NONCOMMUTATIVE SPACES FROM MATRIX MODELS

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Abstract

Noncommutative (NC) spaces commonly arise as solutions to matrix model equations of motion. They are natural generalizations of the ordinary commutative spacetime. Such spaces may provide insights into physics close to the Planck scale, where quantum gravity becomes relevant. Although there has been much research in the literature, aspects of these NC spaces need further investigation. In this dissertation, we focus on properties of NC spaces in several different contexts. In particular, we study exact NC spaces which result from solutions to matrix model equations of motion. These spaces are associated with finite-dimensional Lie-algebras. More specifically, they are two-dimensional fuzzy spaces that arise from a three-dimensional Yang-Mills type matrix model, four-dimensional tensor-product fuzzy spaces from a tensorial matrix model, and Snyder algebra from a five-dimensional tensorial matrix model.

In the first part of this dissertation, we study two-dimensional NC solutions to matrix equations of motion of extended IKKT-type matrix models in three-space-time dimensions. Perturbations around the NC solutions lead to NC field theories living on a two-dimensional space-time. The commutative limit of the solutions are smooth manifolds which can be associated with closed, open and static two-dimensional cosmologies. One particular solution is a Lorentzian fuzzy sphere, which leads to essentially a fuzzy sphere in the Minkowski space-time. In the commutative limit, this solution leads to an induced metric that does not have a fixed signature, and have a non-constant negative scalar curvature, along with singularities at two fixed latitudes. The singularities are absent in the

matrix solution which provides a toy model for resolving the singularities of General relativity. We also discussed the two-dimensional fuzzy de Sitter space-time, which has irreducible representations of $su(1,1)$ Lie-algebra in terms of principal, complementary and discrete series. Field theories on such backgrounds result from perturbations about the solutions. The perturbative analysis requires non-standard Seiberg-Witten maps which depend on the embeddings in the ambient space and the symplectic 2-form. We find interesting properties of the field theories in the commutative limit. For example, stability of the action may require adding symmetry breaking terms to the matrix action, along with a selected range for the matrix coefficients.

In the second part of this dissertation, we study higher dimensional fuzzy spaces in a tensorial matrix model, which is a natural generalization to the three-dimensional actions and is valid in any number of space-time dimensions. Four-dimensional tensor product NC spaces can be constructed from two-dimensional NC spaces and may provide a setting for doing four-dimensional NC cosmology. Another solution to the tensorial matrix model equations of motion is the Snyder algebra. A crucial step in exploring NC physics is to understand the structure of the quantized space-time in terms of the group representations of the NC algebra. We therefore study the representation theory of the Snyder algebra and implementation of symmetry transformations on the resulted discrete lattices. We find the three-dimensional Snyder space to be associated with two distinct Hilbert spaces, which define two reducible representations of the $su(2) \times su(2)$ algebra. This implies the existence of two distinct lattice structures of Snyder space. The difference between the two representations is evident in the spectra of the position operators, which could only be integers in one case and half integers in the other case. We also show that despite the discrete nature of the Snyder space, continuous translations and rotations can be unitarily implemented on the lattices.

List of Abbreviations and Symbols

NC	Noncommutative
NCG	Noncommutative geometry
UV	ultraviolet
YM	Yang-Mills
CS	Chern-Simons
IKKT	Ishibashi, Kawai, Kitazawa and Tsuchiya
BFSS	Banks, Fischler, Shenker and Susskind
S^2	2-sphere
S^3	3-sphere
S^2_F	fuzzy 2-sphere
dS^2	two-dimensional de Sitter
dS^2_F	fuzzy two-dimensional de Sitter
AdS^2_F	fuzzy two-dimensional anti-de Sitter
SYM	Supersymmetric Yang-Mills

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1 Introduction

Noncommutative (NC) physics has been an active area of research in high energy physics. The idea dates back to Schrodinger, who was hoping to tame the ultraviolet (UV) divergence in quantum field theory via a short distance cutoff. This idea was passed on to Pauli, then to Openheimer, and eventually worked out by Snyder, who produced an algebra known as the Snyder algebra in 1947[1]. His algebra introduced a short distance cutoff scale in a Lorentz invariant way. It is a noncommuting algebra generated from space-time coordinate operators as well as angular momentum operators. An immediate consequence of this algebra was a discrete spectra for space-time, which completely diverged from the long existing notion of space-time continuum. This plan for solving the UV divergence problem was abandoned after the success of the renormalization program in quantum field theory. Nevertheless, the axiomatic formulation of noncommutative geometry (NCG) was developed by Connes in the 1980s [2], which laid the foundation for developments which followed. It was not until late 1990s when physicists started to reconsider the idea of NCG seriously. This was in large part due to the progress made in string theory, when Seiberg and Witten first pointed out that space-time noncommutativity emerges naturally from the interactions of open strings [3].

Motivations for studying NC physics appear independent of string theory. For example, NC physics follows naturally from simple, qualitative arguments. From quantum mechanics, we know that in order to resolve smaller and smaller distances one needs to increase the momentum of a test particle probe. On the other hand, general relativity tells us that as the momentum gets larger and larger, the curvature generated by the test particle increases, and when the resolving distance approaches the Planck scale the associated particle momentum will result in a black hole with a horizon radius at the Planck length, defined by $\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35} m$. Here $\hbar$ is the Planck constant, G is Newton's constant, and c is the speed of light. Since no information can be obtained from inside the horizon, there is a lower bound on resolving distances, the bound being the Planck length. More explicitly, Doplicher et. al.[4] derived uncertainty relations for space and time measurements of the form:

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}\theta^{\mu\nu}, \tag{1.1}$$

which result from a commutation relation of the form:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{1.2}$$

where $\hat{x}^\mu$ are operators associated with space-time coordinates x^μ. In the above, $\theta^{\mu\nu}$ is an antisymmetric operator with units of length-squared that is either proportional to the identify (referred to as canonical case) or a function of $\hat{x}^\mu$. According to [4], the magnitude of each component of $\theta^{\mu\nu}$ is around ℓ_p^2. Equation (1.2) defines a NC algebra for the space-time coordinate operators.

The above heuristic argument hints at a connection between the quantum nature of space-time and gravity. In recent decades, NC physics has become an active area of research. Its implications, however, have not been fully explored. In this dissertation, I will mainly focus on studying two aspects of NC physics, one aspect relates to the two-dimensional NC spaces which arise from three-dimensional IKKT type matrix models, and the other relates to the four-dimensional tensor-product NC spaces and Snyder space which can arise from tensorial matrix models.

1.1 Introduction to IKKT type matrix model in 3D

The axiomatic foundations of noncommutative geometry established by Connes are in terms of a set of data called a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, which contains the basic data to encode the NCG information [2]. The set consists of a Hilbert space $\mathcal{H}$, an algebra $\mathcal{A}$ of operators on $\mathcal{H}$ and a self-adjoint operator D. They should satisfy the following properties:

1. $\mathcal{A}$ is a unital, dense $*$-subalgebra of C^*-algebra A;
2. $\mathcal{H}$ is a Hilbert space carrying a faithful representation of $\mathcal{A}$ by bounded operators;
3. D is a self-adjoint Dirac operator on $\mathcal{H}$ encoding the metric structure, with commutator $[D, a]$ bounded $\forall\ a \in \mathcal{A}$.

The above properties of the spectral triple establish the three axioms of NCG. They set up the stage for applications of NCG in physics. A major branch of NC physics deals with applications of NCG in field theories, which is known as NC field theory. The simplest approach of constructing a NC field theory is to replace the point-wise product in the ordinary field theory by a star product. One example of a star product is the Moyal-Weyl star product [5], which assumes a Heisenberg algebra among the coordinate operators. We review the star product approach in section 2.1, where we present two commonly used star products, which are the Moyal-Weyl star product and Voros product.

An alternative way to write the operator algebra is in terms of a matrix representation. There the NC algebra is represented by the NC nature of the matrix multiplication. Dynamics for the matrices is introduced by writing down a matrix action, which defines a matrix model. Matrix models were originally derived from string theory for the purpose of describing non-perturbative effects. There are two main types of matrix actions that have been studied in the literature. Time appears as a continuous parameter in one, which is known as Banks, Fischler, Shenker and Susskind (BFSS) matrix model[6]. Time is a matrix in the other, known as Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) matrix model [7]. Our focus is on the latter. The matrix solutions there may describe families of NC spaces which in some limit yield a smooth space-time manifold. In this sense, spacetime geometry and even gravity can emerge from matrix models. We review the basics of matrix models in section 2.2, where we discussed the development of matrix model actions from a supersymmetric Yang-Mills origin and a string theory origin, respectively.

The simplest matrix model equations of motion in three dimensions can be written in terms of infinite-dimensional matrices X^μ of the form

$$[[X_\mu, X_\nu], X^\nu] = 0. \tag{1.3}$$

This matrix equation admits a simple NC solution, the Moyal plane, which is described by (1.2) where X^μ are identified with $\hat{x}^\mu$, with $\theta^{\mu\nu}$ constant. One approach to obtain less trivial solutions is to include additional terms in (1.3). There one may find fuzzy spaces where the matrices define a finite-dimensional Lie-algebra. Examples are the fuzzy sphere [8],[9],[10],[11],[12], [13],[14],[15], fuzzy AdS^2 [16],[17],[18],[19], [20] and fuzzy cylinder [21] solutions, which can result when a cubic term is added to the matrix model action. These solutions lead to the 2-sphere, two-dimensional anti-de Sitter space, and the cylinder, respectively, in some commutative limit. Taking the commutative limit of a NC theory is similar to taking the classical limit of a quantum theory as the Planck constant $\hbar \to 0$. For the commutative limit the Planck constant is replaced by the NC parameter $\theta^{\mu\nu}$, therefore one considers the $\theta^{\mu\nu} \to 0$ limit.

Except for the cases where solutions are trivial or associated with some finite-dimensional Lie-algebras, finding exact matrix solutions from the matrix equations of motion can be challenging [22]. For this reason, we restrict to Lie-algebra-based NC solutions to matrix model equations of motion. For the interest of this dissertation, we examine two particularly interesting solutions of three-dimensional equations of this kind. One nontrivial solution is the Lorentzian fuzzy sphere [23], which is an embedding of a fuzzy sphere in a three dimensional Minkowski ambient space. This is a solution to the matrix equations of motion when a quadratic term as well as a cubic term are included in the matrix model action. In the commutative limit, one arrives at a 2-sphere with an induced metric that does not have a fixed signature and a non-constant negative scalar curvature that has singularities at two fixed latitudes. Fluctuations about the solution lead to a consistent field theory on the NC space-time. It can be interpreted in terms of an Abelian gauge theory after making a non-standard Seiberg-Witten map. In order to get a tachyon-free theory in the commutative limit, the theory requires a selected range of coefficients for the matrix action.

We prelude our investigations with a brief review of concepts in noncommutative geometry, noncommutative field theories and matrix models in chapter 2. The outline for chapter 3 is as follows. In section 3.1 we first review the well-known example of fuzzy 2-sphere solution in three-dimensional Euclidean space. In section 3.2 we discuss the Lorentzian fuzzy sphere solution resulted from fuzzy 2-spheres embedded in a Minkowski ambient space. In section 3.3 we discuss fuzzy dS^2 solutions [20].

1.2 Introduction to tensorial matrix model in 5D

The three-dimensional matrix action in chapter 3 described in terms of vector-valued matrices can be extended to a matrix model constructed from rank-2 tensor-valued matrices. Unlike in the former case, the cubic term in the latter is valid in any number of dimensions D. Both the vector and tensorial matrix models are equivalent when the dimension of the embedding space is $D = 3$. Tensor products of two-dimensional NC spaces appear as solutions when $D = 5$. We give an explicit example of the NC field theory on the tensor-product NC space $dS^2_F \otimes S^2_F$ in the appendix B.

The Snyder algebra also appears as a solution to the tensor theory when $D = 5$. A crucial step in exploring NC physics is to understand the group representations of the NC algebra. For this, we investigate the representation theory for Snyder space in the rest of chapter 4. The spectra for the spatial coordinates of the Snyder algebra is discrete while time for us remains as a continuous parameter. Angular momentum operators are defined in terms of the position and momentum operators in the usual way, and in three dimensions they generate SO(3) rotation group. Together with the three position operators, they generate SO(4) rotation group. We find the three-dimensional Snyder algebra has two distinct Hilbert space representations. They are associated with two reducible representations of the $SU(2) \times SU(2)$ algebra. This implies the existence of two distinct lattice structures of Snyder space. The difference between the two representations appear in the spectra of the position operators, which can only be integers in one case and half-integers in the other case. Despite the discrete nature of the Snyder space, continuous translations and rotations can still be unitarily implemented on these lattices [24].

This part is outlined as follows. In section 4.1, we show how tensor-product NC spaces can arise as solutions to tensorial matrix model equations of motion. In section 4.2, we show the Snyder algebra can arise as a solution to a tensor-valued matrix action. In section 4.3, we review the three-dimensional (Euclidean) Snyder algebra and obtain the spectra for the position operators. In section 4.4, we discuss the action of the $SO(4)$ group on momentum space. In section 4.5, we obtain basis functions associated with two Hilbert spaces $\mathcal{H}_B$ and $\mathcal{H}_F$. We also discuss momentum eigenfunctions of Cartesian coordinate operators. In section 4.6, we write down unitary transformations on the lattices associated with the translation and rotation group.

2 NC gauge theory and matrix models

In this chapter, we give a brief review of NC field theory, first using the star product approach and then matrix model approach. Here we assume that the NC algebra is given by (1.2), where the NC parameter $\theta^{\mu\nu}$ is a constant antisymmetric matrix.

2.1 Star product approach

2.1.1 Symbol maps and star product

We interpret the NC coordinates $\hat{x}^\mu$ in (1.2) as self-adjoint operators acting on some Hilbert space. Any operator function $\hat{A} = f_A(\hat{x}^\mu)$ can be represented by functions on a commutative manifold spanned by coordinates x^μ via a map $\Phi(\hat{A}) \to f_A(x^\mu)$. The function $f_A(x^\mu)$ is called the symbol of $\hat{A}$. For each symbol map there exists a star product $\star$ such that

$$\Phi(\hat{A}\hat{B}) \to f_A(x^\mu) \star f_B(x^\mu) \tag{2.1}$$

The star product is in general noncommuting. For any given NC algebra, the star product is not unique and an equivalence class of star products can be defined.

2.1.2 Moyal-Weyl star product

The most common example of a star product which is appropriate for the NC algebra (1.2) (where $\theta^{\mu\nu}$ is independent of $\hat{x}^\mu$) is the Moyal-Weyl star product[5]. To define it we use the Weyl-ordered operator W, which associates a NC operator with a classical function f of the commutative variables x^μ. Take space-time dimension $d = 4$ for example, W is defined as

$$W(f) = \frac{1}{(2\pi)^4} \int d^4k \;\; e^{-ik_\mu \hat{x}^\mu} \tilde{f}(k), \tag{2.2}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$

$$\tilde{f}(k) = \int d^4x \;\; e^{ik_\mu x^\mu} f(x), \tag{2.3}$$

$$f(x) = \frac{1}{(2\pi)^4} \int d^4k \;\; e^{-ik_\mu x^\mu} \tilde{f}(k). \tag{2.4}$$

In the above we used $\hat{x}^\mu$ for elements in the NC algebra $\mathcal{A}$ which correspond to commutative coordinates x^μ. Here and hereafter in this subsection we follow the discussions in [25, 26, 27] regarding the derivations of the Moyal-Weyl star product. A useful relation obtained from (2.2) and (2.3) is

$$\begin{aligned} W(e^{iq_\mu x^\mu}) &= \frac{1}{(2\pi)^4} \int d^4k \;\; e^{-ik_\mu \hat{x}^\mu} \Big(\int d^4x \;\; e^{ik_\nu x^\nu} e^{iq_\lambda x^\lambda} \Big) \\ &= \int d^4k \;\; e^{-ik_\mu \hat{x}^\mu} \Big(\frac{1}{(2\pi)^4} \int d^4x \;\; e^{i(k_\nu + q_\nu) x^\nu} \Big) \\ &= \int d^4k \;\; e^{-ik_\mu \hat{x}^\mu} \delta(k_\nu + q_\nu) \\ &= e^{iq_\mu \hat{x}^\mu}. \end{aligned} \tag{2.5}$$

Therefore W maps $e^{iq_\mu x^\mu}$ to $e^{iq_\mu \hat{x}^\mu}$, and we also have the inverse map W^{-1}

$$W^{-1}(e^{iq_\mu \hat{x}^\mu}) = e^{iq_\mu x^\mu}. \tag{2.6}$$

The Moyal-Weyl star product is defined by the correspondence

$$W(f\star g) = W(f)W(g) \tag{2.7}$$

where the product of two operators $W(f)$ and $W(g)$ gives

$$W(f)W(g) = \frac{1}{(2\pi)^8}\int d^4k_1 d^4k_2 \;\; e^{-ik_{1\mu}\hat{x}^\mu} e^{-ik_{2\nu}\hat{x}^\nu} f(k_1)g(k_2). \tag{2.8}$$

To obtain an expression for the Moyal-Weyl star product, we make use of the Becker-Hausdorff-Campbell formula on the operator product $e^{-ik_{1\mu}\hat{x}^\mu}e^{-ik_{2\nu}\hat{x}^\nu}$, which for the case of the canonical NC algebra (1.2) takes the form

$$\begin{aligned} e^{-ik_{1\mu}\hat{x}^\mu} e^{-ik_{2\nu}\hat{x}^\nu} &= e^{-i(k_{1\mu}+k_{2\mu})\hat{x}^\mu - \frac{1}{2}k_{1\mu}k_{2\nu}[\hat{x}^\mu,\hat{x}^\nu]} \\ &= e^{-i(k_{1\mu}+k_{2\mu})\hat{x}^\mu - \frac{i}{2}\theta^{\mu\nu}k_{1\mu}k_{2\nu}}. \end{aligned} \tag{2.9}$$

Then from (2.7) we have

$$\begin{aligned} f\star g &= W^{-1}[W(f)W(g)] \\ &= \frac{1}{(2\pi)^8}\int d^4k_1 d^4k_2 \;\; W^{-1}(e^{-i(k_{1\mu}+k_{2\mu})\hat{x}^\mu - \frac{i}{2}\theta^{\mu\nu}k_{1\mu}k_{2\nu}})\tilde{f}(k_1)\tilde{g}(k_2) \\ &= \frac{1}{(2\pi)^8}\int d^4k_1 d^4k_2 \;\; e^{-i(k_{1\mu}+k_{2\mu})x^\mu - \frac{i}{2}\theta^{\mu\nu}k_{1\mu}k_{2\nu}}\tilde{f}(k_1)\tilde{g}(k_2) \\ &= e^{-\frac{i}{2}\theta^{\mu\nu}k_{1\mu}k_{2\nu}}\Big(\frac{1}{(2\pi)^4}\int d^4k_1 e^{-ik_{1\mu}x^\mu}\tilde{f}(k_1)\Big)\Big(\frac{1}{(2\pi)^4}\int d^4k_2 e^{-ik_{2\nu}x^\nu}\tilde{g}(k_2)\Big), \end{aligned} \tag{2.10}$$

where from the second line to the third line we have used (2.6). Rewriting the two integrals in the last line using the inverse Fourier transform (2.4) and replacing the momenta k_μ by the differential operators $-i\frac{\partial}{\partial x^\mu}$, we finally arrive at the formal expression for Moyal-Weyl star product[5]

$$[f\star g](x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x}. \tag{2.11}$$

It is usually convenient to express the Moyal-Weyl star product in series expansion and only work with the first order contributions in $\theta^{\mu\nu}$, i.e.,

$$[f \star g](x) \approx f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + ... \ , \quad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{2.12}$$

One can then define the $\star$-commutator using the difference between $[f \star g](x)$ and $[g \star f](x)$:

$$\begin{aligned} [f(x), g(x)]_\star &= [f \star g](x) - [g \star f](x) \\ &= \frac{i}{2}\theta^{\mu\nu}\Big(\partial_\mu f(x)\partial_\nu g(x) - \partial_\nu f(x)\partial_\mu g(x)\Big) \\ &= \frac{i}{2}\theta^{\mu\nu}\{f(x), g(x)\}_{PB} + ... \end{aligned} \tag{2.13}$$

where $\{,\}_{PB}$ denotes the Poisson bracket, which is defined as

$$\{f(x), g(x)\}_{PB} = \partial_\mu f(x)\partial_\nu g(x) - \partial_\nu f(x)\partial_\mu g(x). \tag{2.14}$$

For the rest of the dissertation, we will drop the subscript "PB" for the Poisson bracket and simply denote the Poisson brackets by the curly bracket. Using the space-time coordinate x^μ in (2.12), we get

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}. \tag{2.15}$$

which realizes (1.2). A feature of the Moyal-Weyl star product that can be easily checked using the first order form in (2.12) is associativity, namely

$$(f \star g) \star h = f \star (g \star h). \tag{2.16}$$

Differentiation and integration are linear operations that satisfy the following properties:[27]

1. ∂_i denotes a derivation on the NC algebra $\mathcal{A}$. It can be defined using an element $d_i \in \mathcal{A}$ which satisfies $\partial_i A = [d_i, A]$. Derivatives that can be written in this way is referred to as an inner derivation on $\mathcal{A}$. For the NC algebra (1.2), we can get an explicit form of the derivative:

$$i\theta^{\mu\nu}\partial_\nu = [\hat{x}^\mu, \ \cdot\,], \tag{2.17}$$

   which leads to $\partial_\mu \hat{x}^\nu = \delta^\nu_\mu$ upon acting on $\hat{x}^\nu$.

2. The integral on NC space-time $\int$ is required to satisfy the usual trace property for ordinary integrals, namely

$$\int d^n x \ \ [f, g]_\star = \int d^n x \ \ (f \star g - g \star f) = 0. \tag{2.18}$$

   This also results in the integral of a total derivative is zero:

$$\int \ \ \partial_i f = 0. \tag{2.19}$$

   At last, we have a very useful identity

$$\int \ \ f \star g = \int \ \ fg, \tag{2.20}$$

   which can be easily verified to all orders in $\theta^{\mu\nu}$ using (2.11). In both (2.19) and (2.20) it is assumed that f and g vanish sufficiently fast at infinity.

2.1.3 Voros star product

Another example of NC star product is the Voros product[28]. It belongs to the same equivalence class as the Moyal-Weyl star product and follows from the coherent state approach. Here, we follow the discussions in [31] regarding the properties of coherent states and derivations of the Voros star product. We consider the two-dimensional NC plane, i.e., $[\hat{x}, \hat{y}] = i\theta$, where θ is independent of $\hat{x}$ and $\hat{y}$. In this space, we can construct the raising and lowering operators as

$$\hat{a}^\dagger = \hat{x} - i\hat{y}, \quad \hat{a} = \hat{x} + i\hat{y}, \tag{2.21}$$

respectively, and they satisfy

$$[\hat{a}, \hat{a}^\dagger] = 2\theta. \tag{2.22}$$

Coherent states $\{|\alpha\rangle, \alpha \in \mathbb{C}\}$ are eigenstates of the lowering operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and can be expressed in terms of the Fock space basis $\{|n\rangle, n \in \mathbb{N}_0\}$ where $\mathbb{N}_0$ stands for non-negative integers:

$$|\alpha\rangle = e^{-|\alpha|^2/4\theta} \sum_{n=0}^{\infty} \frac{(\alpha/\sqrt{2\theta})^n}{\sqrt{n!}} |n\rangle . \tag{2.23}$$

The norm of scalar product between two coherent states $|\alpha\rangle$ and $\langle\beta|$ is

$$|\langle\beta\,|\,\alpha\rangle| = e^{-\frac{|\alpha-\beta|^2}{4\theta}} . \tag{2.24}$$

We can also obtain the following useful relation

$$\frac{\langle\beta|\,\hat{A}\,|\alpha\rangle}{\langle\beta\,|\,\alpha\rangle} = e^{-\beta\frac{\partial}{\partial\alpha}} e^{\alpha\frac{\partial}{\partial\beta}} \frac{\langle\beta|\,\hat{A}\,|\beta\rangle}{\langle\beta\,|\,\beta\rangle} = e^{-\beta\frac{\partial}{\partial\alpha}} e^{\alpha\frac{\partial}{\partial\beta}} \langle\beta|\,\hat{A}\,|\beta\rangle , \tag{2.25}$$

where $\hat{A}$ is a Hermitian operator. Taking the complex conjugate of (2.25), we get

$$\frac{\langle\alpha|\,\hat{A}\,|\beta\rangle}{\langle\alpha\,|\,\beta\rangle} = e^{-\bar{\beta}\frac{\partial}{\partial\bar{\alpha}}} e^{\bar{\alpha}\frac{\partial}{\partial\bar{\beta}}} \langle\beta|\,\hat{A}\,|\beta\rangle , \tag{2.26}$$

where $\bar{\alpha}$ denotes the complex conjugate of α.

The symbol $\Phi(\hat{A})$ of an operator $\hat{A}$, known as the Berezin symbol[30], is defined as the expectation value of $\hat{A}$ with respect to a coherent state

$$\Phi(\hat{A}) = f_A(\alpha, \bar{\alpha}) = \langle\alpha| \, \hat{A} \, |\alpha\rangle \,. \tag{2.27}$$

The construction of the Berezin symbol implies an ordering in the operators, which is different from the symmetric ordering in the Weyl symbol. The star product of two Berezin symbols, $\Phi(\hat{A})$ and $\Phi(\hat{B})$, of operators $\hat{A}$ and $\hat{B}$, satisfies

$$\Phi(\hat{A}) \star \Phi(\hat{B}) = \Phi(\hat{A}\hat{B}), \tag{2.28}$$

from which we can obtain an explicit expression of the Voros star product

$$\begin{aligned}
[f_A \star f_B](\beta, \bar{\beta}) &= \langle\beta| \, \hat{A}\hat{B} \, |\beta\rangle \\
&= \frac{1}{2\pi\theta} \int d\alpha d\bar{\alpha} \, \langle\beta| \, \hat{A} \, |\alpha\rangle \, \langle\alpha| \, \hat{B} \, |\beta\rangle \\
&= \frac{1}{2\pi\theta} \int d\alpha d\bar{\alpha} |\langle\beta \, | \, \alpha\rangle \, |^2 \Big(e^{-\beta\frac{\partial}{\partial\alpha}} e^{\alpha\frac{\partial}{\partial\beta}} \, \langle\beta| \, \hat{A} \, |\beta\rangle \, \Big) \Big(e^{-\bar{\beta}\frac{\partial}{\partial\bar{\alpha}}} e^{\bar{\alpha}\frac{\partial}{\partial\bar{\beta}}} \, \langle\beta| \, \hat{B} \, |\beta\rangle \, \Big) \\
&= \frac{1}{2\pi\theta} \int d\alpha d\bar{\alpha} \Big(e^{-\beta\frac{\partial}{\partial\alpha} - \bar{\beta}\frac{\partial}{\partial\bar{\alpha}}} |\langle\beta \, | \, \alpha\rangle \, |^2 \Big) \Big(e^{\alpha\frac{\partial}{\partial\beta}} f_A(\beta, \bar{\beta}) \Big) \Big(e^{\bar{\alpha}\frac{\partial}{\partial\bar{\beta}}} f_B(\beta, \bar{\beta}) \Big) \\
&= \frac{1}{2\pi\theta} \int d\alpha d\bar{\alpha} |\langle\beta \, | \, \alpha - \beta\rangle \, |^2 \Big(e^{\alpha\frac{\partial}{\partial\beta}} f_A(\beta, \bar{\beta}) \Big) \Big(e^{\bar{\alpha}\frac{\partial}{\partial\bar{\beta}}} f_B(\beta, \bar{\beta}) \Big) \\
&= \frac{1}{2\pi\theta} \int d\alpha d\bar{\alpha} \;\; e^{-\frac{\alpha\bar{\alpha}}{2\theta}} \Big(e^{\alpha\frac{\partial}{\partial\beta}} f_A(\beta, \bar{\beta}) \Big) \Big(e^{\bar{\alpha}\frac{\partial}{\partial\bar{\beta}}} f_B(\beta, \bar{\beta}) \Big) \\
&= \int d\alpha \Big(\frac{1}{2\pi\theta} \int d\bar{\alpha} \;\; e^{i\bar{\alpha}(\frac{i\alpha}{2\theta} - i\frac{\partial}{\partial\bar{\beta}})} \Big) \Big(e^{\alpha\frac{\partial}{\partial\beta}} f_A(\beta, \bar{\beta}) f_B(\beta, \bar{\beta}) \Big) \\
&= \int d\alpha \;\; \delta(\alpha - 2\theta\frac{\partial}{\partial\bar{\beta}}) \Big(e^{\alpha\frac{\partial}{\partial\beta}} f_A(\beta, \bar{\beta}) f_B(\beta, \bar{\beta}) \Big) \\
&= e^{2\theta\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\bar{\eta}}} f_A(\beta + \zeta, \bar{\beta}) f_B(\beta, \bar{\beta} + \bar{\eta})|_{\zeta = \bar{\eta} = 0} \quad .
\end{aligned} \tag{2.29}$$

where from the first line to the second line we have used the completeness relation for the coherent states $\{|\alpha\rangle\}$, i.e., $\frac{1}{2\pi\theta} \int d\alpha d\bar{\alpha} \, |\alpha\rangle \, \langle\alpha| = 1$.

2.1.4 NC field theory and Seiberg-Witten map

The Moyal-Weyl star product and Voros product provide for a simple prescription for constructing NC field theories on the noncommutative manifold with $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$. One simply replaces all the point-wise product in ordinary field theory by one of the star products. For example, the NC action for a real massless scalar field Φ in four dimensions is

$$S_\phi = \frac{1}{2}\int d^4x \;\; D^\mu\Phi \star D_\mu\Phi \tag{2.30}$$

where the ordinary derivative ∂_μ appearing in the commutative scalar field action got replaced by the NC covariant derivative D_μ so that the action is invariant under the NC gauge transformation. An infinitesimal gauge variation of a real scalar field Φ and a $U(1)$ gauge potential A_μ is given by

$$\delta\Phi = i[\Lambda, \Phi]_\star, \tag{2.31}$$

$$\delta A_\mu = \partial_\mu\Lambda + i[\Lambda, A_\mu]_\star, \tag{2.32}$$

where Λ is an infinitesimal NC gauge parameter. The NC covariant derivative has the following form when acting on a real scalar field Φ

$$D_\mu\Phi = \partial_\mu\Phi - i[A_\mu, \Phi]_\star. \tag{2.33}$$

It follows that $D_\mu\Phi$ transforms covariantly under the NC gauge transformations.

Similarly, for the NC gauge field Yang-Mills action, we have

$$S_A = -\frac{1}{4}\int d^4x \;\; F^{\mu\nu} \star F_{\mu\nu}, \tag{2.34}$$

where the NC field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star \tag{2.35}$$

One can define covariant coordinates as

$$X^\mu = x^\mu + \theta^{\mu\nu} A_\nu. \tag{2.36}$$

To see that it transforms covariantly, one uses (2.32), and gets

$$\begin{aligned} \delta X^\mu &= \delta(x^\mu + \theta^{\mu\nu} A_\nu) \\ &= \theta^{\mu\nu}(\partial_\nu \Lambda + i[\Lambda, A_\nu]_\star) \\ &= i([\Lambda, x^\mu]_\star + [\Lambda, \theta^{\mu\nu} A_\nu]_\star) \\ &= i[\Lambda, X^\mu]_\star. \end{aligned} \tag{2.37}$$

The actions for NC field theories should reduce to the ordinary commutative field theory actions when we take the commutative limit. This corresponds to taking the limit that the NC parameter $\theta^{\mu\nu}$ is zero. NC fields can be mapped to the commutative fields in a manner that maps the NC gauge transformations to commutative gauge transformations. This map is invertible and is known as the Seiberg-Witten (SW) map[3].

To be more specific, we consider NC $U(1)$ gauge field theory, where we express the NC gauge potentials A and gauge parameter Λ as functions of commutative gauge potentials $\mathcal{A}$, gauge parameter λ, and their derivatives

$$A = A(\mathcal{A}, \partial\mathcal{A}, \partial^2\mathcal{A}, ...; \theta), \tag{2.38}$$

$$\Lambda = \Lambda(\lambda, \partial\lambda, \partial^2\lambda, ...; \theta). \tag{2.39}$$

Then the SW-map requires the following consistency condition be satisfied

$$A(\mathcal{A}) + \delta_\Lambda A(\mathcal{A}) = A(\mathcal{A} + \delta_\lambda \mathcal{A}), \tag{2.40}$$

where on the left hand side (LHS) the NC gauge variation $\delta_\Lambda A$ has the form

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda - i[A_\mu, \Lambda]_\star, \tag{2.41}$$

and on the right hand side (RHS) of (2.40) the commutative gauge variation $\delta_\lambda A$ is

$$\delta_\lambda \mathcal{A}_\mu = \partial_\mu \lambda. \tag{2.42}$$

The consistency condition (2.40) can be solved order by order in the NC parameter. For example, for the case of the Moyal-Weyl star product, up to first order, the SW-map is given by

$$A_\mu = \mathcal{A}_\mu - \frac{1}{2}\theta^{\alpha\beta}\mathcal{A}_\alpha(\partial_\beta \mathcal{A}_\mu + \mathcal{F}_{\beta\mu}) \tag{2.43}$$

$$\Phi = \phi - \theta^{\alpha\beta}\mathcal{A}_\alpha \partial_\beta \phi \quad (\phi : \text{real scalar field}) \tag{2.44}$$

$$\Psi = \psi - \frac{1}{2}\theta^{\alpha\beta}\mathcal{A}_\alpha \partial_\beta \psi \;\; (\psi : \text{complex scalar field}) \tag{2.45}$$

$$\Lambda = \lambda - \frac{1}{2}\theta^{\alpha\beta}\mathcal{A}_\alpha \partial_\beta \lambda \tag{2.46}$$

where the commutative U(1) gauge field strength is $\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha$. The aforementioned NC actions (2.30) and (2.34) can be expanded in $\theta^{\alpha\beta}$ using the star product. Using the SW-map, it can then be expressed in terms of commutative scalar fields and gauge fields. The lowest order in $\theta^{\alpha\beta}$ is the commutative action and higher order terms describes the NC corrections to the commutative action.

2.2 Introduction to Matrix models

Here we introduce matrix models in various contexts.

2.2.1 Fock space point of view

The star product provides a convenient tool for constructing NC field theories. It can then be expanded for "small" $\theta^{\alpha\beta}$. Alternatively, one can often construct NC field theories without relying on a perturbative expansion. This can be achieved by realizing the NC algebra in the Fock-space representation.

Let us consider the two-dimensional NC plane again. As shown in the subsection 2.1.3, the two-dimensional NC plane can be described by the raising and lowering operators $\hat{a}^\dagger$ and $\hat{a}$, which satisfy the commutation relation (2.22). $\hat{a}^\dagger$ and $\hat{a}$ act on the Fock-space $\mathcal{H} = \{|n\rangle, n \in \mathbb{N}_0\}$ in the following way

$$\hat{a}^\dagger |n\rangle = \sqrt{2\theta}\sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}\,|n\rangle = \sqrt{2\theta}\sqrt{n}\,|n-1\rangle \tag{2.47}$$

Since the matrix representations of the raising and lowering operators are infinite-dimensional, we now have an infinite-dimensional representation of the NC coordinate operators $\hat{x}$ and $\hat{y}$ in the Fock space. This construction can be generalized to any $2N$-dimensional space-time, in which one only needs to define N pairs of raising and lowering operators.

(2.22) and its higher dimensional generalizations can result as a classical solution to a matrix model equation of motion. For this one can introduce a matrix model action in $2N$ dimensions in terms of the infinite-dimensional matrices

$$X^\mu = \hat{x}^\mu + \theta^{\mu\nu} A_\nu, \quad \mu,\nu = 0, 1, ..., 2N-1, \tag{2.48}$$

where the generalized complex (raising or lowering) operators $\hat{x}^\mu$ is made gauge covariant with the inclusion of gauge connections A_μ, which are also infinite-dimensional matrices and they transform under unitary gauge transformations according to

$$A_\mu \to U[\hat{x}_\mu, U^\dagger] + U A_\mu U^\dagger, \tag{2.49}$$

where U is an infinite-dimensional unitary matrix. Matrices in the form of (2.48) are referred to as "covariant coordinates" as they transform covariantly under unitary gauge transformations, i.e.,

$$X^\mu \to U X^\mu U^\dagger. \tag{2.50}$$

The matrices X^μ can be used to define a matrix model action

$$\begin{aligned}
S_{YM} &= -\frac{1}{4g^2}\mathrm{Tr}\Big([X^\mu, X^\nu] - i\theta^{\mu\nu}\mathbb{1}\Big)^2 \\
&= -\frac{1}{4g^2}\mathrm{Tr}\Big([\hat{x}^\mu + \theta^{\mu\alpha}A_\alpha\ ,\ \hat{x}^\nu + \theta^{\nu\beta}A_\beta] - i\theta^{\mu\nu}\mathbb{1}\Big)^2 \\
&= -\frac{1}{4g^2}\mathrm{Tr}\Big((-i\theta^{\mu\alpha}\theta^{\nu\beta})([-i\theta^{-1}_{\alpha\lambda}\hat{x}^\lambda, A_\beta] - [-i\theta^{-1}_{\beta\sigma}\hat{x}^\sigma, A_\alpha] + i[A_\alpha, A_\beta])\Big)^2 \\
&= \frac{1}{4g^2}\mathrm{Tr}F_{\alpha\beta}F^{\alpha\beta},
\end{aligned} \tag{2.51}$$

which leads to the form of a Yang-Mills action (up to a minus sign) for a gauge theory. In the above derivation we have used the NC algebra (1.2). In this construction, A_μ in the covariant coordinates (2.48) plays the role of a gauge potential, corresponding to the field strength tensor

$$\begin{aligned}
F_{\alpha\beta} &= [-i\theta^{-1}_{\alpha\lambda}\hat{x}^\lambda, A_\beta] - [-i\theta^{-1}_{\beta\sigma}\hat{x}^\sigma, A_\alpha] + i[A_\alpha, A_\beta] \\
&= \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta],
\end{aligned} \tag{2.52}$$

where we have used the derivative defined in (2.17). $F_{\alpha\beta}$ transform covariantly under unitary gauge transformations, i.e.,

$$F_{\alpha\beta} \to U F_{\alpha\beta} U^{\dagger} \tag{2.53}$$

Starting with the matrix model action as defined in the first line of (2.51), the NC algebra (1.2) can be obtained as a classical solution of the system.

2.2.2 String/membrane point of view

The matrix action (2.51) can also be obtained from the Nambu-Goto action of a bosonic string

$$S_{NG} = T \int d^2\sigma \sqrt{-det\, G_{ab}}, \tag{2.54}$$

written in a D-dimensional background, where $G_{ab} = \partial_a X^\mu \partial_b X_\mu$ is identified with the worldsheet metric[1] and is assumed to have Lorentzian signature. Here $X^\mu, \mu = 0, ..., D-1$ denote the string coordinates in the ambient space-time which are functions of $\sigma^a, a = 1, 2$ that parametrize the string worldsheet. σ^0 is a time-like parameter, and σ^1 is space-like. The worldsheet integration measure is $d^2\sigma = d\sigma^0 d\sigma^1$. G_{ab} defines the induced metric on the string worldsheet, where $\partial_a = \frac{\partial}{\partial\sigma^a}$ denotes the differentiation with respect to the worldsheet coordinates.

The Nambu-Goto action is equivalent (classically) to the following action[32]

$$S_{NGP} = \int d^2\sigma \Big(-\frac{1}{4\eta}\{X_\mu, X_\nu\}^2 + \eta T^2 \Big), \tag{2.55}$$

where the Poisson bracket on the worldsheet becomes $\{X^\mu, X^\nu\} = \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu$. η here is a Lagrange multiplier and can be eliminated using its equation of motion,

$$\eta = \frac{\sqrt{-\{X_\mu, X_\nu\}^2}}{2T}. \tag{2.56}$$

Substituting (2.56) back into the action (2.55) one gets

$$\begin{aligned} S_{NGP} &= \int d^2\sigma \Big(-\frac{1}{4}\frac{2T}{\sqrt{-\{X_\mu, X_\nu\}^2}}\{X_\mu, X_\nu\}^2 + \frac{\sqrt{-\{X_\mu, X_\nu\}^2}}{2T}T^2 \Big) \\ &= T\int d^2\sigma \sqrt{-\{X_\mu, X_\nu\}^2} = T\int d^2\sigma \sqrt{-\epsilon^{ab}\epsilon^{cd}\partial_a X^\mu \partial_b X^\nu \partial_c X_\mu \partial_d X_\nu} \\ &= \sqrt{2}T \int d^2\sigma \sqrt{-detG_{ab}}, \end{aligned} \tag{2.57}$$

[1] This identification is only for the sake of simplicity, in general, the worldsheet metric and the induced metric of the target space are related to each other by a conformal factor.

which is equivalent to the Nambu-Goto action in (2.54) up to a constant. Applying in (2.55) the standard quantization rules, where the Poisson bracket is replaced by the commutator, $\{,\} \rightarrow -i[,]$, and integration is replaced by the trace $\int d^2\sigma\sqrt{-\mathtt{G}} \rightarrow \mathrm{Tr}$, $\mathtt{G} = det(G_{ab})$, the action (2.55) turns into

$$S_{NGP} = \alpha \mathrm{Tr}\frac{1}{4}[X_\mu, X_\nu]^2 + \beta \mathrm{Tr}\mathbb{I}, \tag{2.58}$$

which is similar to (2.51) when we identify X^μ with the covariant coordinates in (2.48). The cross term in (2.51) vanishes in the trace as it is proportional to a commutator. The term $\beta\mathrm{Tr}\mathbb{I}$ seems to be problematic for an infinite-dimensional representation of the matrices X^μ, which produces an infinite constant shift to the action. But it shall be canceled by the $[\hat{x}^\mu, \hat{x}^\nu]^2$ contribution from the first term in the expansion and leaves the action finite [27]. The constant parameters α and β are coefficients of the matrix model, which will be determined once we identify (2.58) with a matrix model action.

The similarity between (2.58) and (2.51) is no coincidence. Matrix models were originally obtained from dimensional reductions of 10-dimensional supersymmetric Yang-Mills (SYM) theory [33]. The latter has been considered as a low energy effective theory for M-theory. There are two families of matrix models that can be obtained in this way, which are called Ishibashi, Kawai, Kitazawa and Tsuchiya (IKKT) matrix model[7] and Banks, Fischler, Shenker and Susskind (BFSS) matrix model[6], respectively. We briefly discuss them below.

2.2.3 IKKT matrix model

The IKKT model is a dimensional reduction of 10-dimensional SYM theory 0-dimension, or a point. The $\mathcal{N}=1$, $D=10$, SYM actions can be written as [7]

$$S_{SYM} = \frac{N}{g_0^2 a^6}\int d^{10}x \mathrm{Tr}\Big(\frac{1}{4}\, F_{IJ}F^{IJ} + \frac{i}{2}\bar{\psi}\Gamma^I D_I\psi\Big), \tag{2.59}$$

where a is the cutoff length and g_0 is the dimensionless coupling constant. The vector indices I, J run from 0 to 9, ψ is a 16-component Grassmann variable that describes a Majorana-Weyl spinor, which transforms under the adjoint action of the $U(N)$ gauge group. Γ^I are the 10-dimensional 32×32 Gamma matrices and $D_I = \partial_M + i[A_I, \cdot\,]$ are the covariant derivatives. The 10-dimensional field strength tensor F_{IJ} is defined in terms of the gauge potential A_I as

$$F_{IJ} = \partial_I A_J - \partial_J A_I - i[A_I, A_J]. \tag{2.60}$$

In general, to perform the dimensional reduction of the 10-dimensional SYM theory to d dimensions, one considers d flat dimensions along $x^m, m=1,...,d$ and the extra $(10-d)$ dimensions along $x^\mu, \mu = 0, d+1, ..., 9$ to be compactified. Correspondingly, one has a separation of the 10 bosonic degrees of freedom in A_I into d gauge field degrees of freedom $A_m(x^1,...,x^d)$ and $10-d$ scalar degrees of freedom $\Phi_\mu = A_\mu(x^1,...,x^d)$, which are both assumed to dependent of the reduced d-dimensional space-time. Then (2.59) turns into

$$\begin{aligned} S_{SYM} \sim \int d^{10}x\mathrm{Tr}\Big(\frac{1}{4}\, F_{mn}F^{mn} + \frac{1}{2}D_m\Phi_\mu D^m\Phi^\mu - \frac{1}{4}[\Phi_\mu, \Phi_\nu]^2 \\ + \frac{i}{2}\bar{\psi}\Gamma^m D_m\psi - \frac{1}{2}\bar{\psi}\Gamma^\mu[\Phi_\mu, \psi]\Big). \end{aligned} \tag{2.61}$$

When $d=0$, space-time dependence in $A_m(x^1,...,x^d)$ are completely removed and all the terms involving space-time derivatives in (2.61) vanish. We then arrive at the IKKT matrix model action

$$S_{IKKT} = -\alpha \mathrm{Tr}\Big(\frac{1}{4}[X_\mu, X_\nu]^2 + \frac{1}{2}\bar{\psi}\Gamma^\mu[X_\mu, \psi]\Big), \tag{2.62}$$

where we have changed notation with $\Phi^\mu \to X^\mu$ and identify them with the covariant coordinates X^μ in (2.36). The coefficient α will be determined in terms of fundamental constants in string theory later. The IKKT matrix model in (2.62) is defined in terms of Hermitian matrices X_μ and ψ that do not depend on space-time coordinates, where the bosonic part, i.e., the first term already appears in (2.51) and (2.58).

The IKKT action is invariant under unitary gauge transformation

$$X^\mu \to U X^\mu U^\dagger \tag{2.63}$$

$$\psi \to U\psi U^\dagger \tag{2.64}$$

where $U \in U(N)$. The forms of the infinitesimal gauge variations are (2.37) and

$$\delta\psi = i[\Lambda, \psi], \tag{2.65}$$

where $U = \mathbb{1} + i\Lambda$ and Λ is "small".

The IKKT action is also invariant under supersymmetry transformations

$$\delta_1 X_\mu = \bar{\epsilon}\gamma_\mu\psi, \tag{2.66}$$

$$\delta_1\psi = \frac{1}{2}[X_\mu, X_\nu][\Gamma^\mu, \Gamma^\nu]\epsilon, \tag{2.67}$$

and a second one that corresponds to the shift of the fermion

$$\delta_2 X_\mu = 0 \tag{2.68}$$

$$\delta_2 \psi = \zeta \tag{2.69}$$

where ϵ and ζ are infinitesimal 16-dimensional Grassmann spinors.

The quantization of the above action starts with writing down the grand-canonical partition function

$$Z[\beta] = \sum_{n=1}^{\infty} \int dX d\Psi e^{-S(\beta)} \tag{2.70}$$

where the action $S(\beta)$ is

$$S(\beta) = -\alpha \mathrm{Tr}\Big(\frac{1}{4}[X_\mu, X_\nu]^2 + \frac{1}{2}\bar{\psi}\Gamma^\mu[X_\mu, \psi]\Big) + \beta \mathrm{Tr}\mathbb{1} \tag{2.71}$$

Here X^μ and ψ are $N \times N$ Hermitian matrices. We also have $\mathrm{Tr}\mathbb{1} = N$ so β can be interpreted as the chemical potential dual to the matrix size N. If we regard the matrices as coordinates describing dynamics of a bosonic string, we can obtain the classical limit of the above action by applying the de-quantization map, i.e., $\mathrm{Tr} \to \int d^2\sigma\sqrt{-\mathsf{g}}$, and $[,] \to i\{,\}$, where g is the determinant of the string world-sheet metric, and Poisson bracket is defined by $\{X, Y\} = \frac{1}{\sqrt{-\mathsf{g}}}\epsilon^{ab}\partial_a X \partial_b Y$. then action (2.71) can be related to the Schild-type Green-Schwarz action for type IIB strings[34]

$$S_{GS} = \int d^2\sigma[\sqrt{-\mathsf{g}}\alpha(\frac{1}{4}\{X_\mu, X_\nu\}^2 + \frac{i}{2}\bar{\psi}\Gamma^\mu\{X_\mu, \psi\}) + \beta\sqrt{-\mathsf{g}}]. \tag{2.72}$$

This identification allows us to determine the coefficients α and β in terms of fundamental constants in string theory [7]

$$\alpha = \frac{4\pi^{5/2}}{\sqrt{6G_N}}, \tag{2.73}$$

$$\beta = \frac{12\pi^{9/2}\ell_s^4}{\sqrt{6G_N}}, \tag{2.74}$$

where G_N is the gravitation constant and ℓ_s is the fundamental string length. The identification of the IKKT action and the Green-Schwarz action shows that matrix model action gives a non-perturbative description of the type IIB superstring theory.

2.2.4 BFSS matrix model

The BFSS matrix model is the dimensional reduction of $\mathcal{N} = 1$, $D = 10$, SYM theory to a time-line, or a $D0$-brane. Using the similar procedure, one obtains the reduced action [6]

$$S_{BFSS} = \frac{1}{g_{BFSS}} \int dt \mathrm{Tr}\Big\{\frac{1}{2}(D_0 X_i)^2 + \bar{\psi} D_0 \psi - \frac{1}{4}[X_i, X_j]^2 - \bar{\psi}\Gamma_i[X_i, \psi]\Big\}, \tag{2.75}$$

where g_{BFSS} stands for some coupling constant of the BFSS model. The matrices $X_i(t), (i = 1, ..., 9)$ are time-dependent in this case, and they are argued to describe dynamics of $D0$-branes in type IIA string theory.

We will not go into details of the BFSS matrix models as this dissertation deals exclusively with the IKKT-type matrix models.

3 Fuzzy spaces from three-dimensional IKKT-type matrix models

Various aspects of IKKT matrix models, in a background space with either Euclidean or Lorentzian signatures, have been discussed in the literature. One aspect deals with classical solutions, their implications for cosmology, and perturbation theories [18],[21],[35],[36],[37],[38],[39],[40],[41]. The solutions were generally written in terms of infinite- or finite-dimensional matrices, and they may or may not be associated with finite dimensional Lie-algebras. In this chapter, we explore the properties of two-dimensional NC spaces that are solutions to three-dimensional IKKT matrix models. In particular, we examine the bosonic sector of the three-dimensional IKKT matrix model introduced in the previous chapter.

As the first step, we consider the simplest case where the action only contains the Yang-Mills term

$$S(X) = \frac{1}{g^2}\mathrm{Tr}(-\frac{1}{4}[X_\mu, X_\nu]^2), \tag{3.1}$$

where $\mu, \nu = 0, 1, 2$ or $1, 2, 3$, for Lorentzian or Euclidean signature, respectively. X_μ are infinite-dimensional matrices. The equations of motion follow from extremizing the action are

$$[[X_\mu, X_\nu], X^\nu] = 0 \tag{3.2}$$

The matrix nature of the equations of motion makes it difficult in general to find solutions, especially analytical ones. The exceptional cases are solutions that correspond to finite-dimensional Lie-algebras, among which the most trivial solution corresponds to commuting matrices

$$[\bar{X}_\mu, \bar{X}_\nu] = 0. \tag{3.3}$$

This equation is automatically satisfied for diagonal matrices

$$\bar{X}^\mu = \begin{pmatrix} x_1^\mu & & \\ & x_2^\mu & \\ & & \ddots \end{pmatrix}, \tag{3.4}$$

which can either be finite- or infinite-dimensional. Here we denote the matrix solutions with a bar, i.e., $\bar{X}^\mu$ to distinguish them from the matrices X^μ that appear in the action and equations of motion. A less trivial solution of (3.2) is of the form

$$[\bar{X}^\mu, \bar{X}^\nu] = i\theta^{\mu\nu}, \tag{3.5}$$

with $\theta^{\mu\nu}$ a constant antisymmetric matrix, i.e., $\bar{X}^\mu$ span the Moyal-Weyl space-time. Note that $\theta^{\mu\nu}$ is only invertible for an even number of dimensions.

Lie-algebra based matrix solutions can be obtained by including a cubic term in the original Yang-Mills action, which is sometimes referred to as a Chern-Simons (CS) term. This addition is only allowed in an odd number of dimensions. As stated previously, our discussion in chapter 3 deals with three-dimensional embedding space. We generalize to four and five dimensions in the later chapter 4 and 5, respectively.

One well-known example of two-dimensional NC space is the fuzzy 2-sphere (S_F^2), where the term "fuzzy" has been used to denote finite-dimensional NC spaces. Fuzzy spheres are expressed in terms of $N \times N$ matrices, upon taking $N \to \infty$, corresponding to the commutative limit, they reduce to ordinary (commutative) 2-spheres S^2. The S_F^2 solutions can arise from both Euclidean and Lorentzian IKKT matrix models. In the former case, they are the well known. S_F^2 solutions are less familiar in the latter case. We refer to them as Lorentzian fuzzy spheres [23]. These solutions describe compact NC space-times. Furthermore, such solutions could serve as toy models for closed cosmologies which resolve cosmological singularities. Big bang/crunch singularities appear in the commutative (large N) limit, while the finite dimensional matrix description is singularity free.

In section 3.1 we give a brief review of the fuzzy 2-sphere solution of the three-dimensional Euclidean IKKT-type matrix model where a cubic term is added to (3.1). Following the approach in [15], we show that by fluctuating about the solutions we get a NC gauge theory. We also show how to obtain the commutative gauge theory from the commutative limit of the NC gauge theory. These techniques will also be used for other NC spaces.

In section 3.2, we write down a fuzzy 2-sphere solution to a three-dimensional Lorentzian IKKT-type matrix model. Here cubic and quadratic terms are added to (3.1). We show that the solution yields a closed (two-dimensional) universe in the commutative limit. While the commutative limit of the solution is topologically a two-sphere, there are a number of novel features, arising from the fact that it is embedded in the three-dimensional Minkowski space. The induced metric does not agree with the standard metric on the sphere, and, moreover, it does not have a fixed signature. The curvature computed from the induced metric is not constant and it is negative. It is singular at two fixed latitudes (which are not located at the poles) and time-like geodesics originate and terminate at these latitudes. Thus in this toy model, the big bang/crunch singularities occur at nonzero spatial size. We continue our discussion in this section by examining perturbations around the Lorentzian fuzzy sphere solution. In the commutative limit, the perturbations are described by a scalar field coupled to a gauge field. The latter can be eliminated yielding a scalar field which can propagate in the Lorentzian region of the two-dimensional surface. Depending on the choice of parameters, the scalar field can be massive, massless or tachyonic.

3.1 Review: Fuzzy 2-sphere

In this section, we review the well-known example of the fuzzy 2-sphere S^2_F, and the NC gauge theory that is constructed on it. Here we mainly follow the approach in [15]. The starting point is the matrix model action in three-dimensional Euclidean embedding space

$$S(X) = \frac{1}{g^2}\text{Tr}\left(-\frac{1}{4}[X_i, X_j]^2 + \frac{2}{3}i\alpha\epsilon_{ijk}X_iX_jX_k\right), \quad i,j,k = 1,2,3 \tag{3.6}$$

where we have the cubic term in addition to the Yang-Mills term. Here the coordinates X_i are infinite-dimensional Hermitian matrices that transform covariantly under infinite-dimensional unitary gauge transformation (2.63). α is a dimensionful parameter which has the units of length. ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol where we adopt the convention $\epsilon_{123} = 1$. The action is invariant under three-dimensional translations, rotations and unitary gauge transformations (2.63). The equations of motion contain extra contribution not present in (3.2) that is due to the CS term,

$$[[X_i, X_j], X_j] + i\alpha\epsilon_{ijk}[X_j, X_k] = 0. \tag{3.7}$$

The equations of motion have discrete symmetries that they remain unaltered under the sign flips of any two of three components of X_i, namely proper reflections. An example of this discrete symmetry is

$$(X^1, X^2, X^3) \to (-X^1, -X^2, X^3). \tag{3.8}$$

(3.7) has the solution:

$$[\bar{X}_i, \bar{X}_j] = i\alpha\epsilon_{ijk}\bar{X}_k, \tag{3.9}$$

which defines the algebra of S_F^2. A constraint can be imposed on the Casimir:

$$\bar{X}_i\bar{X}_i = R^2, \tag{3.10}$$

where R is a dimensionful parameter that has the units of length.

For the N-dimensional irreducible representation, one has

$$R^2 = \alpha^2\frac{N^2-1}{4}. \tag{3.11}$$

The commutative limit is

$$\alpha \to 0, \qquad N \to \infty, \tag{3.12}$$

while keeping R fixed. In this limit, the matrices X_i becomes the commutative coordinates x_i, $i = 1, 2, 3$ in the three-dimensional embedding space that span the commutative 2-sphere S^2 with radius R, also in that limit the commutation relations (3.9) yields the Poisson brackets

$$\{x_i, x_j\} = \epsilon_{ijk}x_k. \tag{3.13}$$

In terms of the spherical coordinates θ, ϕ, one can write

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = R\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{3.14}$$

The induced metric on S^2 is given by the pullback of the Euclidean metric of x_i, i.e., $g_{ab} = \partial_a x_i \partial_b x_i$, and leads to

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.15}$$

A Poisson structure can be introduced on the two-dimensional manifold

$$\{f(\theta,\phi),h(\theta,\phi)\} = \frac{1}{\sin\theta}(\partial_\theta f\partial_\phi h - \partial_\phi f\partial_\theta h), \tag{3.16}$$

which is consistent with (3.13).

In the matrix model approach to NC field theory, the gauge field content of the theory can be introduced by considering fluctuations around the solutions

$$X^\mu = \bar{X}^\mu + \theta^{\mu\nu}A_\nu, \tag{3.17}$$

where $\theta^{\mu\nu}$ is constant. X^μ here correspond to the covariant coordinates (2.36) for the Moyal-Weyl plane. The gauge field A_μ plays the role of fluctuations around the classical solution $\bar{X}_\mu$. The matrix action can be expanded in powers of θ

$$S(X) = S_0(\bar{X}) + S_2(\bar{X}, A; \theta^2) + \mathcal{O}(\theta^3, \theta^4), \tag{3.18}$$

which is a finite order expansion that stops at $\mathcal{O}(\theta^4)$. In this expansion, $S_0(\bar{X})$ is the action evaluated at the classical solution. The first order contribution vanishes upon evaluated over the equations of motion, therefore the lowest order NC contribution arises as the second order expansion in θ. One can study properties of NC gauge theory in this manner.

For the corresponding expansion about S_F^2, we write

$$X_i = \bar{X}_i + \alpha R A_i, \tag{3.19}$$

where R is fixed and α is the expansion parameter. A_i, $i = 1, 2, 3$ are NC $U(1)$ gauge potentials on S_F^2 since they transform under the unitary gauge transformation U according to

$$A_i \to -\frac{1}{\alpha R}\bar{X}_i + \frac{1}{\alpha R}U\bar{X}_i U^\dagger + UA_i U^\dagger. \tag{3.20}$$

For "small" gauge parameter Λ, $U = \mathbb{1} + i\alpha R\Lambda$ and we have the gauge variations

$$\delta A_i = i[\Lambda, \bar{X}_i] + i\alpha R[\Lambda, A_i]. \tag{3.21}$$

The NC field strength can be defined in the following way

$$F_{ij}(X) = \frac{1}{\alpha^2 R^2}[X_i, X_j] - \frac{i}{\alpha R^2}\epsilon_{ijk}X_k. \tag{3.22}$$

It gauge transforms according to $F_{ij} \to UF_{ij}U^{-1}$. The definition by construction guarantees that F_{ij} vanishes when being evaluated upon the equations of motion (3.7), i.e., $F_{ij}(\bar{X}) = 0$. Substituting (3.19) and (3.22) into (3.6) leads to

$$S(X) - S(\bar{X}) = \frac{\alpha^4 R^2}{g^2}\mathrm{Tr}\Big\{ -\frac{R^2}{4}F_{ij}F_{ij} - \frac{iR}{6}\epsilon_{ijk}F_{ij}A_k - \frac{i}{6\alpha}\epsilon_{ijk}[A_i, \bar{X}_j]A_k - \frac{1}{6}A_iA_i\Big\}. \tag{3.23}$$

Next we examine the commutative limit $\alpha \to 0$ of the action (3.23). It gives a commutative gauge theory living on S^2. Similar to the standard de-quantization prescription, in the commutative limit, the trace on functions of the fuzzy sphere is replaced by the corresponding integration on the sphere and commutators are replaced by Poisson brackets, i.e.,

$$\frac{1}{N}\mathrm{Tr} \to \int \frac{d\Omega(\theta,\phi)}{4\pi}, \tag{3.24}$$

$$[f(\bar{X}), h(\bar{X})] \to \frac{i\alpha}{R}\{f(x), h(x)\}, \tag{3.25}$$

$$f(X) \to f(x), \tag{3.26}$$

where $d\Omega(\theta,\phi)$ is the integration measure in S^2. The relevant integration measure $d\Omega(\theta,\phi)$ should be such that the standard trace identities survive in the limit, i.e., for any three functions G, H and K on the sphere we want $\int d\Omega(\theta,\phi)\{G,H\}K = \int d\Omega(\theta,\phi)G\{H,K\}$.

In addition to the above prescription, the NC gauge potentials $A_i(X)$ can be replaced by their symbols. We write these three degrees of freedom in terms of two tangential degrees of freedom $\mathcal{A}_\theta(\theta,\phi), \mathcal{A}_\phi(\theta,\phi)$, plus one normal degree of freedom $\Phi(\theta,\phi)$. From the SW-map, $\mathcal{A}_\theta(\theta,\phi)$ and $\mathcal{A}_\phi(\theta,\phi)$ transform as commutative $U(1)$ gauge potentials while $\Phi(\theta,\phi)$ behaves as a scalar. The commutative limit of the NC gauge potentials can be expressed as linear combinations of these degrees of freedom, as the lowest order SW-map,

$$A_i \to \mathcal{A}_\theta(\theta,\phi)K_i^\theta + \mathcal{A}_\phi(\theta,\phi)K_i^\phi + \frac{1}{R}\Phi(\theta,\phi)x_i, \tag{3.27}$$

where K_i are the components of the Killing vector on S^2 and can be summarized in the following expression:

$$K_i^a = g^{ab}\epsilon_{ijk}x_j\partial_b x_k. \tag{3.28}$$

Under these maps (3.23) results in

$$S(X) - S(\bar{X}) \to \frac{C\alpha^4 R^2}{4\pi g^2} \int d\theta d\phi \frac{1}{2\sin\theta}[(\mathcal{F}_{\theta\phi})^2 + \sin^2\theta(\partial_\theta\Phi)^2 + (\partial_\phi\Phi)^2 - 4\sin\theta\Phi\mathcal{F}_{\theta\phi} + 2\sin^2\theta\Phi^2], \quad (3.29)$$

where C is a constant factor that will be absorbed into g later in the definition of the commutative coupling g_c. $\mathcal{F}_{\theta\phi}$ is the commutative U(1) field strength $\mathcal{F}_{\theta\phi} = \partial_\theta\mathcal{A}_\phi - \partial_\phi\mathcal{A}_\theta$.

What we have reviewed so far is the fuzzy 2-sphere solution and the NC $U(1)$ gauge theory living on this NC space. The $U(1)$ gauge theory can be generalized to a $U(M)$ gauge theory in a fairly straightforward way. For this we use the obvious result that $\bar{X}_i \to \bar{X}_i \otimes \mathbb{1}_M$, where $\bar{X}_i$ satisfies (3.9) and (3.10), and $\mathbb{1}_M$ is the M-dimensional unit matrix, also a solution to the equations of motion (3.7). The gauge group is now NC $U(M)$, and thus we replace the $U(1)$ potential A_i by $A_i^m \otimes T^m, m = 1, ..., M^2$, where T^m are the generators of the $U(M)$ gauge group. The same procedure applies for the perturbation theory, which gives

$$S(X) - S(\bar{X}) \to \frac{\alpha^4 R^2}{g_c^2} \int d\theta d\phi [\frac{1}{2\sin\theta}(\mathcal{F}_{\theta\phi}^m)^2 + \frac{\sin\theta}{2}(D_\theta\Phi^m)^2 + \frac{1}{2\sin\theta}(D_\phi\Phi^m)^2 - 2\Phi^m\mathcal{F}_{\theta\phi}^m + \sin\theta(\Phi^m)^2], \quad (3.30)$$

where the indices "m" label a basis in the adjoint representation of $U(M)$. Notice that (3.30) almost shares the same form as the action of the $U(1)$ gauge theory, except that all the partial derivatives are replaced by $U(M)$ covariant derivatives, e.g., $(D_a\Phi)^m = (\partial_a\Phi)^m - C_{pq}^m\mathcal{A}^p\Phi^q$, where C_{pq}^m is the structure constant of $u(M)$, i.e., $[T_p, T_q] = iC_{pq}^m T_m$.

Another generalization of the NC gauge theory is the supersymmetric generalization when fermions are included in the matrix model action (3.6). The supersymmetric action can be written as

$$S(X,\Psi) = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[X_i,X_j]^2 + \frac{2}{3}i\alpha\epsilon_{ijk}X_iX_jX_k + \frac{1}{2}\bar{\Psi}\sigma_i[X_i,\Psi] + \alpha\bar{\Psi}\Psi\right). \tag{3.31}$$

where Ψ is an infinite-dimensional Hermitian matrix, its matrix elements Ψ_{AB} are three-dimensional Majorana-Weyl spinors. σ^i $(i = 1, 2, 3)$ denote Pauli matrices. The third term in (3.31) is the usual supersymmetric counterpart of the Yang-Mills term as appeared in the IKKT action (2.62), and the fourth term in (3.31) is the supersymmetric counterpart of the CS term.

The action (3.31) is now gauge invariant under (2.63) and (2.64). It also respects $\mathcal{N} = 2$ supersymmetries:

$$\delta_1 X_i = i\bar{\epsilon}\sigma_i\Psi, \quad \delta_1\Psi = \frac{1}{2}[X_i,X_j][\sigma^i,\sigma^j]\epsilon \tag{3.32}$$

and

$$\delta_2 X_i = 0, \quad \delta_2\psi = \zeta. \tag{3.33}$$

where ϵ and ζ are infinitesimal Grassmann spinors.

3.2 Lorentzian fuzzy 2-sphere

The fuzzy sphere solutions of the previous section, have been applied in particle physics to make extra dimensions noncommutative[42],[43],[44],[45],[46],[47],[48],[49],[50],[51],[52],[53],[54]. But the fuzzy 2-sphere does not necessarily have to live in a Euclidean ambient space. In this section, we will show that S_F^2 can also be a solution to an IKKT-type matrix model with a Minkowski background metric tensor [23]. This means that in addition to making extra dimensions noncommutative, S_F^2 can be used to make space-time noncommutative. Moreover, they can serve as toy models for noncommutative cosmological space-times. The finite-dimensional matrix description has the advantage of resolving cosmological singularities. These singularities only appear after taking the commutative limit, $N \to \infty$.

3.2.1 Fuzzy sphere solution to Lorentzian IKKT matrix models

The setting here is the bosonic sector of a Lorentzian IKKT-type matrix model in three space-time dimensions. The dynamical degrees of freedom for the matrix model are contained in three infinite-dimensional Hermitian matrices X^μ, $\mu = 0, 1, 2$, with $\mu = 0$ indicating a time-like direction. In addition to the standard Yang-Mills term and cubic term, now we include a quadratic term in the action (all three terms are necessary for obtaining fuzzy sphere solutions):

$$S(X) = \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{2}{3}i\alpha\epsilon_{\mu\nu\lambda}X^\mu X^\nu X^\lambda + \frac{\beta}{2}X_\mu X^\mu\Big) , \qquad (3.34)$$

where g, α and β are real coefficients. Our conventions are $\epsilon_{012} = 1$, and we raise and lower indices $\mu, \nu, ...$ with the flat metric $[\eta_{\mu\nu}]$ =diag$(-1, 1, 1)$. The resulting equations of motion are

$$[[X_\mu, X_\nu], X^\nu] + i\alpha\epsilon_{\mu\nu\lambda}[X^\nu, X^\lambda] = -\beta X_\mu. \qquad (3.35)$$

The dynamics is invariant under three-dimensional Lorentz transformations, $X^\mu \rightarrow L^\mu{}_\nu X^\nu$, where L is a 3×3 Lorentz matrix, and unitary 'gauge' transformations, $X^\mu \rightarrow U X^\mu U^\dagger$, where U is an infinite-dimensional unitary matrix. Similar to the case when the quadric term absent, we still have the discrete proper reflection symmetry in the equation of motion (3.35). An example was (3.8). Translation invariance in the three-dimensional Minkowski space is broken when $\beta \neq 0$.

When $\beta \neq 0$, there exist finite dimensional matrix solutions to the equations of motion (3.35), which are associated with the $su(2)$ algebra. Say that a basis for the latter are $N\times N$ Hermitian matrices $J_i,\ i=1,2,3$, satisfying $[J_i,J_j] = i\alpha\epsilon_{ijk}J_k$.[1] Let us set

$$X^0 = \frac{w_3}{\alpha}J_3 \qquad X^1 = \frac{w_1}{\alpha}J_1 \qquad X^2 = \frac{w_2}{\alpha}J_2\ , \tag{3.36}$$

where w_i are real. Upon substituting this expression into the equations of motion (3.35) one gets

$$\begin{aligned}
(w_1^2 + w_2^2 + \beta)w_3 + 2\alpha w_1 w_2 &= 0 \\
(w_2^2 - w_3^2 + \beta)w_1 - 2\alpha w_2 w_3 &= 0 \\
(w_1^2 - w_3^2 + \beta)w_2 - 2\alpha w_1 w_3 &= 0\ ,
\end{aligned} \tag{3.37}$$

which has nontrivial solutions for w_i. Lorentz symmetry is broken by the solutions (unlike the case with de Sitter and anti-de Sitter solutions[18],[21]). The $su(2)$ Casimir operator for any of the solutions can be written as $\frac{1}{w_3^2}(X^0)^2 + \frac{1}{w_1^2}(X^1)^2 + \frac{1}{w_2^2}(X^2)^2$, which has the value $\frac{1}{4}(N^2-1)$ in the $N-$dimensional irreducible representation, thereby defining a fuzzy sphere or, actually, a fuzzy ellipsoid, since rotational invariance in the (X^0, X^1, X^2) space does not in general hold.

[1] The Levi-Civita symbol here is associated with Euclidian space, unlike the one appearing in (3.34) which is associated with Minkowski space.

In the special case where $w_1^2 = w_2^2 = w_3^2$, the solution is invariant under the full three-dimensional rotation group. Let us more generally restrict to the case of rotational invariance in the (X^1, X^2) plane, which means $w_1^2 = w_2^2$. Two simple solutions exist in this case:

$$X^0 = 2J_3 \qquad X^1 = \frac{\sqrt{-\beta}}{\alpha}\, J_1 \qquad X^2 = -\frac{\sqrt{-\beta}}{\alpha}\, J_2 \ , \tag{3.38}$$

and

$$X^0 = -2J_3 \qquad X^1 = \frac{\sqrt{-\beta}}{\alpha}\, J_1 \qquad X^2 = \frac{\sqrt{-\beta}}{\alpha}\, J_2 \ , \tag{3.39}$$

In order for them to be nontrivial one requires the presence of both the cubic and quadratic terms in (3.34), $\alpha \neq 0$ and $\beta < 0$. Solutions (3.38) and (3.39) are equivalent due to the discrete symmetry similar to (3.8). For the sake of definiteness we choose to work with the former, (3.38). The $su(2)$ Casimir operator for this solution can be written as

$$-\frac{\beta}{4\alpha^2}(X^0)^2 + (X^1)^2 + (X^2)^2 \ , \tag{3.40}$$

having the value $-\frac{\beta}{4}(N^2-1)$ in the $N-$dimensional irreducible representation. The 'time' matrix X^0 then has discrete eigenvalues $2\alpha m$, where $m = \frac{-N+1}{2}, \frac{-N+3}{2}, ..., \frac{N-1}{2}$. For any m defining a time-slice we can also define a spatial size. Call A the 'space' matrix, where $A^2 = X_+X_-$ and $X_\pm = X^1 \pm iX^2$. We can identify it with $-\frac{\beta}{\alpha^2}(\vec{J}^2 - J_3^2 - J_3)$ for the solution (3.38). A^2 then commutes with X^0 and has eigenvalues $-\beta\Big(\frac{N^2-1}{4} - m^2 - m\Big)$. Thus the time and the spatial size are discrete. Examples of spectra for (X^0, A^2) for some $N-$dimensional representations are

$N = 2 \qquad (-\alpha, -\beta),\ (\alpha, 0)$

$N = 3 \qquad (-2\alpha, -2\beta),\ (\alpha, -\beta),\ (2\alpha, 0)$

$$N = 4 \qquad (-3\alpha, -3\beta),\ (-\alpha, -4\beta),\ (\alpha, -3\beta),\ (3\alpha, 0)$$
$$N = 5 \qquad (-4\alpha, -4\beta),\ (-2\alpha, -6\beta),\ (\alpha, -6\beta),\ (2\alpha, -4\beta),\ (4\alpha, 0) \quad (3.41)$$

Say $\alpha > 0$. Then for large N, the spatial size operator A has eigenvalue a_n equal to $\sqrt{-\beta N}$ for the lowest time eigenvalue $\sim -\alpha N$, i.e., the initial state. a_n increases to a maximum value of $\sqrt{-\beta}\,N/2$ as the time goes to zero, and then decreases to zero upon approaching the highest time eigenvalue $\sim \alpha N$, i.e., the final state. This solution can thus be regarded as a discrete analogue of a closed cosmological space-time. The eigenvalues of X^0 versus those of A are plotted for $N = 100$ in Fig. 3.1.

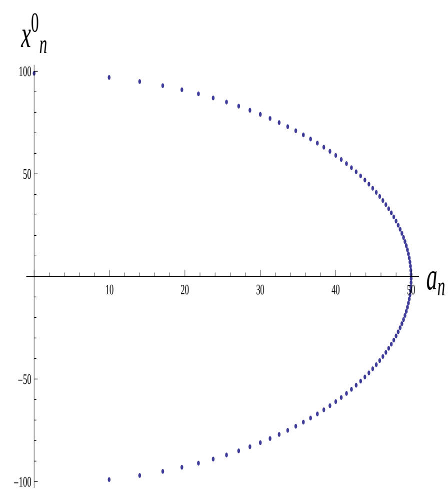

Fig. 3.1. Fuzzy closed universe solution. Plot of the eigenvalues x^0_n of the time matrix X^0 versus the eigenvalues a_n of the space matrix A for $N = 100$, $\alpha = 1$ and $\beta = -1$.

Just as with the fuzzy sphere in a Euclidean background, the commutative limit of the matrix solution here is obtained by taking $N \to \infty$ and $\alpha \to 0$, with αN finite in the limit. Here we also need $\beta \to 0$, with $\sqrt{-\beta}\,N$ finite in the limit. The commutative limit of the solution is then characterized by two real parameters, which we denote by a_0 (not to be confused with an eigenvalue of A)

and r^2,

$$\frac{\sqrt{-\beta}}{2\alpha} \to a_0 \qquad\qquad \frac{\sqrt{-\beta}N}{2} \to r \tag{3.42}$$

Similar to the procedure introduced in section 3.1, one typically defines the commutative limit in analogous fashion to the classical limit of a quantum theory, where α plays an analogous role to $\hbar$. In this limit one replaces the matrices X^μ by commuting space-time coordinates which we denote by x^μ, where x^0 and x^i, $i = 1, 2$, denote the time and space coordinates, respectively. The constraint on the $su(2)$ Casimir operator (3.40) means that in the commutative limit the solution satisfies

$$a_0^2(x^0)^2 + (x^1)^2 + (x^2)^2 = r^2 \tag{3.43}$$

While real a_0 means that the solution is topologically a two sphere, there are a number of novel features, which we show below, due to the fact that this 'sphere' is embedded in Minkowksi space-time.

The commutative limit requires replacing the commutator of functions of X^μ, evaluated for the solution (3.38), by $i\alpha$ times the Poisson bracket of the same functions of the coordinates x^μ. The commutators of X^μ in (3.38) lead to the following Poisson brackets of the coordinates:

$$\{x^0, x^1\} = -2x^2 \qquad \{x^2, x^0\} = -2x^1 \qquad \{x^1, x^2\} = -2a_0^2 x^0 \tag{3.44}$$

We can express x^μ in terms of angular momenta j_i, $i = 1, 2, 3$, which satisfies the $su(2)$ Poisson bracket algebra $\{j_i, j_j\} = \epsilon_{ijk} j_k$, using

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = 2 \begin{pmatrix} j_3 \\ a_0 j_1 \\ -a_0 j_2 \end{pmatrix} \tag{3.45}$$

and from (3.43), $j_1^2 + j_2^2 + j_3^2 = (\frac{r}{2a_0})^2$. For simplicity, we set $r = 2a_0$ so that j_i spans a sphere of unit radius. We can introduce standard spherical coordinates

(θ, ϕ), $0 < \theta < \pi,\ 0 \leq \phi < 2\pi$, and write

$$j_1 = \sin\theta\cos\phi \qquad j_2 = \sin\theta\sin\phi \qquad j_3 = \cos\theta \tag{3.46}$$

The $su(2)$ Poisson bracket algebra for j_i is recovered upon defining the Poisson brackets on the sphere to again be (3.16).

The induced metric $\mathtt{g}_{ab} = \partial_a x^\mu \partial_b x_\mu$, $a, b, ... = \theta, \phi$, computed from (3.45) and (3.46) does not agree with the standard metric on the sphere as shown in (3.15), and, moreover, it does not have a fixed signature. Moreover, the curvature computed from the induced metric is not constant, and it is negative. The invariant interval constructed from the induced metric is

$$-\,d\tau^2 = 4\Big(a_0^2\cos^2\theta - \sin^2\theta\Big)\,d\theta^2 + 4a_0^2\sin^2\theta\,d\phi^2 \tag{3.47}$$

$\mathtt{g}_{\theta\theta}$ vanishes at two latitudes $\theta = \theta_\pm$ on the sphere defined by $\tan\theta_\pm = \pm a_0$. Say that $\theta = \theta_+$ is contained in the northern hemisphere, $0 < \theta_+ < \frac{\pi}{2}$, while $\theta = \theta_-$ is contained in the southern hemisphere, $\frac{\pi}{2} < \theta_- < \pi$. The signature on the sphere is Euclidean for $0 < \theta < \theta_+$ and $\theta_- < \theta < \pi$, while it is Lorentzian for $\theta_+ < \theta < \theta_-$. We can regard θ as a time-like variable for the latter, with $2a_0\sin\theta$ being the spatial radius at any time-slice. $\theta = \theta_\pm$ correspond to singularities in the curvature, as opposed to coordinate singularities. The Ricci scalar computed from the induced metric is

$$R = -\frac{1}{2(a_0^2\cos\theta^2 - \sin^2\theta)^2}\ , \tag{3.48}$$

and thus it is singular at the latitudes $\theta = \theta_\pm$. Equation (3.48) shows that the curvature in the nonsingular regions is everywhere negative. The singularities of the Ricci tensor are analogous to big bang/crunch singularities, with the distinction that they occur at a nonzero spatial radius $2a_0 \sin\theta_\pm = \frac{2a_0^2}{\sqrt{a_0^2+1}}$. Time-like longitudinal geodesics exist in the Lorentzian region which originate and terminate at the singular latitudes $\theta = \theta_\pm$. This is because their tangent vectors $(\frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}) = \Big(\frac{1}{\sqrt{\sin^2\theta - a_0^2\cos\theta^2}}\,, 0\Big)$ are well defined in the Lorentzian region, $\theta_+ < \theta < \theta_-$, while they are imaginary in the Euclidean regions, $0 < \theta < \theta_+$ and $\theta_- < \theta < \pi$. The total elapsed proper time along these geodesics is finite and given by the elliptic integral $2\int_{\tan^{-1}a_0}^{\pi-\tan^{-1}a_0} d\theta \sqrt{\sin^2\theta - a_0^2\cos\theta^2}$.

3.2.2 Emergent field dynamics

Here we perturb around the matrix solution (3.38). Similar to (3.22), we find it useful to define noncommutative field strengths $F_{\mu\nu}$ on the fuzzy sphere. Here we take

$$\begin{aligned} F^{01} &= \frac{1}{\alpha}[X^0, X^1] + 2iX^2 \\ F^{02} &= \frac{1}{\alpha}[X^0, X^2] - 2iX^1 \\ F^{12} &= \frac{1}{\alpha}[X^1, X^2] - \frac{i\beta}{2\alpha}X^0 \,, \end{aligned} \tag{3.49}$$

which like (3.22) transform covariantly under unitary gauge transformations, $F_{\mu\nu} \to UF_{\mu\nu}U^\dagger$, and vanish when evaluated on the fuzzy sphere solutions (3.36). The matrix action (3.34) can then be re-expressed in terms of the noncommutative field strengths

$$\begin{aligned} g^2 S(X) = \mathrm{Tr}\Big\{ &-\frac{\alpha^2}{4}F_{\mu\nu}F^{\mu\nu} - \frac{4}{3}i\alpha^2(F^{01}X^2 + F^{20}X^1) + i\alpha^2\Big(\frac{2}{3} - \frac{\beta}{2\alpha^2}\Big)F^{12}X^0 \\ &+\Big(\frac{\beta}{2} - \frac{2\alpha^2}{3}\Big)\Big((X^1)^2 + (X^2)^2\Big) + \beta\Big(\frac{\beta}{8\alpha^2} - \frac{5}{6}\Big)(X^0)^2\Big\}. \end{aligned} \tag{3.50}$$

Now we perturb around the matrix solution (3.38) using

$$X^0 = 2\Big(J_3 + \frac{\alpha^2}{\sqrt{-\beta}}A^0\Big) \quad X^1 = \frac{\sqrt{-\beta}}{\alpha}\,J_1 + \alpha A^1 \quad X^2 = -\frac{\sqrt{-\beta}}{\alpha}\,J_2 - \alpha A^2\ , \tag{3.51}$$

where the perturbations are functions on the fuzzy sphere, $A^\mu = A^\mu(J_1, J_2, J_3)$. If we write infinitesimal unitary gauge transformations using $U = \mathbb{1} - \frac{i\alpha}{\sqrt{-\beta}}\Lambda$, where Λ is a Hermitian matrix with infinitesimal elements, then the infinitesimal variations of A^μ read

$$\delta A^\mu \ = \ -i\Big(\frac{1}{\alpha}[\Lambda, J^\mu] + \frac{\alpha}{\sqrt{-\beta}}[\Lambda, A^\mu]\Big)\ , \tag{3.52}$$

where we identify (J^0, J^1, J^2) with (J_3, J_1, J_2). Substituting (3.51) into (3.50) gives

$$\begin{aligned} S(X) = \frac{\alpha^2}{g^2}\mathrm{Tr}\Big\{ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{4}{3}i\alpha(F^{01}A^2 + F^{20}A^1) + \frac{2i\alpha^2}{\sqrt{-\beta}}\Big(\frac{2}{3} - \frac{\beta}{2\alpha^2}\Big)F^{12}A^0 \\ &+ \frac{8i\alpha}{3}\,([J_1, A^2] - [J_2, A^1])A^0 - 2i\alpha\Big(\frac{2}{3} - \frac{\beta}{2\alpha^2}\Big)[A^1, A^2]J_3 \\ &+ \Big(\frac{\beta}{2} - \frac{2\alpha^2}{3}\Big)\Big((A^1)^2 + (A^2)^2\Big) - 2\alpha^2\Big(\frac{\beta}{4\alpha^2} - \frac{5}{3}\Big)(A^0)^2\Big\} + S(\bar{X}) \end{aligned} \tag{3.53}$$

As stated previously, the commutative limit is obtained by taking $N \to \infty$, along with $\alpha, \beta \to 0$ and both αN and $\sqrt{-\beta}\,N$ are finite in the limit. Upon using (3.42) and (3.16), the commutative limit of the field strengths (3.49) is

$$\begin{aligned} F^{01} \ &\to \ 2i\alpha\Big(\{j_3, A^1\} - \{j_1, A^0\} - A^2\Big) \\ F^{02} \ &\to \ -2i\alpha\Big(\{j_3, A^2\} - \{j_2, A^0\} + A^1\Big) \\ F^{12} \ &\to \ -2i\alpha a_0\Big(\{j_1, A^2\} - \{j_2, A^1\} - A^0\Big)\ , \end{aligned} \tag{3.54}$$

where A^μ are now functions on the commutative sphere. From (3.16) we need to choose the standard integration measure on the sphere $d\mu(\theta,\phi) = d\Omega(\theta,\phi) = \sin\theta d\theta d\phi$ (rather than say $\sqrt{-\mathsf{g}}\,d\theta d\phi$, where g is the determinant of the induced metric). Then the action (3.53) reduces to

$$S(X) - S(\bar{X}) \to \frac{2\alpha^4}{g_c^2}\int \sin\theta d\theta d\phi \Big\{ -\Big(\{j_3, A^1\} - \{j_1, A^0\}\Big)^2 - \Big(\{j_3, A^2\} - \{j_2, A^0\}\Big)^2 + a_0^2\Big(\{j_1, A^2\} - \{j_2, A^1\}\Big)^2 + 2(a_0^2+1)\{j_3, A^1\}A^2 + (A^0)^2 - a_0^2\Big((A^1)^2 + (A^2)^2\Big)\Big\} , \qquad (3.55)$$

where g_c is the commutative limit of the constant g. Once again, we follow the treatment of the previous subsection and write the perturbations A^μ in terms of commutative gauge potentials $(\mathcal{A}_\theta, \mathcal{A}_\phi)$ and a scalar field ψ on the sphere, now replacing (3.27) by

$$\begin{aligned} A^0 &= \mathcal{A}_\phi + j_3\psi \\ A^1 &= -\sin\phi\mathcal{A}_\theta - \cot\theta\cos\phi\mathcal{A}_\phi + j_1\psi \\ A^2 &= \cos\phi\mathcal{A}_\theta - \cot\theta\sin\phi\mathcal{A}_\phi + j_2\psi \; . \end{aligned} \qquad (3.56)$$

Then from the fundamental Poisson bracket (3.16), gauge variations $(\delta\mathcal{A}_\theta, \delta\mathcal{A}_\phi) = (\partial_\theta\lambda, \partial_\phi\lambda)$ agree with the commutative limit of (3.52), where λ is now an infinitesimal function on the commutative sphere. Substituting (3.56) in (3.55) gives

$$S(X) - S(\bar{X}) \to \frac{2\alpha^4}{g_c^2}\int \sin\theta d\theta d\phi \Big\{(a_0^2\cot^2\theta - 1)\mathcal{F}_{\theta\phi}^2 - \csc^2\theta(\partial_\phi\psi)^2 + \Big(a_0^2\sin^2\theta - \cos^2\theta\Big)(\partial_\theta\psi)^2 - \Big(3 - 2(a_0^2+1)\sin^2\theta\Big)\psi^2 + 2\csc\theta\Big((a_0^2+1)\sin^2\theta - 2a_0^2 + 1\Big)\mathcal{F}_{\theta\phi}\psi - 2\cos\theta(a_0^2+1)\mathcal{F}_{\theta\phi}\partial_\theta\psi\Big\} , \qquad (3.57)$$

where again $\mathcal{F}_{\theta\phi} = \partial_\theta \mathcal{A}_\phi - \partial_\phi \mathcal{A}_\theta$ is the commutative $U(1)$ field strength on the surface. We remark that the gauge field and scalar field kinetic energies have opposite signs, a feature that is present in similar two-dimensional systems[21]. However, gauge fields are nondynamical in two-dimensions. We can solve for $\mathcal{F}_{\theta\phi}$ from the field equations, yielding

$$\mathcal{F}_{\theta\phi} = \frac{\cos\theta(a_0^2+1)\partial_\theta\psi - \Big((a_0^2+1)\sin^2\theta - 2a_0^2+1\Big)\csc\theta\;\psi}{a_0^2\cot^2\theta - 1} + \text{constant}\;, \quad (3.58)$$

and substitute back into the action. Upon setting the constant equal to zero, we get

$$\begin{aligned} S(X) - S(\bar{X}) \to \frac{2\alpha^4 a_0^2}{g_c^2}\int \sin\theta d\theta d\phi \left\{ \frac{(\partial_\theta\psi)^2}{(a_0^2+1)\sin^2\theta - a_0^2} - \frac{\csc^2\theta}{a_0^2}(\partial_\phi\psi)^2 - 4m_{\text{eff}}^2\psi^2 \right\} \\ = \frac{16\alpha^4 a_0^2}{g_c^2}\int \sin\theta d\theta d\phi \left\{ -\frac{1}{2}\partial^a\psi\partial_a\psi \;-\; \frac{1}{2}m_{\text{eff}}^2\psi^2 \right\}, \end{aligned} \quad (3.59)$$

where the index $a = (\theta, \phi)$ is raised and lowered using the induced metric given in (3.47). The effective mass squared of the scalar field is $\theta-$dependent

$$m_{\text{eff}}^2 = \frac{(a_0^2-1)\Big((a_0^2+1)\sin^2\theta - 3a_0^2\Big)}{4a_0^2\Big((a_0^2+1)\sin^2\theta - a_0^2\Big)^2} \quad (3.60)$$

As stated before, the signature of the induced metric is Euclidean when $\sin^2\theta < \frac{a_0^2}{a_0^2+1}$, and Lorentzian when $\sin^2\theta > \frac{a_0^2}{a_0^2+1}$. Therefore (3.59) describes a Euclidean field theory for the former and a Lorentzian field theory for the latter. There are three different possibilities for the Lorentzian field theory:

a) The action describes a tachyon when $a_0^2 > 1$. This is because the factor $(a_0^2+1)\sin^2\theta - 3a_0^2$ in (3.60) is negative in this case.

b) The scalar field is massless when $a_0^2 = 1$.

c) The effective mass-squared for the scalar field is positive when

$$a_0^2 < 1 \qquad \text{and} \qquad \frac{a_0^2}{a_0^2+1} < \sin^2\theta < \frac{3a_0^2}{a_0^2+1} \tag{3.61}$$

It follows that the action (3.59) describes a massive scalar field throughout the entire Lorentzian region when $\frac{1}{2} \leq a_0^2 < 1$. On the other hand, when $a_0^2 < \frac{1}{2}$ the scalar field becomes tachyonic in the region where $\sin^2\theta > \frac{3a_0^2}{a_0^2+1}$.

3.3 Fuzzy dS^2

We continue our discussion of two-dimensional NC spaces in a three-dimensional Lorentzian background. Here we examine the two-dimensional fuzzy de Sitter (dS^2) space-time, which can also be obtained as a solution of the Lorentzian IKKT-type model introduced in section 3.2. The study of fuzzy dS^2 (or AdS^2) space-time and the associated field theories have appeared in the literature[16],[17],[55],[56], [18],[21],[19]. These works have focused on exploring aspects of fuzzy dS^2 such as its representation theory, NC gauge theory and emergent gravity from the matrix model. However, there remains open topics regarding the cosmological interpretations and the stability issue of the commutative gauge theory.

Starting with the same bosonic IKKT action (3.34), one can easily find the fuzzy dS^2 to be a solution of the equations of motion, only here the solution can be obtained regardless of the presence of the quadratic term in the action. The fuzzy dS^2 solution is associated with an $su(1,1)$ algebra, for which one recovers the principal, supplementary and discrete series representations[18]. One feature of the discrete series is that there is a state that is killed by lowering (raising) operator which leads to minimum (maximum) time eigenvalue associated with the minimum radial eigenvalue. In the commutative limit it implies an initial (final) space-time singularity. Thus the discrete series solution provides a noncommutative resolution of a big bang (crunch) singularity.

As is in the previous subsection, we consider small perturbations about the NC solution and then take the commutative limit of the action. Again, the result is a scalar field theory on the space-time manifold associated with the commutative limit of the solution. The quadratic term will turn out beneficial to ensure the stability of the resulting commutative field theory. It is worth pointing out that the procedure used here is only valid for NC spaces associated with a Lie-algebra. For solutions which are not of this type, a general analysis of the perturbation theory is required. It involves obtaining a nontrivial Seiberg-Witten map [19] on the NC space defined by the solution. In Appendix A we give the result for the Seiberg-Witten map on a general two-dimensional manifold and show that the two approaches give the same results for the fuzzy dS^2 solution. We show that for an appropriate range of the coefficients the effective mass squared of the scalar field can be positive ensuring the stability of the field theory in the commutative limit.

3.3.1 Fuzzy dS^2 solution to a matrix model

Once again, we examine the Lorentzian bosonic IKKT action in (3.34),

$$S(X) = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{4}[X_\mu, X_\nu]^2 + \frac{2}{3}i\tilde{\alpha}\epsilon_{\mu\nu\lambda}X^\mu X^\nu X^\lambda + \frac{\beta}{2}X_\mu X^\mu\right), \tag{3.62}$$

where we replace α in (3.34) by $\tilde{\alpha}$ in the cubic term. The equations of motion are the same as (3.35) with α replaced by $\tilde{\alpha}$,

$$[[X_\mu, X_\nu], X^\nu] + i\tilde{\alpha}\epsilon_{\mu\nu\lambda}[X^\nu, X^\lambda] = -\beta X_\mu. \tag{3.63}$$

which admits the fuzzy dS^2 space-time as a solution $X_\mu = \bar{X}_\mu$

$$[\bar{X}_\mu, \bar{X}_\nu] = i\alpha\epsilon_{\mu\nu\lambda}\bar{X}^\lambda. \tag{3.64}$$

Notice that in (3.64), α is not the same as $\tilde{\alpha}$ appeared in (3.62) and (3.63). In general, we have solutions (3.64) with α given by

$$\alpha_\pm = \frac{1}{2}(\tilde{\alpha} \pm \sqrt{\tilde{\alpha}^2 + 2\beta}). \tag{3.65}$$

When $\tilde{\alpha}^2 + 2\beta > 0$, there exist two distinct solutions to the equations of motion (3.63) . They are associated with two distinct fuzzy dS^2 solutions. In the commutative limit, they go to two distinct dS^2 space-times. In both cases $\bar{X}^\mu$ span an $su(1,1)$ Lie-algebra (3.64) with NC parameters $\alpha_\pm$. When $\beta = 0$, one of the solutions becomes that of $\alpha_+ = \tilde{\alpha}$ (this is the only situation when one can identify α with $\tilde{\alpha}$), while the other solution becomes the vacuum solution $\alpha_- \to 0$. When $\tilde{\alpha}^2 + 2\beta = 0$, the two solutions coincide and $\alpha_\pm = \frac{1}{2}\tilde{\alpha}$. Therefore we see that the quadratic term in (3.62) is not necessary for the sole purpose of obtaining the fuzzy dS^2 solution, which is unlike the case of the Lorentzian fuzzy sphere, where $\beta \neq 0$ is required.

To define the fuzzy dS^2 solution one fixes the quadratic Casimir by $\bar{X}_\mu \bar{X}^\mu = R^2$, where R is the length scale of the fuzzy dS^2 space, thus giving an irreducible representation of $su(1,1)$. To obtain the representations of the $su(1,1)$ Lie-algebra, and thereby representations of the fuzzy dS^2 space-time, it is conventional to re-express the matrices $\bar{X}^\mu$ in terms of the laddering operators $\bar{X}_\pm = \bar{X}_1 \pm i\bar{X}_2$. Then

$$[\bar{X}^0, \bar{X}_+] = \alpha \bar{X}_+, \quad [\bar{X}_+, \bar{X}_-] = -2\alpha \bar{X}^0 \ . \tag{3.66}$$

Irreducible representations of the $su(1,1)$ Lie-algebra are well-known and classified by values of the central operator

$$R^2 = \frac{1}{2}(\bar{X}_+\bar{X}_- + \bar{X}_-\bar{X}_+) - (\bar{X}^0)^2 \ \ , \tag{3.67}$$

which we denote by $-\alpha^2 j(j+1)$, and an additional parameter ϵ_0. States $|j, \epsilon_0, n\rangle$ in any irreducible representation can be taken to eigenvectors of $\bar{X}^0$, with $\bar{X}_+$ and

$\bar{X}_-$ behaving as *lowering* and *raising* operators acting on these states, respectively,

$$\begin{aligned}\bar{X}^0 \,|j,\epsilon_0,n\rangle &= -\alpha(\epsilon_0+n)\,|j,\epsilon_0,n\rangle \\ \bar{X}_+ \,|j,\epsilon_0,n\rangle &= i\alpha(j+\epsilon_0+n)\,|j,\epsilon_0,n-1\rangle \\ \bar{X}_- \,|j,\epsilon_0,n\rangle &= i\alpha(j-\epsilon_0-n)\,|j,\epsilon_0,n+1\rangle \,. \end{aligned} \tag{3.68}$$

It follows that

$$\bar{X}_+\bar{X}_- \,|j,\epsilon_0,n\rangle = \alpha^2\Big((\epsilon_0+n)(\epsilon_0+n+1)-j(j+1)\Big)\,|j,\epsilon_0,n\rangle \ . \tag{3.69}$$

We call $\bar{X}_+\bar{X}_-$ the square of spatial size of the fuzzy dS^2 space, with eigenvalues a_n^2, and call x_n^0 the eigenvalues of the time operator $\bar{X}^0$. The eigenvalues of $\bar{X}_+\bar{X}_-$ should be positive-definite, i.e., $a_n^2 \geq 0$. This condition along with (3.69) leads to the following inequality relation

$$\Big(\epsilon_0+n+\frac{1}{2}\Big)^2 \geq \Big(j+\frac{1}{2}\Big)^2 \ , \ \ \textit{for all}\ \ n. \tag{3.70}$$

Nontrivial representations are known to fall into three categories: principal, supplementary and discrete series. For the principal and supplementary series, neither $j+\epsilon_0$ nor $j-\epsilon_0$ are integers, so that no states $|j,\epsilon_0,n\rangle$ are killed by $\bar{X}_+$ or $\bar{X}_-$. There are then no restrictions on the integers n labeling the states, it then may be positive, negative or zero. One takes $j=-\frac{1}{2}+i\rho$, with ρ real, for the principal series, which identically satisfies (3.70). j is assumed to be real for the supplementary series. Then if we choose $-\frac{\pi}{2}\leq\epsilon_0<\frac{\pi}{2}$, we need that $|j+\frac{1}{2}|\leq|\epsilon_0+\frac{1}{2}|$.

Finally, for the discrete series one has that either $j+\epsilon_0$ or $j-\epsilon_0$ are integers. For the former, we can choose $j+\epsilon_0=0$. Then from (3.68), $\bar{X}_+$ kills $|j,-j,0\rangle$, which then serves the role as the bottom state for the irreducible representation

$D^+(j)$. In this case n is restricted to positive integers, including 0. The inequality (3.70) is satisfied for $j \leq 0$. Then the resulting spectra for $\bar{X}^0$ and $\bar{X}_+\bar{X}_-$ are given by

$$x_n^0 = \alpha(j-n) \qquad a_n^2 = \alpha^2(n+1)(n-2j)\ , \qquad n = 0, 1, 2, ... \tag{3.71}$$

The time takes on only negative eigenvalues, assuming $\alpha > 0$. Similarly, if one chooses $j - \epsilon_0 = 0$, then from (3.68), $\bar{X}_-$ kills $|j, j, 0\rangle$. The latter serves the role as the top state for the irreducible representation $D^-(j)$ and in this case n is restricted to negative integers, including 0. The inequality (3.70) is again satisfied for $j \leq 0$. The spectrum for $\bar{X}_+\bar{X}_-$ is the same as in the previous case (3.71), while there is a sign flip for x_n^0, i.e., the signs are now all positive.

We remark that the existence of a bottom (top) state for the discrete series, corresponding to a minimum value for the radial eigenvalues a_n, means that there is a lowest or highest value for the time eigenvalues. It is thus a discrete analogue of a cosmological singularity. Plots of the eigenvalues x_n^0 of the matrice $\bar{X}^0$, with respect to the square root of the eigenvalues of $\bar{X}_+\bar{X}_-$, i.e., a_n, are given in Fig. 3.2 for the discrete series $D^-(-\frac{1}{2})$, as well as for the principal series with $\rho = 10$ and 20.

In the commutative limit, $\bar{X}^\mu$ are replaced by the coordinates functions x^μ, the NC field theory reduces to an ordinary $U(1)$ gauge theory on two-dimensional de Sitter manifold, with embedding coordinates x^μ satisfying $(x^1)^2 + (x^2)^2 - (x^0)^2 = R^2$. Following [18], we adopt a parametrization for the three-dimensional embedding coordinates x^μ, which is given in terms of coordinates τ and σ on the commutative dS^2 manifold:

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = R \begin{pmatrix} \tan\tau \\ \sec\tau\,\cos\sigma \\ \sec\tau\,\sin\sigma \end{pmatrix} \tag{3.72}$$

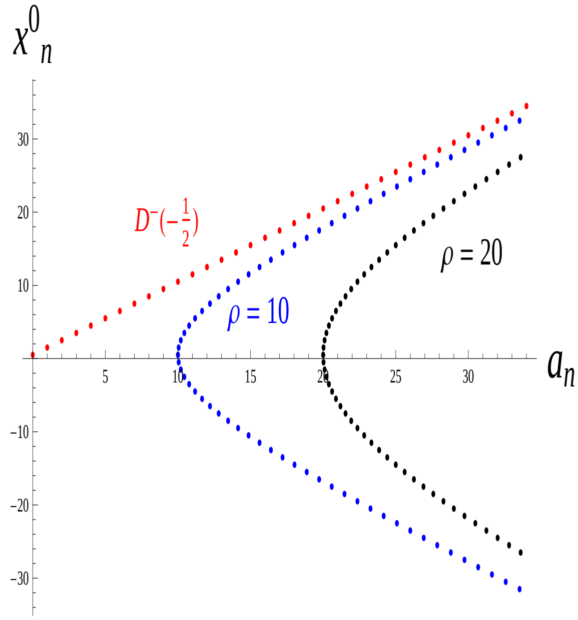

Fig. 3.2. Fuzzy dS^2 solutions. Plots of the eigenvalues x_n^0 and a_n for the discrete series $D^-(-\frac{1}{2})$ and for the principal series with $\rho = 10$ and 20. $\alpha = 1$ and $\epsilon_0 = -\frac{1}{2}$ are chosen in all cases.

where the parameter τ is time-like with $-\frac{\pi}{2} \leq \tau \leq \frac{\pi}{2}$ and σ is space-like with $-\pi \leq \sigma < \pi$. Following from $g_{ab} = \partial_a x_\mu \partial_b x^\mu \; a, b = \tau, \sigma$, the induced metric on the commutative dS^2 manifold in terms of τ and σ is given by

$$-g_{\tau\tau} = g_{\sigma\sigma} = \frac{R^2}{\cos^2\tau} \qquad\qquad g_{\tau\sigma} = 0 \tag{3.73}$$

therefore we have the Poisson bracket given by

$$\{\mathcal{F}(\tau,\sigma), \mathcal{H}(\tau,\sigma)\} = \cos^2\tau(\partial_\tau\mathcal{F}\partial_\sigma\mathcal{H} - \partial_\sigma\mathcal{F}\partial_\tau\mathcal{H}), \tag{3.74}$$

which produces the $su(1,1)$ Lie-algebra $\{x^\mu, x^\nu\} = R\epsilon^{\mu\nu\lambda}x_\lambda$.

3.3.2 NC gauge theory on fuzzy dS^2

In this subsection, we perform small perturbations around the fuzzy dS^2 solution $\bar{X}_\mu$, and the action expansion will give us a NC $U(1)$ gauge theory living on fuzzy dS^2. Like in the previous sections, in the commutative limit, the NC $U(1)$ gauge potentials $A_\mu(\bar{X})$ can be expressed in terms of commutative $U(1)$ gauge potentials $\mathcal{A}_\tau, \mathcal{A}_\sigma$ and a scalar field Φ. In the lowest order in θ, $\mathcal{A}_\tau$ and $\mathcal{A}_\sigma$ are tangential to the two-dimensional surface spanned by x^μ, and Φ is along the normal direction to the surface. The explicit form requires an orthogonal set of basis vectors on dS^2 manifold, and a natural choice is in terms of the normal vector x^μ and the Killing vectors on dS^2. Analogous to (3.28), the Killing vector can be expressed as

$$K^a_\mu = g^{ab}\epsilon_{\mu\nu\lambda}x^\nu\partial_b x^\lambda, \quad a,b=\tau,\sigma\ . \tag{3.75}$$

Moreover, we have the following identities:

$$K^a_\mu K^{b\mu} = R^2\mathsf{g}^{ab} \qquad K^\tau_\mu x^\mu = K^\sigma_\mu x^\mu = 0 \qquad \epsilon_{\mu\nu\lambda}K^{\tau\mu}K^{\sigma\nu}x^\lambda = R\cos^2\tau \tag{3.76}$$

and

$$\frac{\partial K^\tau_\mu}{\partial\tau} = \frac{\partial K^\sigma_\mu}{\partial\sigma} = -\tan\tau K^\tau_\mu - \frac{1}{R}x_\mu \qquad \frac{\partial K^\sigma_\mu}{\partial\tau} = \frac{\partial K^\tau_\mu}{\partial\sigma} = -\tan\tau K^\sigma_\mu$$
$$\frac{\partial x_\mu}{\partial\tau} = -\frac{R}{\cos^2\tau}K^\tau_\mu \qquad \frac{\partial x_\mu}{\partial\sigma} = \frac{R}{\cos^2\tau}K^\sigma_\mu\ . \tag{3.77}$$

We again expand A_μ in the above basis in the commutative limit similar to (3.27)

$$A_\mu \to \mathcal{A}_\tau(\tau,\sigma)K^\tau_\mu + \mathcal{A}_\sigma(\tau,\sigma)K^\sigma_\mu + \frac{1}{R}\Phi(\tau,\sigma)x_\mu \tag{3.78}$$

and apply

$$\text{Tr} \to C \int d\tau d\sigma \sqrt{\mathsf{g}}$$
$$[f(\bar{X}), h(\bar{X})] \to \frac{i\alpha}{R}\{f(x), h(x)\} \tag{3.79}$$

where C is some constant factor that along with the metric determinant g will amount to the coupling constant of the commutative gauge theory, i.e., g_c. In the following we perform the above maps for $\beta = 0$ and $\beta \neq 0$ cases separately.

We once again consider the perturbations of the form (3.17). Following the general argument in (3.18), the perturbation parameter α should scale as proportional to the NC parameter θ in the commutative limit. Similar to the previous sections, we define noncommutative field strengths $F_{\mu\nu}$ on fuzzy dS^2 according to

$$F_{\mu\nu} = \frac{1}{\alpha^2 R^2}[X_\mu, X_\nu] - \frac{i}{\alpha R^2}\epsilon_{\mu\nu\lambda}X^\lambda, \tag{3.80}$$

such that $F_{\mu\nu}$ vanishes when it is evaluated on the fuzzy dS^2 solution $X^\mu = \bar{X}^\mu$, where $\bar{X}^\mu$ satisfies (3.64). We next reexpress the matrix model action in terms of A_μ and $F_{\mu\nu}$.

We first examine the $\beta = 0$ case. When $\beta = 0$, we have $\tilde{\alpha} = \alpha$, and the action expansion gives

$$\begin{aligned} S(X) - S(\bar{X}) = \frac{\alpha^4 R^2}{g^2}\text{Tr}\Big(&- \frac{R^2}{4}F_{\mu\nu}F^{\mu\nu} - \frac{iR}{6}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^\lambda \\ &- \frac{i}{6\alpha}\epsilon_{\mu\nu\lambda}[A^\mu, \bar{X}^\nu]A^\lambda + \frac{1}{6}A_\mu A^\mu\Big) \end{aligned} \tag{3.81}$$

Our goal now is to obtain the commutative limit of the NC action (3.81). The action can be reexpressed on the two-dimensional de Sitter manifold upon using the appropriate star product for de Sitter space[18]. The latter can be expanded in the NC parameter α, and in the limit $\alpha \to 0$, it corresponds to the commutative limit of the NC theory. As stated above, the theory contains the $U(1)$ gauge potentials $\mathcal{A}_\tau$, $\mathcal{A}_\sigma$, that are tangential to the surface, and one scalar degree of freedom Φ that is associated with perturbations normal to the surface. Once again, the gauge fields are non-dynamical and can be eliminated from the theory leaving only the scalar degree of freedom. The stability of the field theory can be examined from the sign of the effective mass term in the resulting scalar field theory action. We will show that stability of the theory requires $\beta \neq 0$ in the matrix action (3.62).

Upon applying the maps (3.79), the action (3.81) may be written as an integral of symbol functions according to

$$S(X) - S(\bar{X}) \rightarrow \frac{\alpha^4 R^2}{g_c^2} \int d\mu \Big\{ \frac{1}{4} \Big(\{A_\mu, x_\nu\} - \{A_\nu, x_\mu\} \Big)^2 - \frac{R}{2} \epsilon_{\mu\nu\lambda} \{A^\mu, x^\nu\} A^\lambda \Big\}, \tag{3.82}$$

where $d\mu$ is the invariant measure over dS^2, which takes the form $d\mu = d\tau d\sigma\, R^2/\cos^2\tau$ in terms of the parametrization (3.72). Using the previously established identities in this subsection, we get

$$\begin{aligned} \{A_\mu, x_\nu\} - \{A_\nu, x_\mu\} &= \mathcal{F}_{\tau\sigma} K^\tau_\mu K^\sigma_\nu + \Phi \epsilon_{\mu\nu\lambda} \frac{x^\lambda}{R} + (\partial_\tau \phi - \mathcal{A}_\sigma) \frac{x_\mu}{R} K^\sigma_\nu \\ &\quad + (\partial_\sigma \phi - \mathcal{A}_\tau) \frac{x_\mu}{R} K^\tau_\nu \quad - \quad (\mu \rightleftharpoons \nu) \,, \end{aligned} \tag{3.83}$$

where $\mathcal{F}_{\tau\sigma} = \partial_\tau \mathcal{A}_\sigma - \partial_\sigma \mathcal{A}_\tau$ is the $U(1)$ field strength on the de Sitter manifold. Substituting (3.78) and (3.83) into (3.82) gives

$$S(X) - S(\bar{X}) \rightarrow \frac{\alpha^4 R^2}{g_c^2} \int d\tau d\sigma \Big\{ -\frac{1}{2}\cos^2\tau\,(\mathcal{F}_{\tau\sigma})^2 + \frac{1}{2}(\partial_\tau \Phi)^2 - \frac{1}{2}(\partial_\sigma \Phi)^2 + 2\Phi\mathcal{F}_{\tau\sigma} - \frac{\Phi^2}{\cos^2\tau} \Big\}, \quad (3.84)$$

$$\rightarrow \frac{\alpha^4 R^2}{g_c^2} \int d\tau d\sigma \sqrt{-\mathrm{g}} \Big\{ \frac{R^2}{4}\,\mathcal{F}_{ab}\mathcal{F}^{ab} - \frac{1}{2}\partial_a \Phi \partial^a \Phi + \frac{2}{\sqrt{-\mathrm{g}}}\Phi\mathcal{F}_{\tau\sigma} - \frac{1}{R^2}\Phi^2 \Big\}\ . \quad (3.85)$$

A careful look at the above field action shows a sign discrepancy between the kinetic terms for the gauge fields and the scalar field. This also occurred in (3.57) and appears to be a generic feature of the Lorentzian matrix model [19],[23]. This is not totally unexpected since the matrix model action, specifically the Yang-Mills term, is not positive definite. Once again, the situation is harmless in two space-time dimensions since the gauge fields are non-dynamical in two dimensions, and can be eliminated from the theory completely by using the equations of motion[2]. We now get

$$\mathcal{F}_{\tau\sigma} = \frac{2\Phi}{\cos^2\tau} + constant. \quad (3.86)$$

Substituting the above back into the action will result in an effective scalar field theory on commutative dS^2, and the sign mismatch between the kinetic terms disappears. The associated effective Lagrangian density is

$$\mathcal{L}_{\text{eff}} \ = \ -\frac{1}{2}\partial_a \Phi \partial^a \phi + \frac{1}{R^2}\Phi^2 \quad (3.87)$$

[2]However, the same does not apply in higher dimensions, and this issue is yet to be resolved.

But we have another problem. The sign in the quadratic term suggests $m_{eff} = -\frac{2}{R^2}$, i.e., the scalar field is tachyonic! This is problematic since a tachyonic mass term leads to an unstable vacuum. The persistence of tachyonic modes appear to be generic features of the Lorentzian IKKT-type matrix model whose dynamics follows from the action (3.62) when $\beta = 0$. A cure of the instability problem is to require $\beta \neq 0$ in the action (3.62). The reason that this could work is because with the additional free parameter β, one has more freedom to tune the parameters such that in a particular range of parameter space, the sign of the scalar mass term will flip so that the stability of the theory is ensured.

Now we examine the $\beta \neq 0$ case. The perturbation expansion of the action (3.62) around the one of the two fuzzy dS^2 solutions (3.65) gives

$$S(X) - S(\bar{X}_{ds}) = \frac{\alpha^4 R^2}{g^2} \mathrm{Tr}\Big(- \frac{R^2}{4} F_{\mu\nu} F^{\mu\nu} - i\gamma R \epsilon_{\mu\nu\lambda} F^{\mu\nu} A^\lambda + \frac{i\gamma}{\alpha} \epsilon_{\mu\nu\lambda} [A^\mu, \bar{X}^\nu] A^\lambda + \gamma A_\mu A^\mu \Big) \qquad (3.88)$$

where $\gamma = \frac{1}{2} - \frac{\tilde{\alpha}}{3\alpha}$ and $F_{\mu\nu}$ is again defined by (3.80). We now repeat the previous procedure to obtain the commutative limit of the action. A simple dimension analysis in the action suggests that in the commutative limit, $\tilde{\alpha}$ should again scale proportionally to the NC parameter as $\tilde{\alpha} \to \upsilon\theta$, while β should scale proportionally to θ^2, i.e., $\beta \to \omega\theta^2$. One now gets

$$S(X) - S(\bar{X}_{ds}) \to \frac{\alpha^4 R^2}{g_c^2} \int d\tau d\sigma \times \Big(-\frac{cos^2\tau}{2} \mathcal{F}_{\tau\sigma}^2 + \frac{1}{2} (\partial_\tau \Phi)^2 - \frac{1}{2} (\partial_\sigma \Phi)^2 + 2(1 + \frac{\beta}{\alpha^2}) \mathcal{F}_{\tau\sigma} \Phi - (1 + \frac{\beta}{\alpha^2}) \frac{\Phi^2}{cos^2\tau} \Big) \qquad (3.89)$$

after eliminating the gauge fields, one gets the Lagrangian density for the effective field theory

$$\mathcal{L}_{eff} = -\frac{1}{2}(\partial_a \Phi)(\partial^a \Phi) - \frac{1}{2}(\partial_\sigma \Phi)^2 + \frac{1}{R^2}(1 + \frac{3\beta}{\alpha^2} + \frac{2\beta^2}{\alpha^4})\Phi^2 \qquad (3.90)$$

Now to ensure the stability of the theory, one simply requires the constant factor $(1 + \frac{3\beta}{\alpha^2} + \frac{2\beta^2}{\alpha^4}) \leq 0$, from which we get a stabilized region in the parameter space:

$$-1 \leq \frac{\beta}{\alpha^2} \leq -\frac{1}{2} \quad or \quad \frac{3}{2} \leq \frac{\tilde{\alpha}}{\alpha} \leq 2. \qquad (3.91)$$

It is worth mentioning here that the inclusion of a quadratic term in the bosonic IKKT action is not the only way of removing the instability from the theory. For example, a quartic term $\delta \mathrm{Tr}(X_\mu X^\mu)^2$ also has the potential of removing the instability, with stabilized region:

$$\frac{1}{4}(-5 - \sqrt{17}) \leq \frac{R^2\delta}{\alpha^2} \leq \frac{1}{4}(-5 + \sqrt{17}).$$

4 Fuzzy spaces and Snyder space from tensorial matrix model

In this chapter, we search for NC solutions with space-time dimensions greater than two by generalizing the IKKT-type matrix action in chapter 3. The new action is constructed in terms matrices with tensor space-time indices. We obtain two types of solutions: one is formed by taking tensor products of the two-dimensional NC spaces, and the other is the Snyder algebra. We devote the majority of this chapter to the discussions of the representation theory of the three-dimensional Snyder space, along with the implementation of continuous symmetry transformations on Snyder space.

4.1 Generalization of the YM matrix model

We introduce a matrix model action of the form

$$S(B) = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{16}[B_{MN}, B_{PQ}]^2 - \frac{2}{3}i\alpha B_{MN}B^{NP}B_P{}^M\right), \tag{4.1}$$

where $B_{MN} = -B_{NM}$, $M, N = 0, 1, ..., D$ are infinite-dimensional matrices. g once again represents a coupling constant for the matrix action, and we raise or lower tensor indices $M, N, ...$ by the D-dimensional flat metric η_{MN} whose signature depends on the solution space under consideration. The equations of motion from extremizing (4.1) give

$$[[B_{MN}, B_{PQ}], B^{PQ}] + 4i\alpha[B_{MP}, B^P{}_N] = 0. \tag{4.2}$$

The form of the action (4.1) resembles the structure of the three-dimensional bosonic IKKT matrix action[1] (3.6), where the anti-symmetric tensor-valued matrices B_{MN} replace the vector-valued matrices X^μ. One nice feature of this action is that the cubic term can be written down in any number of space-time dimensions, not necessarily in an odd number of dimensions as in the three-dimensional IKKT type action.

First, if we restrict to $2+1$ dimensional embedding space-time and identify B^{MN} as the dual of X^M, i.e., $B^{MN} = \epsilon^{MNP} X_P$, the above action will reduce to the familiar three-dimensional matrix model action (3.34) in the $\beta = 0$ case. We can therefore view (4.1) as a generalization of the three-dimensional matrix action to any number of dimensions. As seen in chapter 3, there exist various NC spaces as solutions to the above matrix model equations of motion, such as the fuzzy 2-sphere, fuzzy dS^2 and fuzzy cylinder solutions[21].

For example, the fuzzy dS^2 solutions given by the commutations relations

$$[\bar{X}^0, \bar{X}^1] = i\alpha \bar{X}^2, \quad [\bar{X}^0, \bar{X}^2] = -i\alpha \bar{X}^1, \quad [\bar{X}^1, \bar{X}^2] = i\alpha \bar{X}^0 \tag{4.3}$$

can be re-expressed by the following commutation relations

$$[\bar{B}^{12}, \bar{B}^{20}] = i\alpha \bar{B}^{01}, \quad [\bar{B}^{12}, \bar{B}^{01}] = -i\alpha \bar{B}^{20}, \quad [\bar{B}^{20}, \bar{B}^{01}] = i\alpha \bar{B}^{12}, \tag{4.4}$$

where $\bar{B}_{\mu\nu}\bar{B}^{\mu\nu} = R^2 \mathbb{1}$ is the invariant quadratic Casimir of (4.4), which defines an irreducible representation of the algebra in terms of the constant parameter R.

As another example, we consider the fuzzy cylinder solution from the three-dimensional IKKT type matrix action, which was first examined in [21]. The

[1] A quadratic term $\sim B_{MN}B^{MN}$ can also be added for more general analysis, but here we are considering the simpler case when such a term is not present.

solution is given by the following commutation relations

$$[\bar{X}^0, \bar{X}^1] = 2i\alpha\bar{X}^2, \quad [\bar{X}^0, \bar{X}^2] = -2i\alpha\bar{X}^1, \quad [\bar{X}^1, \bar{X}^2] = 0. \tag{4.5}$$

which in the dual form $B_{\mu\nu} = \epsilon_{\mu\nu\lambda}X^\lambda$ is re-expressed as

$$[\bar{B}^{12}, \bar{B}^{20}] = 2i\alpha\bar{B}^{01}, \quad [\bar{B}^{12}, \bar{B}^{01}] = -2i\alpha\bar{B}^{20}, \quad [\bar{B}^{20}, \bar{B}^{01}] = 0, \tag{4.6}$$

The unique quadratic Casimir invariant of this algebra is $\bar{B}_{20}\bar{B}^{20} + \bar{B}_{01}\bar{B}^{01} = (\bar{X}^1)^2 + (\bar{X}^2)^2 = R^2\mathbb{1}$, which determines an irreducible representation in terms of the radius R of the cylinder.

We also have the well-known fuzzy 2-sphere solution from the three-dimensional Euclidean IKKT type matrix model. The solution is given by

$$[\bar{X}^1, \bar{X}^2] = i\alpha\bar{X}^3, \quad [\bar{X}^1, \bar{X}^3] = -i\alpha\bar{X}^2, \quad [\bar{X}^2, \bar{X}^3] = i\alpha\bar{X}^1. \tag{4.7}$$

which in the dual form $B_{ij} = \epsilon_{ijk}X_k$ reads

$$[\bar{B}^{23}, \bar{B}^{31}] = i\alpha\bar{B}^{12}, \quad [\bar{B}^{23}, \bar{B}^{12}] = -i\alpha\bar{B}^{31}, \quad [\bar{B}^{31}, \bar{B}^{12}] = i\alpha B^{12}, \tag{4.8}$$

along with the quadratic Casimir given by $\bar{B}_{ij}\bar{B}_{ij} = R^2$, with the radius of the sphere R.

One can construct higher dimensional solutions by taking tensor products of the above two-dimensional solutions, such as $dS^2 \otimes dS^2 \otimes ... \otimes dS^2$, $dS^2 \otimes S^2 \otimes ....$ An example of four-dimensional tensor product NC space is $dS^2 \otimes dS^2$ embedded in a five-dimensional Minkowski background with metric $\eta_{MN} = diag(-1, -1, 1, 1, 1)$

$$\begin{aligned}
&[\bar{B}^{12}, \bar{B}^{20}] = i\alpha\bar{B}^{01}, \quad [\bar{B}^{12}, \bar{B}^{01}] = -i\alpha\bar{B}^{20}, \quad [\bar{B}^{20}, \bar{B}^{01}] = i\alpha\bar{B}^{12}, \\
&[\bar{B}^{34}, \bar{B}^{40}] = i\alpha\bar{B}^{03}, \quad [\bar{B}^{34}, \bar{B}^{03}] = -i\alpha\bar{B}^{40}, \quad [\bar{B}^{40}, \bar{B}^{03}] = i\alpha\bar{B}^{34},
\end{aligned} \tag{4.9}$$

and requiring that all the other commutators vanish. This leads to two quadratic Casimir invariants $\bar{B}_{\mu\nu}\bar{B}^{\mu\nu}$, where $\mu, \nu = 0,1,2$ and $\bar{B}_{\mu'\nu'}\bar{B}^{\mu'\nu'}$ where $\mu', \nu' = 0,3,4$, respectively, along with four additional linear Casimir invariants $\bar{B}^{13}, \bar{B}^{14}, \bar{B}^{23}$ and $\bar{B}^{24}$.

A possibly more physical four-dimensional NC space is the fuzzy $dS^2 \otimes S^2$ space-time, because it involves one time-like direction and three space-like directions. The solutions satisfy the following commutation relations in (4+1)-dimensional embedding space-time, in terms of $B_{MN}, M, N = 0,1,2,3,4$

$$
\begin{aligned}
&[\bar{B}^{12}, \bar{B}^{20}] = i\alpha B^{01}, \quad [\bar{B}^{12}, \bar{B}^{01}] = -i\alpha \bar{B}^{20}, \quad [\bar{B}^{20}, \bar{B}^{01}] = i\alpha \bar{B}^{12}, \\
&[\bar{B}^{42}, \bar{B}^{34}] = i\alpha \bar{B}^{23}, \quad [\bar{B}^{42}, \bar{B}^{23}] = -i\alpha \bar{B}^{34}, \quad [\bar{B}^{34}, \bar{B}^{23}] = i\alpha \bar{B}^{42}, \qquad (4.10)
\end{aligned}
$$

and all the other commutators vanish. The two independent quadratic Casimir operators are $\bar{B}_{20}\bar{B}^{20}+\bar{B}_{01}\bar{B}^{01}+\bar{B}_{12}\bar{B}^{12} = R^2$, and $\bar{B}_{42}\bar{B}^{42}+\bar{B}_{34}\bar{B}^{34}+\bar{B}_{23}\bar{B}^{23} = R^2$, along with four linear Casimir invariants $\bar{B}^{03}, \bar{B}^{04}, \bar{B}^{13}$ and $\bar{B}^{14}$.

We can obtain a NC field theory on the fuzzy $dS^2 \otimes S^2$ using the perturbative approach in chapter three, which upon taking the commutative limit will result in a commutative field theory on the four-dimensional $dS^2 \otimes S^2$ manifold. The calculation here becomes rather nontrivial due to the increase in the number of degrees of freedom. The physical interpretation of the resultant field theory becomes less clear. For example, there are issues with the field theory in the commutative limit, such as sign mismatch in the kinetic energy terms and stability problem, which are tamable in the two-dimensional field theory in chapter 3, but the procedures there do not generally apply in higher dimensions, and it remains an open question on how to solve them. Nevertheless, as an example of obtaining NC field theories on the tensor product fuzzy spaces, we show the the details in Appendix B. For the rest of this chapter, we study a simpler solution to the tensorial matrix model, which is the Snyder algebra.

4.2 Snyder algebra from tensorial matrix model

In this section we examine another family of solutions from the tensor-valued matrix model introduced in the previous section. They are the $(N+1)$-dimensional de Sitter group, whose generators satisfy the commutation relation

$$[\bar{B}^{MN}, \bar{B}^{PQ}] = i\alpha(\eta^{MQ}\bar{B}^{NP} + \eta^{NP}\bar{B}^{MQ} - \eta^{MP}\bar{B}^{NQ} - \eta^{NQ}\bar{B}^{MP}), \qquad (4.11)$$

where all the Roman indices run from 0 to N, and $\eta_{MN} = diag(-1, 1, ..., 1)$ is the Minkowski metric in $N+1$ dimensional space-time. As was shown in the original paper by Snyder[1], the four-dimensional covariant Snyder algebra can be obtained from a four-dimensional de Sitter space embedded in $(4+1)$-space-time dimensions. Here we consider the action (4.1) in $(4+1)$-dimensional embedding space and show that the Snyder algebra can be considered as a solution to the matrix model equations of motion when we extend the de Sitter algebra.

The matrix action and equations of motion in $4+1$-dimensional embedding space are (4.1) and (4.2), respectively, with indices $M, N, ...$ replaced by $m, n, ... = 0, 1, 2, 3, 4$. One solution to the equations of motion is the four-dimensional de Sitter group $dS^4 = SO(4,1)$, whose ten generators $B_{mn} = \bar{B}_{mn}$ satisfy the commutation relations (4.11). We now extend the algebra by introducing $\mathcal{P}_m$ with $[\mathcal{P}_m, \mathcal{P}_n] = 0$. The de Sitter manifold is defined by the constraint in terms of $\mathcal{P}_m$

$$\eta^{mn}\mathcal{P}_m\mathcal{P}_n = 1. \qquad (4.12)$$

We assume $\mathcal{P}_m$ transforms as a five-vector with respect to the four-dimensional de Sitter group, i.e.,

$$[B^{mn}, \mathcal{P}^r] = i\alpha(\eta^{rm}\mathcal{P}^n - \eta^{rn}\mathcal{P}^m) \qquad (4.13)$$

To project the above structures to 3 + 1 dimensions, one defines the four-momenta p^μ such that

$$p^\mu = \Delta \frac{\mathcal{P}^\mu}{\mathcal{P}^4}, \tag{4.14}$$

where we have assumed that $\mathcal{P}^4$ has an inverse, and Δ is a constant factor which has units of energy. We decompose the matrices B^{mn} into $B^{4\mu}$ and $B^{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$. We further identify four components in $B^{4\mu}$ as the 4-dimensional position operators $\hat{x}^\mu$, up to a dimensionful parameter, and identify six components in $B^{\mu\nu}$ as an anti-symmetric tensor field living in four-space-time dimensions. A consistent choice was given by Snyder [1]

$$B^{4\mu} \equiv \Delta \hat{x}^\mu \tag{4.15}$$

$$B^{\mu\nu} \equiv \hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu. \tag{4.16}$$

Now all the matrix elements $B^{4\mu}$ and $B^{\mu\nu}$ are dimensionless. Under these identifications, the extended de Sitter group algebra can be brought to the form of the four-dimensional covariant Snyder algebra

$$\begin{aligned}
[\hat{x}^\mu, \hat{x}^\nu] &= \frac{i}{\Delta^2}(\hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu), \\
[\hat{x}^\mu, \hat{p}^\nu] &= i\Big(\eta^{\mu\nu} + \frac{\hat{p}^\mu \hat{p}^\nu}{\Delta^2}\Big), \\
[\hat{p}^\mu, \hat{p}^\nu] &= 0
\end{aligned} \tag{4.17}$$

where we have set $\alpha = 1$, without loss of generality. $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ is the Minkowski metric in $(3 + 1)$-dimensional space-time.

The NC field theory approach adopted in previous sections is problematic to apply to the case of Snyder algebra, due to the fact that an associative star product has not been found for Snyder algebra [67]. Instead of finding a field theory on Snyder space, we devote our discussions in the following sections to representation theory of the Snyder algebra and the implementation of continuous transformations on the discrete Snyder space.

4.3 Snyder space and Position eigenstates

As pointed out by Snyder, the time component $\hat{x}^0$ of the space-time four vector $\hat{x}^\mu$ in Snyder's algebra has a continuous spectrum [1], we therefore focus on the subalgebra generated by the three position and momentum operators, $\hat{x}_i$ and $\hat{p}_i$, $i = 1, 2, 3$, respectively. They generate the three-dimensional (Euclidean) Snyder algebra, which satisfies commutation relations

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= \frac{i}{\Delta^2}\epsilon_{ijk}\hat{L}_k \\ [\hat{x}_i, \hat{p}_j] &= i\Big(\delta_{ij} + \frac{\hat{p}_i\hat{p}_j}{\Delta^2}\Big) \\ [\hat{p}_i, \hat{p}_j] &= 0 \end{aligned} \tag{4.18}$$

This algebra is a deformation of the Heisenberg algebra, where $\Delta \neq 0$ is the deformation parameter with units of energy.[2] The Heisenberg algebra is recovered when $\Delta \to \infty$. $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$ are the angular momenta. They satisfy the usual commutation relations

$$\begin{aligned} [\hat{x}_i, \hat{L}_j] &= i\epsilon_{ijk}\hat{x}_k \\ [\hat{p}_i, \hat{L}_j] &= i\epsilon_{ijk}\hat{p}_k \\ [\hat{L}_i, \hat{L}_j] &= i\epsilon_{ijk}\hat{L}_k \end{aligned} \tag{4.19}$$

[2]Here and throughout the article we assume that $\Delta^2 > 0$. The case of $\Delta^2 < 0$ is relevant for double special relativity[59] and is characterized by an upper bound on the momentum and deformed energy-momentum dispersion relations. In the latter case, $\hat{x}_i$ and $\hat{L}_i$ generate an $SO(3,1)$ algebra and the $\hat{x}_i$ have continuous eigenvalues. Discussion of non-relativistic quantum mechanics for the case $\Delta^2 < 0$ can be found in [58].

and so generate the three-dimensional rotation group.

Next, we enlarge the group to $SO(4)$ by including $\hat{x}_i$ in the set of rotation generators. The bases of the $so(4)$ algebra are the three-dimensional coordinates and angular momenta. Combinations of $so(4)$ generators provide convenient bases for obtaining spectra of the position operators. Here we follow the discussion in [61].

We define $\hat{L}_{AB} = -\hat{L}_{BA}$, with

$$\hat{L}_{ij} = \epsilon_{ijk}\hat{L}_k \qquad \hat{L}_{i4} = \Delta\ \hat{x}_i\ . \tag{4.20}$$

Then (4.18) and (4.19) gives the standard form of the $so(4)$ algebra

$$[\hat{L}_{AB}, \hat{L}_{CD}] \;=\; i(\delta_{AC}\hat{L}_{BD} - \delta_{BC}\hat{L}_{AD} - \delta_{AD}\hat{L}_{BC} + \delta_{BD}\hat{L}_{AC})\ . \tag{4.21}$$

Alternatively, we have the two $SU(2)$ generators

$$\hat{A}_i = \frac{1}{2}(\hat{L}_i + \Delta\ \hat{x}_i) \qquad \hat{B}_i = \frac{1}{2}(\hat{L}_i - \Delta\ \hat{x}_i)\ , \tag{4.22}$$

satisfying

$$\begin{aligned} [\hat{A}_i, \hat{A}_j] &= i\epsilon_{ijk}\hat{A}_k \\ [\hat{B}_i, \hat{B}_j] &= i\epsilon_{ijk}\hat{B}_k \\ [\hat{A}_i, \hat{B}_j] &= 0\ . \end{aligned} \tag{4.23}$$

Since the position operators are perpendicular to the angular operators, one has the identity $\hat{x}_i\hat{L}_i = \hat{L}_i\hat{x}_i = 0$, which implies

$$\hat{A}_i\hat{A}_i = \hat{B}_i\hat{B}_i\ , \tag{4.24}$$

and hence there is only one independent quadratic Casimir operator for $so(4)$.

$\hat{A}_i\hat{A}_i$, $\hat{A}_3$ and $\hat{B}_3$ form a complete set of commuting operators, and so we can write down the three independent eigenvalue equations

$$\begin{aligned}\hat{A}_i\hat{A}_i|j, m_A, m_B> &= j(j+1)|j, m_A, m_B> \\ \hat{A}_3|j, m_A, m_B> &= m_A|j, m_A, m_B> \\ \hat{B}_3|j, m_A, m_B> &= m_B|j, m_A, m_B>, \end{aligned} \tag{4.25}$$

where $m_A, m_B = -j, 1-j, ..., j$, and $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$. Since there is only one independent Casimir operator, we need only one index j to label the irreducible representations of the $so(4)$ algebra. Since we have $\hat{L}_3 = \hat{A}_3 + \hat{B}_3$ and $\hat{x}_3 = (\hat{A}_3 - \hat{B}_3)/\Delta$, the set of all eigenvalues $\{|j, m_A, m_B>\}$ are also eigenstates of $\hat{L}_3$ and $\hat{x}_3$,

$$\begin{aligned}\hat{L}_3\;|j, m_A, m_B> &= (m_A + m_B)\;|j, m_A, m_B> \\ \hat{x}_3\;|j, m_A, m_B> &= \frac{1}{\Delta}\;(m_A - m_B)\;|j, m_A, m_B> . \end{aligned} \tag{4.26}$$

Therefore, the eigenvalues for $\hat{x}_3$ are evenly spaced. The result also holds for $\hat{x}_1$ and $\hat{x}_2$, or any choice of Cartesian coordinates. Consequently, Snyder space corresponds to a cubical lattice with lattice size Δ^{-1}, where no two directions are simultaneously measurable. For a given eigenvalue of x_i, there are infinite number of choices for m_A and m_B. Therefore, the eigenvalues measured in any particular direction are infinitely degenerate. For example, from (4.26), the set of degenerate eigenvectors associated with any particular eigenvalue $n_3/\Delta,\ n_3 =$ integer, of $\hat{x}_3$ is

$$\Big\{|j, k, k-n_3>, \quad k \in \text{integers}, \quad j \geq |k| \text{ and } |k-n_3|\Big\}. \tag{4.27}$$

Alternatively, we can obtain the spectra for the radial coordinates by utilizing another basis for $so(4)$. This basis is associated with the sum of the two $SU(2)$ generators $\hat{A}_i + \hat{B}_i$, which is just the orbital angular momentum L_i. The basis, which we denote by $\{|j,\ell,m\rangle_\circ\}$, diagonalizes $\hat{A}_i\hat{A}_i$, $\hat{L}_i\hat{L}_i$ and $\hat{L}_3$, i.e.,

$$\begin{aligned} \hat{L}_i\hat{L}_i\,|j,\ell,m\rangle_\circ &= \ell(\ell+1)\,|j,\ell,m\rangle_\circ \\ \hat{L}_3\,|j,\ell,m\rangle_\circ &= m\,|j,\ell,m\rangle_\circ\ . \end{aligned} \tag{4.28}$$

From the domain of m_A and m_B in the Cartesian basis, we get ℓ and m are taking integer values, $\ell = 2j, 2j-1, ..., 1, 0$, and $m = m_A + m_B = -\ell, -\ell+1, ...\ell$. $j(j+1)$ is once again the eigenvalue of $\hat{A}_i\hat{A}_i$. Of course, we also have

$$(\hat{L}_1 \pm i\hat{L}_2)\ |j,\ell,m\rangle_\circ = \sqrt{\ell(\ell+1) - m(m\pm 1)}\ |j,\ell,m\pm 1\rangle_\circ\ . \tag{4.29}$$

The basis vectors $|j,\ell,m\rangle_\circ$ are also eigenvectors of $\hat{x}_i\hat{x}_i$. This is because

$$\hat{x}_i\hat{x}_i = \frac{4\hat{A}_i\hat{A}_i - \hat{L}_i\hat{L}_i}{\Delta^2}, \tag{4.30}$$

which acting on the basis ket $|j,\ell,m\rangle_\circ$ gives

$$\hat{x}_i\hat{x}_i\,|j,\ell,m\rangle_\circ \;=\; \frac{4j(j+1) - \ell(\ell+1)}{\Delta^2}\,|j,\ell,m\rangle_\circ\ . \tag{4.31}$$

So the radial coordinate takes the values $\sqrt{4j(j+1) - \ell(\ell+1)}/\Delta$. For a given j, it ranges from $\sqrt{2j}/\Delta$ to $2\sqrt{j(j+1)}/\Delta$ and the eigenvalues are $2\ell+1$ degenerate. Snyder space in this basis corresponds to a set of concentric spheres, and as was observed in [60], the area of the spheres is quantized.

4.4 Momentum space

4.4.1 Differential representations

In the previous section, we only examined the algebra generated by $\hat{x}_i$ and $\hat{L}_i$. Here we include the momentum operators $\hat{p}_i$. The latter are simultaneously diagonalizable. We denote their eigenvalues and eigenvectors by p_i and $|\vec{p}>$, respectively,

$$\hat{p}_i|\vec{p}>= p_i|\vec{p}> \tag{4.32}$$

The eigenvalues are continuous, and the set of all of them defines momentum space. In what follows, we write down differential representations of the $so(4)$ algebra on momentum space. One such representation was given in Snyder's work[1]. Here we extend Snyder's result to a one parameter family of differential representations. An alternative representation is obtained by taking the Fourier transform, on which we briefly comment at the end of this section.

Since the Snyder algebra is a deformation of the Heisenberg algebra, we can write down the deformation map from the Heisenberg algebra to the Snyder algebra. The former is generated by $\hat{q}_i$ and $\hat{p}_i$, satisfying

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij} \qquad [\hat{q}_i, \hat{q}_j] = 0 \ , \tag{4.33}$$

and of course $[\hat{p}_i, \hat{p}_j] = 0$. The map relates the coordinates $\hat{q}_i$ to Snyder's position operators $\hat{x}_i$, and is given by

$$\hat{x}_i = \hat{q}_i + \frac{1}{2\Delta^2}(\hat{p}_i\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j\hat{p}_i) \ , \tag{4.34}$$

under which the Snyder algebra is recovered. On the other hand, the momentum operators are unchanged by the map.

In the differential representation in momentum space, we replace $\hat{q}_i$ by $i\frac{\partial}{\partial p_i}$ and so the operators $\hat{x}_i$ and $\hat{L}_i$ can be realized by

$$\hat{x}_i \rightarrow i\frac{\partial}{\partial p_i} + \frac{ip_i}{\Delta^2}\left(p_j\frac{\partial}{\partial p_j} + 2\right) \tag{4.35}$$

$$\hat{L}_i \rightarrow -i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k} \; , \tag{4.36}$$

which act on functions $\psi(\vec{p}) =< \vec{p}|\psi >$, where $|\psi >$ is a vector in the Hilbert space for the Snyder algebra and we are using Dirac notation. From (4.36), the result found earlier that ℓ is an integer means that wavefunctions in momentum space are single-valued. The operators are symmetric for the scalar product $\langle\phi\,|\,\psi\rangle = \int d^3p\;\phi(\vec{p})^*\psi(\vec{p})$, provided the functions ϕ and ψ vanish sufficiently rapidly at spatial infinity in momentum space.

More generally, we can preserve the Snyder algebra by adding a term proportional to $\hat{p}_i$ in (4.34). So we can generalize the differential operator (4.35) to

$$\hat{x}_i \rightarrow i\frac{\partial}{\partial p_i} + \frac{ip_i}{\Delta^2}\left(p_j\frac{\partial}{\partial p_j} + \eta\right) \; , \tag{4.37}$$

where we restrict η to the reals. A value of η different from 2 deforms the integration measure from d^3p to

$$d\mu(\vec{p}) = \frac{d^3p}{(1+\frac{\vec{p}^2}{\Delta^2})^{2-\eta}} \; . \tag{4.38}$$

The operators are now symmetric for the scalar product

$$< \phi|\psi >= \int d\mu(\vec{p})\;\phi(\vec{p})^*\psi(\vec{p}) \; , \tag{4.39}$$

for functions ϕ and ψ satisfying asymptotic conditions which depend on η. The identity operator on the Hilbert space is given by $\int d\mu(\vec{p})\,|\vec{p}\rangle\,\langle\vec{p}|$. For $\psi(\vec{p})$ to be normalizable it must go like $1/|\vec{p}|^w$, as $|\vec{p}|\to\infty$, where $w > \eta - \frac{1}{2}$. We recover the trivial measure for $\eta = 2$, while $\eta = 0$ was the choice of Snyder[1].[3] For $\eta = 0$, normalizable functions $\psi(\vec{p})$ need not vanish as $|\vec{p}|\to\infty$.

Although in this thesis we shall rely exclusively on representations in momentum space, we here mention representations on the Fourier transform space. The latter is spanned by functions $\tilde{\psi}(\vec{q}) =< \vec{q}|\psi >$, q_i and $|\vec{q}>$, respectively, denoting the eigenvalues and eigenvectors of $\hat{q}_i$, satisfying (4.33). Here we represent $\hat{p}_i$ by $-i\frac{\partial}{\partial q_i}$, and so from (4.34), $\hat{x}_i$ involves second order derivatives. $\hat{x}_i$ and $\hat{L}_i$ are realized by[4]

$$\begin{aligned} \hat{x}_i &\to q_i - \frac{1}{\Delta^2}\left(2\frac{\partial}{\partial q_i} + q_j\frac{\partial^2}{\partial q_i \partial q_j}\right) \\ \hat{L}_i &\to -i\epsilon_{ijk}q_j\frac{\partial}{\partial q_k} \end{aligned} \tag{4.40}$$

The differential operators are symmetric for the scalar product which utilizes the trivial measure d^3q; i.e., the scalar product between two functions $\tilde{\phi}$ and $\tilde{\psi}$ is $< \tilde{\phi}|\tilde{\psi} >= \int d^3q\; \tilde{\phi}(\vec{q})^*\tilde{\psi}(\vec{q})$.

4.4.2 Asymptotic conditions

Here we argue that in order to define a consistent action of $SO(4)$, it is necessary to impose certain asymptotic conditions in momentum space. The result is that momentum space is isomorphic to the $SO(3)$ group manifold. For this, we

[3]The $\eta = 0$ differential representation in [1] was written down for the four-dimensional Minkowski version of (4.18), and the measure that appears there is $d^4p/(1 + \frac{p^\mu p_\mu}{\Delta^2})^{5/2}$.

[4] A similar representation was found in [62], however the differential operators there were not symmetric.

map the three-momentum to four operators $\hat{\mathcal{P}}_A,\ A = 1, 2, 3, 4$, which gives

$$\hat{\mathcal{P}}_i = \frac{\hat{p}_i}{\sqrt{\hat{\vec{p}}^2 + \Delta^2}} \qquad\qquad \hat{\mathcal{P}}_4 = \frac{\Delta}{\sqrt{\hat{\vec{p}}^2 + \Delta^2}} \ . \tag{4.41}$$

This is the inverse map as we discussed in (4.14), only here we are projecting a three-vector to a four-vector in the Euclidean background. Recalling that $\hat{L}_{AB}$ in (4.20) and (4.21) are the $SO(4)$ generators, then $\hat{\mathcal{P}}_A$ rotates as a four vector,

$$[\hat{L}_{AB}, \hat{\mathcal{P}}_C] \ = \ i(\delta_{AC}\hat{\mathcal{P}}_B - \delta_{BC}\hat{\mathcal{P}}_A), \tag{4.42}$$

which is the same algebra as (4.13) only with the four-dimensional Euclidean background metric δ_{AB} replacing the $(4+1)$-dimensional Minkowski metric η_{mn}. The four momentum operators are constrained by

$$\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2 + \hat{P}_4^2 = \mathbb{I} \ , \tag{4.43}$$

where $\mathbb{I}$ is the identity, and so their eigenvalues lie on S^3. It corresponds to a slice of the four-dimensional de Sitter space reviewed in section 5.1. Notice that the eigenvalues of $\hat{P}_A$ do not span all of S^3. This can be seen by choosing $\Delta > 0$, then $\hat{P}_4$ has positive-definite eigenvalues, and so the coordinatization (4.41) gives a restriction to one hemisphere of S^3, while $\Delta < 0$ gives a restriction to the other hemisphere of S^3.

Equivalently, one can define a $\Delta-$dependent map $g^{(\Delta)}$ from $\mathbb{R}^3 = \{\vec{p}\}$ to $SU(2)$ according to

$$g^{(\Delta)}(\vec{p}) = \frac{1}{\sqrt{\vec{p}^2 + \Delta^2}} \begin{pmatrix} \Delta + ip_3 & p_2 + ip_1 \\ -p_2 + ip_1 & \Delta - ip_3 \end{pmatrix} = \frac{(\Delta \mathbb{1} + ip_i\sigma_i)}{\sqrt{\vec{p}^2 + \Delta^2}} \ \ \in \ \ SU(2) \ , \tag{4.44}$$

where, as before, p_i denotes the eigenvalues of $\hat{p}_i$. Applying the differential representations (4.36) and (4.37), one gets

$$\begin{aligned} \hat{L}_i \, g^{(\Delta)}(\vec{p}) &= -\frac{1}{2} \, [\sigma_i, g^{(\Delta)}(\vec{p})] \\ \hat{x}_i \, g^{(\Delta)}(\vec{p}) &= -\frac{1}{2\Delta} \, [\sigma_i, g^{(\Delta)}(\vec{p})]_+ \; , \end{aligned} \tag{4.45}$$

where σ_i are the Pauli matrices, $[\ ,\ ]_+$ denotes the anticommutator and we have chosen $\eta = 0$ for convenience. It follows that $\hat{A}_i$ and $\hat{B}_i$ act, respectively, as left and right generators of $SU(2)$,

$$\begin{aligned} \hat{A}_i \, g^{(\Delta)}(\vec{p}) &= -\frac{1}{2} \, \sigma_i \, g^{(\Delta)}(\vec{p}) \\ \hat{B}_i \, g^{(\Delta)}(\vec{p}) &= \frac{1}{2} \, g^{(\Delta)}(\vec{p}) \, \sigma_i \; . \end{aligned} \tag{4.46}$$

The map (4.44) when applied to all of momentum space, $g^{(\Delta)} \circ \mathbb{R}^3$, does not cover all of $SU(2)$. Rather, assuming $\Delta > 0$, it is a restriction to $SU(2)$ matrices satisfying

$$\mathrm{Re}\; g^{(\Delta)}(\vec{p})_{11} > 0, \;\; \mathrm{Re}\; g^{(\Delta)}(\vec{p})_{22} \; > \; 0 \; . \tag{4.47}$$

Furthermore, $g^{(\Delta)} \circ \mathbb{R}^3$, for fixed Δ, is not invariant under $SO(4)$ because points on one hemisphere of S^3 can be rotated to the opposite hemisphere. That is, the conditions (4.47) are not preserved under $SO(4)$. Thus the action of $\hat{A}_i$ and $\hat{B}_i$ in (4.46) cannot be consistently exponentiated.

Let us next introduce the complementary map $g^{(-\Delta)}$ from $\mathbb{R}^3 = \{\vec{p}\}$ to $SU(2)$. It gives a restriction to $SU(2)$ matrices satisfying

$$\mathrm{Re}\; g^{(-\Delta)}(\vec{p})_{11} < 0, \;\; \mathrm{Re}\; g^{(-\Delta)}(\vec{p})_{22} \; < \; 0 \; . \tag{4.48}$$

Then $[g^{(\Delta)} \circ \mathbb{R}^3] \cup [g^{(-\Delta)} \circ \mathbb{R}^3]$ spans all of $SU(2)$ and is invariant under the action of $SO(4)$. For finite momentum, $|\vec{p}|< \infty$, the two maps identify each point $\vec{p}$ in $\mathbb{R}^3$ with two points in $SU(2)$. Using

$$g^{(-\Delta)}(-\vec{p}) = -g^{(\Delta)}(\vec{p}) \; , \tag{4.49}$$

the two maps are related by $Z_2 = \{1, -1\}$. There is thus a $2-1$ map from $SU(2)$ to $\{\vec{p}, \; |\vec{p}|< \infty\}$.

The restriction to finite momentum can be lifted upon imposing appropriate asymptotic conditions in momentum space. For (4.49) to hold as $|\vec{p}|\rightarrow \infty$, we need to identify opposite points at infinity,

$$\vec{p} \; \leftrightarrow \; -\vec{p} \, , \qquad \text{as} \qquad |\vec{p}|\rightarrow \infty \; . \tag{4.50}$$

These asymptotic conditions mean that momentum space is $SU(2)/Z_2 = SO(3)$. The $SO(3)$ matrices $\{R_{ij}(\vec{p})\}$ are given explicitly by

$$\sigma_i R_{ij}(\vec{p}) = g^{(\Delta)}(\vec{p}) \sigma_j g^{(\Delta)}(\vec{p})^\dagger \; . \tag{4.51}$$

Upon applying (4.44),

$$R(\vec{p}) = \frac{1}{\vec{p}^2 + \Delta^2} \begin{pmatrix} \Delta^2 + p_1^2 - p_2^2 - p_3^2 & 2(p_1 p_2 + \Delta p_3) & 2(p_1 p_3 - \Delta p_2) \\ 2(p_1 p_2 - \Delta p_3) & \Delta^2 - p_1^2 + p_2^2 - p_3^2 & 2(\Delta p_1 + p_2 p_3) \\ 2(\Delta p_2 + p_1 p_3) & 2(p_2 p_3 - \Delta p_1) & \Delta^2 - p_1^2 - p_2^2 + p_3^2 \end{pmatrix} \tag{4.52}$$

From (4.45), the action of $SO(4)$ generators on these matrices is given by

$$\begin{aligned} \hat{L}_i \; R_{jk}(\vec{p}) \;\; &= \;\; - \, [T_i, R(\vec{p})]_{jk} \\ \hat{x}_i \; R_{jk}(\vec{p}) \;\; &= \;\; -\frac{1}{\Delta} \Big([T_i, R(\vec{p})]_+ \Big)_{jk} \, , \qquad (T_i)_{jk} = -i \; \epsilon_{ijk} \; , \end{aligned} \tag{4.53}$$

where we have again chosen $\eta = 0$ for convenience. Equivalently, $\hat{A}_i$ and $\hat{B}_i$ are, respectively, the left and right generators of $SO(3)$,

$$\begin{aligned}\hat{A}_i\, R_{jk}(\vec{p}) &= -\,[T_i R(\vec{p})]_{jk} \\ \hat{B}_i\, R_{jk}(\vec{p}) &= [R(\vec{p}) T_i]_{jk}\ . \end{aligned} \tag{4.54}$$

This action can be consistently exponentiated to $SO(3) \times SO(3)$ acting on momentum space by left and right multiplication.

One can promote R_{ij} to operator-valued matrix elements $\hat{R}_{ij}$. For this we only need to replace p_i by the operators $\hat{p}_i$ in the definition of the $SO(3)$ matrices in (4.51). The result is the set of operators $\hat{R}_{ij} = R_{ij}(\hat{\vec{p}})$, whose eigenvalues are $R_{ij}(\vec{p})$

$$\hat{R}_{ij}|\vec{p}> = R_{ij}(\vec{p})|\vec{p}> \ . \tag{4.55}$$

With the imposition of the asymptotic conditions (4.50) on momentum space, the Snyder algebra is generated by $\hat{A}_i$, $\hat{B}_i$ and $\hat{R}_{ij}$. They satisfy commutation relations (4.23), along with

$$\begin{aligned}[\hat{A}_i, \hat{R}_{jk}] &= -\,[T_i \hat{R}]_{jk} \\ [\hat{B}_i, \hat{R}_{jk}] &= [\hat{R} T_i]_{jk} \\ [\hat{R}_{ij}, \hat{R}_{k\ell}] &= 0\ . \end{aligned} \tag{4.56}$$

This, along with (4.23), is an alternative way to write the three-dimensional Euclidean Snyder algebra, which takes into account the topology of momentum space. In the next section we look at the different representations of this algebra.

4.5 Wavefunctions on momentum space

Here we find two distinct representations of the three-dimensional Euclidean Snyder algebra. We examine them using three different bases in the subsections that follow. The first deals directly with the algebra (4.23) and (4.56), the second are momentum eigenfunctions of the radial coordinate $\sqrt{\hat{x}_i\hat{x}_i}$ and the third are momentum eigenfunctions of $\hat{x}_3$.

4.5.1 Two Hilbert spaces

Multiple connectivity in a classical theory implies that there are multiple quantizations of the system[63]. We can view (4.23) and (4.56) as resulting from quantization on a doubly connected momentum space. This leads to two distinct representations. We denote the corresponding Hilbert spaces by $\mathcal{H}_B$ and $\mathcal{H}_F$. They are as follows:

The Hilbert space $\mathcal{H}_B$, consists of complex functions $\{\Phi_B, \Psi_B, ...\}$ on $SO(3) = \{R\}$. The scalar product between any two such functions Φ_B and Ψ_B is an integral over $SO(3)$,

$$< \Phi_B|\Psi_B >= \int_{SO(3)} d\mu(R)_{SO(3)} \; \Phi_B(R)^* \Psi_B(R) \;, \tag{4.57}$$

where $d\mu(R)_{SO(3)}$ is the invariant measure on $SO(3)$. From the Peter-Weyl theorem, any function Ψ_B in $\mathcal{H}_B$ can be expanded in terms of the $(2j_B+1)\times(2j_B+1)$ irreducible matrix representations $\{D^{j_B}(R),\ j_B = 0, 1, 2, ...\}$ of $SO(3)$, which serve as basis functions,

$$\Psi_B(R) = \sum_{j_B=0}^{\infty} \sum_{m,n=-j_B}^{j_B} a^{j_B}_{mn} D^{j_B}_{mn}(R) \;, \tag{4.58}$$

$a^{j_B}_{mn}$ are constants. $\hat{A}_i$ and $\hat{B}_i$ act on the irreducible matrix representations according to

$$\begin{aligned}\hat{A}_i\ D^{j_B}_{mn}(R) &= -\,[D^{j_B}(T_i)D^{j_B}(R)]_{mn} \\ \hat{B}_i\ D^{j_B}_{mn}(R) &= [D^{j_B}(R)D^{j_B}(T_i)]_{mn}\ ,\end{aligned} \tag{4.59}$$

where $D^{j_B}(T_i)$ denote the $(2j_B+1)\times(2j_B+1)$ matrix representations of the $SO(3)$ generators. $\hat{R}_{ij}\ D^{j_B}_{mn}(R)$, for $j_B \geq 1$, is a linear combination of $D^{j_B+1}(R)$, $D^{j_B}(R)$ and $D^{j_B-1}(R)$. Applying the Casimir operator, one gets

$$\hat{A}_i\hat{A}_i\ D^{j_B}_{mn}(R) = \hat{B}_i\hat{B}_i\ D^{j_B}_{mn}(R) = j_B(j_B+1)\ D^{j_B}_{mn}(R)\ , \tag{4.60}$$

and so j appearing in (4.25) belongs to the set of all integers for the Hilbert space $\mathcal{H}_B$.

The Hilbert space $\mathcal{H}_F$, consists of complex functions $\{\Phi_F, \Psi_F, ...\}$ on $SU(2) = \{g\}$, with scalar product

$$< \Phi_F|\Psi_F >= \int_{SU(2)} d\mu(g)_{SU(2)}\ \Phi_F(g)^*\Psi_F(g)\ , \tag{4.61}$$

where $d\mu(g)_{SU(2)}$ is the invariant measure on $SU(2)$. The basis functions for $\mathcal{H}_F$ are restricted to all half-integer irreducible matrix representations of $SU(2)$, $\{\mathtt{D}^{j_F}(g),\ j_F = \frac{1}{2}, \frac{3}{2}, ...\}$. Thus any Ψ_F in $\mathcal{H}_F$ has the expansion

$$\Psi_F(g) = \sum_{j_F=\frac{1}{2},\frac{3}{2},...}\ \sum_{m,n=-j_F}^{j_F} \mathtt{a}^{j_F}_{mn}\mathtt{D}^{j_F}_{mn}(g)\ , \tag{4.62}$$

$\mathtt{a}^{j_F}_{mn}$ are constants. $\hat{A}_i$ and $\hat{B}_i$ act on the irreducible matrix representations according to

$$\begin{aligned}\hat{A}_i\ \mathtt{D}^{j_F}_{mn}(g) &= -\ [\mathtt{D}^{j_F}(T_i)\mathtt{D}^{j_F}(g)]_{mn}\\ \hat{B}_i\ \mathtt{D}^{j_F}_{mn}(g) &= [\mathtt{D}^{j_F}(g)\mathtt{D}^{j_F}(T_i)]_{mn}\ ,\end{aligned}\tag{4.63}$$

where $\mathtt{D}^{j_F}(T_i)$ denote the $(2j_F+1)\times(2j_F+1)$ matrix representations of the $SU(2)$ generators. Now

$$\hat{A}_i\hat{A}_i\ \mathtt{D}^{j_F}_{mn}(g) = \hat{B}_i\hat{B}_i\ \mathtt{D}^{j_F}_{mn}(g) = j_F(j_F+1)\ \mathtt{D}^{j_F}_{mn}(g)\ ,\tag{4.64}$$

and only half-integer values of $j = j_F$ occur in (4.25) for the Hilbert space $\mathcal{H}_F$.

4.5.2 Eigenfunctions of $\hat{x}_i\hat{x}_i$, $\hat{L}_i\hat{L}_i$ and $\hat{L}_3$

Here we write down the momentum-dependent eigenfunctions $\phi_{j,\ell,m}(\vec{p}) =< \vec{p}|j,\ell,m>_\circ$ of $\hat{x}_i\hat{x}_i$, $\hat{L}_i\hat{L}_i$ and $\hat{L}_3$. (An alternative discussion of momentum eigenfunctions of the radial coordinate can be found in [58].) We show that these eigenfunctions are related to the spherical harmonics of S^3 - restricted to one hemisphere S^3_+. As in the previous subsection, we find two distinct Hilbert spaces, one consisting of $\phi_{j,\ell,m}(\vec{p})$ with j integer and the other j half-integer. Here we do not make any initial assumptions on the domain of the wavefunctions, such as (4.50). For generality, we drop the restriction to $\eta = 0$, which was made in the previous subsection. Using (4.37), the differential representation of $\hat{x}_i\hat{x}_i$ is given by

$$-\Delta^2\hat{x}_i\hat{x}_i \to -\frac{\hat{L}_i\hat{L}_i}{\rho^2} + (1+\rho^2)^2\Big(\partial_\rho^2+\frac{2}{\rho}\partial_\rho\Big) + \eta\Big[2(1+\rho^2)\rho\partial_\rho+(1+\eta)\rho^2+3\Big]\ ,\tag{4.65}$$

where ρ is the rescaled radial component of the momentum, $\rho = |\vec{p}|/\Delta$. Its eigenfunctions are proportional to the spherical harmonics $Y^{\ell}_m(\theta,\phi)$ on S^2,

$$\phi_{j,\ell,m}(\vec{p}) \;=\; \frac{1}{\rho} u_{j,l}(\rho)\, Y^{\ell}_m(\theta,\phi)\ , \tag{4.66}$$

where θ and ϕ are spherical angles in momentum space. From the eigenvalue equation (4.31) we get the following differential equation for the function $u_{j,\ell}(\rho)$

$$\left\{(1+\rho^2)\frac{\partial^2}{\partial\rho^2} + 2\eta\rho\frac{\partial}{\partial\rho} + \frac{4j(j+1)+\eta\Big(1+\rho^2(\eta-1)\Big)}{1+\rho^2} - \frac{\ell(\ell+1)}{\rho^2}\right\} u_{j,l}(\rho) \;=\; 0 \tag{4.67}$$

Solutions for $u_{j,\ell}(\rho)$ involve Gegenbauer polynomials $C^{(b)}_n$, where n is a nonnegative integer and $b > \frac{1}{2}$. They are

$$u_{j,\ell}(\rho) = \;\mathcal{N}_{j,\ell}\,\sin^{\ell+1}\chi\cos^{\eta-1}\chi\, C^{(\ell+1)}_{2j-\ell}(\cos\chi)\ , \tag{4.68}$$

where $\mathcal{N}_{j,\ell}$ are normalization constants. The angle χ is defined by

$$\tan\chi = \rho\ , \tag{4.69}$$

and runs from 0 to $\pi/2$. Then the eigenfunctions of $\phi_{j,\ell,m}(\vec{p})$ are

$$\phi_{j,\ell,m}(\vec{p}) \;=\; \mathcal{N}_{j,\ell}\,\sin^{\ell}\chi\;\cos^{\eta}\chi\, C^{(\ell+1)}_{2j-\ell}(\cos\chi)\, Y^{\ell}_m(\theta,\phi)\ \ . \tag{4.70}$$

Using the measure (4.38), the norm of $\phi_{j,\ell,m}(\vec{p})$ is finite due to the ultraviolet scale Δ. Furthermore, it is independent of the parameter η, as the η dependence in $\phi_{j,\ell,m}(\vec{p})$ is canceled out by the η dependence in the measure.

The eigenfunctions $\phi_{j,\ell,m}(\vec{p})$ are related to spherical harmonics on S^3. If we set $\eta = 0$, they are in fact identical to the spherical harmonics, up to a normalization factor, and are obtainable from $O(4)$ representation matrices.[64] However, their domain is not all of S^3. To see this we can embed the three-sphere in $\mathbb{R}^4$, by defining

$$\begin{aligned} P_1 &= \sin\chi\sin\theta\cos\phi \\ P_2 &= \sin\chi\sin\theta\sin\phi \\ P_3 &= \sin\chi\cos\theta \\ P_4 &= \cos\chi \end{aligned} \tag{4.71}$$

It follows that $P_1^2+P_2^2+P_3^2+P_4^2=1$, and like the eigenvalues of the operators $\hat{P}_A$ defined in (4.41), P_A span a hemisphere of S^3. As before, $P_4 \geq 0$, since $0 \leq \chi \leq \frac{\pi}{2}$. The restriction of the domain of the spherical harmonics to the half-sphere S^3_+ affects their normalization. More significantly, it affects their completeness relations, as we discuss below.

Demanding that $\phi_{j,\ell,m}(\vec{p})$ are orthonormal,

$$< \phi_{j,\ell,m}|\phi_{j',\ell',m'} > = \int d\mu(\vec{p})\ \phi_{j,\ell,m}(\vec{p})^*\phi_{j'\ell',m'}(\vec{p}) \ = \ \delta_{j,j'}\delta_{\ell,\ell'}\delta_{m,m'}\ , \tag{4.72}$$

leads to the following conditions on the Gegenbauer polynomials

$$\Delta^3\mathcal{N}^*_{j,\ell}\mathcal{N}_{j',\ell'}\int_0^{\frac{\pi}{2}} d\chi\ \sin^{2\ell+2}\chi\, C^{(\ell+1)}_{2j-\ell}(\cos\chi)\, C^{(\ell+1)}_{2j'-\ell}(\cos\chi) \ = \ \delta_{j,j'}\ \ , \tag{4.73}$$

using the measure (4.38). Gegenbauer polynomials $\{C^{(\ell+1)}_n(\xi)\}$ are standardly normalized over the domain $-1 \leq \xi \leq 1$, or equivalently $0 \leq \chi \leq \pi$, rather than $0 \leq \chi \leq \pi/2$. The standard normalization condition is

$$\int_0^{\pi} d\chi\ \sin^{2\ell+2}\chi\, C^{(\ell+1)}_n(\cos\chi)\, C^{(\ell+1)}_{n'}(\cos\chi) \ = \ \frac{\pi\ (n+2\ell+1)!}{2^{2\ell+1}\ n!\ (n+\ell+1)(\ell!\,)^2}\ \delta_{n,n'} \tag{4.74}$$

In order to relate this to (4.73), we can use the property

$$C_n^{(\ell+1)}(\xi) = (-1)^n C_n^{(\ell+1)}(-\xi) \ . \tag{4.75}$$

Then $\{C_n^{(\ell+1)}(\xi), n \text{ even}\}$ and $\{C_n^{(\ell+1)}(\xi), n \text{ odd}\}$ form two sets of orthogonal polynomials over the half-domain, $0 \leq \xi \leq 1$, or equivalently $0 \leq \chi \leq \pi/2$. From (4.73) and (4.74), the normalization constants are given by

$$|\mathcal{N}_{j,\ell}|^2 = \frac{2^{2\ell+2}(2j+1)(2j-\ell)!\,(\ell!\,)^2}{\Delta^3\pi(2j+\ell+1)!} \ . \tag{4.76}$$

We note that the set $\{C_n^{(\ell+1)}(\xi), n \text{ even}\}$ is *not* orthogonal to $\{C_n^{(\ell+1)}(\xi), n \text{ odd}\}$ over the half-domain, $0 \leq \xi \leq 1$. Thus there are two distinct sets of orthonormal polynomials. Using $n = 2j - \ell$, they correspond to either j equal to an integer or j equal to a half-integer, and for any given value of ℓ. There are then two distinct sets of orthonormal eigenfunctions $\phi_{j,\ell,m}(\vec{p})$, and two distinct sets of spherical harmonics on S^3_+. As in the previous subsection, we find that there are two representations of the Snyder algebra, $\mathcal{H}_B$ and $\mathcal{H}_F$, the former associated with all integer values of j and the latter associated with all half-integer values of j.

Unlike the derivation in subsection 4.1, here we did not a priori make any assumptions about the topology of momentum space, such as (4.50), which identifies opposite points at infinity. The two bases of eigenfunctions which result here are distinguished by their asymptotic properties. Restricting to $\eta = 0$, we get

$$\phi_{j,\ell,m}(\vec{p}) \sim C_{2j-\ell}^{(\ell+1)}(0)\; Y^\ell_m(\theta,\phi) \qquad \text{as} \quad |\vec{p}| \to \infty \ . \tag{4.77}$$

From (4.75), $C_{2j-\ell}^{(\ell+1)}(0)$ vanishes when $2j - \ell$ equals an odd integer. Using this and the well known property

$$Y^\ell_m(\pi-\theta,\pi+\phi) = (-1)^\ell\; Y^\ell_m(\theta,\phi) \ , \tag{4.78}$$

it follows that the asymptotic expression (4.77) has even parity when j is an integer and odd parity when j is a half-integer. Trivial examples of this are the zero angular momentum, parity even, eigenfunctions

$$\phi_{j,0,0}(\vec{p}) = \frac{\sin(1+2j)\chi}{\Delta^{3/2}\pi\ \sin\chi}\ , \tag{4.79}$$

again assuming $\eta = 0$, which vanish as $|\vec{p}| \to \infty$ when j is half-integer. In general, wavefunctions Ψ_B in $\mathcal{H}_B$ satisfy

$$\Psi_B(\vec{p}) = \Psi_B(-\vec{p})\ , \qquad \text{as} \qquad |\vec{p}| \to \infty\ , \tag{4.80}$$

while Ψ_F in $\mathcal{H}_F$ satisfy

$$\Psi_F(\vec{p}) = -\Psi_F(-\vec{p})\ , \qquad \text{as} \qquad |\vec{p}| \to \infty \tag{4.81}$$

The eigenfunctions (4.70) provide a transform from momentum space to discrete position space, which in this basis is composed of concentric spheres of radii equal to $\sqrt{4j(j+1) - \ell(\ell+1)}/\Delta$. If $\Psi(\vec{p})$ denotes a wavefunction in the former space and $\Psi^{\circ}_{j,\ell,m}$ is the corresponding wavefunction on the discrete space, then the transform and its inverse are given by

$$\Psi^{\circ}_{j,\ell,m} = \int d\mu(\vec{p})\ \phi_{j,\ell,m}(\vec{p})^{*}\Psi(\vec{p}) \tag{4.82}$$

$$\Psi(\vec{p}) = \sum_{\substack{j=0,1,2,\ldots \\ \text{or} \\ j=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots}} \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} \phi_{j,\ell,m}(\vec{p})\ \Psi^{\circ}_{j,\ell,m} \tag{4.83}$$

The sum over integer j is for Hilbert space $\mathcal{H}_B$ and half-integer j is for $\mathcal{H}_F$.

Finally, in addition to (4.68), there are another set of solutions for $u_{j,l}(\rho)$ in (4.67). They can be expressed in terms of hypergeometric functions ${}_2F_1$ according to

$$\frac{\rho^{-\ell}}{(1+\rho^2)^{j+\frac{\eta}{2}}}\ {}_2F_1\left(-j-\frac{\ell}{2}-\frac{1}{2}, -j-\frac{\ell}{2}; \frac{1}{2}-\ell; -\rho^2\right) \tag{4.84}$$

They lead to a complementary set of eigenfunctions $\{\phi'_{j,\ell,m}(\vec{p})\}$ of $\hat{x}_i\hat{x}_i$, $\hat{L}_i\hat{L}_i$, and $\hat{L}_3$, which unlike (4.68), are singular at the origin in momentum space. The singularity is integrable only for the case of zero angular momentum, where the eigenfunction is given by

$$\phi'_{j,0,0}(\vec{p}) = \frac{\cos(1+2j)\chi}{\Delta^{3/2}\pi\ \sin\chi}\ , \tag{4.85}$$

and we again take $\eta = 0$. The solutions (4.79) and (4.85) comprise the spherically symmetric waves for the system.

4.5.3 Eigenfunctions of $\hat{A}_i\hat{A}_i$, $\hat{x}_3$ and $\hat{L}_3$

In the previous section we found eigenfunctions of the radial coordinate operator. Here we examine eigenfunctions of $\hat{x}_3$. More precisely, we study the momentum-dependent basis functions associated with the eigenvectors $\{|j, m_A, m_B\rangle\}$ of Sec 2.2. We denote these basis functions by $\eta_{j,m_A,m_B}(\vec{p}) = \langle\vec{p}\,|\,j, m_A, m_B\rangle$. They are obtained from $\phi_{j,\ell,m}(\vec{p})$ in (4.70) by a change of basis

$$\phi_{j,\ell,m}(\vec{p}) = \sum_{m_A=-j}^{j} \langle j, j; m_A, m-m_A\,|\,\ell, m\rangle\, \eta_{j,m_A,m-m_A}(\vec{p})\ , \tag{4.86}$$

where $\langle j, j; m_A, m_B\,|\,\ell, m\rangle$ are Clebsch-Gordan coefficients. Expressions for $\eta_{j,m_A,m_B}(\vec{p})$ can be given in terms of the cylindrical variables $R_p,\ \phi_p,\ p_3$, [where $p_1 = R_p\cos\phi_p,\ p_2 =$

$R_p \sin\phi_p$ and $0 \le \phi_p < 2\pi$]. The eigenfunctions have the general form

$$\eta_{j,m_A,m_B}(\vec{p}) = f_{j,m_A,m_B}(R_p, p_3)\; e^{i(m_A+m_B)\phi_p}\;, \tag{4.87}$$

where $f_{j,m_A,m_B}(R_p, p_3)$ are eigenfunctions of $\hat{A}_i\hat{A}_i$ and $\hat{x}_3$. We can obtain the precise form of $f_{j,m_A,m_B}(R_p, p_3)$ for the special case of zero angular momentum in the third direction. Then, $\eta_{j,m_A,-m_A}(\vec{p}) = f_{j,m_A,-m_A}(R_p, p_3)$, corresponding to plane waves in the third direction. From the eigenvalue equation

$$\hat{x}_3\; \eta_{j,m_A,-m_A}(\vec{p}) = 2m_A\; \eta_{j,m_A,-m_A}(\vec{p})\;, \tag{4.88}$$

we get

$$\eta_{j,m_A,-m_A}(\vec{p}) = e^{-2i\, m_A\, \tan^{-1}(\frac{p_3}{\Delta})}\;\; \mathcal{F}_{j,m_A}\Big(\frac{\vec{p}^2+\Delta^2}{{p_3}^2+\Delta^2}\Big)\;, \tag{4.89}$$

where we used the differential representation (4.37) with $\eta = 0$. The functions $\mathcal{F}_{j,m_A}(\zeta)$ are determined from the remaining eigenvalue equation

$$\hat{A}_i\hat{A}_i\; \eta_{j,m_A,-m_A}(\vec{p}) = j(j+1)\eta_{j,m_A,-m_A}(\vec{p})\;, \tag{4.90}$$

which leads to

$$\zeta^2\frac{d}{d\zeta}\Big((\zeta-1)\frac{d}{d\zeta}\Big)\mathcal{F}_{j,m_A}(\zeta) + (j^2+j-{m_A}^2\zeta)\mathcal{F}_{j,m_A}(\zeta) = 0 \tag{4.91}$$

Solutions for $\mathcal{F}_{j,m_A}(\zeta)$ can be expressed in terms of hypergeometric functions. Up to a normalization, $\mathcal{F}_{j,m_A}(\zeta)$ is

$$\zeta^{-j}\; {}_2F_1\left(-j-m_A, m_A-j; -2j; \zeta\right) \tag{4.92}$$

Additional solutions are

$$\zeta^{j+1} \, {}_2F_1 \left(j - m_A + 1, j + m_A + 1; 2j + 2; \zeta \right) \ , \tag{4.93}$$

which are singular at $\zeta = 1$. Using $\zeta = (\vec{p}^2 + \Delta^2)/({p_3}^2 + \Delta^2)$, the latter are divergent along p_3-axis.

The eigenfunctions $\eta_{j,m_A,m_B}(\vec{p})$ provide a transform from momentum space to the spatial lattice (with only x_3 determined). If $\Psi(\vec{p})$ is a wavefunction in the former space and $\Psi^{\#}_{j,m_A,m_B}$ is a wavefunction on the lattice, then they are related by

$$\Psi^{\#}_{j,m_A,m_B} = \int d\mu(\vec{p}) \ \eta_{j,m_A,m_B}(\vec{p})^* \ \Psi(\vec{p}) \tag{4.94}$$

$$\Psi(\vec{p}) = \sum_{\substack{j = 0,1,2,\ldots \\ \text{or} \\ j = \frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots}} \ \sum_{m_A,m_B=-j}^{j} \eta_{j,m_A,m_B}(\vec{p}) \ \Psi^{\#}_{j,m_A,m_B} \tag{4.95}$$

Once again, the sum over integer j is for Hilbert space $\mathcal{H}_B$ and half-integer j is for $\mathcal{H}_F$.

4.6 Continuous transformations on Snyder space

It is now straightforward to write down the unitary action of the rotation and translation group on Snyder space. They are generated by $\hat{L}_i$ and $\hat{p}_i$, respectively. The latter is not responsible for discrete translations from one point on the spatial lattice to a neighboring point. Rather, $\hat{p}_i$ generate continuous translations in the basis $\{|\vec{q}>\}$ which diagonalizes the operators $\hat{q}_i$ conjugate to $\hat{p}_i$. [Cf. eq. (4.33).] Differential representations associated with this basis were given in eq. (4.40).

In the basis $\{|j,\ell,m>_\circ\}$, Snyder space corresponds to the set of concentric spheres with radii equal to $\sqrt{4j(j+1)-\ell(\ell+1)}/\Delta$. $m=-\ell,...\ell$ is a degeneracy index. A rotation by $\vec{\theta}$ is given by

$$e^{i\theta_i\hat{L}_i}|j,\ell,m>_\circ = \sum_{m'=-\ell}^{\ell} D^{\ell}_{m',m}(\vec{\theta})\,|j,\ell,m'>_\circ\,, \tag{4.96}$$

where D^ℓ are $SO(3)$ matrix representations, $D^{\ell}_{m',m}(\vec{\theta}) =<\ell,m'|e^{i\theta_i\hat{L}_i}|\ell,m>$ and $|\ell,m>$ are the usual angular momentum eigenstates. A translation by $\vec{a}$ is given by

$$e^{ia_i\hat{p}_i}|j,\ell,m>_\circ = \sum_{\substack{j'=0,1,2,...\\ \text{or}\\ j'=\frac{1}{2},\frac{3}{2},\frac{5}{2},...}} \sum_{\ell'=0}^{2j'} \sum_{m'=-\ell'}^{\ell'} \int d\mu(\vec{p})|j',\ell',m'>_\circ\; e^{ia_ip_i}\;\phi_{j',\ell',m'}(\vec{p})^*\;\phi_{j,\ell,m}(\vec{p})\,, \tag{4.97}$$

and one sums over integer (half-integer) values of j' when j is integer (half-integer), because the translation operator does not take states out of the Hilbert space.

In the basis $\{|j, m_A, m_B>\}$, Snyder space corresponds to a lattice with the eigenvalues of the third coordinate given by $(m_A - m_B)/\Delta$. Using the Clebsch-Gordan coefficients, rotations act according to

$$\begin{aligned} e^{i\theta_i \hat{L}_i}|j, m_A, m_B> &= \\ \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} \sum_{m'_A, m'_B=-j}^{j} & |j, m'_A, m'_B>< j, j; m'_A, m'_B|\ell, m> \times \\ & < j, j; m_A, m_B|\ell, m_A + m_B>^* \; D^{\ell}_{m, m_A+m_B}(\vec{\theta}) \, , \end{aligned} \tag{4.98}$$

while translations are given by

$$\begin{aligned} e^{ia_i \hat{p}_i}|j, m_A, m_B> &= \\ \sum_{\substack{j' = 0,1,2,\ldots \\ \text{or} \\ j' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots}} \sum_{m'_A, m'_B=-j'}^{j'} \int d\mu(\vec{p}) & \quad |j', m'_A, m'_B> e^{ia_i p_i} \; \eta_{j', m'_A, m'_B}(\vec{p})^* \; \eta_{j, m_A, m_B}(\vec{p}) \, , \end{aligned} \tag{4.99}$$

Again, one sums over integer (half-integer) values of j' when j is integer (half-integer). As stated above, $e^{ia_i \hat{p}_i}$ does not, in general, map one point on the spatial lattice to another point. To hop from one eigenvalue of $\hat{x}_3$ to another, one needs an operator that maps (m_A, m_B) to (m'_A, m'_B), with $m_A - m_B \neq m'_A - m'_B$. Such an operator need not be unitary; Examples are the raising and lowering operators $\hat{A}_1 \pm i\hat{A}_2$ and $\hat{B}_1 \pm i\hat{B}_2$.

5 Conclusion

In this dissertation, we have studied NC spaces that are exact solutions to the equations of motion of matrix models in various dimensions. These solutions are associated with finite-dimensional Lie-algebras. We have discussed aspects of these NC spaces regarding their irreducible representations, cosmological interpretations, NC field theories and their commutative limit.

In the first part of this dissertation, we focused on two-dimensional Lie-algebra-based NC spaces as solutions to equations of motion of three-dimensional IKKT-type matrix models. In particular, we have examined the Lorentzian fuzzy sphere and fuzzy dS^2 solutions in different settings. The former is associated with a matrix action that requires both the cubic and quadratic terms in addition to the Yang-Mills term, while in the latter case, the quadratic term is optional for obtaining a fuzzy dS^2 solution.

The Lorentzian fuzzy sphere solutions that arise from a Lorentzian IKKT-type model provide toy models of a noncommutative two-dimensional closed universe, where time and spatial size have discrete values. Singularities in the Ricci tensor appear in the large N (i.e., commutative) limit. They are analogous to big bang/crunch singularities, with the novel feature that they occur at nonzero spatial size. Perturbations around the fuzzy sphere solution are described by a scalar field in the commutative limit which can propagate in the Lorentzian region of the manifold. The scalar field can be massive, massless or tachyonic, the choice depending on the parameter a_0^2 (and also on the range of θ when $a_0^2 < \frac{1}{2}$). For $\frac{1}{2} \leq a_0^2 < 1$ the scalar field is always massive, ensuring the stability of the commutative field theory in this case. Corrections to the commutative limit are obtained by expressing the matrix product in the action (3.53) in terms of the star product on the sphere[11]-[14] and keeping the next order terms in the $1/N$ expansion.

The fuzzy dS^2 solutions arises from the equations of motion of an IKKT-type matrix model action. Depending on the value of the coefficient β of the quadratic term, the number of fuzzy dS^2 solutions varies. In all cases, the fuzzy dS^2 solutions are associated with an $su(1,1)$ Lie-algebra, which leads to principal, complementary and discrete series representations. For the discrete series, there exist top (bottom) states which corresponds to minimum values for a_n^2, which provides a resolution of the cosmological singularities. Perturbative analysis can be performed in the same manner as that of the Lorentzian fuzzy sphere solution, which leads to NC gauge theory on the fuzzy dS^2. In the commutative limit, the NC theory induces ordinary gauge field theory on the commutative dS^2 manifold, where the commutative gauge fields are nondynamic thus can be further eliminated using the equation of motion. We have obtained a scalar field theory with a tachyonic mass term in the absence of the quadratic term. The stability of the theory is restored when the quadratic term is present, provided the values of the coefficients α and β lie within a certain range.

A fuzzy AdS^2 solution can be obtained with a simple change of signature in the embedding space, i.e., $diag[\eta_{\mu\nu}] = (1, -1, -1)$. The perturbation theory and stability analysis can be similarly obtained, which is almost identical to the case of fuzzy dS^2 [20]. One distinction in the scalar field theory resulted from NC field theory on fuzzy AdS^2 solution is that one does not have the instability problem even for the case $\beta = 0$. This is because the switch in the role of space-like and time-like coordinates on the AdS^2 surface. Another potentially interesting aspect of the fuzzy AdS^2 solution is due to the AdS/CFT correspondence, where AdS space in D dimensions provides the gravity content of the correspondence, which can have a field theory interpretation in $D-1$ dimensions[57]. Therefore, a future direction can be the study of fuzzy AdS/CFT, which may provide more implications for the the understanding of the correspondence.

For a more realistic model of a noncommutative cosmological space-time, one can look for fuzzy coset space solutions to Lorentzian IKKT-type matrix models associated with dimension $d > 2$ [14]. One possible example worth consideration is the fuzzy analogue of the four-dimensional coset $SU(3)/U(2)$ or CP^2. For coset spaces with $d > 4$ one may be able to make both four-dimensional space-time and extra dimensions noncommutative. Just as with the example of the fuzzy sphere, the commutative limit may lead to a manifold divided up into regions with different signatures of the metric. Perturbations about such solutions are expected to be described by a coupled gauge-scalar theory in the commutative limit. A common feature of the emergent field theories in previous examples [21] is that scalar field and gauge field kinetic energies can appear with opposite signs, which is also seen in (3.57). This sign discrepancy was harmless for $d = 2$, since the gauge field could be eliminated. On the other hand, it is of concern for $d > 2$, so it would be interesting to see if this discrepancy can be cured upon taking the commutative limit of higher dimensional fuzzy coset space solutions.

In the second part of this dissertation, we proposed to obtain exact NC spaces solutions in higher dimensions by considering a tensorial matrix model. One family of higher dimensional NC spaces are found to be formed by taking tensor product of the two-dimensional NC spaces studied in chapter 3. The NC field theory on the tensor product NC spaces can be similarly constructed by the perturbative approach. More interestingly, we found that the de Sitter group is a generic family of solutions to the equations of motion of the tensorial matrix model. By extending the de Sitter algebra, we further showed that four dimensional covariant Snyder algebra can be a solution to the matrix model equations of motion in $(4 + 1)$-dimensional space-time.

We devote the majority of this part to the investigation of the Snyder algebra, in particular, we examined the three-dimensional Snyder space spanned by the position operators. The position operators are associated with discrete spectra, which easily follows from the discreteness of the $SO(4)$ representations. The corresponding $so(4)$ algebra contains only one independent quadratic Casimir operator, and so unitary irreducible representations of $SO(4)$ that occur for this model are labeled by a single quantum number j, which a priori can have integer or half-integer values. For the case of the three-dimensional Euclidean Snyder algebra, we have found that one gets a consistent action of $SO(4)$ upon making an identification at infinite momentum whereby momentum space is isomorphic to the $SO(3)$ group manifold. Because $SO(3)$ is doubly connected, the quantum theory is not unique. We find two sets of basis functions on momentum space, spanning two distinct Hilbert spaces $\mathcal{H}_B$ and $\mathcal{H}_F$. The basis functions are distinguished by their asymptotic properties and also by the quantum number j, which takes all integer values for $\mathcal{H}_B$, and all half-integer values for $\mathcal{H}_F$. Because the values of j determine the spectra of the position operators, the two Hilbert spaces imply the existence of two different spatial lattices, i.e., two different Snyder spaces. Rotations and translations are implemented as unitary transformations on the lattices. In fact, the full Poincaré group and even the super-Poincaré group can be made to have a consistent action on the lattices[65],[66].

There have been various attempts to write down field theory on Snyder space with the goal of having a consistent Lorentz invariant noncommutative quantum field theory[67],[68]. The approach considered so far is to try to write down star product representations of the Snyder algebra on a commutative space-time manifold. The proposed star products have introduced some confusion, in that they are either nonassociative or lead to a deformation of the Poincaré symmetries[67],[68]. Therefore we have not adopted the perturbative approach towards NC field theory on Snyder space. Instead we give some speculations here. Since Snyder space is, in fact, a lattice, a more appropriate way to proceed with the second quantization of the theory may be to consider lattice field theory. Unlike usual lattice field theory, here fields can be defined in only one direction of the spatial lattice; this would correspond, for example, to the radial direction if we use the basis $\{|j, \ell, m >_\circ\}$, or the third Cartesian direction if we use the basis $\{|j, m_A, m_B >\}$. Assuming the latter, fields are a function of quantum numbers j, m_A and m_B, and some continuous evolution parameter, say λ. It can also depend on spin quantum numbers. In the simplest case of a real scalar field $\Phi(\lambda)_{j,m_A,m_B}$, we can write

$$\Phi(\lambda)_{j,m_A,m_B} = \int d\mu(\vec{p}) \left(\mathbf{a}(\vec{p})\, \eta_{j,m_A,m_B}(\vec{p})\, e^{-i\lambda H} + \mathbf{a}(\vec{p})^\dagger\, \eta_{j,m_A,m_B}(\vec{p})^*\, e^{i\lambda H} \right) , \tag{5.1}$$

where H generates evolution in λ, and we introduced particle creation (annihilation) operators $\mathbf{a}(\vec{p})^\dagger$ $\left(\mathbf{a}(\vec{p})\right)$. The field action $\mathcal{S}[\Phi]$ involves an integral over λ, as well as a sum over the lattice,

$$\mathcal{S}[\Phi] = \int d\lambda \sum_{\mathcal{H}_B \text{ or } \mathcal{H}_F} \mathcal{L}[\Phi(\lambda)] , \tag{5.2}$$

where $\mathcal{L}[\Phi(\lambda)]$ is the field Lagrangian. If the lattice-statistics connection mentioned above applies, then the sum is over the Hilbert space $\mathcal{H}_B$ for bosonic fields and $\mathcal{H}_F$ for fermionic fields. A nontrivial problem is then to find a Lagrangian on the lattice which gives relativistic invariant dynamics and reduces to a familiar model in the limit of zero lattice spacing $\Delta \to \infty$.

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Appendices

A Seiberg-Witten map approach

The perturbative analysis performed in chapter 3 and chapter 4 can also be achieved using the Seiberg-Witten map. The advantage of the SW-map approach is that it does not require that the matrix solution be written in terms of a Lie-algebra. In the latter case one can always define a NC field strength tensor [see for example (3.22), (3.49) and (3.80)] that can be used to construct the perturbed matrix action. In this appendix, we outline a perturbative analysis which does not require the construction of a field strength tensor and show the explicit calculation for the case of NC gauge theory on fuzzy dS^2 and the Lorentzian fuzzy sphere. The end result for the commutative action agree with that in section 3.3.

For the SW-map approach, it is convenient to start from the action (3.34) in its commutative limit. To obtain the commutative action, one again replaces the trace by integral and commutators of the matrix functions by Poisson brackets of the functions of the coordinates on dS^2. The Poisson brackets on the three-dimensional space spanned by x^μ are singular, and a function of the coordinates can be found which is central in the Poisson bracket algebra. Setting that function equal to a constant yields a two-dimensional surface $\mathcal{M}_2$, upon which a nonsingular Poisson bracket can be defined. [In general, similar arguments can be made to recover an even-dimensional manifold starting with a d =odd dimensional matrix model.] Say that τ and σ parametrize the two-dimensional surface, where τ is a time-like parameter and σ is space-like. We will assume that any time slice of $\mathcal{M}_2$ is a circle, $0 \leq \sigma < 2\pi$. In terms of the three embedding coordinates the surface is defined by the functions $y^\mu = y^\mu(\tau, e^{i\sigma})$. We denote solutions to the equations of motion by $y^\mu = x^\mu(\tau, e^{i\sigma})$, and focus on solutions with an $SO(2)$ isometry group, associated with rotations in the 1-2 plane. For this we write the ansatz

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \tau \\ a(\tau)\cos\sigma \\ a(\tau)\sin\sigma \end{pmatrix} \tag{A.1}$$

Here we have introduced a factor $a(\tau)$ which is the radius at any τ-slice. For any two functions $\mathcal{F}(\tau, e^{i\sigma})$ and $\mathcal{G}(\tau, e^{i\sigma})$ on $\mathcal{M}_2$ we can write

$$\{\mathcal{F}, \mathcal{G}\}(\tau, e^{i\sigma}) = h\left(\partial_\sigma\mathcal{F}\partial_\tau\mathcal{G} - \partial_\tau\mathcal{F}\partial_\sigma\mathcal{G}\right) , \tag{A.2}$$

where in general h is some function of τ and $e^{i\sigma}$. The function h in the general definition of the Poisson bracket is fixed when restricted to the Poisson structure on a particular manifold.

Since the matrix model action (3.34) and the equations of motion (3.35) can be expressed in terms of commutators, their commutative limit can be expressed in terms of Poisson brackets. In order that all terms survive in the commutative limit, we need that $\tilde{\alpha}$ vanishes in the limit, or more specifically, that it is proportional to θ. We write as $\tilde{\alpha} \to \upsilon\theta$, with υ finite. The quadratic term in the action will survive in the limit provided that β goes like θ^2, i.e., $\beta \to \omega\theta^2$, with ω finite. Then the commutative limit of the action is

$$S_c(y) = \frac{1}{g_c^2} \int_{\mathcal{M}_2} d\mu(\tau,\sigma)\Big(\frac{1}{4}\{y_\mu, y_\nu\}\{y^\mu, y^\nu\} + \frac{\upsilon}{3}\epsilon_{\mu\nu\lambda}\, y^\mu\{y^\nu, y^\lambda\} + \frac{w}{2} y_\mu y^\mu\Big) \,, \quad \text{(A.3)}$$

where g_c is the commutative limit of the coupling g and $d\mu(\tau,\sigma)$ is the integration measure on $\mathcal{M}_2$. We can then take

$$d\mu(\tau,\sigma) = d\tau d\sigma/h \quad \text{(A.4)}$$

for the Poisson bracket defined in (A.2). The commutative limit of the equations of motion (3.35) is given by

$$\{\{y_\mu, y_\nu\}, y^\nu\} - \upsilon\epsilon_{\mu\nu\rho}\{y^\nu, y^\rho\} - wy_\mu = 0 \quad \text{(A.5)}$$

Substituting the ansatz (A.1) in (A.5) will result in a single differential equation only involving the scale factor $a(\tau)$, which can be solved to obtain either exact solutions or numerical ones. For the dS^2 manifold the solutions are exact and $a(\tau)$ is found to be

$$a(\tau)^2 = 1 + \tau^2 \,, \quad h_\pm = \frac{1}{2}(\upsilon \pm \sqrt{\upsilon^2 + 2w}), \quad \text{(A.6)}$$

corresponding to a de Sitter space-time,

$$(x^1)^2 + (x^2)^2 - (x^0)^2 = 1. \quad \text{(A.7)}$$

Another exact solution we discussed in chapter 3 is the Lorentzian fuzzy sphere, for which one has

$$a(\tau)^2 = 1 - \tau^2 \,, \quad h = 2\upsilon, \quad \text{(A.8)}$$

corresponding to

$$(x^1)^2 + (x^2)^2 + (x^0)^2 = 1 \quad \text{(A.9)}$$

Next, we perturb around the classical solution x^μ

$$y^\mu = x^\mu + \theta A^\mu \tag{A.10}$$

As discussed in chapter 3, A^μ transform as noncommutative gauge potentials. Up to first order in θ, the infinitesimal gauge variations of A^μ are given by

$$\delta A^\mu = \{\Lambda, x^\mu\} + \theta\{\Lambda, A^\mu\}, \tag{A.11}$$

where the Poison bracket is defined in (A.2), while the commutative gauge variations are

$$\delta\mathcal{A}_\tau = \partial_\tau\lambda, \;\; \delta\mathcal{A}_\sigma = \partial_\sigma\lambda, \;\; \delta\Phi = 0. \tag{A.12}$$

As reviewed in chapter 2, for NC gauge potentials and gauge parameters expanded in powers of θ, given in (2.38), they are required to satisfy consistency condition (2.40). First, we rewrite (2.40) as

$$\delta_\Lambda A(\mathcal{A};\theta) = A(\mathcal{A} + \delta_\lambda\mathcal{A};\theta) - A(\mathcal{A};\theta) = \delta_\lambda A(\mathcal{A};\theta) \tag{A.13}$$

Then the SW-map can be obtained by solving the condition (A.13) simultaneously for A and Λ order by order in terms of the θ-ordered expansions of the NC gauge parameter and NC gauge potentials:

$$\begin{aligned} \Lambda &= \Lambda^{(0)} + \theta\Lambda^{(1)} + \theta^2\Lambda^{(2)} + ... + \theta^n\Lambda^{(n)} + ... \\ A &= A^{(0)} + \theta A^{(1)} + \theta^2 A^{(2)} + ... + \theta^n A^{(n)} + ... \end{aligned} \tag{A.14}$$

To zeroth order in θ, we get

$$\begin{aligned} \Lambda^{(0)} &= \lambda \\ \delta A^0 &= \{\Lambda, x^0\} + \theta\{\Lambda, A^0\} \\ &= -h\partial_\sigma\lambda - \theta h\partial_\sigma\Lambda^{(1)} + \mathcal{O}(\theta^2) \end{aligned} \tag{A.15}$$

which gives

$$\delta A^{(0)0} = -h\delta\mathcal{A}_\sigma \tag{A.16}$$

Following this procedure one can find at the zeroth order:

$$\begin{aligned}
\Lambda^0 &= \lambda \\
A^{0(0)} &= -h\mathcal{A}_\sigma + ha'a\Phi \\
A^{1(0)} &= h(a\cos\sigma\,\mathcal{A}_\tau - a'\sin\sigma\,\mathcal{A}_\sigma + a\sin\sigma\,\Phi) \\
A^{2(0)} &= h(-a\sin\sigma\,\mathcal{A}_\tau - a'\cos\sigma\,\mathcal{A}_\sigma + a\cos\sigma\,\Phi)
\end{aligned} \tag{A.17}$$

The first order expansion terms leads to

$$\begin{aligned}
\Lambda^{(1)} &= -h\mathcal{A}_\sigma\partial_\tau\lambda, A^{(1)0} = h\Big(\frac{1}{2}\partial_\tau(h\mathcal{A}_\sigma^2) - \partial_\tau(haa'\Phi)\mathcal{A}_\sigma + \partial_\sigma(haa'\Phi)\mathcal{A}_\tau\Big) \\
A^{(1)1} &= h\Big(\frac{1}{2}\partial_\tau(\mathcal{A}_\sigma^2 ha'\sin\sigma) + \frac{1}{2}\partial_\sigma(\mathcal{A}_\tau^2 ha\cos\sigma) - \mathcal{A}_\sigma\partial_\tau(\Phi ha\sin\sigma) \\
&\quad + \mathcal{A}_\tau\partial_\sigma(\Phi ha\sin\sigma) + \cos\sigma\partial_\tau(\mathcal{A}_\tau\mathcal{A}_\sigma ha)\Big) \\
A^{(1)2} &= h\Big(\frac{1}{2}\partial_\tau(\mathcal{A}_\sigma^2 ha'\cos\sigma) - \frac{1}{2}\partial_\sigma(\mathcal{A}_\tau^2 ha\sin\sigma) - \mathcal{A}_\sigma\partial_\tau(\Phi ha\cos\sigma) \\
&\quad + \mathcal{A}_\tau\partial_\sigma(\Phi ha\cos\sigma) + \sin\sigma\partial_\tau(\mathcal{A}_\tau\mathcal{A}_\sigma ha)\Big)
\end{aligned} \tag{A.18}$$

Substituting these results into the commutative action (A.3), we get

$$S_c(y) = -\frac{\theta^2}{g_c^2}\int d\tau d\sigma\, h(\tau)^3\mathsf{g}^2\left(\frac{1}{2}\mathcal{F}_{\tau\sigma}\mathcal{F}^{\tau\sigma} - \frac{1}{2}\partial_{\mathsf{a}}\phi\partial^{\mathsf{a}}\phi + \gamma(\tau)\mathcal{F}_{\tau\sigma}\phi - \frac{1}{2}m^2(\tau)\phi^2\right) + S_c(x)\,, \tag{A.19}$$

where indices $\mathsf{a}, \mathsf{b}, ... = \tau, \sigma$. The time-dependent coupling coefficient $\gamma(\tau)$ and mass $m(\tau)$ are given by

$$\begin{aligned}
\gamma(\tau) &= \frac{2a(\tau)^2}{h(\tau)^2\mathsf{g}^2}\Big(-\upsilon h(\tau)\mathsf{g}_{\tau\tau} + \omega a(\tau)\Big(\frac{\tau}{a(\tau)}\Big)'\Big) \\
m(\tau)^2 &= \frac{a(\tau)^2}{h(\tau)^2\mathsf{g}^2}\left(\mathsf{g}_{\tau\tau}\Big(2h(\tau)^2 - 4\upsilon h(\tau) - \omega\Big) + 2\omega a(\tau)\Big(\frac{\tau}{a(\tau)}\Big)'\right)
\end{aligned} \tag{A.20}$$

Here we see a common feature for these systems, which is that the kinetic energies of the gauge and scalar fields have opposite signs. Upon eliminating the gauge field using its equation of motion, $\mathcal{F}^{\tau\sigma} + \gamma(\tau)\phi = \text{constant}/(h(\tau)^3\mathsf{g}^2)$, and substituting back into (A.19), we now get the effective action

$$S_c^{\text{eff}}(y) = -\frac{\theta^2}{g_c^2}\int d\tau d\sigma\, h(\tau)^3\mathsf{g}^2\left(-\frac{1}{2}\partial_{\mathsf{a}}\phi\partial^{\mathsf{a}}\phi - \frac{1}{2}m_{\text{eff}}(\tau)^2\phi^2\right) + S_c^{\text{eff}}(x)\,, \tag{A.21}$$

where the mass-squared for the scalar field gets modified to

$$\begin{aligned} m_{\text{eff}}(\tau)^2 \; &= \; m(\tau)^2 + \mathsf{g}\gamma(\tau)^2 \\ &= \frac{2}{h(\tau)^2\mathsf{g}}\Bigg(\upsilon^2 + (h(\tau){-}\upsilon)^2 - \frac{\omega}{2} \\ &+ \frac{\omega a(\tau)}{\mathsf{g}_{\tau\tau}^2 h(\tau)^2}\Big(\frac{\tau}{a(\tau)}\Big)'\Big(\mathsf{g}_{\tau\tau}h(\tau)(h(\tau){-}4\upsilon) + 2\omega a(\tau)\Big(\frac{\tau}{a(\tau)}\Big)'\Big)\Bigg) \end{aligned} \tag{A.22}$$

Now we apply the general results to dS^2 and Lorentzian S^2 cases separately.

For dS^2, $a(\tau)$ and $h(\tau)$ are given in (A.6), the result for m^2_{eff} is in general a function of the time parameter τ and it can be positive, negative or zero. It is negative when $\omega = 0$ for all spherically symmetric solutions. This follows from the Lorentzian signature of the metric, $\mathsf{g} < 0$. γ, m^2 and m^2_{eff} are constants for the de Sitter solution (A.6) with υ and ω finite. In this case (A.20) and (A.22) yield

$$\gamma = m^2 = 2\Big(2{-}\frac{\upsilon}{h}\Big) \qquad\qquad m^2_{\text{eff}} = 2\Big(2{-}\frac{\upsilon}{h}\Big)\Big({-}3{+}2\frac{\upsilon}{h}\Big) \tag{A.23}$$

The result agrees with the mass squared appearing in (3.90) and the range for coefficients α and β in (3.91), under which the stability of the theory is restored.

For sphere embedded in Minkowski space-time, $a(\tau)$, $h(\tau)$ are given in (A.8), and w has to be nonzero and negative, i.e., $w = -4\upsilon^2$. The function $\gamma(\tau)$ and mass squared (A.20) and (A.22) are now given by

$$\begin{aligned} \gamma &= -\frac{1+2\tau^2}{(1-2\tau^2)^2}, \\ m^2 &= -\frac{3-2\tau^2}{(1-2\tau^2)^2}, \\ m^2_{\text{eff}} &= 4\frac{1-\tau^2+2\tau^4}{(1-2\tau^2)^3}. \end{aligned} \tag{A.24}$$

B NC field theory on $dS^2_F \otimes S^2_F$

In this appendix, we obtain the NC field theory on fuzzy $dS^2 \otimes S^2$ following the same procedure as described in chapter 3. To simplify the analysis, here we define X^μ $(\mu = 0, 1, 2)$ and Y_k $(k = 1, 2, 3)$:

$$\begin{aligned} &B^{01} = X_2, B^{20} = X_1, B^{12} = X_0 \\ &B_{42} = Y_1, B_{34} = Y_2, B_{23} = Y_3 \end{aligned} \tag{B.1}$$

Then (4.10) simplifies to the familiar form of commutation relations for fuzzy dS^2 and fuzzy 2-sphere, respectively

$$\begin{aligned} [\bar{X}_\mu, \bar{X}_\nu] &= i\alpha\epsilon_{\mu\nu\lambda}\bar{X}^\lambda \\ [\bar{Y}_i, \bar{Y}_j] &= i\alpha\epsilon_{ijk}\bar{Y}_k, \end{aligned} \tag{B.2}$$

where $\mu, \nu\ \lambda = 0, 1, 2$ and $i, j, k = 1, 2, 3$, also we use the convention $\epsilon^{012} = -\epsilon_{012} = 1$, for the former set of indices and $\epsilon^{123} = \epsilon_{123} = 1$ for the latter. Setting the remaining four components in B^{MN} to zero

$$\bar{B}^{03} = \bar{B}^{04} = \bar{B}^{13} = \bar{B}^{14} = 0, \tag{B.3}$$

it follows that $[\bar{X}_\mu, \bar{Y}_i] = 0$, so the two solution spaces are disconnected, and their tensor product is a four-dimensional fuzzy space-time $dS^2_F \otimes S^2_F$ embedded in (4+1)-dimensional embedding space, for which we choose the metric $\eta_{MN} = diag(-1, 1, 1, 1, 1)$.

In terms of the matrices X_μ, Y_i and B_{MN}, the action (4.1) can be rewritten as

$$\begin{aligned} S = \frac{1}{g^2}\mathrm{Tr}\Big\{ &-\frac{1}{4}[X_\mu, X_\nu]^2 - \frac{1}{4}[Y_i, Y_j]^2 + \frac{2}{3}i\alpha\epsilon_{\mu\nu\lambda}X^\mu X^\nu X^\lambda + \frac{2}{3}i\alpha\epsilon_{ijk}Y_iY_jY_k + \frac{1}{2}[X_\mu, Y_j]^2 \\ &-\frac{1}{2}[X_\mu, B_{03}]^2 - \frac{1}{2}[X_\mu, B_{04}]^2 + \frac{1}{2}[X_\mu, B_{13}]^2 + \frac{1}{2}[X_\mu, B_{14}]^2 \\ &+\frac{1}{2}[Y_i, B_{03}]^2 + \frac{1}{2}[Y_i, B_{04}]^2 - \frac{1}{2}[Y_i, B_{13}]^2 - \frac{1}{2}[Y_i, B_{14}]^2 \\ &+2i\alpha X_2[B_{13}, B_{03}] + 2i\alpha X_2[B_{14}, B_{04}] - 2i\alpha Y_2[B_{04}, B_{03}] - 2i\alpha Y_2[B_{13}, B_{14}] \\ &-2i\alpha B_{03}[X_1, Y_3] + 2i\alpha B_{04}[X_1, Y_1] + 2i\alpha B_{13}[X_0, Y_3] - 2i\alpha B_{14}[X_0, Y_1]\Big\} \end{aligned} \tag{B.4}$$

From section 3.1 and 2.3, we can define field strengths as

$$\alpha^2 R^2 F_{\mu\nu} = [X_\mu, X_\nu] - i\alpha\epsilon_{\mu\nu\lambda}X^\lambda, \tag{B.5}$$
$$\alpha^2 R^2 F_{ij} = [Y_i, Y_j] - i\alpha\epsilon_{ijk}Y_k. \tag{B.6}$$

We can then expand about the $dS^2_F \otimes S^2_F$ solution using

$$\begin{aligned} X_\mu &= \bar{X}_\mu + \alpha R A_\mu, \\ Y_i &= \bar{Y}_i + \alpha R B_i, \\ B_{03} &= \alpha\Phi_1, B_{04} = \alpha\Phi_2, \\ B_{13} &= \alpha\Phi_3, B_{14} = \alpha\Phi_4 \end{aligned} \tag{B.7}$$

The matrix action under these perturbations gives a NC gauge theory on $dS^2_F \otimes S^2_F$

$$\begin{aligned} S(B) = \frac{\alpha^4 R^2}{g^2}\mathrm{Tr}\Big\{ &- \frac{R^2}{4}F_{\mu\nu}F^{\mu\nu} - \frac{iR}{6}\epsilon_{\mu\nu\lambda}F^{\mu\nu}A^\lambda - \frac{1}{6\alpha}\epsilon_{\mu\nu\lambda}[A^\mu, \bar{X}^\nu]A^\lambda + \frac{1}{6}A_\mu A^\mu \\ &- \frac{R^2}{4}F_{ij}F^{ij} - \frac{iR}{6}\epsilon_{ijk}F_{ij}B_k - \frac{1}{6\alpha}\epsilon_{ijk}[B_i, \bar{X}_j]B_k - \frac{1}{6}B_iB_i \\ &+ \frac{1}{2\alpha^2}[\bar{X}_\mu, B_i]^2 + \frac{1}{2\alpha^2}[\bar{Y}_i, A_\mu]^2 - \frac{1}{\alpha^2}[\bar{X}_\mu, B_i]\bar{Y}_i, A^\mu] \\ &+ \frac{1}{2\alpha^2 R^2}(-[\bar{X}_\mu, \Phi_1]^2 - [\bar{X}_\mu, \Phi_2]^2 + [\bar{X}_\mu, \Phi_3]^2 + [\bar{X}_\mu, \Phi_4) \\ &+ \frac{1}{2\alpha^2 R^2}([\bar{Y}_i, \Phi_1]^2 + [\bar{Y}_i, \Phi_2]^2 - [\bar{Y}_i, \Phi_3]^2 - [\bar{Y}_i, \Phi_4]^2) \\ &+ \frac{2i}{\alpha R^2}(\bar{X}_2[\Phi_3, \Phi_1] + \bar{X}_2[\Phi_4, \Phi_2] - \bar{Y}_2[\Phi_2, \Phi_1] - \bar{Y}_2[\Phi_3, \Phi_4]) \\ &+ \frac{2i}{\alpha R}([\bar{X}_1, \Phi_1]B_3 - [\bar{Y}_3, \Phi_1]A_1 - [\bar{X}_1, \Phi_2]B_1 + [\bar{Y}_1, \Phi_2]A_1 \\ &- [\bar{X}_0, \Phi_3]B_3 + [\bar{Y}_3, \Phi_3]A_0 + [\bar{X}_0, \Phi_4]B_1 - [\bar{Y}_1, \Phi_4]A_0)\Big\} + S(\bar{B}). \end{aligned} \tag{B.8}$$

In the commutative limit, the tensor product fuzzy space corresponds to the four-dimensional manifold $dS^2 \otimes S^2$, which can be parametrized by the coordinates x^μ and y_i in the same way defined in chapter 3

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = R\begin{pmatrix} \tan\tau \\ \sec\tau\,\cos\sigma \\ \sec\tau\,\sin\sigma \end{pmatrix} \tag{B.9}$$

for dS^2 and

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = R \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{B.10}$$

for S^2. The nonvanishing components of the induced metric tensors are

$$g_{\tau\tau} = -g_{\sigma\sigma} = -\frac{R^2}{\cos^2\tau}, \quad g_{\tau\sigma} = 0 \tag{B.11}$$

$$g_{\theta\theta} = R^2, g_{\phi\phi} = R^2\sin^2\theta, g_{\theta\phi} = 0 \tag{B.12}$$

Given two functions f and h on the manifold, we introduce their Poisson bracket

$$\{f(\tau,\sigma,\theta,\phi), h(\tau,\sigma,\theta,\phi)\} = \cos^2\tau(\partial_\tau f\partial_\sigma h - \partial_\sigma f\partial_\tau h) + \frac{1}{\sin\theta}(\partial_\theta f\partial_\phi h - \partial_\phi f\partial_\theta h) \tag{B.13}$$

from which one recovers $\{x^\mu, x^\nu\} = R\epsilon^{\mu\nu\lambda}x_\lambda$ and $\{y_i, y_j\} = R\epsilon_{ijk}y_k$.

In the commutative limit, we can separate out the degrees of freedom in the symbols $A_\mu(\tau,\sigma,\theta,\phi)$ and $B_i(\tau,\sigma,\theta,\phi)$, which in the lowest order of the expansion are the tangential components $\mathcal{A}_\tau(\tau,\sigma,\theta,\phi), \mathcal{A}_\sigma(\tau,\sigma,\theta,\phi), \mathcal{B}_\theta(\tau,\sigma,\theta,\phi), \mathcal{B}_\phi(\tau,\sigma,\theta,\phi)$ and normal components $\phi_A(\tau,\sigma,\theta,\phi)$ and $\phi_B(\tau,\sigma,\theta,\phi)$ using the Killing vectors of the manifold $K^\tau_\mu, K^\sigma_\mu, K^\theta_i$ and K^ϕ_i, which were defined in (3.28), and (3.75) and then use the decomposition

$$\begin{aligned} RA_\mu(\tau,\sigma,\theta,\phi) &= \mathcal{A}_\sigma(\tau,\sigma,\theta,\phi)K^\tau_\mu + \mathcal{A}_\tau(\tau,\sigma,\theta,\phi)K^\sigma_\mu + \frac{1}{R}\phi_A(\tau,\sigma,\theta,\phi)x_\mu, \\ RB_i(\tau,\sigma,\theta,\phi) &= \mathcal{B}_\theta(\tau,\sigma,\theta,\phi)K^\theta_i + \mathcal{B}_\phi(\tau,\sigma,\theta,\phi)K^\phi_i + \frac{1}{R}\phi_B(\tau,\sigma,\theta,\phi)y_i. \end{aligned} \tag{B.14}$$

Some useful relations for K^a_i $(a = \theta, \phi)$ are

$$\begin{aligned} & K^a_i K^b_i = R^2 g^{ab}, K^a_i y_i = 0, \epsilon_{ijk}K^\theta_i K^\phi_j y_k = \frac{R}{\sin\theta} \\ & \partial_\theta K^\theta_i = 0, \partial_\theta K^\phi_i = -\cot\theta K^\phi_i + \frac{y_i}{R\sin\theta}, \\ & \partial_\phi K^\theta_i = \sin\theta\cos\theta K^\phi_i - \frac{y_i\sin\theta}{R}, \quad \partial_\phi K^\phi_i = -\cot\theta K^\theta_i \\ & \partial_\theta y_i = -R\sin\theta K^\phi_i, \partial_\phi y_i = R\sin\theta K^\theta_i \end{aligned} \tag{B.15}$$

Similar identities regarding K^a_μ can be found in section 3.3 in chapter 3.

To obtain the commutative action, one again replaced the trace by integrals and commutators by Poisson brackets

$$[f(\bar{B}), h(\bar{B})] \rightarrow \frac{i\alpha}{R}\{f(x,y), h(x,y)\} \tag{B.16}$$

$$\mathrm{Tr} \rightarrow \varpi\kappa \int d\tau d\sigma d\theta d\phi (\frac{R^2}{\cos^2\tau})(R^2 \sin\theta) \tag{B.17}$$

where ϖ, κ are constants. Then (B.8) gives

$$S(B) - S(\bar{B}) \rightarrow S(B)_{ke} + S(B)_{int} + S(B)_{scalar} \tag{B.18}$$

where the pure kinetic terms are

$$\begin{aligned} S(B)_{ke} =& \frac{\varpi\kappa\alpha^4 R^2}{g^2} \int d\tau d\sigma d\theta d\phi \sqrt{-\mathrm{g}} \Big\{ \frac{R^2}{4}\mathcal{F}_{ab}\mathcal{F}^{ab} - \frac{1}{2}\partial_a\phi_A\partial^a\phi_A + \frac{1}{2}\partial_a\phi_B\partial^a\phi_B \\ & - \frac{1}{2}\partial_a\phi_1\partial^a\phi_1 - \frac{1}{2}\partial_a\phi_2\partial^a\phi_2 + \frac{1}{2}\partial_a\phi_3\partial^a\phi_3 + \frac{1}{2}\partial_a\phi_4\partial^a\phi_4 \Big\}. \end{aligned} \tag{B.19}$$

The interactions terms are

$$\begin{aligned} S(B)_{int} = & \frac{\varpi\kappa\alpha^4 R^4}{g^2} \int d\tau d\sigma d\theta d\phi \times \Big\{ + 2\sin\theta \mathcal{F}_{\tau\sigma}\phi_A - \frac{2}{\cos^2\tau}\mathcal{F}_{\theta\phi}\phi_B \\ & - \frac{2\sin\theta\cos\sigma\tan\tau}{\cos\tau}\phi_1\mathcal{F}_{\sigma\phi} - \frac{2\sin\theta\sin\sigma}{\cos\tau}\phi_1\mathcal{F}_{\tau\phi} \\ & - \frac{2\sin\theta\cos\sigma\tan\tau\sin\phi}{\cos\tau}\phi_2\mathcal{F}_{\sigma\theta} - \frac{2\cos\theta\cos\sigma\tan\tau\cos\phi}{\cos\tau}\phi_2\mathcal{F}_{\sigma\phi} \\ & - \frac{2\cos\theta\sin\sigma\cos\phi}{\cos\tau}\phi_2\mathcal{F}_{\tau\phi} - \frac{2\sin\theta\sin\sigma\sin\phi}{\cos\tau}\phi_2\mathcal{F}_{\tau\theta} \\ & - \frac{2\sin\theta}{\cos^2\tau}\phi_3\mathcal{F}_{\sigma\phi} - \frac{2\sin\theta\sin\phi}{\cos^2\tau}\phi_4\mathcal{F}_{\sigma\theta} - \frac{2\cos\theta\cos\phi}{\cos^2\tau}\phi_4\mathcal{F}_{\sigma\phi}\Big\}. \end{aligned} \tag{B.20}$$

At last, pure scalar terms are

$$
\begin{aligned}
S(B)_{scalar} = \frac{\varpi\kappa\alpha^4}{2g^2} \int d\tau d\sigma d\theta d\phi \, \sqrt{-\mathrm{g}} \Big\{ & -\frac{1}{2}\Phi_A^2 + \frac{1}{2}\Phi_B^2 \\
& + (\sin\tau\sin\sigma)\Phi_1\partial_\sigma\Phi_3 - (\cos\sigma\cos\tau)\Phi_1\partial_\tau\Phi_3 + (\sin\tau\sin\sigma)\Phi_2\partial_\sigma\Phi_4 \\
& - (\cos\sigma\cos\tau)\Phi_2\partial_\tau\Phi_4 - (\cot\theta\sin\phi)\Phi_2\partial_\phi\Phi_1 + (\cos\phi)\Phi_2\partial_\theta\Phi_1 \\
& - (\cot\theta\sin\phi)\Phi_3\partial_\phi\Phi_4 + (\cos\phi)\Phi_3\partial_\theta\Phi_4 - (\sin\tau\cos\sigma\cos\theta)\Phi_1\partial_\sigma\Phi_B \\
& - (\cos\tau\sin\sigma\cos\theta)\Phi_1\partial_\tau\Phi_B + \frac{\cos\sigma}{\cos\tau}\Phi_1\partial_\phi\Phi_A \\
& + (\sin\tau\cos\sigma\sin\theta\cos\phi)\Phi_2\partial_\sigma\Phi_B + (\cos\tau\sin\sigma\sin\theta\cos\phi)\Phi_2\partial_\tau\Phi_B \\
& + \frac{\cos\sigma\cot\theta\cos\phi}{\cos\tau}\Phi_2\partial_\phi\Phi_A + \frac{\cos\sigma\sin\phi}{\cos\tau}\Phi_2\partial_\theta\Phi_A \\
& - \cos\theta\Phi_3\partial_\sigma\Phi_B - \tan\tau\Phi_3\partial_\phi\Phi_A + (\sin\theta\cos\phi)\Phi_4\partial_\sigma\Phi_B \\
& - (\tan\tau\cot\theta\cos\phi)\Phi_4\partial_\phi\Phi_A - (\tan\tau\sin\phi)\Phi_4\partial_\theta\Phi_A \Big\}
\end{aligned}
\tag{B.21}
$$

where we have six commutative $U(1)$ field strength tensor

$$
\begin{aligned}
&\mathcal{F}_{\tau\sigma} = \partial_\tau\mathcal{A}_\sigma - \partial_\sigma\mathcal{A}_\tau, \;\; \mathcal{F}_{\theta\phi} = \partial_\theta\mathcal{B}_\phi - \partial_\phi\mathcal{B}_\theta, \;\; \mathcal{F}_{\tau\theta} = \partial_\tau\mathcal{B}_\theta - \partial_\theta\mathcal{A}_\tau, \\
&\mathcal{F}_{\tau\phi} = \partial_\tau\mathcal{B}_\phi - \partial_\phi\mathcal{A}_\tau, \;\; \mathcal{F}_{\sigma\theta} = \partial_\sigma\mathcal{B}_\theta - \partial_\theta\mathcal{A}_\sigma, \;\; \mathcal{F}_{\sigma\phi} = \partial_\sigma\mathcal{B}_\phi - \partial_\phi\mathcal{A}_\sigma.
\end{aligned}
\tag{B.22}
$$