Z^k) \cong \ker\left[\check{H}^0(X, \Gamma^k) \to \check{H}^0(X, B^{k+1})\right] \cong \ker\left[\check{H}^0(X, \Gamma^k) \to \check{H}^0(X, \Gamma^{k+1})\right]$ and $\mathrm{im}\left[\check{H}^0(X, \Gamma^k) \to \check{H}^0(X, \Gamma^{k+1})\right] \cong \ker\left[\check{H}^0(X, B^{k+1}) \to \check{H}^1(X, Z^k)\right]$. Since $H^k(\hat{\Gamma}^*(X)) = \ker\left[\check{H}^0(X, \Gamma^k) \to \check{H}^0(X, \Gamma^{k+1})\right] / \mathrm{im}\left[\check{H}^0(X, \Gamma^{k-1}) \to \check{H}^0(X, \Gamma^k)\right]$, the desired sequence follows. □

Corollary 3.7. *If $\Gamma^*$ is a cochain complex of presheaves of modules on a paracompact Hausdorff space $X$, then for any $k$ there is the following short exact sequence, functorial in $\Gamma^*$:*

$$0 \to im\left[\check{H}^0(X, B^k) \to \check{H}^1(X, Z^{k-1})\right] \to H^k(\hat{\Gamma}^*(X))$$

$$\to \ker\left[\check{H}^0(X, H^k(\Gamma^*)) \to \check{H}^1(X, B^k)\right] \to 0$$

*If $\Gamma^{k-1}$ is fine, the first term in this sequence may be replaced with $\check{H}^1(X, Z^{k-1})$.*

Proof. From the short exact sequence of presheaves $0 \to B^k \to Z^k \to H^k(\Gamma^*(X)) \to 0$ defining $H^k(\Gamma^*(X))$, Lemma 1.12 gives an isomorphism $\check{H}^0(X, Z^k)/\check{H}^0(X, B^k) \cong \ker\left[\check{H}^0(X, H^k(\Gamma^*(X)) \to H^1(X, B^k)\right]$. From Lemma 3.6 there is an isomorphism

$$\check{H}^0(X, Z^k)/\ker\left[\check{H}^0(X, B^k) \to \check{H}^1(X, Z^{k-1})\right] \cong H^k(\hat{\Gamma}^*(X))$$

Therefore, $H^k(\hat{\Gamma}^*(X))$ maps surjectively onto $\ker\left[\check{H}^0(X, H^k(\Gamma^*(X)) \to H^1(X, B^k)\right]$ with kernel isomorphic to $\check{H}^0(X, B^k)/\ker\left[\check{H}^0(X, B^k) \to \check{H}^1(X, Z^{k-1})\right]$, which is isomorphic to $\operatorname{im}\left[\check{H}^0(X, B^k) \to \check{H}^1(X, Z^{k-1})\right]$. This gives the first exact sequence. For the second, there is a short exact sequence of presheaves $0 \to Z^{k-1} \to \Gamma^{k-1} \to B^k \to 0$, and if $\Gamma^{k-1}$ is fine, Lemma 1.12 and Lemma 3.5 together give $\operatorname{im}\left[\check{H}^0(X, B^k) \to \check{H}^1(X, Z^{k-1})\right] = \check{H}^1(X, Z^{k-1})$, as needed. □

Lemma 3.8. *Let $\Gamma^*$ be a nonnegative cochain complex of fine presheaves of modules on a paracompact Hausdroff space $X$. If for some integers $0 \leq m < n$, $H^k(\Gamma^*(X))$ is locally zero for $k < m$ and for $m < k < n$, then there are functorial isomorphisms $\check{H}^{k-m}(X, H^k(\Gamma^*)) \cong H^k(\hat{\Gamma}^*(X))$ for all $k < n$ and a functorial injection $\check{H}^{n-m}(X, H^m(\Gamma^*)) \to H^n(\hat{\Gamma}^*(X))$.*

Proof. For each $k$ there is a short exact sequence of presheaves $0 \to Z^k \to \Gamma^k \to B^{k+1} \to 0$. Since $\Gamma^k$ is fine, $\check{H}^p(X, B^{k+1}) \cong \check{H}^{p+1}(X, Z^k)$ for all $p \geq 1$ by

Lemma 1.12 and Lemma 3.5. For each $k$ there is also a short exact sequence of presheaves $0 \to B^k \to Z^k \to H^k(\Gamma^*(X)) \to 0$. Since $H^k(\Gamma^*(X))$ is locally zero for $k < m$ and $m < k < n$, it follows from Lemma 1.12 and Lemma 1.13 that $\check{H}^p(X, B^k) \cong \check{H}^p(X, Z^k)$ for $k < m$ or $m < k < n$ and all $p$. Since $B^0$ is the zero presheaf, it follows by induction on $k$ from the above isomorphisms that for $k < m$ and all $p \geq 1$, $\check{H}^p(X, Z^k) = 0 = \check{H}^p(X, B^{k+1})$. From this and Corollary 3.7, $H^i(\hat{\Gamma}^*(X)) = 0$ for all $i < m$, so the result holds for $k < m$. If $k = m$, then $H^m(\hat{\Gamma}^*(X)) \cong \check{H}^0(X, H^m(\Gamma^*(X)))$ and the result holds in this case also.

For $m < k \leq n$, note that $\check{H}^p(X, B^m) = 0$ for all $p \geq 1$. From the short exact sequence of presheaves $0 \to B^m \to Z^m \to H^m(\Gamma^*(X)) \to 0$ it follows that $\check{H}^p(X, Z^m) \cong \check{H}^p(X, H^m(\Gamma^*(X)))$. For $m < i < n$, Corollary 3.7 gives $\check{H}^1(X, Z^{i-1}) \cong H^i(\hat{\Gamma}^*(X))$ and for $i = n$ it gives an injective map $\check{H}^1(X, Z^{n-1}) \to H^n(\hat{\Gamma}^*(X))$. Therefore, for all $m < i \leq n$:

$$\check{H}^1(X, Z^{i-1}) \cong \check{H}^1(X, B^{i-1}) \cong \check{H}^2(X, Z^{i-2}) \cong \ldots$$

$$\cong \check{H}^{i-m}(X, Z^m) \cong \check{H}^{i-m}(X, H^m(\Gamma^*(X)))$$

This gives the result for $m < k \leq n$. □

Theorem 3.9. *Let $X$ be a paracompact Hausdorff space and $\tau : \Gamma_1^* \to \Gamma_2^*$ a cochain map between nonnegative cochain complexes of fine presheaves of modules on $X$. Assume that for some $m \geq 0$, $\tau_* : H^n(\Gamma_1^*) \to H^n(\Gamma_2^*)$ is a local isomorphism for $n < m$ and a local injection for $n = m$. Then the induced map $\hat{\tau}_* : H^n(\hat{\Gamma}_1^*(X)) \to H^n(\hat{\Gamma}_2^*(X))$ is an isomorphism for $n < m$ and an injection for $n = m$.*

Proof. Let $\Gamma_\tau^*$ be the mapping cone of $\tau$. Then $\Gamma_\tau^n = \Gamma_1^{n+1} \oplus \Gamma_2^n$, and for $\gamma_1 \in \Gamma_1^{n+1}(U)$, $\gamma_2 \in \Gamma_2^n(U)$, $\partial(\gamma_1, \gamma_2) = (-\partial(\gamma_1), \tau(\gamma_1) + \partial(\gamma_2))$ by definition. Note that $\Gamma_\tau^*$ is a nonnegative cochain complex of fine presheaves on $X$, and for any open $U \subset X$ there is the exact sequence

$$... \to H^n(\Gamma_2^*(U)) \to H^n(\Gamma_\tau^*(U)) \to H^{n+1}(\Gamma_1^*(U)) \xrightarrow{\tau_*} H^{n+1}(\Gamma_2^*(U)) \to ...$$

Taking the direct limit as $U$ varies over open neighborhoods of each $x \in X$, $\tau_* : H^k(\Gamma_1^*(X)) \to H^k(\Gamma_2^*(X))$ is a local isomorphism for $k < n$ and a local injection for $k = n$ if and only if $H^k(\Gamma_\tau^*)$ is locally zero for $k < n$. By Lemma 3.8 it follows that $H^k(\hat{\Gamma}_\tau^*(X)) = 0$ for $k < n$; if $n = 0$ this is trivially true, and if $n > 0$ choose $m = 0$ in the statement of that lemma.

Clearly, $\hat{\Gamma}_\tau^*$ is the mapping cone $\Gamma_{\hat{\tau}}^*$ of the induced map $\hat{\tau} : \hat{\Gamma}_1^* \to \hat{\Gamma}_2^*$ between the completions. Therefore, the following sequence is exact:

$$... \to H^k(\hat{\Gamma}_2^*(X)) \to H^k(\hat{\Gamma}_\tau^*(X)) \to H^{k+1}(\hat{\Gamma}_1^*(X)) \xrightarrow{\hat{\tau}_*} H^{k+1}(\hat{\Gamma}_2^*(X)) \to ...$$

Since $H^k(\hat{\Gamma}_\tau^*(X))$ was shown above to be zero for $k < n$, the result follows from this. □

## 3.3 THE MAP $\mu$

Let $\mathscr{U}$ be an open covering of a topological space $X$. Then there is a canonical chain transformation $C_*^{\text{sing}}(\mathscr{U}) \to C_*(\mathscr{U})$ taking $\sigma$ to the ordered tuple $(\sigma(v_0), \sigma(v_1), ...\sigma(v_n))$, which is clearly in $C_*(\mathscr{U})$ since the image of $\sigma$ is contained in some $U \in \mathscr{U}$. This induces a cochain map $C^*(\mathscr{U}, A) \to C^*_{\text{sing}}(\mathscr{U}, A)$, which further induces morphisms $H^n(\mathscr{U}, A) \to H^n_{\text{sing}}(\mathscr{U}, A)$. Taking this map to the limit of each direct system gives a map $\varinjlim H^n(\mathscr{U}, A) \to \varinjlim H^n_{\text{sing}}(\mathscr{U}, A)$, and by precomposing this with the isomorphism from Corollary 1.9 we obtain a map $\mu' : \overline{H}^n(X, A) \to \varinjlim H^n_{\text{sing}}(\mathscr{U}, A)$. Furthermore $\varinjlim H^n_{\text{sing}}(\mathscr{U}, A) \cong H^n(X, A)$ also, by an isomorphism we will call $\mu''$, so we may define a morphism $\mu : \overline{H}^n(X, A) \to H^n(X, A)$ to be the composition $\mu''\mu'$, which is a natural transformation of cohomologies.

We may define a presheaf map $\tau : \Phi^*(\bullet, A) \to C^*(\bullet, A)$ such that if $f \in \Phi^n(U, A)$ and $\sigma : |\Delta^n| \to U$, then $\tau f(\sigma) := f(\sigma(v_0), ..., \sigma(v_n))$.

This induces a map $\hat{\tau} : \hat{\Phi}^*(\bullet, A) \to \hat{C}^*(\bullet, A)$ such that the following square is commutative:

$$\begin{array}{ccc} \Phi^*(\bullet, A) & \xrightarrow{\tau} & C^*(\bullet, A) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \hat{\Phi}^*(\bullet, A) & \xrightarrow{\hat{\tau}} & \hat{C}^*(\bullet, A) \end{array}$$

From the discussion in Section 3.2 we have isomorphisms $\alpha^* : H^*(C^*(\bullet, A)) \cong H^*(\hat{C}^*(\bullet, A))$ and $\overline{C}^*(\bullet, A) \cong \hat{\Phi}^*(\bullet, A)$, the latter of which gives an isomorphism $H^*(\overline{C}^*(\bullet, A)) \cong H^*(\hat{\Phi}^*(\bullet, A))$; furthermore $H^*(X, A) \cong H^*(C^*(\bullet, A)(X))$ and $\overline{H}^*(X, A) \cong H^*(\overline{C}^*(\bullet, A)(X))$. Using these, we have the following commutative diagram:

$$\begin{array}{ccc} \overline{H}^*(X, A) & \xrightarrow{\quad\mu\quad} & H^*(X, A) \\ {\scriptstyle\cong}\updownarrow & & \updownarrow{\scriptstyle\cong} \\ H^*(\hat{\Phi}^*(\bullet, A)(X)) & \xrightarrow{\hat{\tau}_*} & H^*(\hat{C}^*(\bullet, A)(X)) \end{array}$$

Therefore, $\mu$ is an isomorphism or an injection if and only if $\hat{\tau}_*$ is.

For a subspace $Y \subset X$ and given $n$, the cohomology groups $\{H^n(U, A) : U \text{ is a neighborhood of } Y\}$ form a direct system ordered by inclusion of neighborhoods. The restriction maps $H^n(U, A) \to H^n(Y, A)$ define a natural homomorphism $i : \varinjlim H^n(U, A) \to H^n(Y, A)$. This is also true for Alexander-Spanier cohomology.

Theorem 3.10. *Let $X$ be a paracompact Hausdorff space and suppose that for some $n \geq 0$, each $x \in X$ has the natural map $i : \varinjlim H^k(U, A) \to H^k(\{x\}, A)$ as an isomorphism for all $k < n$. Then the map $\mu : \overline{H}^k(X, A) \to H^k(X, A)$ is an isomorphism for all $k < n$ and an injection for $k = n$.*

Proof. Both $C^*(\bullet, A)$ and $\Phi^*(\bullet, A)$ are nonnegative cochain complexes of fine presheaves. The map $i$ is an isomorphism for all $x \in X$, which implies that

$\tau_* : H^k(\Phi^*(\bullet, A)) \to H^k(C^*(\bullet, A))$ is a local isomorphism for $k < n$ and a local injection for $k = n$ (in fact, $\tau_*$ is always locally injective for all $k$). By Theorem 3.9, this makes $\hat{\tau}_* : H^k(\hat{\Phi}^*(\bullet, A)) \to H^k(\hat{C}^*(\bullet, A))$ an isomorphism for $k < n$ and an injection for $k = n$. Therefore, the same is true of $\mu$. □

A space $X$ is *homologically locally connected in dimension* $n$ if for every $x \in X$ and neighborhood $U$ of $x$ there exists a neighborhood $V \subset U$ of $x$ such that the map $\tilde{H}_k(V) \to \tilde{H}_k(U)$ induced by inclusion is trivial for $k \leq n$. A space is *homologically locally connected* if it is homologically locally connected in dimension $n$ for all $n$. Any locally contractible space, in particular any CW complex, is homologically locally connected.

Lemma 3.11. *If $X$ is homologically locally connected in dimension $n$, then $\tilde{H}^k(C^*(\bullet, A)) := H^k(Hom(\tilde{C}_*(\bullet), A))$ is locally zero for $k \leq n$ and all $A$.*

Proof. Let $f^* \in \mathrm{Hom}(\tilde{C}_n(U), A)$ be a cocycle with $0 \leq k \leq n$ and $x \in U$. If $k = 0$, let $V \subset U$ be a neighborhood of $x$ such that $\tilde{H}_0(V) \to \tilde{H}_0(U)$ is trivial. If $c \in \tilde{C}_0(V)$, then there exists $c' \in C_1(U)$ such that $c = dc'$. Thus, $f^*(c) = f^*(dc') = (\partial f)(c') = 0$, so $f^*|\tilde{C}_0(V) = 0$, which shows $\tilde{H}^0(C^*(\bullet, A))$ is locally zero.

If $k > 0$, let $V \subset V' \subset U$ be neighborhoods of $x$ with $\tilde{H}_{k-1}(V) \to \tilde{H}_{k-1}(V')$ and $H_k(V') \to H_k(U)$ both trivial. If $c$ is a reduced singular $(k-1)$-cycle of $V$, let $c'$ be a $k$-chain of $V'$ such that $dc' = c$. Then $f^*(c') \in A$ is independent of the choice of $c'$ since if $c''$ also has $dc'' = c$, $c' - c'' = de$ for some $(k+1)$-chain $e$ in $U$, and $f^*(c' - c'') = f^*(de) = (\partial f^*)(e) = 0$. Thus there is a homomorphism $\tilde{f}^* : \tilde{Z}_{k-1}(V) \to A$ such that $\tilde{f}^*(c) = f^*(c')$ if $dc' = c$. Since $C_{k-1}(V)/\tilde{Z}_{k-1}(V)$ is isomorphic to a subgroup of $C_{k-2}(V)$ if $k > 1$ or to $\mathbb{Z}$ if $k = 1$, it is free, so there exists a homomorphism $d^* : C_{k-1}(V) \to A$ which is an extension of $\tilde{f}^*$. Then $f^*|C_k(V) = \partial d^*$, so $\tilde{H}^k(C^*(\bullet, A))$ is locally zero. □

Using Theorem 3.10 and Lemma 3.11, we obtain the following:

Corollary 3.12. *If $X$ is a paracompact Hausdorff space which is homologically locally connected in dimension $n$, then $\mu : \overline{H}^n(X, A) \to H^n(X, A)$ is an isomorphism for $k \leq n$ and an injection for $k = n + 1$.*

## 3.4 COHOMOLOGY WITH LOCAL COEFFICIENTS

Let $X$ be a locally contractible, path-connected, paracompact, and Hausdorff space with the universal cover $p : \tilde{X} \to X$, so $\tilde{X}$ is also locally contractible, paracompact, and Hausdorff, and let $\pi = \pi_1(X)$ be the fundamental group of $X$. Then $\pi$ acts on $C_n(\tilde{X})$ by deck transformation; in particular if $g \in \pi$ and $\sigma : |\Delta^n| \to \tilde{X}$, then $g \cdot \sigma : |\Delta^n| \to \tilde{X}$ is just $\sigma$ followed by the action of $g$ on its image. Similarly, if both $X$ and $\tilde{X}$ are CW-complexes and $p$ is a cellular map, then $\pi$ acts on $C_n^{\text{cell}}(\tilde{X})$ by taking $g \cdot e_\alpha^n$ to be precisely the $n$-cell which is the image of $e_\alpha^n$ under the action of $g$. In this way both $C_n(\tilde{X})$ and $C_n^{\text{cell}}(\tilde{X})$ are $\pi$-modules for each $n$. If $A$ is any $\pi$-module, define a cochain complex as follows: Let $C_\pi^n(X, A) := \text{Hom}_{\mathbb{Z}[\pi]}(C_n(\tilde{X}), A)$, with coboundary maps defined by precomposition with the boundary map of $C_n(\tilde{X})$; in the cellular case the chain equivalence between $C_n(\tilde{X})$ and $C_n^{\text{cell}}(\tilde{X})$ gives $\text{Hom}_{\mathbb{Z}[\pi]}(C_n(\tilde{X}), A) \cong \text{Hom}_{\mathbb{Z}[\pi]}(C_n^{\text{cell}}(\tilde{X}), A)$, so identify $C_\pi^n(X, A)$ as the latter with coboundaries given by precomposition with the cellular boundary maps instead. Then the *cohomology of $X$ with local coefficients* is defined to be the cohomology of the cochain complex $C_\pi^n(X, A)$. This will be denoted by $H_\pi^n(X, A)$.

Theorem 3.13. *If $G$ is a discrete group and $A$ is any $G$-module, then $\pi = \pi_1(BG, A) \cong G$ and $H_\pi^n(BG, A) \cong H^n(G, A)$ by a natural isomorphism.*

Proof. That $\pi \cong G$ follows from the definition of $BG$ in Section 2.4, so it suffices to give a natural chain equivalence between $C^*(G, A)$ and $C_\pi^*(BG, A)$. Let $q : \ker(r) \to C_{\text{cell}}^*(EG, A)$ be the natural injective cochain map from Lemma 2.18, whose image is precisely the set of those elements $f \in C_{\text{cell}}^n(EG, A)$ with $f(g[g_0, ..., g_n]) = gf([g_0, ..., g_n])$. Clearly $C_\pi^n(BG, A) = \text{Hom}_{\mathbb{Z}[\pi]}(C_n^{\text{cell}}(EG), A)$ is a subcomplex of $C^*_{\text{cell}}(EG, A)$ consisting of the same functions, so composing $q$ with the natural isomorphism between the cohomologies of $C^*(G, A)$ and $\ker(r)$ from Corollary 2.16 gives the desired isomorphisms. $\square$

Now, let $A_{\tilde{X}}$ be the constant presheaf of $\tilde{X}$ associated with $A$, and $\hat{A}_{\tilde{X}}$ its completion, the constant *sheaf* on $\tilde{X}$. Then define the sheaf $\mathcal{A}$ on $X$ as follows: For any open set $U$,

$$\mathcal{A}(U) := \{\gamma \in \hat{A}_{\tilde{X}}(p^{-1}(U)) : \text{for every } x \in p^{-1}(U) \text{ and } g \in \pi, \gamma(g \cdot x) = g \cdot \gamma(x)\}.$$

Note that for any $x \in X$, there exists an open neighborhood $U_x$ of $x$ such that $p^{-1}(U_x)$ is a union of disjoint open sets in $\tilde{X}$ which are each mapped homeomorphically onto $U_x$ by $p$. Then $\mathcal{A}|U_x$ is a constant sheaf on $U_x$ since for any $V \subset U_x$, any $\gamma \in \hat{A}_{\tilde{X}}(p^{-1}(V))$ satisfies $\gamma(g \cdot y) = g \cdot \gamma(y)$ for all $y \in p^{-1}(V)$ and $g \in G$. Thus $\mathcal{A}(V) = \hat{A}_{\tilde{X}}(p^{-1}(V))$ and restriction maps between such subsets are also constant. In the sense that each $x$ has an open neighborhood $U_x$ with $\mathcal{A}|U_x$ a constant sheaf, $\mathcal{A}$ is called *locally constant.* We wish to show the following:

THEOREM 3.14. *If $X$ is a locally contractible, path-connected, paracompact, and Hausdorff space, then there is a natural isomorphism $\check{H}^*(X, \mathcal{A}) \cong H^*_\pi(X, A)$.*

Proving this will require some intermediary steps.

**The Local Singular Presheaf:**

Let the local singular presheaf on $p : \tilde{X} \to X$, denoted $C^*_\pi(\bullet, A)$, assign to each open subset $U$ of $X$ the cochain complex $C^*_\pi(U, A) = \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(p^{-1}(U)), A)$. Now, fix $n$ and consider the presheaf $C^n_\pi(\bullet, A)$; we wish to show that this presheaf is fine. If $\mathscr{U}$ is a locally finite open cover of $X$ and for each $x \in X$, $U_x \in \mathscr{U}$ is chosen such that $x \in U_x$, define

$e_U : C^n_\pi(V, A) \to C^n_\pi(V, A)$ by the formula:

$$(e_U f)(\sigma) := \begin{cases} f(\sigma) & U = U_{p(\sigma(v_0))} \\ 0 & U \neq U_{p(\sigma(v_0))} \end{cases}$$

If $V' \subset V$, the following commutative square shows that $e_U$ is an endomorphism of $C^n_\pi(\bullet, A)$:

$$\begin{array}{ccc} C^n_\pi(V, A) & \xrightarrow{e_U} & C^n_\pi(V, A) \\ \downarrow & & \downarrow \\ C^n_\pi(V', A) & \xrightarrow{e_U} & C^n_\pi(V', A) \end{array}$$

Also, if $\sigma : |\Delta^n| \to \tilde{X}$ satisfies $p\left(\sigma\left(|\Delta^n|\right)\right) \subset V - \overline{U} \subset V - U$ for some $U \in \mathscr{U}$, then $U \neq U_{p(\sigma(v_0))}$, so $(e_U f)(\sigma) = 0$, satisfying condition (a) for $C^n_\pi(\bullet, A)$ to be fine. If $V$ is an open subset of $X$, for any $\sigma$ with image in $p^{-1}(V)$, we have $U_{p(\sigma(v_0))}$ as the unique $U$ such that $(e_U f)(\sigma) \neq 0$ for all $f \in C^n_\pi(V, A)$. Hence, whenever $V$ has nonempty intersection with only finitely many $U \in \mathscr{U}$, it follows that $\left(\sum e_U f\right)(\sigma) = (e_{U_{p(\sigma(v_0))}} f)(\sigma) = f(\sigma)$ for every $f \in C^n_\pi(V, A)$, satisfying condition (b) also.

Now, for any open cover $\mathscr{U}$ of $X$, consider the collection of Čech cochains $\check{C}^p(\mathscr{U}, C^q_\pi(\bullet, A))$ where both $p$ and $q$ range over the nonnegative integers. For fixed $q$, let the Čech coboundary map $\check{C}^p(\mathscr{U}, C^q_\pi(\bullet, A)) \to \check{C}^{p+1}(\mathscr{U}, C^q_\pi(\bullet, A))$ be denoted by $\check{\partial}$. On the other hand, for fixed $p$ the coboundary maps $C^q_\pi(U, A) \to C^{q+1}_\pi(U, A)$ induce a map $\partial : \check{C}^p(\mathscr{U}, C^q_\pi(\bullet, A)) \to \check{C}^p(\mathscr{U}, C^{q+1}_\pi(\bullet, A))$ given by $\left[\partial f(U_0, ..., U_p)\right](\sigma) = \partial\left[f(U_0, ..., U_p)\right](\sigma) = [f(U_0, ..., U_p)](d\sigma)$. Clearly, these two coboundary maps have $\partial\check{\partial} = \check{\partial}\partial$ as a map $\check{C}^p(\mathscr{U}, C^q_\pi(\bullet, A)) \to \check{C}^{p+1}(\mathscr{U}, C^{q+1}_\pi(\bullet, A))$. We define a new cochain complex $\mathrm{Tot}\check{C}(\mathscr{U}, C_\pi(\bullet, A))$, with $(\mathrm{Tot}\check{C}(\mathscr{U}, C_\pi(\bullet, A)))^n := \bigoplus_{p+q=n} \check{C}^p(\mathscr{U}, C^q_\pi(\bullet, A))$ and the new coboundary map $\partial^{\mathrm{Tot}}$ defined on a term $\check{C}^p(\mathscr{U}, C^q_\pi(\bullet, A))$ to be $\partial^{\mathrm{Tot}}\gamma := \check{\partial}\gamma + (-1)^p\partial\gamma$. This is a chain complex since:

$$\partial^{\mathrm{Tot}}\partial^{\mathrm{Tot}}\gamma = \partial^{\mathrm{Tot}}(\check{\partial}\gamma + (-1)^p\partial\gamma) = \check{\partial}\check{\partial}\gamma + \check{\partial}(-1)^p\partial\gamma + (-1)^{p+1}\partial\check{\partial}\gamma + (-1)^p\partial(-1)^p\partial\gamma$$

$$= 0 + (-1)^p\check{\partial}\partial\gamma + (-1)^{p+1}\partial\check{\partial}\gamma + 0 = (-1)^p(\check{\partial}\partial\gamma - \check{\partial}\partial\gamma) = 0$$

Hence, applying $\partial^{\mathrm{Tot}}\partial^{\mathrm{Tot}}$ to the entire sum also gives 0. We will abbreviate this cochain complex by $\mathrm{Tot}^*(\mathscr{U})$, with cohomology $H^*(\mathrm{Tot}(\mathscr{U}))$. For any refinement $\mathscr{V}$ of $\mathscr{U}$, the maps $\check{C}^p(\mathscr{U}, C^q_\pi(\bullet, A)) \to \check{C}^p(\mathscr{V}, C^q_\pi(\bullet, A))$ for each pair $p, q$ clearly extend to a map $\mathrm{Tot}^*(\mathscr{U}) \to \mathrm{Tot}^*(\mathscr{V})$, and these maps form a direct system for essentially the same reason as in Čech cohomology; hence we may define cohomology groups $H^*(\mathrm{Tot}(X)) := \varinjlim H^*(\mathrm{Tot}(\mathscr{U}))$. Furthermore, by taking $\mathscr{U}$ to be covers indexed by $X$, $\mathrm{Tot}^*(X)$ may be viewed as $\bigoplus_{p+q=n} \check{C}^p(X, C^q_\pi(\bullet, A))$ instead, and, similar to the case of Čech cohomology, $H^*(\mathrm{Tot}^*(X)) \cong \varinjlim H^*(\mathrm{Tot}(\mathscr{U})) = H^*(\mathrm{Tot}(X))$.

This complex has two natural filtrations, namely the Čech filtration $F^m_{\check{\mathrm{C}}\mathrm{ech}}(\mathrm{Tot}^n(\mathscr{U})) = \bigoplus_{p\geq m} \check{C}^p(\mathscr{U}, C^{m-p}_\pi(\bullet, A))$ and the singular filtration $F^m_{\mathrm{sing}}(\mathrm{Tot}^n(\mathscr{U})) = \bigoplus_{q\geq m} \check{C}^{m-q}(\mathscr{U}, C^q_\pi(\bullet, A))$, as follows:

$$\check{C}^n(\mathscr{U}, C^0_\pi(\bullet, A)) = F^n_{\check{\mathrm{C}}\mathrm{ech}}(\mathrm{Tot}^n(\mathscr{U})) \hookrightarrow \ldots \hookrightarrow F^1_{\check{\mathrm{C}}\mathrm{ech}}(\mathrm{Tot}^n(\mathscr{U})) \hookrightarrow F^0_{\check{\mathrm{C}}\mathrm{ech}}(\mathrm{Tot}^n(\mathscr{U})) = \mathrm{Tot}^n(\mathscr{U})$$

$$\check{C}^0(\mathscr{U}, C^n_\pi(\bullet, A)) = F^n_{\mathrm{sing}}(\mathrm{Tot}^n(\mathscr{U})) \hookrightarrow \ldots \hookrightarrow F^1_{\mathrm{sing}}(\mathrm{Tot}^n(\mathscr{U})) \hookrightarrow F^0_{\mathrm{sing}}(\mathrm{Tot}^n(\mathscr{U})) = \mathrm{Tot}^n(\mathscr{U})$$

These give rise to the two spectral sequences ${}^{\check{C}}E^{p,q}_r$, ${}^{s}E^{p,q}_r$ respectively, which may be used to compute the cohomology groups of the complex. Of particular note is that ${}^{\check{C}}E^{p,q}_1 = \check{C}^p(\mathscr{U}, H^q_\pi(\bullet, A))$ and similarly, ${}^{s}E^{p,q}_1 = \check{H}^p(\mathscr{U}, C^q_\pi(\bullet, A))$. These filtrations may also be applied to the cochain complex $\mathrm{Tot}^n(X) = \bigoplus_{p+q=n} \check{C}^p(X, C^q_\pi(\bullet, A))$ of the entire space.

Lemma 3.15. *There is a natural isomorphism $H^*_\pi(X, A) \to H^*(Tot(X))$.*

PROOF. For any open cover $\mathscr{U}$ of $X$, let $C^*_\pi(\mathscr{U}, A) := \mathrm{Hom}_{\mathbb{Z}[\pi]}(C^{\mathrm{sing}}_*(p^{-1}(\mathscr{U})), A)$; then $C^*_\pi(\mathscr{U}, A)$ is cochain equivalent to $C^*_\pi(X, A)$ since $C^{\mathrm{sing}}_*(p^{-1}(\mathscr{U}))$ is chain equivalent to $C_*(\tilde{X})$ by Theorem 3.4. Also, for any $q$, we may identify $C^q_\pi(\mathscr{U}, A)$ with $\check{C}^0(\mathscr{U}, C^q_\pi(\bullet, A))$ in the following way: Given any $f \in C^q_\pi(\mathscr{U}, A)$, $U \in \mathscr{U}$ and $\sigma \in C_q(U)$, let $[\iota_U f(U)](\sigma) := f(\sigma)$; this defines $\iota_U f \in C^q_\pi(U, A)$ for every $U \in \mathscr{U}$. Clearly $\iota_U f = 0$ for all $U \in \mathscr{U}$ only if $f = 0$, and for any collection $\{g_U \in C^q_\pi(U, A) : U \in \mathscr{U}\}$ we may define $f(\sigma) := g_U(U)(\sigma)$ for each $\sigma \in C_q(U)$ in order to have $\iota_U f = g_U$ for all $U \in \mathscr{U}$. Hence we have an inclusion $\iota : C^n_\pi(\mathscr{U}, A) \to \mathrm{Tot}^n(\mathscr{U})$ given by $f \to \{\iota_U f\}_{U \in \mathscr{U}}$. In fact this is a chain map: First, for any $\sigma \in C_*(U_0 \cap U_1)$, we have

$$\left[\check{\partial}\iota f(U_0, U_1)\right](\sigma) = [\iota f(U_0) - \iota f(U_1)](\sigma) = [\iota f(U_0)](\sigma) - [\iota f(U_1)](\sigma) = f(\sigma) - f(\sigma) = 0$$

Second, $[\partial \iota f(U)](\sigma) = (\partial f)(\sigma)$ by definition. Therefore, $\partial^{\mathrm{Tot}} \iota f = \check{\partial}\iota f + (-1)^0 \partial \iota f = 0 + \partial f = \iota \partial f$. This induces a map $\iota : H^*_\pi(X, A) \to H^*(\mathrm{Tot}(X))$, and it remains to see that this map is an isomorphism.

By Lemma 3.5, since $C^q_\pi(\bullet, A)$ is fine for each fixed $q$, it follows that $\check{H}^p(X, C^q_\pi(\bullet, A)) = 0$ for all $p > 0$. Therefore,

$${}^sE_1^{p,q} = \begin{cases} \check{H}^0(X, C^q_\pi(\bullet, A)), & p = 0 \\ 0, & p > 0 \end{cases}$$

So the sequence given by the singular filtration collapses, and the cohomology of the entire complex is isomorphic to that of the complex with terms $\check{H}^0(X, C^n_\pi(\bullet, A))$ and the coboundary $\partial$; this is $\check{H}^0(X, H^n_\pi(\bullet, A))$, and $\iota$ is clearly a natural isomorphism as a map $H^n_\pi(X, A) \to \check{H}^0(X, H^n_\pi(\bullet, A))$. □

LEMMA 3.16. *There is a natural isomorphism* $\check{H}(X, \mathcal{A}) \to H^*(\mathit{Tot}(X))$.

PROOF. Let $\mathscr{U} := \{U_x : x \in X\}$ where $U_x$ are chosen such that for each $x$, $p^{-1}(U_x)$ is a union of disjoint open sets in $\tilde{X}$ which are each mapped homeomorphically onto $U_x$ by $p$.

Call this a $p$-cover of $X$. If $\mathscr{V}$ is a refinement of $\mathscr{U}$ indexed by $X$, then for each $x \in X$, $V_x \subset U_x$ so that $p^{-1}(V_x)$ is also a disjoint union of open sets in $\tilde{X}$ and the homeomorphisms to $U_x$ also give homeomorphisms to $V_x$; that is, $\mathscr{V}$ is also a $p$-cover. Hence the collection of $p$-covers is cofinal in the collection of all open covers indexed by $X$. Next, if $\mathscr{U}$ is a $p$-cover, then since $X$ is locally contractible, each $U_x \in \mathscr{U}$ has some contractible subset $V_x$ such that $x \in V_x$; hence the collection $\mathscr{V} := \{V_x : x \in X\}$ of these subsets is a refinement of $\mathscr{U}$ all of whose elements are contractible. Then, since $X$ is paracompact, we may take a locally finite refinement $\mathscr{W}$ of $\mathscr{V}$, which is again indexed by $X$. If for any $x \in X$ some intersection $W_{y_1} \cap ... \cap W_{y_n}$ of elements of $\mathscr{W}$ contains $x$ and is not contractible, choose $x \in W'_x \subset W_{y_1} \cap ... \cap W_{y_n}$ so that $W'_x$ is contractible; if there is no such intersection, let $W'_x = W_x$. This can occur only finitely many times for each $x$ since $\mathscr{W}$ is locally finite. Hence we finally obtain a refinement $\mathscr{W}' = \{W'_x : x \in X\}$ of $\mathscr{U}$ which is indexed by $X$ such that every finite intersection $\bigcap\limits_{i=0}^{n} W'_{x_i}$ is either empty or contractible; we will call this a *good cover* of $X$. Since every $p$-cover has such a refinement, good covers form a cofinal family among the open covers of $X$, and the limits of both the system $\{\check{H}^*(\mathscr{U}, \mathcal{A})\}$ and $\{H(\mathrm{Tot}^*(\mathscr{U}))\}$ may be computed using only good covers.

Now, let $\mathscr{U}$ be a good cover of $X$, and define a map $\psi_{\mathscr{U}} : \check{C}^t(\mathscr{U}, \mathcal{A}) \to \check{C}^t(\mathscr{U}, C^0_\pi(\bullet, A))$ by the formula

$$[(\psi_{\mathscr{U}} f)(U_{x_0}, ..., U_{x_t})]\,(\sigma) := [f(U_{x_0}, ..., U_{x_t})]\,(p(\sigma(v_0)))$$

We claim this gives a cochain map $\check{C}^*(\mathscr{U}, \mathcal{A}) \to \mathrm{Tot}^*(\mathscr{U})$ by identifying $\check{C}^n(\mathscr{U}, C^0_\pi(\bullet, A))$ as a summand of $\mathrm{Tot}^n(\mathscr{U})$. First note that, if $U_{x_0} \cap ... \cap U_{x_{t+1}} \neq \emptyset$, $\sigma \in C_0(U_{x_0} \cap ... \cap U_{x_{t+1}})$, and $f \in \check{C}^t(\mathscr{U}, \mathcal{A})$, then:

$$\left[(\check{\partial}\psi_{\mathscr{U}} f)(U_{x_0}, ..., U_{x_{t+1}})\right](\sigma) = \sum_{i=0}^{t+1} (-1)^i \left[\psi_{\mathscr{U}} f(U_{x_0}, ..., \hat{U}_{x_i}, ..., U_{x_{t+1}})\right](\sigma)$$

$$= \sum_{i=0}^{t+1} (-1)^i \left[f(U_{x_0}, ..., \hat{U}_{x_i}, ..., U_{x_{t+1}})\right](p(\sigma(v_0))) = \left[(\partial f)(U_{x_0}, ..., U_{x_{t+1}})\right](p(\sigma(v_0)))$$

$$= \left[(\psi_{\mathscr{U}} \partial f)(U_{x_0}, ..., U_{x_{t+1}})\right](\sigma)$$

Next, for the same $f$ and for $U_{x_0} \cap ... \cap U_{x_t} \neq \emptyset$, any $\sigma \in C_1(p^{-1}(U_{x_0} \cap ... \cap U_{x_t}))$ defines a path from $\sigma(v_0)$ to $\sigma(v_1)$. Since $U_{x_0} \cap ... \cap U_{x_t}$ is contractible, $p^{-1}(U_{x_0} \cap ... \cap U_{x_t})$ is a disjoint union of sets each of which is homeomorphic to $U_{x_0} \cap ... \cap U_{x_t}$ by $p$, so in particular each is contractible. Thus $H_1(p^{-1}(U_{x_0} \cap ... \cap U_{x_t})) \cong 0$, which means $\sigma = d\eta$ for some $\eta \in C_2(p^{-1}(U_{x_0} \cap ... \cap U_{x_t}))$, and any cocycle $h \in C^1_\pi(U_{x_0} \cap ... \cap U_{x_t}, A)$ satisfies $h(\sigma) = h(d\eta) = \partial h'(\eta)$ for some $h' \in C^2_\pi(U_{x_0} \cap ... \cap U_{x_t}, A)$. Therefore, there exists some $h' \in C^2_\pi(U_{x_0} \cap ... \cap U_{x_t}, A)$ such that $[(\partial \psi_{\mathscr{U}} f)(U_{x_0}, ..., U_{x_t})](\sigma) = \partial\partial h'(\sigma) = 0$. Thus, $\partial^{\text{Tot}}\psi_{\mathscr{U}} = \check{\partial}\psi_{\mathscr{U}} = \psi_{\mathscr{U}}\partial f$, and $\psi_{\mathscr{U}}$ is a cochain map.

Furthermore, $\psi_{\mathscr{U}} f = 0$ only if $[f(U_{x_0}, ..., U_{x_n})](p(\sigma(v_0))) = 0$ for every collection $\{x_0, ..., x_n\}$ and every $\sigma \in C_0(p^{-1}(U_{x_0} \cap ... \cap U_{x_n}))$, that is, if $[f(U_{x_0}, ..., U_{x_n})](x) = 0$ for every $x \in p^{-1}(U_{x_0} \cap ... \cap U_{x_n})$. In this case, $f = 0$. On the other hand, for any element $h \in \check{C}^n(\mathscr{U}, C^0_\pi(\bullet, A))$ and collection $\{x_0, ..., x_n\}$, if $x \in U_{x_0} \cap ... \cap U_{x_n}$, let $\sigma_x$ be the element of $C_0(U_{x_0} \cap ... \cap U_{x_n})$ so that $\sigma_x(v_0) = x$ and let $f \in \check{C}(\mathscr{U}, \mathcal{A})$ be given by $[f(U_{x_0}, ..., U_{x_n})]\, p((x)) := [h(U_{x_0}, ..., U_{x_n})](\sigma_x)$; then clearly $\psi_{\mathscr{U}} f = h$. Therefore, $\psi_{\mathscr{U}}$ is an *isomorphism* identifying $\check{C}^n(\mathscr{U}, \mathcal{A})$ with the summand $\check{C}^n(\mathscr{U}, C^0_\pi(\bullet, A))$ of $\text{Tot}^n(\mathscr{U})$. This gives a further isomorphism between $\check{H}^n(\mathscr{U}, \mathcal{A})$ and $\check{H}^n(\mathscr{U}, C^0_\pi(\bullet, A))$, and since for each intersection $U_{x_0} \cap ... \cap U_{x_t}$, $p^{-1}(U_{x_0} \cap ... \cap U_{x_t})$ is either empty or a disjoint union of contractible sets, $H^q_\pi(U_{x_0} \cap ... \cap U_{x_t}, A) = \begin{cases} C^0_\pi(U_{x_0} \cap ... \cap U_{x_t}, A), & q = 0 \\ 0, & q > 0 \end{cases}$, so

$$\check{C}^t(\mathscr{U}, \mathcal{A}) \cong \check{C}^t(\mathscr{U}, C^0_\pi(\bullet, A)) = \check{C}^t(\mathscr{U}, H^0_\pi(\bullet, A)).$$

This makes $\check{H}^t(\mathscr{U}, \mathcal{A}) \cong \check{H}^t(\mathscr{U}, C^0_\pi(\bullet, A))$, and $\check{C}^t(\mathscr{U}, H^q_\pi(\bullet, A)) = 0$ whenever $q > 0$. Therefore,

$${}^{\check{C}}E_1^{t,q} = \begin{cases} \check{H}^t(\mathscr{U}, C^0_\pi(\bullet, A)), & q = 0 \\ 0, & q > 0 \end{cases}$$

So the sequence given by the Čech filtration collapses and the cohomology of the entire complex is isomorphic to the cohomology groups $\check{H}^*(\mathscr{U}, C^0_\pi(\bullet, A))$. Thus, $H^n(\mathrm{Tot}(\mathscr{U})) \cong \check{H}^n(\mathscr{U}, C^0_\pi(\bullet, A)) \cong \check{H}^n(\mathscr{U}, \mathcal{A})$ by natural isomorphisms, and carrying this isomorphism to the direct limit gives the claimed isomorphism. □

Combining these two lemmas gives $\check{H}(X, \mathcal{A}) \cong H^*(\mathrm{Tot}(X)) \cong H^*_\pi(X, A)$, completing the proof of Theorem 3.14.

# CHAPTER 4

# COHOMOLOGICAL CONTINUITY

As seen in Corollary 1.6, the cohomology of a profinite group may be computed using the cohomologies of the finite groups forming the inverse system whose limit it is. Alexander-Spanier and Čech cohomology both have a similar property regarding the cohomology of an inverse limit of spaces, provided those spaces satisfy certain properties. Here we will describe this property in detail and give proof that these cohomology theories possess it.

## 4.1 CONTINUITY OF ALEXANDER-SPANIER COHOMOLOGY

This section is adapted from parts of Spanier [11].

Let $H_o^*(\bullet)$ be some cohomology theory on topological spaces. For an inverse system $\{X_i, \varphi_{ij}\}_{i,j\in I}$ of topological spaces with the limit $X = \varprojlim X_i$, the projection maps $\varphi_{ij} : X_j \to X_i$ induce maps $\varphi_{ij}^* : H_o^*(X_i) \to H_o^*(X_j)$ which form a direct system of cohomology groups or modules $\{H_o^*(X_i), \varphi_{ij}^*\}_{i,j\in I}$. Furthermore, the natural projection maps $\varphi_i : X \to X_i$ induce maps $\varphi_i^* : H_o^*(X_i) \to H_o^*(X)$ which are compatible with this system, so the universal property of direct limits gives a map $\varphi^* : \varinjlim H_o^*(X_i) \to H_o^*(X)$ which commutes with the maps $\varphi_i^*$ and $\varphi_{ij}^*$ in the usual way. However, this map is not always an isomorphism for all cohomology theories. If $H_o^*(\bullet)$ is a cohomology theory for which the map $\varphi^*$ is an isomorphism whenever the spaces $X_i$ are all compact and Hausdorff, then $H_o^*(\bullet)$ has the continuity property. We wish to show that the Alexander-Spanier cohomology theory $\overline{H}^*(\bullet, A)$ has this property for all modules $A$.

If the system $\{X_i, \varphi_{ij}\}_{i,j\in I}$ consists of compact Hausdorff subspaces $X_i$ of some ambient space $Y$ and the maps $\varphi_{ij} : X_j \to X_i$ are inclusion maps $X_j \subset X_i$, then the system is called a *nested system*; in this situation $X \cong \bigcap_{i\in I} X_i$. A cohomology theory $H^*_o(\bullet)$ in which $\varphi^*$ is an isomorphism for nested systems has the weak continuity property.

LEMMA 4.1. *Let $\{X_i, \varphi_{ij}\}_{i\in I}$ be an inverse system of compact Hausdorf spaces and $X = \varprojlim X_i$. Then there exists a nested system $\{Y_i, p_{ij}\}_{i\in I}$ in an ambient space $Y$ along with homotopy equivalence maps $R_i : Y_i \to X_i$ which give an equivalence of the two inverse systems such that $\varprojlim Y_i = \varprojlim X_i = X$.*

PROOF. Any compact Hausdorff space can be embedded into a cube; let $e_i : X_i \hookrightarrow C_i$ be such embeddings for each $i$. Let $Y = \prod_{k\in I} C_k$ with natural projection maps $\pi_k : Y \to C_k$ and define $Y_k \subset Y$ to be the set of tuples $(c_i)$ with $c_k \in X_k$ and $c_i = \varphi_{ik}(c_k)$ whenever $i \preceq k$ (so the coordinates $c_i$ are arbitrary whenever $i \npreceq k$). Then for each $i$ let $P_i := \prod_{k\preceq i} C_k$ with $\pi_{i,k} : P_i \to C_k$ the natural projections, and let $Q_i := \prod_{k\npreceq i} C_k$.

Define the map $\phi_i : X_i \to P_i$ such that $\phi_i(x)$ is the tuple $(c_k) \in P_i$ with $\pi_{i,k}(c_k) = e_k\varphi_{ki}(x)$ for each $k \preceq i$. Since $\pi_{i,i}\phi_i = e_i\varphi_{ii} = e_i$, this map is injective; since $X_i$ is compact and $P_i$ is Hausdorff, this makes $\phi_i$ an embedding, giving a homeomorphism $\phi'_i : X_i \to \phi_i(X_i) \subset P_i$. The inverse of this homeomorphism is the restriction of the projection map $r_i = \pi_{i,i}|\phi_i(X_i)$, and by definition $Y_i = \phi_i(X_i) \times Q_i$. Since $Q_i$ is contractible (as a product of cubes), the map $R_i = r_iq_i : Y_i \to X_i$, where $q_i$ is the natural projection $Y_i \to \phi_i(X_i)$, is a homotopy equivalence.

Whenever $i \preceq j$, it follows that $Y_j \subset Y_i$ since any tuple $(c_k) \in Y_j$ satisfies $c_j \in X_j$ and $c_k = \varphi_{kj}(c_j)$ when $k \preceq j$, so $c_i = \varphi_{ij}(c_j)$. Also, if $k \preceq i \preceq j$ then $c_k = \varphi_{ki}\varphi_{ij}(c_j) = \varphi_{ki}(c_i)$, placing $(c_k) \in Y_i$ again. Thus, we may define $p_{ij} : Y_j \to Y_i$ to be this inclusion and obtain a nested system $\{Y_i, p_{ij}\}_{i\in I}$. If $i \preceq j$, then for any tuple $(c_k) \in Y_j$,

$$R_ip_{ij}((c_k)) = R_i((c_k)) = q_i((c_k)) = c_i = \varphi_{ij}(c_j) = \varphi_{ij}(q_j((c_k))) = \varphi_{ij}(R_j((c_k))).$$

Thus the maps $R_i$ are compatible and form a map between inverse systems. Furthermore, the limit of the nested system $\{Y_i, p_{ij}\}_{i\in I}$ is

$$\varprojlim Y_i = \bigcap_{i\in I} Y_i = \{(c_j) : c_j \in X_j \text{ for all } j, \varphi_{kj}(c_j) = c_k \text{ for all } k \preceq j\}$$

$$= \{(c_j) \in \prod_{j\in I} X_j : \varphi_{kj}(c_j) = c_k \text{ for all } k \preceq j\} = \varprojlim X_i$$

by the construction of the inverse limit of $\{X_i, \varphi_{ij}\}_{i\in I}$. Therefore, $\varprojlim Y_i = X$. □

COROLLARY 4.2. *If a cohomology theory $H_o^*(\bullet)$ has the weak continuity property, then it also has the continuity property.*

PROOF. Let $\{X_i, \varphi_{ij}\}_{i\in I}$ be an inverse system of compact Hausdorff spaces with $X = \varprojlim X_i$, and let $\{Y_i, p_{ij}\}_{i\in I}$ and $R_i : Y_i \to X_i$ be as in Lemma 4.1. Then the induced maps $R_i^* : H_o^*(X_i) \to H_o^*(Y_i)$ are isomorphisms compatible with the direct systems $\{H_o^*(X_i), \varphi_{ij}^*\}$ and $\{H_o^*(Y_i), p_{ij}^*\}$ such that $H_o^*(X) \cong \varinjlim H_o^*(Y_i)$ by the weak continuity of $H_o^*(\bullet)$ and $\varinjlim H_o^*(Y_i) \cong \varinjlim H_o^*(X_i)$ by taking the isomorphisms $R_i^*$ to the limit. □

Thus, it will suffice to show that $\overline{H}^*(\bullet, A)$ has the weak continuity property.

LEMMA 4.3. *Let $A$ be a subset of a topological space $X$ and $\mathscr{V}$ an open covering of $X$. Then there exists a neighborhood $N$ of $A$ and a function $f : N \to A$ (not necessarily continuous) such that*

**(a):** *For all $x \in A$, $f(x) = x$*

**(b):** *If $V \in \mathscr{V}$, then $f(V \cap N) \subset V^*$*

PROOF. If $A$ is empty, let $N = A$ and $f$ be the identity map. If not, let $N = \cup\{V \in \mathscr{V} : V \cap A \neq \emptyset\}$ and define $f : N \to A$ by $f(x) = x$ for all $x \in A$. If $x \notin A$ choose $f(x) \in A$ so that there is some $V \in \mathscr{V}$ with both $x$ and $f(x)$ in $V$. Such a choice of $f(x)$ is always possible by the definition of $N$. Clearly, if $x \in V \cap N$, there exists

$V' \in \mathscr{V}$ with both $x$ and $f(x)$ in $V'$. Therefore $x \in V \cap V'$ and $V' \subset V^*$, so $f(V \cap N) \subset V^*$. Therefore, both **(a)** and **(b)** are satisfied. $\square$

LEMMA 4.4. *For any closed subspace $Y$ of a paracompact Hausdorff space $X$, the natural homomorphism $i : \varinjlim \overline{H}^n(U, A) \to \overline{H}^n(Y, A)$ is an isomorphism.*

PROOF. Let $\phi \in \Phi^n(Y, A)$ be a cochain such that $\partial\phi$ is zero on $\mathscr{W}^{n+2}$, where $\mathscr{W}$ is an open covering of $A$. Let $\mathscr{U} = \{W \cup (X - Y) : W \in \mathscr{W}\}$ and note that $\mathscr{U}$ is an open cover of $X$ since $Y$ is closed in $X$. Let $\mathscr{V}$ be an open star refinement of $\mathscr{U}$ and let $N$, $f : N \to Y$ be the neighborhood of $Y$ and function for $\mathscr{V}$ as given by Lemma 4.3. Then $f^\#\phi \in \Phi^n(N, A)$, and we claim $\partial f^\#\phi = f^\#\partial\phi$ is zero on $\mathscr{V}^{n+2} \cap N^{n+2}$. By **(b)** in Lemma 4.3, for any $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ with $f(V \cap N) \subset U$. Then $f(V \cap N) \subset U \cap Y \subset W$ for some $W \in \mathscr{W}$, which means $\partial f^\#\phi$ is zero on $(V \cap N)^{n+2}$. Therefore, $f^\#\phi$ represents a cocycle of $\overline{C}^n(N)$ and, by **(a)** in Lemma 4.3, $(f^\#\phi)|Y = \phi$. Thus the cohomology class $[\phi] \in \overline{H}^n(Y, A)$ is the image under restriction of the cohomology class $[f^\#\phi] \in \overline{H}^n(N, A)$, which means $i : \varinjlim \overline{H}^n(N, A) \to \overline{H}^n(Y, A)$ is surjective.

To prove that $i$ is an injection, let $N'$ be a paracompact neighborhood of $Y$. Let $\mathscr{W}$ be an open cover of $N'$ and $\mathscr{W}'$ an open covering of $Y$. Let $\phi \in \Phi^n(N', A)$ be such that $\partial\phi$ is zero on $\mathscr{W}^{n+2}$ and $\phi|Y = \partial\phi'$. Then let $\mathscr{U} = \{W' \cup (N' - Y) : W' \in \mathscr{W}'\}$ and note that $\mathscr{U}$ is an open covering of $N'$ since $Y$ is closed. Let $\mathscr{V}$ be an open star refinement of both $\mathscr{W}$ and $\mathscr{U}$ (which covers $N'$). Let $N$, $f : N \to Y$ be a neighborhood of $Y$ contained in $N'$ and function for $\mathscr{V}$ as given by Lemma 4.3. If $V \in \mathscr{V}$, then $f(V \cap N) \subset W'$ for some $W' \in \mathscr{W}'$, so $f^\#(\phi|Y) = \partial f^\#\phi'$ on $V^{n+1} \cap N^{n+1}$.

To see that the class $[f^\#(\phi|Y)] \in \overline{H}^n(N, A)$ is equal to the class $[\phi|N]$, for $\psi \in C^k(N, A)$ define $D\psi \in C^{k-1}(N, A)$ by

$$(D\psi)(x_0, ..., x_{k-1}) = \sum_{j=0}^{k-1} (-1)^j \psi(x_0, ..., x_j, f(x_j), ..., f(x_{k-1}))$$

This makes $\partial D\psi + D\partial\psi = f^{\#}(\psi|Y) - \psi$. For every $V \in \mathscr{V}$, $(V \cap N) \cup f(V \cap N) \subset W$ for some $W \in \mathscr{W}$ by **(b)** in Lemma 4.3, and since $\partial\phi$ is zero on $\mathscr{W}^{n+2}$, it follows that $\partial D(\phi|N) = f^{\#}(\phi|Y) - \phi|N$ on $\mathscr{V}^{n+1} \cap N^{n+1}$. Therefore the cohomology class $[\phi] \in \overline{H}^n(N', A)$ maps to zero in $\overline{H}^n(N, A)$. This suffices to show that $i$ is injective, so it is an isomorphism. □

THEOREM 4.5. *The Alexander-Spanier cohomology theory is weakly continuous. That is, if $\{X_k, i_{kj}\}$ is a nested system of compact Hausdorff subspaces of an ambient space $Y$, then the maps $i_k$ induce an isomorphism $i^* : \varinjlim \overline{H}^n(X_k, A) \cong \overline{H}^n(X, A)$.*

PROOF. If $F$ is a closed subset of $X_j$ for some $j$, the collection $\{X_k \cap F\}_k$ consists of compact sets directed downward by inclusion, and $X \cap F = \cap(X_k \cap F)$. Therefore, if $X \cap F = \emptyset$, there exists some $k$ with $X_k \cap F = \emptyset$. So if $U$ is any neighborhood of $X$ in $X_j$, there exists some $k$ with $X_k \subset U$. Let $\varphi_k : \overline{H}^n(X_k, A) \to \varinjlim \overline{H}^n(X_j, A)$ be the natural injection given by the universal property of the direct limit.

To see that $i^*$ is surjective, let $u \in \overline{H}^n(X, A)$. By Lemma 4.4 for any $j$ there is a neighborhood $U$ of $X$ in $X_j$ and an element $v \in \overline{H}^n(U, A)$ so that $V|X = u$. Let $k$ be such that $X_k \subset U$ and $v_k := v|X_k$; then $v_k \in \overline{H}^n(X_k, A)$ and $i^*\varphi_k(v_k) = u$ .

To see that $i^*$ is injective, let $u \in \overline{H}^n(X_j, A)$ satisfy $i^*\varphi_j(u) = 0$. By Lemma 4.4 there is a neighborhood $U$ of $X$ in $X_j$ such that $u|U = 0$. Choose $k$ so that $X_k \subset U$; then $u|X_k = 0$, so $i^*$ is injective, and thus an isomorphism. □

COROLLARY 4.6. *The Alexander-Spanier cohomology theory has the continuity property.*

## 4.2 CONTINUITY OF ČECH COHOMOLOGY

While it is true that Čech cohomology satisfies the continuity property as described in the previous section, it actually satisfies an even *stronger* version of that property which can only be defined for a cohomology theory over sheaves of modules. Here we will establish this stronger property, but a few more definitions are needed first.

Let $\{X_i, \varphi_{ij}\}_{i\in I}$ be an inverse system of compact Hausdorff topological spaces, with $X := \varprojlim X_i$. We define a *system of sheaves* $\{\Gamma_i\}_{i\in I}$ on this inverse system to consist of a sheaf of abelian groups or modules $\Gamma_i$ on each $X_i$ along with injective presheaf maps $f_{ij} : \Gamma_i \to \varphi_{ij}\Gamma_j$ whenever $i \preceq j$ such that:

**a.** $f_{ii} = \mathrm{id}$ for all $i \in I$

**b.** $f_{jk}f_{ij} = f_{ik}$ whenever $i \preceq j \preceq k$

Note that whenever $i \preceq j \preceq k$ and $U \subset X_i$, we have

$$\varphi_{ij}\Gamma_j(U) = \Gamma_j(\varphi_{ij}^{-1}(U)) \overset{f_{jk}}{\to} \varphi_{jk}\Gamma_k(\varphi_{ij}^{-1}(U)) = \Gamma_k(\varphi_{ik}^{-1}(U)) = \varphi_{ik}\Gamma_k(U)$$

This defines a morphism $\varphi_{ij}\Gamma_j(U) \to \varphi_{ik}\Gamma_k(U)$ which will also be denoted $f_{jk}$; then for each $i$, the conditions on the presheaf maps above make the collection $\{\varphi_{ij}\Gamma_j : i \preceq j\}$ a direct system.

Let $\mathcal{B} = \{\varphi_i^{-1}(U) : i \in I \text{ and } U \subset X_i \text{ open}\}$ be the standard basis on $X$ as an inverse limit; then a *limiting partial presheaf* for the system of sheaves above is a presheaf $\Gamma$ on $\mathcal{B}$ together with a map of presheaves $f_i : \Gamma_i \to \varphi_i\Gamma$ for each $i \in I$ such that $f_j f_{ij} = f_i$ whenever $i \preceq j$, $\varphi_i\Gamma = \varinjlim_{i\preceq j} \varphi_{ij}\Gamma_j$, and $f_j : \varphi_{ij}(\Gamma_j) \to \varphi_i(\Gamma)$ is the canonical map from this limit. Such a presheaf exists and is unique up to isomorphisms of presheaves on $\mathcal{B}$. Similar to the above, whenever $i \preceq j$, $f_j$ gives a map $\varphi_{ij}\Gamma_j \to \varphi_i\Gamma$ of presheaves on $X_i$, which will also be denoted $f_j$.

Fix $i \in I$ and let $\mathscr{U}$ be an open cover of $X_i$. Whenever $i \preceq j$, $\varphi_{ij}^{-1}(\mathscr{U}) := \{\varphi_{ij}^{-1}(U) : U \in \mathscr{U}\}$ is an open cover of $X_j$, so we may define a map $\varphi_{ij}^{\#} : \check{C}^*(\mathscr{U}, \varphi_{ij}\Gamma_j) \to \check{C}^*(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j)$ by $(\varphi_{ij}^{\#}\theta)(\varphi_{ij}^{-1}(U_0), ..., \varphi_{ij}^{-1}(U_n)) := \theta(U_0, ..., U_n)$. This is well-defined since $\varphi_{ij}^{-1}(U_0 \cap ... \cap U_n) = \varphi_{ij}^{-1}(U_0) \cap ... \cap \varphi_{ij}^{-1}(U_n)$, so $\varphi_{ij}\Gamma_j(U_0 \cap ... \cap U_n) = \Gamma_j(\varphi_{ij}^{-1}(U_0) \cap ... \cap \varphi_{ij}^{-1}(U_n))$. Indeed, $\varphi_{ij}^{\#}$ is clearly an isomorphism of chain complexes. Then, if $i \preceq j \preceq k$ the map $f_{jk} : \varphi_{ij}\Gamma_j \to \varphi_{ik}\Gamma_k$ induces a cochain map

$f_{jk} : \check{C}^*(\mathscr{U}, \varphi_{ij}\Gamma_j) \to \check{C}^*(\mathscr{U}, \varphi_{ik}\Gamma_k)$. This determines a map $\psi_{jk} : \check{C}^*(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j) \to \check{C}^*(\varphi_{ik}^{-1}(\mathscr{U}), \Gamma_k)$ so that the following diagram commutes:

$$\begin{array}{ccc} \check{C}^*(\mathscr{U}, \varphi_{ij}\Gamma_j) & \xrightarrow{f_{jk}} & \check{C}^*(\mathscr{U}, \varphi_{ik}\Gamma_k) \\ \varphi_{ij}^{\#}\downarrow & & \downarrow\varphi_{ik}^{\#} \\ \check{C}^*(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j) & \xrightarrow[\psi_{jk}]{} & \check{C}^*(\varphi_{ik}^{-1}(\mathscr{U}), \Gamma_k) \end{array}$$

These cochain maps form isomorphic direct systems $\{\check{C}^*(\mathscr{U}, \varphi_{ij}\Gamma_j), f_{jk}\}_{i\preceq j}$ and $\{\check{C}^*(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j), \psi_{jk}\}_{i\preceq j}$. For similar definitions of $\varphi_i^{\#}$ and $\psi_j$, we obtain cochain maps forming another commutative diagram for $\Gamma$ and $X$ whenever $i \preceq j$:

$$\begin{array}{ccc} \check{C}^*(\mathscr{U}, \varphi_{ij}\Gamma_j) & \xrightarrow{f_{j}} & \check{C}^*(\mathscr{U}, \varphi_{i}\Gamma) \\ \varphi_{ij}^{\#}\downarrow & & \downarrow\varphi_{i}^{\#} \\ \check{C}^*(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j) & \xrightarrow[\psi_{j}]{} & \check{C}^*(\varphi_{i}^{-1}(\mathscr{U}), \Gamma) \end{array}$$

Here $\varphi_i^{\#}$ is again an isomorphism, as before. Moreover, for $i \preceq j$ the maps $f_j$ are compatible with the first direct system above, so they determine a cochain map $f : \varinjlim_{i\preceq j} \check{C}^n(\mathscr{U}, \varphi_{ij}\Gamma_j) \to \check{C}^n(\mathscr{U}, \varphi_i(\Gamma))$. Similarly, the maps $\psi_j$ are compatible with the second direct system and give a cochain map $\psi : \varinjlim_{i\preceq j} \check{C}^n(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j) \to \check{C}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma)$ such that $f$ is an isomorphism if and only if $\psi$ is.

LEMMA 4.7. *For each $i \in I$ and finite open cover $\mathscr{U}$ of $X_i$, the maps $f$ and $\psi$ are isomorphisms of chain complexes.*

PROOF. It suffices to show that $f$ is an isomorphism. Suppose that $f([\theta]) = 0$ where $[\theta]$ is represented by some $\theta \in \check{C}^n(\mathscr{U}, \varphi_{ij}\Gamma_j)$. Then $f_j(\theta) = 0$, so for every $(n+1)$-tuple $U := (U_0, ..., U_n)$ of elements of $\mathscr{U}$, $(f_j\theta)(U) = 0$ in $f_j(\varphi_{ij}\Gamma_j(U_0 \cap ... \cap U_n)) \subset \varphi_i\Gamma(U_0 \cap ... \cap U_n)$. But $\varphi_i\Gamma = \varinjlim_{i\preceq j} \varphi_{ij}\Gamma_j$, so for each such $U \in \mathscr{U}^{n+1}$ there exists some $k_U \succeq j$ with $(f_{jk_U}\theta)(U) = 0$ in $\varphi_{ik_U}\Gamma_{k_U}(U_0 \cap ... \cap U_n)$. Since $\mathscr{U}$

is finite, so is $\mathscr{U}^{n+1}$, so there exists $k \in I$ with $k \succeq k_U$ for *every* $U \in \mathscr{U}^{n+1}$. Then $(f_{jk}\theta)(U) = (f_{k_U k} f_{jk_U}\theta)(U) = 0$ for *all* $U \in \mathscr{U}^{n+1}$, so that $f_{jk}(\theta) = 0$ in $\check{C}(\mathscr{U}, \varphi_{ik}\Gamma_k)$. Thus $[\theta] = 0$ in $\varinjlim_{i \preceq j} \check{C}^n(\mathscr{U}, \varphi_{ij}\Gamma_j)$, and $\ker f = 0$.

Next, let $\zeta \in \check{C}^n(\mathscr{U}, \varphi_i\Gamma)$. For each $U \in \mathscr{U}^{n+1}$, $\zeta(U) \in \varphi_i\Gamma(U_0 \cap ... \cap U_n)$ and $\varphi_i\Gamma = \varinjlim_{i \preceq j} \varphi_{ij}\Gamma_j$. Thus there exists $j_U \succeq i$ and $\gamma_U \in \varphi_{ij_U}\Gamma_{j_U}(U_0 \cap ... \cap U_n)$ with $f_{j_U}(\gamma_U) = \zeta(U)$. Again, since $\mathscr{U}^{n+1}$ is finite, we may choose $k \in I$ with $k \succeq j_U$ for all $U \in \mathscr{U}^{n+1}$. Then let $\theta \in \check{C}(\mathscr{U}, \varphi_{ik}\Gamma_k)$ be defined by $\theta(U) := f_{j_U k}(\gamma_U) \in \varphi_{ik}\Gamma_k(U_0 \cap ... \cap U_n)$, which makes $(f_k\theta)(U) = f_k f_{j_U k}(\gamma_U) = f_{j_U}(\gamma_U) = \zeta(U)$ for every $U \in \mathscr{U}^{n+1}$. By definition, then, $f_k(\theta) = \zeta$, so $\zeta$ is in the image of $f$. Hence $f$ is surjective, and therefore an isomorphism. □

The cochain maps $\psi_{jk}$ induce maps on cohomology $\psi_{jk} : \check{H}^n(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j) \to \check{H}(\varphi_{ik}^{-1}(\mathscr{U}), \Gamma_k)$ which form a direct system of abelian groups $\{\check{H}^*(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j), \psi_{jk}\}_{i \preceq j}$. Similarly, the cochain maps $\psi_j$ induce a compatible family of morphisms $\psi_j : \check{H}^n(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j) \to \check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma)$ which determine a map $\psi : \varinjlim_{i \preceq j} \check{H}^n(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j) \to \check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma)$. From Lemma 4.7 we immediately obtain:

Corollary 4.8. *The map $\psi$ on cohomology is an isomorphism for all $n \in \mathbb{N}$.*

For each $j \succeq i$ and open cover $\mathscr{U}$ of $X_i$, define $h_{ij} : \check{H}^n(X_i, \Gamma_i) \to \check{H}^n(X_j, \Gamma_j)$ to be the direct limit of the maps $\psi_{ij} = \varphi_{ij}^* f_{ij} : \check{H}(\mathscr{U}, \Gamma_i) \to \check{H}^n(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j)$ over the collection of all open covers $\mathscr{U}$ of $X_i$. These maps form a direct system of abelian groups $\{\check{H}^n(X_i, \Gamma_i), h_{ij}\}_{i \in I}$. Furthermore, by taking the direct limit of the morphisms $\psi_i = \varphi_i^* f_i : \check{H}^n(\mathscr{U}, \Gamma_i) \to \check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma)$ over the same collection we obtain a compatible family of morphisms $h_i : \check{H}^n(X_i, \Gamma_i) \to \check{H}(X, \Gamma)$ for each $i \in I$. This family of maps determines a homomorphism $h : \varinjlim \check{H}^n(X_i, \Gamma_i) \to \check{H}^n(X, \Gamma)$.

Theorem 4.9. *If $\{X_i, \varphi_{ij}\}_{i \in I}$ is a surjective inverse system of compact Hausdorff spaces, then $h$ is an isomorphism for all $n$.*

Proof. To see that $h$ is injective, let $\theta \in \check{H}^n(X_i, \Gamma_i)$ satisfy $h_i(\theta) = 0$. Since $X_i$ is compact, the collection of finite open covers is cofinal for the system determining $\check{H}^n(X_i, \Gamma_i)$, so $\theta$ is represented by a cohomology class $[\theta_i] \in \check{H}^n(\mathscr{U}, \Gamma_i)$ for some finite open cover $\mathscr{U}$ of $X_i$. Therefore, $\psi_i([\theta_i]) \in \check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma)$ is taken to 0 in the limit $\check{H}^n(X, \Gamma)$, so there exists some $j \in I$ and finite open cover $\mathscr{V}$ of $X_j$ such that $\varphi_j^{-1}(\mathscr{V})$ is a refinement of $\varphi_i^{-1}(\mathscr{U})$ and the restriction map $\check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma) \to \check{H}^n(\varphi_j^{-1}(\mathscr{V}), \Gamma)$ takes $\psi_i([\theta_i])$ to 0.

Now, choose some $k \succeq i, j$. Since $\varphi_{ik}^{-1}(\mathscr{U})$ and $\varphi_{jk}^{-1}(\mathscr{V})$ are open covers of $X_k$ and $X_k$ is compact, there exists a finite open cover $\mathscr{W}$ of $X_k$ which is a common refinement of both $\varphi_{ik}^{-1}(\mathscr{U})$ and $\varphi_{jk}^{-1}(\mathscr{V})$. Thus $\varphi_k^{-1}(\mathscr{W})$ is a refinement of $\varphi_k^{-1}(\varphi_{jk}^{-1}(\mathscr{V})) = \varphi_k^{-1}(\mathscr{V})$, hence a refinement of $\varphi_i^{-1}(\mathscr{U})$ also, giving the following commutative diagram, where $r$ is the restriction map:

$$
\begin{array}{ccc}
 & \check{H}^n(\mathscr{U}, \Gamma_i) & \\
 {\scriptstyle \psi_{ik}} \swarrow & & \searrow {\scriptstyle \psi_i} \\
\check{H}^n(\varphi_{ik}^{-1}(\mathscr{U}), \Gamma_k) & \xrightarrow{\psi_k} & \check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma) \\
\downarrow {\scriptstyle r} & & \downarrow {\scriptstyle r} \\
\check{H}^n(\mathscr{W}, \Gamma_k) & \xrightarrow{\psi_k} & \check{H}^n(\varphi_k^{-1}(\mathscr{W}), \Gamma)
\end{array}
$$

In this diagram, the restriction $\check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma) \to \check{H}(\varphi_k^{-1}(\mathscr{W}), \Gamma)$ maps $\psi_i([\theta_i])$ to 0. Let $[\theta_k]$ be the image under the restriction map $\check{H}^n(\varphi_{ik}^{-1}(\mathscr{U}), \Gamma_k) \to \check{H}(\mathscr{W}, \Gamma_k)$ of $\psi_{ik}([\theta_i])$. Then $\psi_k([\theta_k]) = 0$ in $\check{H}^n(\varphi_k^{-1}(\mathscr{W}), \Gamma)$. However, by Corollary 4.8, $\check{H}^n(\varphi_k^{-1}(\mathscr{W}), \Gamma) \cong \varinjlim_{k \preceq \ell} \check{H}^n(\varphi_{k\ell}^{-1}(\mathscr{W}), \Gamma_\ell)$. Thus, for some $\ell \succeq k$, $\psi_{k\ell}([\theta_k]) = 0$ in $\check{H}(\varphi_\ell^{-1}(\mathscr{W}), \Gamma_\ell)$, so $h_{i\ell}(\theta) = 0$ in $\check{H}^n(X_\ell, \Gamma_\ell)$. Therefore, $\ker(h) = 0$ and $h$ is injective.

To see that $h$ is surjective, note that each element $\zeta$ of $\check{H}^n(X, \Gamma)$ is represented by a cohomology class $[\zeta_i] \in \check{H}^n(\varphi_i^{-1}(\mathscr{U}), \Gamma)$ for some $i \in I$ and some finite open cover $\mathscr{U}$ of $X_i$. By Corollary 4.8, there exists some $j \succeq i$ and $[\theta_i] \in \check{H}(\varphi_{ij}^{-1}(\mathscr{U}), \Gamma_j)$ such that $\psi_j([\theta_i]) = [\zeta_i]$. Then the equivalence class $\theta \in \check{H}^n(X_j, \Gamma_j)$ represented by $[\theta_i]$ has $h_j(\theta) = [\psi_j([\theta_i])] = \zeta$. Therefore, $h$ is surjective, and so an isomorphism. □

Define the *limiting sheaf* of the system of sheaves to be the sheaf $\hat{\Gamma}$ on $X$ which is the completion of $\Gamma$. By composing $h$ with the isomorphism $\check{H}^*(X, \Gamma) \cong \check{H}^*(X, \hat{\Gamma})$ described in the discussion following Corollary 1.15, we obtain the following:

Corollary 4.10. *For a surjective inverse system of spaces with a system of sheaves* $\Gamma_i$, *if* $\hat{\Gamma}$ *is the limiting sheaf, then* $\varinjlim \check{H}^n(X_i, \Gamma_i) \cong \check{H}^n(X, \hat{\Gamma})$.

# CHAPTER 5

# A CLASSIFYING SEQUENCE OF SPACES

We are now prepared to prove the main results of this thesis.

Let $G = \varprojlim G_i$ be a profinite group, specifically the inverse limit of a surjective system of finite discrete groups $G_i$ indexed by a directed partially ordered set $I$.

THEOREM 5.1. *There exists a sequence of topological spaces $\{BG^n\}_{n\geq 0}$ such that, for any discrete $G$-module $A$ on which $G$ acts trivially and for all $k < n$,*
*$\overline{H}^k(BG^n, A) \cong H^k(G, A)$.*

PROOF. Let $NG_i$ be the nerve of $G_i$, as defined in Section 2.4. Then let $NG_i^{\leq n}$ be the $n$-truncation of $NG_i$, and note that since $G_i$ is finite, $NG_i[k]$ is also finite for every $k$, so that each $NG_i^{\leq n}$ is in $\mathbf{sFSet}_n$. The inverse system $\{G_i, \varphi_{ij}\}$ of which $G$ is a limit induces surjective maps $\varphi_{ij} : NG_j \to NG_i$ which clearly restrict to maps $\varphi_{ij} : NG_j^{\leq n} \to NG_i^{\leq n}$, and in both cases satisfy the necessary conditions to form a surjective inverse system, so in particular we have the inverse system $\{NG_i^{\leq n}, \varphi_{ij}\}$ in $\mathbf{sFSet}_n$. For each pair $i$, $n$, the geometric realization $|NG_i^{\leq n}|$ is the $n$-skeleton $BG_{i,n}$ of the classifying space $BG_i = |NG_i|$; in particular it is a finite CW-complex. From Section 2.4, $H^k(BG_i, A) \cong H^k(G_i, A)$ where $G_i$ is given the trivial action on $A$, and if $k < n$ then
$H^k(|NG_i^{\leq n}|, A) = H^k(BG_{i,n}, A) = H^k(BG_i, A)$; therefore, $H^k(|NG_i^{\leq n}|, A) \cong H^k(G_i, A)$ in this case. Furthermore, since $|NG_i^{\leq n}|$ is a finite CW-complex, it is compact (hence paracompact), Hausdorff, and homologically locally connected, so the map
$\mu : \overline{H}^k(|NG_i^{\leq n}|, A) \to H^k(|NG_i^{\leq n}|, A)$ is an isomorphism by Corollary 3.12.

Define $NG^{\leq n} := \varprojlim NG_i^{\leq n}$. Then $|NG^{\leq n}| = \varprojlim |NG_i^{\leq n}|$ by Theroem 2.1, and we may without ambiguity define $BG^n := |NG^{\leq n}|$. Then, since each $|NG_i^{\leq n}|$ is compact and Hausdorff, Corollary 4.6 gives the isomorphism $\overline{H}^k(BG^n, A) \cong \varinjlim \overline{H}^k(|NG_i^{\leq n}|, A)$, and furthermore $\varinjlim \overline{H}^k(|NG_i^{\leq n}|, A) \cong \varinjlim H^k(|NG_i^{\leq n}|, A)$ by the isomorphisms $\overline{H}^k(|NG_i^{\leq n}|, A) \cong H^k(|NG_i^{\leq n}|, A)$ which commute with the maps of each direct system. When $k < n$, this last term is isomorphic to $\varinjlim H^k(G_i, A)$. Now, let $A$ be the limit of the constant direct system given by $A_i = A$, $f_{ij} = \mathrm{id}_A$. Then clearly $\varphi_{ij}$ and $f_{ij}$ are compatible in the sense needed for Lemma 1.4, so by Corollary 1.5 $\varinjlim H^k(G_i, A)$ is isomorphic to $H^k(G, A)$. □

Now, if $A$ is *any* discrete $G$-module, then since $G = \varprojlim G_i$, $\varphi_i : G \to G_i$ are surjective, and each $G_i$ is finite, the subgroups $\ker(\varphi_i)$ form a cofinal family among the open normal subgroups $K$ of $G$ which give $G$ as an inverse limit $\varprojlim G/K$. Denote $A_i := A^{\ker(\varphi_i)}$, so that each $A_i$ is a $G_i$-module and $A = \varinjlim A_i$, and so that the injective maps $f_{ij} : A_i \to A_j$ are compatible with the maps $\varphi_{ij} : G_j \to G_i$. By Corollary 1.6, then, $H^k(G, A) \cong \varinjlim H^k(G_i, A_i)$.

Next $|NG_i^{\leq n}|$ is a finite CW complex for each $i \in I$, so it is compact, Hausdorff, locally contractible, and path-connected. Hence if $n > 1$, it follows that $\pi_1(|NG_i^{\leq n}|) \cong G_i$ and $H^k_{G_i}(|NG_i^{\leq n}|, A_i) \cong H^k_{G_i}(|NG_i|, A_i) \cong H^k(G_i, A_i)$ for all $k < n$ by Theorem 3.13. On the other hand, the action of $G_i$ on $A_i$ determines a sheaf $\mathcal{A}_i^n$ on $|NG_i^{\leq n}|$ such that, by Theorem 3.14, $\check{H}^k(|NG_i^{\leq n}|, \mathcal{A}_i^n) \cong H^k_{G_i}(|NG_i^{\leq n}|, A_i)$.

Lemma 5.2. *For a fixed $n$, the sheaves $\{\mathcal{A}_i^n\}_{i\in I}$ form a system of sheaves on the inverse system of spaces $\{|NG_i^{\leq n}|\}_{i\in I}$.*

Proof. First note that, since each $|NG_i^{\leq n}|$ has the universal cover $EG_i^{\leq n} = |\mathcal{E}G_i^{\leq n}|$, the maps $\varphi_{ij}$ further determine maps $\varphi_{ij} : EG_j^{\leq n} \to EG_i^{\leq n}$ such that $\varphi_{ij} p_j = p_i \varphi_{ij}$ where $p_i : EG_i^{\leq n} \to |NG_i^{\leq n}|$ are the universal covering maps. Let $\overline{A}_i^n$ be the constant presheaf associated with $A_i$ on the space $EG_i^{\leq n}$ and let $\hat{A}_i^n$ be its completion, the constant sheaf.

Then for any open subset $U$ of $EG_i^{\leq n}$, the map $f_{ij} : A_i \to A_j$ determines an injective map $f_{ij} : \overline{A}_i^n(U) \to \varphi_{ij}\overline{A}_j^n(U)$ as follows: If $U$ is empty, this is just the 0 map; otherwise the source is $A_i$ and the target is $A_j$, so the map is precisely $f_{ij} : A_i \to A_j$. Since the restriction maps to nonempty subsets are just the identity, these maps clearly commute with restriction maps, as needed. This in turn induces an injective presheaf map $\hat{A}_i^n \to \widehat{\varphi_{ij}A_j^n}$ between the completions of the sheaves (on $EG_i^{\leq n}$); compose this with the natural map $\widehat{\varphi_{ij}A_j^n} \to \varphi_{ij}\hat{A}_j^n$ to obtain an injective presheaf map $\hat{f}_{ij} : \hat{A}_i^n \to \varphi_{ij}\hat{A}_j^n$.

Next, by definition, if $U \subset |NG_i^{\leq n}|$, then

$$\mathcal{A}_i^n(U) = \{\gamma \in \hat{A}_i^n(p_i^{-1}(U)) : \text{for every } x_i \in p_i^{-1}(U) \text{ and } g_i \in G_i, \gamma(g_i \cdot x_i) = g_i \cdot \gamma(x_i)\}$$

and, similarly,

$$\mathcal{A}_j^n(\varphi_{ij}^{-1}(U)) = \{\gamma \in \hat{A}_j^n(p_j^{-1}\varphi_{ij}^{-1}(U)) : \text{for every } x_j \in p_j^{-1}\varphi_{ij}^{-1}(U)$$

$$\text{and } g_j \in G_j, \gamma(g_j \cdot x_j) = g_j \cdot \gamma(x_j)\}$$

Since $\varphi_{ij}p_j = p_i\varphi_{ij}$, $p_j^{-1}\varphi_{ij}^{-1}(U) = \varphi_{ij}^{-1}p_i^{-1}(U)$, so $\hat{f}_{ij}$ descends to an injective map $f_{ij} : \mathcal{A}_i^n(U) \to \varphi_{ij}\hat{A}_j^n(p_i^{-1}(U)) = \hat{A}_j^n(\varphi_{ij}^{-1}p_i^{-1}(U)) = \hat{A}_j^n(p_j^{-1}\varphi_{ij}^{-1}(U))$; we claim that the image of this map is contained in $\varphi_{ij}\mathcal{A}_j^n(U)$. Let $x_i$ be any element of $|NG_i^{\leq n}|$ and $U_i$ an open neighborhood of $x_i$ such that $p_i^{-1}(U_i)$ is a union of disjoint open sets in $EG_i^{\leq n}$ which are each mapped homeomorphically onto $U_i$ by $p_i$. Then, similarly, every element $x_j \in \varphi_{ij}^{-1}(x_i)$ (which is nonempty since $\varphi_{ij}$ is surjective) has $\varphi_{ij}^{-1}(U_i)$ as an open neighborhood with $p_j^{-1}\varphi_{ij}^{-1}(U_i)$ a union of disjoint open sets in $EG_j^{\leq n}$ each of which is mapped homeomorphically onto $\varphi_{ij}^{-1}(U_i)$ by $p_j$. Hence, $\mathcal{A}_i^n$ restricts to a constant sheaf on $U_i$, and $f_{ij}(\mathcal{A}_i^n|U_i)$ is a constant sheaf on $\varphi_{ij}^{-1}(U_i)$, but also $f_{ij}(\mathcal{A}_i^n(U_i)) \leq \hat{A}_j^n(p_j^{-1}\varphi_{ij}^{-1}(U_i)) = \mathcal{A}_j^n(\varphi_{ij}^{-1}(U_i)) = \varphi_{ij}\mathcal{A}_j^n(U_i)$ since $\mathcal{A}_j^n$ restricts to a constant sheaf on $\varphi_{ij}^{-1}(U_i)$.

Next, let $U$ be *any* open subset of $|NG_i^{\leq n}|$. From the proof of Lemma 3.16, for any open cover $\mathscr{U}$ of $U$ there exists a $p$-cover refinement $\mathscr{V}$ of $\mathscr{U}$, and by the above discussion each $V \in \mathscr{V}$ satisfies $f_{ij}(\mathcal{A}_i^n(V)) \leq \mathcal{A}_j^n(\varphi_{ij}^{-1}(V))$. Since $\hat{A}_j^n$ is a sheaf, each element $f_{ij}(\gamma) \in f_{ij}(\mathcal{A}_i^n(U))$ may be identified with a compatible $\mathscr{V}$-family $\{\eta_V\}$ where each $\eta_V \in f_{ij}(\mathcal{A}_i^n(V)) \leq \mathcal{A}_j^n(\varphi_{ij}^{-1}(V))$. Since $\mathcal{A}_j^n$ is *also* a sheaf, this family uniquely determines an element $\eta \in \mathcal{A}_j^n(\varphi_{ij}^{-1}(V))$ with $f_{ij}(\gamma) = \eta$. Thus $f_{ij}(\mathcal{A}_i^n(U)) \leq \mathcal{A}_j^n(\varphi_{ij}^{-1}(U))$ also, as needed.

Finally, the conditions that $f_{ii}$ is the identity and $f_{jk}f_{ij} = f_{ik}$ whenever $i \preceq j \preceq k$ follow directly from the same conditions holding on the system $\{A_i\}_{i\in I}$ of modules. □

Let $\mathcal{A}^n$ be the limiting sheaf of this system on $BG^n$. Then we have:

THEOREM 5.3. *Let $\{BG^n\}$ be the spaces from Theorem 5.1. Then, for all $k < n$,*

$$\check{H}^k(BG^n, \mathcal{A}^n) \cong \varinjlim \check{H}^k(|NG_i^{\leq n}|, \mathcal{A}_i^n) \cong \varinjlim H^k(G_i, A_i) \cong H^k(G, A)$$

PROOF. Since by Lemma 5.2 the sheaves $\{\mathcal{A}_i^n\}_{i\in I}$ form a system of sheaves on the inverse system of spaces $\{|NG_i^{\leq n}|\}_{i\in I}$ and $\mathcal{A}^n$ is that system's limiting sheaf, Corollary 4.10 gives an isomorphism $\check{H}^k(BG^n, \mathcal{A}^n) \cong \varinjlim \check{H}^k(|NG_i^{\leq n}|, \mathcal{A}_i^n)$. Next, by Theorem 3.13 each $\check{H}^k(|NG_i^{\leq n}|, \mathcal{A}_i^n)$ is naturally isomorphic to $H^k(G_i, A_i)$; hence these isomorphisms commute with both direct systems, so they give an isomorphism $\varinjlim \check{H}^k(|NG_i^{\leq n}|, \mathcal{A}_i^n) \cong \varinjlim H^k(G_i, A_i)$ between their limits. Finally, Corollary 1.5 provides the isomorphism $\varinjlim H^k(G_i, A_i) \cong H^k(G, A)$. □

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