THE CORONA THEOREM FOR THE MULTIPLIER ALGEBRAS ON WEIGHTED DIRICHLET SPACES

by

BERHANU TEKLE KIDANE

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ABSTRACT

In this dissertation we give a proof of "The Corona Theorem for Infinitely Many Functions for the Multiplier Algebras on Weighted Dirichlet Spaces", and we obtain explicit estimates on the size of the solution.

We denote the open unit disc of the complex plane by $\mathbb{D}$, and for $\alpha \in (0,1)$ we denote by $\mathcal{D}_\alpha$ the Weighted Dirichlet Spaces of all holomorphic functions on $\mathbb{D}$, and by $\|f\|_{\mathcal{D}_\alpha}$ 0r $\|f\|_\alpha$ the weighted Dirichlet norm.

To prove the $\mathcal{D}_\alpha$ *Corona Theorem* and find bounds for the solution we use a similar technique employed by Trent [Tr 2] in his proof of the corona theorem for Dirichlet space.

The first challenge is to find a norm that is expressed on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. In this dissertation we first establish such a norm. The new norm basically enables us to extend multipliers on the weighted Dirichlet spaces to multipliers on a related weighted harmonic Dirichlet spaces. Finally using Schur's Theorem and Cauchy's Transform we are able to prove the $\mathcal{D}_\alpha$ corona theorem and give bounds for the solution.

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CONTENTS

1. INTRODUCTION

1.1. Historical Background

In Mathematics the word Theorem is a major statement that is proved; we can say the word Theorem is synonymous to the word "Main Result". Theorems tend to be given names, The Fundamental Theorem of Calculus, Fermat's Little Theorem, The Corona Theorem and etc. In contrast, a Conjecture is a statement that has not been proven. A conjecture is not the same as a theorem. Famous conjectures tend to be given names too: The Poincare hypothesis, The Riemann hypothesis etc.

When we come to the Corona problem, it was the Japanese Mathematician Kakutani who first stated the conjecture in the beginning of the 1940's.

The word, "Corona " denotes the set of maximal ideals in $H^\infty(\mathbb{D})$ which are not in the W^* (weak star) closure of the maximal ideals in the unit disk $\mathbb{D}$. Carleson proved that the corona is empty.

Lennart Carleson has made many fundamental contributions to Harmonic Analysis, Complex Analysis, dynamical system, and other areas of mathematics. Standing out among them is his 1962 solution of the famous Corona Problem.

The corona theorem for finitely many functions is referred as the Carlson Corona Theorem or the $H^\infty(\mathbb{D})$ Corona Theorem. Carleson's proof of the corona theorem is very much illustrative (for example; in his proof Carleson introduced what is now know as Carleson's measure, a fundamental tool in complex and harmonic analysis), but at the same time highly complicated. The need to come up with a simpler proof and immediate generalization of the Carleson's Corona Theorem (say for infinitely many functions) are perhaps the first steps to take followed by a multitude set of corona problems on different spaces.

Hormander introduced the $\overline{\partial}$-technique for the corona problem and Wolff simplified Carleson's proof of the corona theorem using the $\overline{\partial}$-technique.

Rosenblum [R] and Tolokonnikov [T] independently proved the Carleson's Corona Theorem for the infinite version. Fuhrman and Vasyunin proved the Matrix version of the $H^\infty(\mathbb{D})$ Corona Theorem.

The H^p $(1 \leq p \leq \infty)$ corona Theorem for a finite number of bounded analytic functions on polydisks was established independently by Li [Li] and Lin [L], the infinite version was done by Trent [Tr3]

The Corona theorem on Dirichlet space for finite set of functions is done by Tolokonnikov; while the corresponding theorem for infinitely many functions from the multiplier algebra of the Dirichlet space is due to Trent[Tr2].

In this thesis, we extend Trent's result on Dirichlet space to weighted Dirichlet spaces for the weight $\alpha \in (0,\ 1)$. Our proof is done in a similar fashion to that of what Trent did for the Dirichlet space.

The main problem is, we don't know how to prove the weighted corona theorem and give the estimates based on known norms. That is, based on norms that come directly from the definition of the weighted Dirichlet spaces. Let's say a little more about this.

By definition the norm for $f \in \mathcal{D}_\alpha$ is given by

$$\|f\|_\alpha = \left(\sum_{n=0}^{\infty} (n+1)^\alpha \, |a_n|^2 \right)^{1/2}$$

For $\alpha = 1$, $\mathcal{D}$ is the Dirichlet space, and

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1) \, |a_n|^2 \tag{1.1.1}$$

Or equivalently the norm can be given by

$$\|f\|_{\mathcal{D}}^2 = \|f\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \tag{1.1.2}$$

or

$$\|f\|_{\mathcal{D}}^2 = \|f\|_{H^2(\mathbb{D})}^2 + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left|f(e^{it}) - f(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^2} d\sigma(t) d\sigma(\theta) \tag{1.1.3}$$

In the proof of the Dirichlet Corona Theorem ($\mathcal{D}$ Corona Theorem) for Dirichlet space, T. Trent used the norm in (1.1.3). Directly proving the $\mathcal{D}$ Corona Theorem using the norms (1.1.1) and (1.1.2) is still an open problem.

Since the technique employed by Trent [Tr2] can easily be extended to the weighted case, our major task is to find a norm for the weighted Dirichlet spaces that is similar to the norm given in (1.1.3).

In this thesis, for the weight $\alpha \in (0, 1)$, we succeed in finding such a norm, which is most crucial for our work.

The next important thing to do is to give bounds for the solution; our approach for this case is different from what is given in [Tr2]. We use Schur's theorem and the Cauchy transform and some other techniques to get the estimates.

2. REPRODUCING KERNEL HILBERT SPACES

In this section we explain the idea of a reproducing kernel Hilbert space in general and on the unit disk. We state some facts and give some standard examples of reproducing kernel Hilbert spaces. Good references for this sections are N. K. Nikolski [Ni], J. Agler and J.E. McCarthy [AM], and V. I. Paulsen [P].

2.1. Definitions and Properties

Let $\mathbb{R}$ and $\mathbb{C}$ denotes the set of real numbers and complex numbers respectively. We will use $\mathbb{F}$ to denote either $\mathbb{R}$ *or* $\mathbb{C}$. Given a non - empty set E, $\mathcal{F}(E, \mathbb{F})$ denotes the set of all functions from E to $\mathbb{F}$.

$\mathcal{F}(E, \mathbb{F})$ with the usual operations of addition and scalar multiplication is a vector space over $\mathbb{F}$.

Definition 2.1.1. Given a set E we will say that $\mathcal{H} \equiv \mathcal{H}(E)$ is a reproducing kernel Hilbert space (RKHS) on E over $\mathbb{F}$ if

(i) $\mathcal{H}$ is a subspace of $\mathcal{F}(E, \mathbb{F})$

(ii) $\mathcal{H}$ is endowed with an inner product $\langle \, , \, \rangle$ which makes $\mathcal{H}$ into a Hilbert space

(iii) for every $x \in E$, the linear evaluational functional $\mathcal{L}_x : \mathcal{H} \to \mathbb{F}$ defined by $\mathcal{L}_x(f) = f(x)$, is bounded.

If $\mathcal{H}$ is a RKHS on E, then by Riesz Representation Theorem every bounded linear functional is given by the inner product with a unique vector in $\mathcal{H}$; thus for every $y \in E$, there exists a unique vector, $k_y \in \mathcal{H}$, such that for every $f \in \mathcal{H}$, $f(y) = \langle f,\ k_y \rangle$.

Definition 2.1.2. The function k_y is called the Reproducing Kernel for the point y. The 2-variable function $K(x,\ y)$, defined by $K(x,\ y) = k_y(x)$ is called the Reproducing Kernel for $\mathcal{H}$.

Note that $K :\ E \times E \to \mathbb{F}$, and so $K(.,\ y)$ is in $\mathcal{H}$.

The following properties are immediate consequences of the definition of RKHS.

Property 2.1.3. The reproducing kernel of a Hilbert space is unique.

Property 2.1.4. Suppose $\mathcal{H}$ is a RKHS on E with reproducing kernel $K(x,\, y)$ then

(i) $K(x,\ y) = k_y(x) = \langle k_y,\, k_x \rangle$

(ii) $K(x,\, x) \geq 0$

(iii) $K(x,\, y) = \overline{K(y,\, x)}$

(iv) $|\, K(x,\, y)\,|^2 \leq K(x,\, x)K(y,\, y)$.

Suppose $\mathcal{H}$ is a RKHS on E with reproducing kernel $K(x,\, y)$. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then for every $y \in E$, $k_y \in \mathcal{H}$ and

$$k_y = \sum_{n=0}^{\infty} \alpha_n e_n, \qquad \text{where } \sum_{n=0}^{\infty} |\,\alpha_n|^{\,2} < \infty, \quad \text{and} \quad \langle e_n,\, k_y \rangle = \overline{\alpha_n}\ .$$

This shows that, $\alpha_n = \langle k_y, e_n \rangle = \overline{e_n(y)}$. Thus, $\sum_{n=0}^{\infty} |\, e_n(y)\,|^2 < \infty$ for all $y \in E$.

Conversely, we obtain the following Theorem.

Theorem 2.1.5. Suppose $\mathcal{H}$ is a Hilbert space of functions on E with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Assume that $\forall x \in E$, $\sum_{n=0}^{\infty} |\, e_n(x)\,|^2 < \infty$. Then $\mathcal{H}$ is a reproducing kernel Hilbert space, RKHS, with reproducing kernel given by:

$$K(x,\, y) = \sum_{n=0}^{\infty} \overline{e_n(y)}\, e_n(x).$$

Proof. Let $y \in \mathcal{H}$ be fixed. Define $\widetilde{k}_y$ by $\widetilde{k}_y = \sum_{n=0}^{\infty} \overline{e_n(y)}\, e_n$

since $\sum_{n=0}^{\infty} |\, e_n(y)\,|^2 < \infty$, $\widetilde{k}_y \in \mathcal{H}$

Now let $f \in \mathcal{H}$, which implies $f = \sum_{n=0}^{\infty} \langle f,\, e_n \rangle\, e_n$. Thus

$$\begin{aligned} \left\langle f,\, \widetilde{k}_y \right\rangle_{\mathcal{H}} &= \left\langle \sum_{j=0}^{\infty} \langle f,\, e_j \rangle\, e_j,\ \sum_{n=0}^{\infty} \overline{e_n(y)}\, e_n \right\rangle \\ &= \sum_{n=0}^{\infty} \langle f,\, e_n \rangle\, e_n(y) \\ &= f(y) \\ &\leq \|\, f \,\| \left\| \widetilde{k}_y \right\| \end{aligned}$$

Thus, evaluation at y is a bounded linear functional;so, $\mathcal{H}$ is a reproducing kernel Hilbert Space, and by uniqueness $\widetilde{k}_y = k_y$.

□

2.2. Examples

In this section we will consider some standard examples of reproducing kernel Hilbert spaces.

Example 2.2.1. $H^2(\mathbb{D})$**, Hardy Space**

$H^2(\mathbb{D}) = \{f\ : f \in Hol(\mathbb{D}),\ f(z) = \sum_{n=0}^{\infty} a_n z^n\ \ and \sum_{n=0}^{\infty} |\, a_n\,|^2\ < \infty\}$

If $f \in H^2(\mathbb{D})$, then its norm is defined as: $\|\, f\,\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |\, a_n\,|^2\ < \infty.$

Let $\{e_n(z)\}_{n=0}^{\infty} = \{z^n\}_{n=0}^{\infty}$, then $\|\, e_n\| = 1$, and $\langle e_n,\, e_m\rangle = 0\ for\ n \neq m$

Thus $\{z^n\}_{n=0}^{\infty}$ is an orthonormal system which forms a basis for $H^2(\mathbb{D})$.

Furthermore $\sum_{n=0}^{\infty} |\, e_n(a)\,|^2 = \sum_{n=0}^{\infty} |\, a\,|^{2n} < \infty,\ for\ a \in \mathbb{D}$ implies that

$H^2(\mathbb{D})$ is a reproducing kernel Hilbert space, with Reproducing

kernel given by:

$$\begin{aligned} K(z,\, \omega) &= \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\omega)} \\ &= \sum_{n=0}^{\infty} (z\,\overline{\omega})^n \\ &= 1/(1 - z\,\overline{\omega}). \end{aligned}$$

Example 2.2.2. $A^2(\mathbb{D})$, **Bergman Space**

$A^2(\mathbb{D}) = \{f\ : f \in Hol(\mathbb{D}),\ f(z) = \sum_{n=0}^{\infty} a_n z^n\ for\ and \sum_{n=0}^{\infty} |\, a_n\,|^2/(n+1)\ < \infty\}$

If $f \in A^2(\mathbb{D})$, then its norm is defined as: $\|\, f\,\|^2_{A^2(\mathbb{D})} = \sum_{n=0}^{\infty} |\, a_n\,|^2/(n+1)\ < \infty.$

Polynomials belong to $f \in A^2(\mathbb{D})$, and so $\|z^n\| = 1/(n+1)$. Thus,

$$\{e_n\}_{n=0}^{\infty},\ \text{with} : e_n(z) = (\sqrt{n+1}\,)z^n$$

forms an orthonormal basis for $A^2(\mathbb{D})$.

Thus $A^2(\mathbb{D})$ is a reproducing kernel Hilbert space with reproducing kernel:

$$K(z,\,\omega) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\omega)} = \sum_{n=0}^{\infty}(1/(n+1))(\overline{\omega}\,z)^n$$
$$= 1/(1 - z\overline{\omega})^2$$

Example 2.2.3. $\mathcal{D}(\mathbb{D})$ or $\mathcal{D}$, **Dirichlet Space**

$\mathcal{D} = \{f \in Hol(\mathbb{D})\ :\ f(z) = \sum_{n=0}^{\infty} a_n z^n\ and\ \sum_{n=0}^{\infty}(n+1)|\,a_n\,|^2 < \infty\}$

For $f \in \mathcal{D}$ define the norm of f by: $\|\,f\,\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty}(n+1)|\,a_n\,|^2 < \infty\}$.

Then it is not hard to see that $\left\{z^n/(\sqrt{n+1})\right\}_{n=0}^{\infty}$ forms an orthonormal basis for $\mathcal{D}$.

The reproducing kernel for $\mathcal{D}$ is given by: $k_\omega(z) = \sum_{n=0}^{\infty} \frac{\overline{\omega}^n}{\sqrt{n+1}} \frac{z^n}{\sqrt{n+1}}$

$$= \sum_{n=0}^{\infty} \frac{(z\,\overline{w})^n}{n+1}$$

$$= \frac{1}{\overline{\omega}\,z} log \frac{1}{1-\overline{\omega}\,z}$$

Example 2.2.4.: **Weighted Hilbert Spaces on** $\mathbb{D}$, generalized

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\alpha_n > 0$ for all n be such that $\liminf_{n\to\infty}(\alpha_n)^{(1/n)} = 1$.

Then define $H_\alpha(\mathbb{D})$, Hilbert spaces of functions on $\mathbb{D}$ by:

$$H_\alpha(\mathbb{D}) = \{f\ :\ f \in Hol(\mathbb{D}),\ f(z) = \sum_{n=0}^{\infty} f_n z^n,\ and\ \sum_{n=0}^{\infty}|\,f_n\,|^2\alpha_n < \infty\}$$

- If f and g are in $H_\alpha(\mathbb{D})$, then define $\langle f,\,g\rangle := \sum_{n=0}^{\infty} f_n\overline{g_n}\,\alpha_n$

- If $f \in H_\alpha(\mathbb{D})$ then the norm of f is defined by: $\|\,f\,\|_\alpha^2 = \sum_{n=0}^{\infty}|\,f_n\,|^2\alpha_n$

We claim, $H_\alpha(\mathbb{D})$ is a reproducing kernel Hilbert space on $\mathbb{D}$. To show this, set $e_n(z) = (1/\sqrt{\alpha_n})z^n$. Then it is not hard to verify that $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis for $H_\alpha(\mathbb{D})$.

And for $z \in \mathbb{D}$, since $\liminf\limits_{n\to\infty}(\alpha_n)^{(1/n)} = 1$ $\sum\limits_{n=0}^{\infty} |\, e_n(z)\,|^2 = \sum\limits_{n=0}^{\infty} \frac{1}{\alpha_n} |\, z\,|^{2n} < \infty$.

Hence, $H_\alpha(\mathbb{D})$ is a reproducing kernel Hilbert space with reproducing kernel given by:

$$\begin{aligned} K(z,\,\omega) &= \sum_{n=0}^{\infty} e_n(z)\overline{e_n(\omega)} \\ &= \sum_{n=0}^{\infty} (1/\alpha_n)(z\,\overline{\omega}\,)^n \end{aligned}$$

Special cases of **Example 2.2.4.**

(i) The case $\alpha_n = 1$ gives Hardy space

(ii) The case $\alpha_n = 1/(n+1)$ gives Bergman space

(iii) The case $\alpha_n = (n+1)$ gives Dirichlet space

(iv) The case $\alpha_n = 1/(n+1)^\alpha,\;\; \alpha > 0$ gives weighted Bergman space

with Reproducing kernel given by: $K(z,\,\omega) = \sum\limits_{n=0}^{\infty} (n+1)^\alpha (z\,\overline{\omega}\,)^n$

(v) The case $\alpha_n = (n+1)^\alpha,\;\; \alpha > 0$ gives weighted Dirichlet space

with reproducing kernel given by $K(z,\,\omega) = \sum\limits_{n=0}^{\infty} 1/(n+1)^\alpha (z\,\overline{\omega}\,)^n$

2.3. Characterization of Reproducing Kernels

In this section we consider necessary and sufficient conditions for a function $K(x,\ y)$ to be a reproducing kernel for some RKHS.

Definition 2.3.1. Let $A = (a_{ij})$ be a $n \times n$ complex matrix. Then A is **positive definite** if and only if for every $\alpha_1,\ \alpha_2,\ \ldots, \alpha_n$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\alpha_i}\, \alpha_j a_{ij} \geq 0.$$

Definition 2.3.2. Let X be a set and $K:\ X \times X \to \mathbb{C}$ be a function of two variables. Then K is called a **kernel function** (written $K \geq 0$) if for every n and for every choice of n distinct points $\{x_1,\ ,\ \ldots\ , x_n\} \subseteq X$, the matrix $(K(x_i,\ x_j))$ is positive definite.

We state next, a proposition and a theorem which completes our characterization.

Proposition 2.3.3. Let E be a set and let $\mathcal{H}$ be a RKHS on E with reproducing kernel K. Then K is a kernel function.

Proof. Fix $\{x_1,\ \ldots\ , x_n\} \subseteq E$ and $\alpha_1,\ \ldots\ , \alpha_n \in \mathbb{C}$. Then it holds that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\alpha_i}\, \alpha_j K(x_i,\ x_j) = \left\langle \sum_{j=1}^{n} \alpha_j k_{x_j},\ \sum_{i=1}^{n} \alpha_i k_{x_i} \right\rangle = \left\| \sum_{j=1}^{n} \alpha_j k_{x_j} \right\| \geq 0,$$

and the result follows.

□

Theorem 2.3.4. (Moore) Let X be a set and $K : X \times X \to \mathbb{C}$ be a function. If K is a kernel function, then there exists a reproducing kernel Hilbert space of functions on X such that K is the reproducing kernel for $\mathcal{H}$.

PROOF. See Wikipedia, *Reproducing Kernel Hilbert Spaces*, and [Tr1].

3. MULTIPLIER ALGEBRAS

In the formulation of the corona theorems, the functions of interest are those functions which come from the multiplier algebras of the corresponding spaces. So, in this section we define and state some properties of multiplier algebras.

Definition 3.1. A *Complex algebra* is a vector space $\mathbf{U}$ over $\mathbb{C}$ in which an associative and distributive multiplication is defined, i.e.,

(i) $(xy)z = x(yz), \quad (x+y)z = xz + yz, \quad x(y+z) = xy + xz$, for all $x,\ y,$ and $z \in \mathbf{U}$, and

(ii) $\alpha(xy) = x(\alpha y) = (\alpha x)y$, for all x and $y \in \mathbf{U}$, α scalar.

$\mathbf{U}$ is *a normed complex algebra* if

$$\| xy \| \leq \|x\| \, \|y\| \,, \quad \text{for all } x, y \in \mathbf{U}$$

$\mathbf{U}$ is a *Banach algebra* if $\mathbf{U}$ is a Banach space with respect to this norm.

For more details see W. Rudin [Ru].

Definition 3.2. Let $E \subseteq \mathbb{C}$, and $\mathcal{H}(E)$ be Hilbert space of functions on E.

For any $\phi \in \mathcal{H}(E)$, define : $M_\phi : \mathcal{H}(E) \to \mathcal{H}(E)$ by

$$M_\phi(f) = \phi f, \quad \text{for all } f \text{ in } \mathcal{H}(E).$$

If $\phi f \in \mathcal{H}(E)$ for all f in $\mathcal{H}(E)$, then ϕ or M_ϕ is called a *multiplier* for $\mathcal{H}(E)$.

If $\phi f \in \mathcal{H}(E)$ for all f in $\mathcal{H}(E)$, then by the closed graph theorem, $M_\phi \in \mathcal{B}(\mathcal{H}(E))$ (space of bounded operators on $\mathcal{H}(E)$).

Let $M(\mathcal{H}(E))$ denotes the set of all multipliers on $\mathcal{H}(E)$; that is:

$$M(\mathcal{H}(E)) = \{M_\phi \in \mathcal{B}(\mathcal{H}(E)) : M_\phi f = \phi f \in \mathcal{H}(E) \text{ for all } f \in \mathcal{H}(E)\}.$$

and let

$$\mathcal{M}(\mathcal{H}(E)) = \{\phi \in \mathcal{H}(E) : \phi f \in \mathcal{H}(E) \text{ for all } f \in \mathcal{H}(E)\}$$

Proposition 3.3. Let $\mathcal{H}(E)$ be a RKHS. If M_ϕ is a multiplier for $\mathcal{H}(E)$, then $M_\phi^*(k_\omega) = \overline{\phi(\omega)}\, k_\omega$.

Proof. Let $f \in \mathcal{H}(E)$ be any, then

$$\begin{aligned} \left\langle f,\, M_\phi^* k_\omega \right\rangle &= \langle M_\phi f,\, k_\omega\rangle \\ &= \langle \phi f,\, k_\omega\rangle \\ &= \phi(\omega) f(\omega) \\ &= \left\langle f,\, \overline{\phi(\omega)}\, k_\omega \right\rangle, \quad \forall f \in \mathcal{H}(E) \end{aligned}$$

□

In passing we note a theorem and some important properties of $H^2(\mathbb{D})$.

A multiplier on $H^2(\mathbb{D})$ is denote by T_ϕ and is called an "*Analytic Toeplitz operator*".

Proposition 3.4. Let $f,\, g \in H^\infty(\mathbb{D})$, $\alpha \in \mathbb{C}$, then

(i) $T_{f+\alpha g} = T_f + T_{\alpha g}$

(ii) $T_f^* = T_{\overline{f}}$

Theorem 3.5. $M(\mathcal{H}^2(\mathbb{D})) = H^\infty(\mathbb{D})$

For the proof see J. Agler and J.E. McCarthy [AM].

4. WEIGHTED DIRICHLET SPACES

In this section we will define the weighted analytic spaces on the unit disk. For more detail see Z. Wu [Wu], C. Cowen and B. Maccluer [CM].

We also define our main ideas which are weighted Dirichlet spaces and weighted harmonic Dirichlet spaces and their corresponding norms for the weight $\alpha \in (0, 1)$.

Definition 4.1. For $\alpha \in \mathbb{R}$, $\mathcal{A}_\alpha(\mathbb{D})$ or just simply $\mathcal{A}_\alpha$ will denote the weighted analytic space on the unit disk $\mathbb{D}$. $\mathcal{A}_\alpha$ is defined by:

$$\mathcal{A}_\alpha = \left\{ f \in Hol(\mathbb{D}) : \; f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ , } and \sum_{n=0}^{\infty} (n+1)^\alpha |a_n|^2 < \infty \right\}$$

For any $f \in \mathcal{A}_\alpha$ the norm of f is defined by

$$\| f \|_{\mathcal{A}_\alpha}^2 = \sum_{n=0}^{\infty} (n+1)^\alpha \, |a_n|^2$$

Note For $\alpha = -1$, $\mathcal{A}_{-1} = \mathcal{A}$ is the Bergman space.

For $\alpha < 0$, $\mathcal{A}_\alpha$ is a weighted Bergman space

For $\alpha = 0$, $\mathcal{A}_0 = H^2(\mathbb{D})$ is the Hardy Space

For $\alpha = 1$, $\mathcal{A}_1 = \mathcal{D}$ is the Dirichlet space

For $\alpha > 0$, $\mathcal{A}_\alpha = \mathcal{D}_\alpha$ is the weighted Dirichlet space

In this thesis we consider weighted Dirichlet spaces only for $\alpha \in (0, 1)$. From now on, $\mathcal{D}_\alpha$ or $\mathcal{D}_\alpha(\mathbb{D})$, represents weighted Dirichlet space for the weight $\alpha \in (0, 1)$.

Definition 4.2. Let $\alpha \in \mathbb{R}$. We denote the **Weighted Harmonic Dirichlet Space** by $\mathcal{HD}_\alpha$ and define by:

$$\mathcal{HD}_\alpha = \left\{ f \in L^2(\mathbb{T}, d\lambda) : f(e^{it}) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{int} \ and \ \sum_{n=-\infty}^{\infty} (1+|n|)^\alpha |\widehat{f}(n)|^2 < \infty \right\}.$$

The norm, $\| f \|^2_{\mathcal{HD}_\alpha}$, of a function f in $\mathcal{HD}_\alpha$ is defined by:

$$\| f \|^2_{\mathcal{HD}_\alpha} = \sum_{n=-\infty}^{\infty} (1+|n|)^\alpha |\widehat{f}(n)|^2$$

4.1. Algebra of Operators on $\mathcal{D}_\alpha$

The algebra of operators we consider here are the multipliers algebra on weighted Dirichlet spaces and weighted harmonic Dirichlet spaces. For a fixed α we denote the multiplier algebra on $\mathcal{D}_\alpha$ by $\mathcal{M}(\mathcal{D}_\alpha)$ and on $\mathcal{HD}_\alpha$ by $\mathcal{M}(\mathcal{HD}_\alpha)$, where

$$\mathcal{M}(\mathcal{D}_\alpha) = \{ \phi \in \mathcal{D}_\alpha : g\phi \in \mathcal{D}_\alpha \text{ , for all } g \in \mathcal{D}_\alpha \}, \text{ and}$$

$$\mathcal{M}(\mathcal{HD}_\alpha) = \{ \phi \in \mathcal{HD}_\alpha : g\phi \in \mathcal{HD}_\alpha, \text{ for all } g \in \mathcal{HD}_\alpha \}$$

Note that $\mathcal{M}(\mathcal{HD}_\alpha)$ is defined only on the boundary of $\mathbb{D}$.

Furthermore we denote and define:

$$M(\mathcal{D}_\alpha) := \{ M_\phi \in \mathcal{B}(\mathcal{D}_\alpha) : M_\phi f = \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha \}.$$

$$M(\mathcal{HD}_\alpha) := \{ M_\phi \in \mathcal{B}(\mathcal{HD}_\alpha) : M_\phi f = \phi f \in \mathcal{HD}_\alpha \text{ for all } f \in \mathcal{HD}_\alpha \}.$$

Note that

$$\phi \in \mathcal{M}(E) \text{ implies that } \|\,\phi\,\| \leq \ \|\,M_\phi\,\|$$

Next we consider what we call row and column operators for the weighted Dirichlet spaces. These are important operators for our version of the corona theorem on the weighted Dirichlet space.

Let $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$ and $F = (f_1,\ f_2,\ .\ .\ .\ )$. Then define

M_F^R, a Row Operator, $M_F^R : \bigoplus_{n=1}^{\infty} \mathcal{D}_\alpha \longrightarrow \mathcal{D}_\alpha$ by

$$M_F^R\left(\{h_j\}_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} f_j h_j, \text{ for } \{h_j\}_{j=1}^{\infty} \in \bigoplus_{n=1}^{\infty} \mathcal{D}_\alpha, \text{ and}$$

M_F^C, a Column Operator, $M_F^C : \mathcal{D}_\alpha \longrightarrow \bigoplus_{n=1}^{\infty} \mathcal{D}_\alpha$ by

$$M_F^C(h) = \left(f_1 h,\ f_2 h,\ f_3 h\ \ .\ .\ .\ \right)^t, \text{ for } h \in \mathcal{D}_\alpha$$

Once these operators are defined, then we consider the question:

How big is $\mathcal{M}(\mathcal{D}_\alpha)$?

To answer this question we recall $\mathcal{M}(H^2(\mathbb{D})) = \mathcal{H}^\infty(\mathbb{D})$, and the relationship between multiplier spaces and reproducing kernels.

The relationship between multiplier spaces and reproducing kernels is expressed for $\phi \in \mathcal{M}(\mathcal{D}_\alpha)$ and $z \in \mathbb{D}$, as $M_\phi^* k_z = \overline{\phi(z)}\, k_z$, which implies

$$\text{and so } \left\|M_\phi^*\right\| \geq |\phi(z)|\,, \forall z \in \mathbb{D}$$

$$\|\phi\|_\infty \leq \|M_\phi\|$$

Thus $M_\phi \in \mathcal{B}(\mathcal{D}_\alpha)$ implies that $\phi \in \mathcal{H}^\infty(\mathbb{D})$ which gives, $\mathcal{M}(\mathcal{D}_\alpha) \subset \mathcal{H}^\infty(\mathbb{D})$.

Similarly for $\phi_{ij} \in \mathcal{M}(\mathcal{D}_\alpha)$ and matrix valued operator $M_{[\phi_{ij}]}$, where

$$M_{[\phi_{ij}]} = \begin{pmatrix} M_{\phi_{11}} & M_{\phi_{12}} & ... & ... \\ M_{\phi_{21}} & M_{\phi_{22}} & ... & ... \\ ... & ... & ... & ... \\ ... & ... & ... & ... \end{pmatrix},$$

we have that $\mathcal{M}(\bigoplus_1^\infty \mathcal{D}_\alpha) \subset H^\infty_{\mathcal{B}(l^2)}(\mathbb{D})$. Note that $M_{[\phi_{ij}]} : \bigoplus_1^\infty \mathcal{D}_\alpha \longrightarrow \bigoplus_1^\infty \mathcal{D}_\alpha$ and for each ij, $M_{\phi_{ij}} \in \mathcal{B}(\mathcal{D}_\alpha)$.

We claim $\mathcal{M}(\mathcal{D}_\alpha) \subseteq \mathcal{H}^\infty(\mathbb{D})$ is a strict containment. To show this, for each $\alpha \in (0,1)$, consider the sum:

$$\sum_{n=1}^{\infty} \frac{z^{n^{4m+1}}}{n^{2m\alpha}}, \quad \textit{where } m = [1/\alpha] + 1,\ z \in \mathbb{D}$$

and $[\,.\,]$ denotes the greatest integer function, thus

$$2m\alpha = 2\left([1/\alpha] + 1\right)\alpha \;=\; 2\alpha[1/\alpha] + 2\alpha \;>\; 2$$

Let

$$g(z) \; = \; \sum_{n=1}^{\infty} \frac{z^{n^{4m-1}}}{n^{2m\alpha}} \; .$$

Then

$$|\, g(z)\, | \;\; \leq \;\; \sum_{n=1}^{\infty} \frac{|\, z^{n^{4m-1}}\, |}{n^{2m\alpha}} \;\; \leq \sum_{n=1}^{\infty} \frac{1}{n^2}, \; \forall z \in \mathbb{D}$$

implies that $g \in \mathcal{H}^{\infty}(\mathbb{D})$.

Question: Is $g \in \mathcal{D}_\alpha$? Answer is **NO.**

Recall $f \in \mathcal{D}_\alpha$ implies that $f(z) = \sum\limits_{n=o}^{\infty} \; f_n z^n$, and $\|f\|_\alpha^2 = \sum\limits_{n=o}^{\infty} (n+1)^\alpha \, |f_n|^2 < \infty$

For g to be in $\mathcal{D}_\alpha$ it must hold that $\sum\limits_{n=o}^{\infty} (n+1)^\alpha \, |g_n|^2 < \infty$

$$\begin{aligned} \text{But, } \| \, g \, \|_\alpha^2 = \sum_{n=o}^{\infty} (n+1)^\alpha \, |g_n|^2 &= \sum_{n=o}^{\infty} (n^{4m-1} + 1)^\alpha \tfrac{1}{n^{4m\alpha}} \\ &= \sum_{n=o}^{\infty} (\tfrac{n^{4m-1}+1}{n^{4m}})^\alpha \\ &= \sum_{n=o}^{\infty} (\tfrac{1}{n} + \tfrac{1}{n^{4m}})^\alpha \geq \sum_{n=1}^{\infty} \tfrac{1}{n^\alpha} \end{aligned}$$

Since for $0 < \alpha < 1$, $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges to ∞, we have $g \notin \mathcal{D}_\alpha$.

Thus, $\mathcal{M}(\mathcal{D}_\alpha)$ is strictly contained in $\mathcal{H}^{\infty}(\mathbb{D})$.

5. CORONA THEOREMS

In this chapter first we will see some known corona theorems, then we will consider the weighted corona theorem on $\mathcal{D}_\alpha$.

5.1. Carleson's Corona Theorem

The original Carleson's Corona Problem can be stated in purely analytical terms as: Refer [NE]

Theorem 5.1.1 Let $f_1, \ f_2, \ \ldots \ , f_n$ be analytic and be bounded in $\mathbb{D}$ and suppose that $|\, f_1(z)\,| + |\, f_2(z)\,| + \ \ldots \ + |\, f_n(z)\,| \geq \ 1$ for all $z \in \mathbb{D}$, then there exist $g_1, \ g_2, \ \ldots \ , g_n$ which are analytic and bounded in $\mathbb{D}$, such that $f_1 g_1 + f_2 g_2 + \ldots \ + f_n g_n \equiv 1$ for all $z \in \mathbb{D}$.

Carleson was the first to prove this and so the theorem is named *Carleson Corona Theorem*. Note that the functions f_i and g_j for each i and j are from $H^\infty(\mathbb{D})$, which is the multiplier algebra for this case.

We note, in passing, a more general conjecture which is suggested by **Theorem 5.1.1** Consider the following problem for finite collection of functions:

Problem (5.1.2.) Let $f_1, \ f_2, \ \ldots \ f_n$ and g be functions in $H^\infty(\mathbb{D})$ satisfying

$$|\, g(z)\,| \leq |\, f_1(z)\,| + |\, f_2(z)\,| + \ \ldots \ + |\, f_n(z)\,|, \text{ for all } z \in \mathbb{D} \tag{5.1.1.1}$$

It is natural to ask if (5.1.1.1) implies the existence of $g_1, \ g_2, \ \ldots \ g_n \in H^\infty$ such that $f_1(z)g_1(z) + f_2(z)g_2(z) + \ldots \ + f_n(z)g_n(z) \equiv g(z), \forall z \in \mathbb{D}$.

Note first in **Problem (5.1.2.)** if $g(z) = 1$ for all $z \in \mathbb{D}$, then **Problem (5.1.2.)** is the Carleson Corona Theorem.

Rao [Ra] shows that the answer to this question, in general, is negative. Treil has recently shown that g^2 does not belong to I. On the other hand, Wolff has proved that g^3 belongs to I. For more detail see G. Gentili and D. Struppa [GS]

We will not consider this general question in this thesis.

5.2. $H^\infty(\mathbb{D})$ Corona Theorem

Next we consider the infinite version of Carleson's Corona Theorem. This version was proved by Rosenblum [R] and Tolokonnikov [T], and the best estimate for the solution is given by Ukiyama.

Theorem 5.2.1. ($H^\infty(\mathbb{D})$ Corona Theorem)

Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{H}^\infty(\mathbb{D})$ with $0 < \epsilon^2 < \sum\limits_{j=1}^{\infty} |f_j(z)|^2 \leq 1$ for all $z \in \mathbb{D}$. Then there exists a positive number $k(\epsilon) < \infty$ and $\{g_j\}_{j=1}^{\infty} \subset \mathcal{H}^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^{\infty} f_j g_j = 1 \text{ and } \quad sup\,\{\sum_{j=1}^{\infty} |g_j(z)|^2\} \leq k(\epsilon).$$

Here also the multiplier algebra is $H^\infty(\mathbb{D})$.

Now let's consider the row and column operators T_F^R and T_F^C.

Let $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(H^2(\mathbb{D}))$ and $F \ = \ (f_1,\ f_2,\ .\ .\ .\ )$. Then define $T_F^R \ : \ \bigoplus\limits_1^{\infty} H^2(\mathbb{D}) \ \longrightarrow \ H^2(\mathbb{D})$,a row operator given by

$$T_F^R\,(\{h_j\}_{j=1}^{\infty}) \ = \ \sum_{j=1}^{\infty} f_j h_j,\ where\ \{h_j\}_{j=1}^{\infty} \in \bigoplus_1^{\infty} H^2(\mathbb{D}),\ and$$

$T_F^C \ : \ H^2(\mathbb{D}) \longrightarrow \bigoplus\limits_1^{\infty} H^2(\mathbb{D})$, a column operator given by

$$T_F^C\,(h) \ = \ (f_1 h,\ f_2 h,\ f_3 h\ \ .\ .\ .\ )^T,\ where\ h \in H^2(\mathbb{D})$$

From part of the point-wise hypothesis of the $H^2(\mathbb{D})$ Corona Theorem, we have that:

$$\left\|T_F^R\right\|^2 \;=\; \left\|T_F^C\right\|^2 \;=\; Sup\left\{\sum_{j=1}^{\infty}|f_j(z)|^2 :\; z\in\mathbb{D}\right\}.$$

Thus, one part of the point-wise hypothesis of the $H^\infty(\mathbb{D})$ Corona Theorem gives the boundedness of the operators T_F^R and T_F^C.

In terms of the operators T_F^R and T_F^C Theorem 5.2.1. can be stated as:

Theorem 5.2.2. Let $\{f_j\}_{j=1}^{\infty}\subset\mathcal{H}^\infty(\mathbb{D})$ with $0<\epsilon^2<\;\left\|T_F^R\right\|^2\leq 1$.

Then there exist a positive $k(\epsilon)$ and $\{g_j\}_{j=1}^{\infty}\subset\mathcal{H}^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^{\infty} f_j g_j = 1 \;\text{ and }\; \left\|\,T_G^C\,\right\|^2 \leq k(\epsilon).$$

5.3. The Dirichlet Corona Theorem

Consider the Dirichlet space, $\mathcal{D}$. $\mathcal{M}(\mathcal{D})$ is the multiplier algebra and we know that $\mathcal{M}(\mathcal{D})$ is strictly contained in $\mathcal{H}^\infty(\mathbb{D})$.

Note $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$, and $\sum_{j=1}^{\infty} |f_j(z)|^2 < \infty$ need not imply the boundedness of $\| M_F^C \|$, where M_F^C is a column operator. But the boundedness of $\| M_F^C \|$ always gives the boundedness of $\| M_F^R \|$, see [Tr2]. Thus, the replacement of $\sum_{j=1}^{\infty} |f_j(z)|^2 \leq 1$ for $z \in \mathbb{D}$ in the $H^\infty(\mathbb{D})$ Corona Theorem will be $\| M_F^C \| \leq 1$. As a result of this the $\mathcal{D}$ Corona theorem is stated as:

Theorem 5.3.1 (The $\mathcal{D}$ Corona Theorem)

Let $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$. Assume that , $\| M_F^C \| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^\infty | f_j(z) |^2$ for all $z \in \mathbb{D}$. Then there exist $C(\epsilon) < \infty$ and $\{g_j\}_{j=1}^\infty \subseteq \mathcal{M}(\mathcal{D})$ such that:

$$\begin{aligned} (i) \ \sum_{j=1}^{\infty} f_j g_j &= 1 \\ (ii) \ \| M_G^C \| &\leq C(\epsilon) \end{aligned}$$

For the proof of the $\mathcal{D}$ Corona theorem refer to Trent [Tr 2].

In the proof of the Dirichlet Corona theorem, Trent used a norm that is expressed on the boundary of $\mathbb{D}$. We need a similar representation for the weighted Dirichlet case.

5.4. Norm Estimates

Recall that for $f \in \mathcal{D}_\alpha$ its norm is given by

$$(1) \quad \| f \|^2_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha \, |a_n|^2$$

Two other norms that are equivalent to the above norm are

$$(2) \quad \| f \|^2_{\mathcal{D}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{\mathbb{D}} |f'(z)|^2 \, (1-|z|^2)^{1-\alpha} dA(z)$$

$$(3) \quad \| f \|^2_{\mathcal{D}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left| f(e^{it}) - f(e^{i\theta}) \right|^2}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} d\sigma(t) d\sigma(\theta)$$

(1) is equivalent to (2) can be seen easily, but we need to verify the equivalence between (1) and (3). We need this equivalence because our estimates are on the boundary of $\mathbb{D}$.

First let's note the following. For $f \in \mathcal{D}_\alpha$, $f(z) = \sum\limits_{n=0}^{\infty} f_n \, z^n$, and if (3) holds, we have

$$\| f \|^2_{\mathcal{D}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left| f(e^{it}) - f(e^{i\theta}) \right|^2}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} d\sigma(t) d\sigma(\theta)$$

$$\| f \|^2_{\mathcal{D}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left| \sum_{n=0}^{\infty} f_n (e^{in\theta} - e^{int}) \right|^2}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} d\sigma d\sigma$$

$$= \|f\|^2_{H^2(\mathbb{D})} + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f_n \overline{f_k} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{(e^{in\theta} - e^{int})\,(e^{-ik\theta} - e^{-ikt})}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} d\sigma d\sigma$$

$$= \|f\|^2_{H^2(\mathbb{D})} + \sum_{n=0}^{\infty} |\, f_n \,|^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left| e^{in\theta} - e^{int} \right|^2}{\left| e^{it} - e^{i\theta} \right|^{\alpha+1}} d\sigma d\sigma$$

In particular if $f(z) = z^n$, then

$$\| f \|_\alpha^2 = \|f\|_{H^2(\mathbb{D})}^2 + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\, e^{in\theta} - e^{int} \,|^2}{|e^{it} - e^{i\theta}|^{\alpha+1}} d\sigma d\sigma$$

Thus, we are done if we show for large N

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\, e^{iN\theta} - e^{iNt} \,|^2}{|\, e^{it} - e^{i\theta} \,|^{\alpha+1}} \, d\sigma(t) d\sigma(\theta) \sim (N+1)^\alpha$$

where $(N+1)^\alpha$ is the α *Dirichlet* norm of $f(z) = z^N$, i.e. $\| f \|_{\mathcal{D}_\alpha}^2 = (N+1)^\alpha$. This means we are done, if we show for large N there exists a constant $C(\alpha)$, depending only on α, such that

$$\frac{1}{C(\alpha)}(N+1)^\alpha \leq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\, e^{iN\theta} - e^{iNt} \,|^2}{|\, e^{it} - e^{i\theta} \,|^{\alpha+1}} \, d\sigma(t) d\sigma(\theta) \leq C(\alpha)(N+1)^\alpha$$

In other words we want a norm for the function f in the weighted Dirichlet space that is expressed as a norm on the boundary of the unit disk $\mathbb{D}$, $\partial\mathbb{D}$.

So, let $\beta = (1+\alpha)/2$, $f(z) = z^N$ and consider

$$\begin{aligned} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\, f(e^{it}) - f(e^{i\theta}) \,|^2}{|e^{it} - e^{i\theta}|^{2\beta}} d\sigma d\sigma &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \frac{e^{iNt} - e^{iN\theta}}{(e^{it} - e^{i\theta})^\beta} \right|^2 d\sigma d\sigma \\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \frac{1 - e^{iN(t-\theta)}}{(1 - e^{i(t-\theta)})^\beta} \right|^2 d\sigma(t) d\sigma(\theta) \end{aligned}$$

Fix θ and set $s = t - \theta \ \Rightarrow \ d\sigma(s) = d\sigma(t)$ ($d\sigma$ is normalized measure), then

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|f(e^{it})-f(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{2\beta}}d\sigma d\sigma &= \int_{-\pi}^{\pi}\int_{-\pi-\theta}^{\pi-\theta}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s)d\sigma(\theta)\\ &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s)d\sigma(\theta)\\ &= \int_{-\pi}^{\pi}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s)\end{aligned}$$

We want to show:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{1-e^{iN(t-\theta)}}{(1-e^{i(t-\theta)})^{\beta}}\right|^2 d\sigma(t)d\sigma(\theta) \ = \ \int_{-\pi}^{\pi}\left|\frac{1-e^{iNs}}{(1-e^{is})^{\beta}}\right|^2 d\sigma(s) \ \sim \ (N+1)^{\alpha}.$$

To this end let $0 \ < \ r \ < \ 1$, and

$$E(r) = \int_{-\pi}^{\pi}\left|\frac{(1-r^N e^{iNs})}{(1-re^{is})^{\beta}}\right|^2 d\sigma(s).$$

We need the following results from K Zhu [Zh] in our approximation process.

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{1}{|1-we^{i(\theta)}|^{2\beta}}\,d\sigma(\theta) &= \int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)}{\Gamma(\beta)k!}(\omega e^{i\theta})^k\right|^2 d\sigma(\theta)\\ &= \sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)^2}{\Gamma(\beta)^2(k!)^2}\mid\omega\mid^{2k} \qquad (5.4.0)\end{aligned}$$

By (5.4.0), it can be easily shown that:

$$\int_{-\pi}^{\pi}\left|\frac{(1-r^Ne^{iNs})}{(1-re^{is})^\beta}\right|^2 d\sigma = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\beta+k)\Gamma(\beta+n)}{\Gamma(\beta)^2 k!n!}r^{n+k}\int_{-\pi}^{\pi}e^{is(k-n)}\left|(1-r^Ne^{iNs})\right|^2 d\sigma$$

$$= \sum_{k,n=0}^{\infty}\frac{\Gamma(\beta+k)\Gamma(\beta+n)}{\Gamma(\beta)^2 k!n!}r^{n+k}[(1+r^{2n})\delta_{n,k}-r^N\delta_{n,k+N}-r^N\delta_{k,n+N}]$$

Denote the above equation by (*)

Note, from the right hand side of (*) we see that the values of the integrals in the summation are 0, except when $n=k$, $n=k+N$, and $k=n+N$. So, to determine what values $E(r)$ takes we first prove the following Lemma.

Lemma 5.4.1. Consider the double series $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n a_k$. Assume that $a_n \geq 0$ for all n, the series converges uniformly, N is fixed and large positive integer, and the terms of the series are all 0 except for the cases $n=k$, $n=k+N$ and $k=n+N$. Then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n a_k = \sum_{k=0}^{N} a_k^2 + 2\sum_{k=0}^{N} a_k a_{k+N} + \sum_{k=1}^{N}(a_{k+N}^2 + a_{k+N}a_{k+2N}) + \sum_{k=N+1}^{\infty}(a_{k+N}^2 + a_k a_{k+N} + a_{k+N}a_{k+2N})$$

Proof. :

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n a_k = \underbrace{\sum_{n=0}^{N}\sum_{k=0}^{N} a_n a_k}_{(1)} + \underbrace{\sum_{n=N+1}^{\infty}\sum_{k=0}^{N} a_n a_k}_{(2)} + \underbrace{\sum_{n=0}^{N}\sum_{k=N+1}^{\infty} a_n a_k}_{(3)} + \underbrace{\sum_{n=N+1}^{\infty}\sum_{k=N+1}^{\infty} a_n a_k}_{(4)}$$

We consider the right hand side of the above equation, for the cases $n=k$, $n=k+N$ and $k=n+N$.

For (1), since $n,\ k \leq N$ the case $n = k + N$ and $k = n + N$ happen only for $n = N\ ,\ k = 0$ and $k = N\ ,\ n = 0$, thus

$$(1) = \quad \sum_{n=0}^{N} \sum_{k=0}^{N} a_n a_k = a_0 a_0 + a_1 a_1 + ... + a_N a_N + a_0 a_N + a_N a_0 = \sum_{k=0}^{N} a_n^2 + 2\, a_0 a_N$$

For (2), since k starts from $N + 1$ and n varies from 0 to N we do not have the cases $n = k$ and $n = k + N$. So, the only option is $k = N + n$. Thus,

$$(2) = \sum_{n=0}^{N} \sum_{k=N+1}^{\infty} a_n a_k = a_0 \sum_{k=N+1}^{\infty} a_k + a_1 \sum_{k=N+1}^{\infty} a_k + ... + a_N \sum_{k=N+1}^{\infty} a_k$$

$$= a_1 a_{N+1} + a_2 a_{N+2} + ... + a_N a_{2N} = \sum_{k=1}^{N} a_k a_{N+k}$$

(3) is handled like (2) to get $\displaystyle\sum_{k=0}^{N} \sum_{n=N+1}^{\infty} a_n a_k = \sum_{k=1}^{N} a_k a_{N+k}$.

Using (1), (2), and (3) we get

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n a_k \quad = \quad \sum_{k=0}^{N} a_k^2 + 2 \sum_{k=0}^{N} a_k a_{k+N} + \sum_{n=N+1}^{\infty} \sum_{k=N+1}^{\infty} a_k a_n$$

Next we will consider the sum (4).

$$(4) = \sum_{n=N+1}^{\infty} \sum_{k=N+1}^{\infty} a_k a_n = (a_{N+1} + a_{N+2} + a_{N+3} + ...)(a_{N+1} + a_{N+2} + a_{N+3} + ...)$$

$$= a_{N+1}(a_{N+1} + a_{N+2} + a_{N+3} +,\ \ldots\ )$$

$$+\ a_{N+2}(a_{N+1} + a_{N+2} + a_{N+3} +,\ \ldots\ )$$

$$+\ a_{N+3}(a_{N+1} + a_{N+2} + a_{N+3} +,\ \ldots\ )$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots$$

For $j = 1, 2 \ldots$ consider

$$a_{N+j}\,(a_{N+1} + a_{N+2} + a_{N+3}+,\ \ldots\ ) \qquad (**)$$

There are two cases for $(**)$

(i) $\quad j \in \{1, 2, \ \ldots\ N\}$, and

(ii) $\quad j \in \{N+1,\ N+2,\ \ldots\ \}$.

We want to find the values of $a_{N+j}\, a_{N+m}$, when $n = k$, $n = N + k$, $k = N + n$ and 0 for all other cases, which are easily seen to be

$$a_{N+j}\, a_{N+m} = \begin{cases} (a_{N+j})^2,\ if\ \ j = m, \\ a_{N+j}a_{2N+j},\ \ if\ m = j + N, \\ a_j a_{N+j},\ \ j = m + N \end{cases}$$

Thus,

For case (i) since $j \le N$ the case $a_j a_{N+j},\ j = m + N$ cannot happen, hence

$$a_{N+j}\, a_{N+m} = \begin{cases} (a_{N+j})^2,\ if\ \ j = m, \\ a_{N+j}a_{2N+j},\ \ if\ m = j + N, \end{cases}$$

For case (ii) since $j \ge N + 1$ we have

$$a_{N+j}\, a_{N+m} = \begin{cases} (a_{N+j})^2,\ if\ \ j = m, \\ a_{N+j}a_{2N+j},\ \ if\ m = j + N, \\ a_j a_{N+j},\ \ j = m + N \end{cases}$$

Hence,

$$\sum_{n=N+1}^{\infty} \sum_{k=N+1}^{\infty} a_k a_n = \sum_{k=1}^{N} (a_{N+k}^2 + a_{N+k} a_{2N+k}) + \sum_{k=N+1}^{\infty} (a_{N+k}^2 + a_{N+k} a_k + a_{N+k} a_{2N+k})$$

Now from (1), (2), (3), and (4) we get that:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n a_k = \sum_{k=0}^{N} a_k^2 + 2 \sum_{k=0}^{N} a_k a_{k+N} + \sum_{k=1}^{N} (a_{k+N}^2 + a_{k+N} a_{k+2N}) + \sum_{k=N+1}^{\infty} (a_{k+N}^2 + a_k a_{k+N} + a_{k+N} a_{k+2N})$$

This proves the Lemma.

□

Lemma 5.4.2. Let $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n a_k$ be a non negative series that converges uniformly.

Let N be a fixed positive integer and $\{C_{nk}\}_{n,k=0}^{\infty}$ be a sequence such that

$$C_{nk} = \begin{cases} 1, & if \;\; k = n, \\ -1, & if \; k = n + N, \;\; \text{or} \;\; n = k + N \\ 0, & \text{otherwise} \end{cases}$$

Then $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n a_k C_{nk} = \sum_{k=0}^{N} (a_k^2 - 2\, a_k a_{k+N}) + \sum_{k=1}^{N} (a_{k+N}^2 - a_{k+N} a_{k+2N}) + \sum_{k=N+1}^{\infty} (a_{k+N}^2 - a_k a_{k+N} - a_{k+N} a_{k+2N})$

PROOF: Similar to Lemma 5.4.1.

Thus, by setting $a_k = \frac{\Gamma(\beta+k)}{\Gamma(\beta)k!}$ we can express $E(r)$ in terms the $a_k's$ as follows.

$$
\begin{aligned}
E(r) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma(\beta+k)\Gamma(\beta+n)}{\Gamma(\beta)^2 k!n!} r^{n+k} \int_{-\pi}^{\pi} e^{is(k-n)} \left|(1-r^N e^{iNs})\right|^2 d\sigma \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_k a_n\, r^{n+k} \int_{-\pi}^{\pi} e^{is(k-n)} \left|(1-r^N e^{iNs})\right|^2 d\sigma \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_k a_n r^{n+k} \int_{-\pi}^{\pi} [\,(1+r^{2N})e^{is(k-n)} - r^N e^{is(k+N-n)} - r^N e^{is(k-N-n)}] d\sigma \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} a_k a_n r^{n+k} [(1+r^{2n})\delta_{n,k} - r^N \delta_{n,k+N} - r^N \delta_{k,n+N}]
\end{aligned}
$$

Now using Lemma 5.4.2. $E(r)$ can be expressed as:

$$
\begin{aligned}
E(r) &= \frac{1}{\Gamma(\beta)^2}\left(\sum_{k=0}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2}(1+r^{2N})r^{2k} - 2\sum_{k=0}^{N} \frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{(k!)(k+N)!} r^{2k+2N}\right) + \\
&\quad \frac{1}{\Gamma(\beta)^2}\sum_{k=1}^{N}\left(\frac{\Gamma(\beta+k+N)^2}{[(k+N)!]^2}(1+r^{2N})r^{2(k+N)} - \frac{\Gamma(\beta+k+N)\Gamma(\beta+k+2N)}{(k+N)!(k+2N)!} r^{2k+4N}\right) \\
&+ \frac{1}{\Gamma(\beta)^2}\sum_{k=N+1}^{\infty}\{(\,\frac{\Gamma(\beta+k+N)^2}{[(k+N)!]^2}(1+r^{2N})r^{2(k+N)} - \frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{(k!)(k+N)!} r^{2k+2N} \\
&\qquad - \frac{\Gamma(\beta+k+N)\Gamma(\beta+k+2N)}{(k+N)!(k+2N)!} r^{k+N} r^{k+2N} r^{N}\ )\}
\end{aligned}
$$

Let's denote by:

$$S_1(r) = \sum_{k=0}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2}(1+r^{2N})r^{2k} - 2\sum_{k=0}^{N} \frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{(k!)(k+N)!} r^{2k+2N}$$

$$= \sum_{k=0}^{N} \left(\frac{\Gamma(\beta+k)^2}{(k!)^2}(1+r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{(k!)(k+N)!} r^{2N-k} \right) r^{2k}$$

$$S_2(r) = \sum_{k=1}^{N} \left(\frac{\Gamma(\beta+k+N)^2}{[(k+N)!]^2}(1+r^{2N})r^{2(k+N)} - \frac{\Gamma(\beta+k+N)\Gamma(\beta+k+2N)}{(k+N)!(k+2N)!} r^{2k+4N} \right)$$

$$= \sum_{k=1}^{N} \left(\frac{\Gamma(\beta+k+N)^2}{[(k+N)!]^2}(1+r^{2N}) - \frac{\Gamma(\beta+k+N)\Gamma(\beta+k+2N)}{(k+N)!(k+2N)!} r^{2N} \right) r^{2k+2N}$$

$$S_3(r) = \sum_{k=N+1}^{\infty} \{(\frac{\Gamma(\beta+k+N)^2}{[(k+N)!]^2}(1+r^{2N})r^{2(k+N)} - \frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{(k!)(k+N)!} r^{2k+2N}$$
$$- \frac{\Gamma(\beta+k+N)\Gamma(\beta+k+2N)}{(k+N)!(k+2N)!} r^{2k+4N})\}$$

Then $E(r) = S_1(r) + S_2(r) + S_3(r)$

Lemma 5.4.3. For $0 < r < 1$

$$\lim_{r\to 1^-} E(r) = E(1) = \int_{-\pi}^{\pi} \left| \frac{1-e^{iNs}}{(1-e^{is})^\beta} \right|^2 d\sigma(s)$$

PROOF. : $\lim_{r\to 1^-} E(r) = \lim_{r\to 1^-} [S_1(r) + S_2(r) + S_3(r)]$

Since $S_1(r)$ and $S_2(r)$ have finite terms

$\lim_{r\to 1^-} [S_1(r) + S_2(r) + S_3(r)] = \lim_{r\to 1^-} S_1(r) + \lim_{r\to 1^-} [S_2(r) + S_3(r)]$

Recall, $a_k = \frac{\Gamma(\beta+k)}{\Gamma(\beta)k!}$, then

$$\begin{aligned} S_1(r) &= \sum_{k=0}^{N} \left[a_k^2(1+r^{2N})r^{2k} - 2r^N a_k a_{k+N} r^{2k+N}\right] \\ &= \sum_{k=0}^{N} a_k r^{2k} \left[a_k(1+r^{2N}) - 2a_{k+N} r^{2N}\right] \\ &= \underbrace{r^{2N} \sum_{k=0}^{N} a_k r^{2k} \left[2a_k - 2a_{k+N}\right]}_{S_{11}(r)} + \underbrace{(1-r^{2N}) \sum_{k=0}^{N} a_k^2 r^{2k}}_{S_{12}(r)}. \end{aligned}$$

which gives, $S_1(r) = S_{11}(r) + S_{12}(r)$, and so,

$$\lim_{r\to 1^-} S_1(r) = \lim_{r\to 1^-} [S_{11}(r) + S_{12}(r)].$$

By definition it is clear that $\lim_{r\to 1^-} S_{12}(r) = 0$. Also since $a_k - a_{k+m} \geq 0$ for all $m = 0, 1, 2, \ldots, N$, by Lebesgue Dominated Convergence Theorem (LDCT), we see that $\lim_{r\to 1^-} S_{11}(r) = S_{11}(1)$ which implies that $\lim_{r\to 1^-} S_1(r) = S_1(1)$.

Next we will consider $S_3(r)$. In terms of the $a_k's$, $S_3(r)$ is given by:

$$\begin{aligned}
S_3(r) &= \sum_{k=N+1}^{\infty} a_{k+N} r^{2k+2N} \left[(1+r^{2N})a_{k+N} - a_k - r^{2N} a_{k+2N}\right] \\
&= \sum_{k=N+1}^{\infty} a_{k+N} a_{k+N} r^{2k+2N}(1+r^{2N}) - \sum_{k=N+1}^{\infty} a_{k+N} a_k r^{2k+2N} - \\
&\quad \sum_{k=N+1}^{\infty} a_{k+N} a_{k+2N} r^{2k+2N} r^{2N} \\
&= \sum_{k=N+1}^{\infty} a_{k+N} a_{k+N} r^{2k+2N}(1+r^{2N}) - \sum_{k=N+1}^{2N} a_{k+N} a_k r^{2k+2N} - \\
&\quad \sum_{k=2N+1}^{\infty} a_{k+N} a_k r^{2k+2N} - \sum_{k=N+1}^{\infty} a_{k+N} a_{k+2N} r^{2k+2N} r^{2N} \\
&= \sum_{k=N+1}^{\infty} (a_{k+N})^2 r^{2k+2N}(1+r^{2N}) - \sum_{k=1}^{N} a_{k+2N} a_{k+N} r^{2k+2N} - \\
&\quad \sum_{k=N+1}^{\infty} a_{k+2N} a_{k+N} r^{2k+4N} - \sum_{k=N+1}^{\infty} a_{k+N} a_{k+2N} r^{2k+2N} r^{2N} \\
&= \underbrace{\sum_{k=N+1}^{\infty} a_{k+N} r^{2k+2N} \left[a_{k+N}(1+r^{2N}) - 2a_{k+2N} r^{2N}\right]}_{S_3'(r)} \underbrace{- \sum_{k=1}^{N} a_{k+2N} a_{k+N} r^{2k+2N}}_{S_{22}'(r)}
\end{aligned}$$

Thus now $S_3(r)$=$S_3'(r) + S_{22}'(r)$, and so $E(r)$ can also be expressed as

$$E(r) = S_1(r) \ + \ S_2(r) \ + \ S_{22}'(r) \ + \ S_3'(r),$$

and setting $S_2'(r) = S_{22}'(r) \ + \ S_2(r)$ we write

$$E(r) = S_1(r) \ + \ S_2'(r) \ + \ S_3'(r)$$

This implies that, $\lim_{r\to 1^-} E(r) \ = \ \lim_{r\to 1^-} S_1(r) \ + \ \lim_{r\to 1^-} [\, S_2'(r) + S_3'(r)]$

To complete the proof of the lemma we need to show

$$\lim_{r\to 1^-} [\,S_2'(r) + S_3'(r)] = S_2'(1) \ + \ S_3'(1) = S_2(1) \ + \ S_3(1).$$

To this end, first we will show that $S_2^{'}(r)$ and $S_3^{'}(r)$ are both non negative. Note that because of the similarity in the terms of $S_2^{'}(r)$ and $S_3^{'}(r)$ it suffices to show $S_2^{'}(r)$ is non-negative.

$$\begin{aligned}
S_2^{'}(r) &= S_2(r) \ + \ S_{22}^{'}(r) \\
&= \sum_{k=1}^{N} a_{k+N} r^{2k+2N} \left[a_{k+N}(1+r^{2N}) - a_{k+2N} r^{2N}\right] + \sum_{k=1}^{N} a_{k+2N} a_{k+N} r^{2k+2N} \\
&= \sum_{k=1}^{N} a_{k+N} r^{2k+2N} \left[a_{k+N}(1+r^{2N}) - 2a_{k+2N} r^{2N}\right] \\
&= \sum_{k=1}^{N} (a_{k+N})^2 r^{2k+2N} \left[(1+r^{2N}) - 2\frac{a_{k+2N}}{a_{k+N}} r^{2N}\right] \\
&= \sum_{k=1}^{N} (a_{k+N})^2 r^{2k+2N} \left[1 + (1 - 2\frac{a_{k+2N}}{a_{k+N}}) r^{2N}\right]
\end{aligned}$$

From the right hand side of the above equation to show that $S_2^{'}(r)$ is non negative it suffice to show

$$1 + (1 - 2\frac{a_{k+2N}}{a_{k+N}}) r^{2N} \geq \ 0$$

So, we consider

$$
\begin{aligned}
(1-2\frac{a_{k+2N}}{a_{k+N}})r^{2N} &= \left[1-2\frac{\Gamma(\beta+k+2N)}{(k+2N)!}\frac{(k+N)!}{\Gamma(\beta+k+N)}\right]r^{2N} \\
&= \left(1-2(\frac{\beta+k+2N-1}{k+2N})(\frac{\beta+k+2N-2}{k+2N-1})(\frac{\beta+k+N}{k+N+1})\right)r^{2N} \\
&\leq \left[1-2\left(\frac{\beta+k+N}{k+N+1}\right)^{N}\right]r^{2N} \\
&\leq \left[1-2\left(1-\frac{1-\beta}{k+N+1}\right)^{N}\right]r^{2N} \ \in \ (-1,\ 0)
\end{aligned}
$$

Similarly it can can be shown that

$$
(1-2\frac{a_{k+2N}}{a_{k+N}})r^{2N} \ \geq \ \left[1-2\left(1-\frac{1-\beta}{k+2N}\right)^{N}\right]r^{2N} \ \in \ (-1,\ 0)
$$

And from the above results we obtain $(1-2\frac{a_{k+2N}}{a_{k+N}})r^{2N} \ \in \ (-1,\ 0)$, which implies $1+(1-2\frac{a_{k+2N}}{a_{k+N}})r^{2N} > 0$, and so proving ${S_2}'(r) \geq 0$ and ${S_3}'(r) \geq 0$

Now since, $S_2'(r)$ and $S_3'(r)$, by LDCT we get

$\lim_{r\to 1^-} S_2'(r) \ = \ S_2'(1)$ and $\lim_{r\to 1^-} S_3'(r) \ = \ S_3'(1)$ which in turn implies

$$
\begin{aligned}
\lim_{r\to 1^-} E(r) \ &= \ \lim_{r\to 1^-} S_1(r) \ + \ \lim_{r\to 1^-} [\, S_2'(r) \ + \ S_3'(r)] \\
&= \ S_1(1) \ + \ S_2'(1) \ + \ S_3'(r) \\
&= E(1)
\end{aligned}
$$

This proves Lemma 5.4.3.

□

Next we will sate and prove our basic approximation theorem. This theorem will make it possible for the norm equivalence formula on the boundary of the unit circle to work.

Theorem 5.4.4.. For Large N

$$\int_{-\pi}^{\pi} \left| \frac{1-e^{iNs}}{(1-e^{is})^{\alpha+1}} \right|^2 d\sigma(s) \sim (N+1)^{\alpha}$$

Proof. : By Lemma 5.4.3

$E(r) = [S_1(r) + S_2(r) + S_3(r)] = S_1(r) + [S_2'(r) + S_3'(r)]$, and

$$\lim_{r\to 1^-} E(r) \quad = \quad E(1) = \int_{-\pi}^{\pi} \left| \frac{1-e^{iNs}}{(1-e^{is})^{\beta}} \right|^2 d\sigma(s)$$

So, we want to show $E(1) \sim (N+1)^{\alpha}$ for large N. We do this by estimating $S_1(1),\ S_2{}'(1)$ and $S_3{}'(1)$

Recall

$$\begin{aligned} S_1(r) \quad &= \quad \sum_{k=0}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2}(1+r^{2N})r^{2k} - \ 2\sum_{k=0}^{N} \frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{(k!)(k+N)!} r^{2k+2N} \\ &= \quad \sum_{k=0}^{N} \left(\frac{\Gamma(\beta+k)^2}{(k!)^2}(1+r^{2N}) - 2\frac{\Gamma(\beta+k)\Gamma(\beta+k+N)}{(k+N)!k!} r^{2N} \right) r^{2k} \end{aligned}$$

Taking limit as $r \nearrow 1$ we get that (See the proof of Lemma 5.4.3)

$$\begin{aligned} S_1(1) \quad &= \quad 2\sum_{k=0}^{N} \frac{\Gamma(\beta+k)}{k!} \left(\frac{\Gamma(\beta+k)}{k!} - \frac{\Gamma(\beta+k+N)}{(N+k)!} \right) \\ &= \quad \sum_{k=0}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \left[1 - (\frac{\beta+N+k-1}{k+N})(\frac{\beta+N+k-2}{k+N-1})...(\frac{\beta+k}{k+1}) \right] \end{aligned}$$

Note the following are readily seen

$$\begin{aligned} \frac{\beta+j+k-1}{k+j} &= \frac{\beta+j+k-1}{k+j+1-1} < 1 \ \text{ and} \\ \frac{\beta+j+k-1}{k+j} &- \frac{\beta+j+k-2}{k+j-1} = 1-\frac{1-\beta}{k+j} - 1 + \frac{1-\beta}{k+j-1} > 0, \end{aligned}$$

these imply that

$$\frac{\beta+j+k-1}{k+j} \quad > \quad \frac{\beta+j+k-2}{k+j-1}, \ \text{ for all } \ j=1, \ \ldots N$$

Thus, for $n = 1, \ \ldots, \ N$ we get

$$\frac{\beta+N+k-1}{k+N} > \frac{\beta+N+k-2}{k+N-1} > \ ... \ > \frac{\beta+k}{k+1} \tag{5.4.4.1}$$

Hence,

$$\begin{aligned} S_1(1) &= \sum_{k=0}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \left[1 - (\frac{\beta+N+k-1}{k+N})(\frac{\beta+N+k-2}{k+N-1})...(\frac{\beta+k}{k+1})\right] \\ &\leq \quad D_1 + C_1 \sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \left[1-(\frac{\beta+k}{k+1})^N\right], \ \text{ for some large M} \end{aligned}$$

and by(5.4.4.1) we get

$$S_1(1) \leq \ C + C \sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2}, \ \text{ for some constant C}$$

and by Stirling's formula

$$\sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \ \sim \ \sum_{k=M}^{N} k^{2\beta-2} \ \sim N^{2\beta-1},$$

which implies $S_1(1) \leq \ C_0 N^{2\beta-1}$

Conversely, we want to show $S_1(1) \geq D_0 N^{2\beta-1}$ for some constant D_0. So, consider

$$\begin{aligned} S_1(1) &= \sum_{k=0}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \left[1-(\frac{\beta+N+k-1}{k+N})(\frac{\beta+N+k-2}{k+N-1})...(\frac{\beta+k}{k+1})\right] \\ &\geq D_2 + C_2 \sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \left[1-(\frac{\beta+N+k-1}{k+N})^N\right], \quad \text{for some large M} \end{aligned}$$

by (5.4.4.1) we get

$$\sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \left[1-(1-\frac{1-\beta}{k+N})^N\right] \quad \geq \quad D_3 \sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2}$$

for some constant D_3, and by Stirling's formula

$$D_3 \sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \sim \sum_{k=M}^{N} \frac{\Gamma(\beta+k)^2}{(k!)^2} \sim N^{2\beta-1}.$$

Thus, $S_1(1) \geq D_0 N^{2\beta-1}$ for some constant D_0.

Next we approximate $S_2'(1)$. Recall

$$\begin{aligned} S_2'(r) &= S_2(r) + S_{22}'(r) \\ &= \sum_{k=1}^{N} a_{k+N} r^{2k+2N} \left[(1+r^{2N})a_{k+N} - 2r^{2N} a_{k+2N}\right] \end{aligned}$$

By Lemma 5.4.3. as $r \nearrow 1$, $S_2'(r)$ converges to $S_2'(1)$ and $S_2'(r) \leq S_2'(1)$.

So,

$$\begin{aligned}
S_2'(1) &= \sum_{k=1}^{N} a_{k+N}\left[2a_{k+N} - 2a_{k+2N}\right] \\
&= \sum_{k=1}^{N}(a_{k+N})^2\left[2 - 2\frac{a_{k+2N}}{a_{k+N}}\right] \\
&\leq \sum_{k=1}^{N}\frac{\Gamma(\beta+k+N)^2}{\Gamma(\beta)^2((k+N)!)^2}\left[2 - 2\left(1-\frac{1-\beta}{k+N+1}\right)^N\right] \\
&\leq \sum_{k=1}^{N}\frac{\Gamma(\beta+k+N)^2}{\Gamma(\beta)^2((k+N)!)^2} \sim \sum_{k=1}^{N}(k+N)^{2\beta-2} \sim N^{2\beta-1}
\end{aligned}$$

which gives $S_2'(1) \leq C_3 N^{2\beta-1}$. Finally we will estimste $S_3'(1)$

$$\begin{aligned}
S_3'(1) &= \sum_{k=N+1}^{\infty} a_{k+N}\left[2a_{k+N} - 2a_{k+2N}\right] \\
&= \sum_{k=N+1}^{\infty} 2(a_{k+N})^2\left[1 - \frac{a_{k+2N}}{a_{k+N}}\right] \\
&\leq \sum_{k=N+1}^{\infty} 2(a_{k+N})^2\left[1 - \left(1-\frac{1-\beta}{k+N+1}\right)^N\right] \quad \text{by } (5.4.4.1) \\
&\leq \sum_{k=N+1}^{\infty} 2(a_{k+N})^2\left[1 - \left(1-\frac{NC_{00}(1-\beta)}{k+N+1}\right)\right]
\end{aligned}$$

for some constant C_{00} independent of N and k. Replacing the $a_{k+N}'s$ in the above inequality we get

$$\begin{aligned}
S_3'(1) &\leq \frac{N(1-\beta)C_{00}}{\Gamma(\beta)^2}\sum_{k=N+1}^{\infty}\frac{\Gamma(\beta+k+N)^2}{(k+N)(k+N)!^2}\left(\frac{1}{1+1/(k+N)}\right) \\
&\sim \sum_{k=N+1}^{\infty}(k+N)^{2\beta-3} \quad \text{by Stirling formula} \\
&\sim N^{2\beta-1}
\end{aligned}$$

Thus, $S_3'(1) \leq C_4 N^{2\beta-1}$.

Now putting together all the estimates obtained above that is:

$S_1(1) \leq \ C_0 N^{2\beta-1}$, $S_1(1) \geq D_0 N^{2\beta-1}$, $S_2^{'}(1) \leq C_3 N^{2\beta-1}$, and $S_3^{'}(1) \leq C_4 N^{2\beta-1}$, and using the fact that $E(1) = S_1(1) + S_2^{'}(1) + S_3^{'}(1)$, we get an upper estimate for $E(1)$.

$$
\begin{aligned}
E(1) &= S_1(1) + S_2^{'}(1) + S_3^{'}(1) \\
&\leq C_0 N^{2\beta-1} + C_3 N^{2\beta-1} + C_4 N^{2\beta-1} = [C_0 + C_3 + C_4] N^{2\beta-1} \\
&\leq CN^{2\beta-1}, \quad \text{for some constant } \ C > 0
\end{aligned}
$$

On the other hand

$D_0 N^{2\beta-1} \leq S_1(1)$ implies that

$D_0 N^{2\beta-1} \leq E(1) \leq CN^{2\beta-1}$, this implies

$E(1) \sim N^{2\beta-1}$, since $S^(1)$, $S_2^{'}(1)$ and $S_3^{'}(1)$ are non-negative

This proves Theorem 5.4.4.

□

Next we consider $\bigoplus_1^\infty \mathcal{D}_\alpha$, l^2-valued Dirichlet Space. Let $f \in \bigoplus_1^\infty \mathcal{D}_\alpha$, then

$$
\| f \|_{\mathcal{D}_\alpha}^2 = \int_{-\pi}^{\pi} \| f \|^2 \ d\sigma \ + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\left\| f(e^{it}) - f(e^{i\theta}) \right\|^2}{|e^{it} - e^{i\theta}|^{\alpha+1}} d\sigma d\sigma
$$

We do also need weighted harmonic Dirichlet Spaces, $\mathcal{HD}_\alpha$ and $\bigoplus_1^\infty \mathcal{HD}_\alpha$ (l^2 valued), which are defined on the boundary of $\mathbb{D}$.

Note also that for $f \in \mathcal{HD}_\alpha$ and $f \in \bigoplus_1^\infty \mathcal{HD}_\alpha$, the norms are given by:

$$\| f \|^2_{\mathcal{HD}_\alpha} = \|f\|^2_{H^2(\mathbb{D})} + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(e^{it}) - f(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma, \text{ and}$$

$$\| f \|^2_{\bigoplus_1^\infty \mathcal{HD}_\alpha} = \int_{-\pi}^{\pi} \| f \|^2 \; d\sigma \; + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|f(e^{it}) - f(e^{i\theta})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma$$

Before passing to the the next section, which is the main result of this dissertation, we would like to make the following general observation.

A RKHS with non-empty multiplier algebra need not necessarily guarantee a "Corona type problem". The following is an example of a RKHS for which its multiplier algebra does not satisfy a Corona Theorem.

Let $\mathcal{H}_E$ denotes a RKHS on a set E. For $g \in \mathcal{H}_E$, define M_g by:

$M_g(f) = g(x)f(x)$ for $x \in E$ and $f \in \mathcal{H}_E$. and let

$\mathcal{M}(\mathcal{H}_E) = \{f \in \mathcal{H}_E \; : M_f \in \mathcal{B}(\mathcal{H}_E)\}$, the multiplier algebra for $\mathcal{H}_E$.

We are interested whether an abstract corona theorem can hold for the general algebra $\mathcal{M}(\mathcal{H}_E)$. The answer to this question is a **NO**.

In [Tr4] it is shown that $\mathcal{H}_E$ is a RKHS and there is a a function $g \in \mathcal{M}(\mathcal{H}_E)$, so that $1/g \in \mathcal{H}_E$ and $g(x) \geq \epsilon > 0$, $\forall x \in E$, but $1/g \notin \mathcal{M}(\mathcal{H}_E)$.

5.5. The Weighted Corona Theorem

In this section we will give the proof of the $\mathcal{D}_\alpha$ Corona theorem. In our proof we will use the "*Commutant Lifting Theorem*" and the "*boundary* " norm, that naturally extends multipliers on weighted Dirichlet spaces to multipliers on weighted harmonic Dirichlet spaces.

In Section 4.1., we showed that $\mathcal{M}(\mathcal{D}_\alpha) \subsetneqq \mathcal{H}^\infty(\mathbb{D})$. Using a similar argument for the weighted Dirichlet space, we observe that a point-wise upper bound hypothesis may not be sufficient to give the boundedness of column operators in the weighted space. However, we will justify that boundedness of the "Column Operators" implies the boundedness of the "Row Operators".

Lemma 5.5.1. Let $M_F^C \in B(\mathcal{D}_\alpha, \bigoplus_1^\infty \mathcal{D}_\alpha)$. Then $M_F^R \in B(\bigoplus_1^\infty \mathcal{D}_\alpha,\ \mathcal{D}_\alpha)$ and $\left\|M_F^R\right\| \le \sqrt{10}\left\|M_F^C\right\|$.

Proof. Take $\left\|M_F^C\right\| \le 1$ Note that since $Sup_{z\in\mathbb{D}} \sum_{j=1}^\infty |f_j(z)|^2 \le \left\|M_F^C\right\|^2 \le 1$, we have

$$\sum_{j=1}^\infty |f_j(z)|^2 = F(z)F(z)^* \le 1, \text{ for all } z \in \mathbb{D}$$

Now let $\underline{U} = \{u_k\}_{k=1}^{\infty} \in \bigoplus_{1}^{\infty} \mathcal{D}_\alpha$, then

$$\begin{aligned}\left\| M_F^R(\{u_k\}_{k=1}^{\infty}) \right\|^2 &= \left\| \sum_{k=1}^{\infty} f_k u_k \right\|^2 \\ &= \int_{\partial\mathbb{D}} |\sum_{k=1}^{\infty} f_k u_k|^2 d\sigma + \int_{\mathbb{D}} |(\sum_{k=1}^{\infty} f_k u_k)'|^2 (1 - |z|^2)^{1-\alpha}\, dA\end{aligned}$$

First note that

$$\begin{aligned}\int_{\partial\mathbb{D}} |\sum_{k=1}^{\infty} f_k u_k|^2 d\sigma &\leq \int_{\partial\mathbb{D}} \sum_{k=1}^{\infty} |f_k u_k|^2 d\sigma \leq \int_{\partial\mathbb{D}} \sum_{k=1}^{\infty} |f_k|^2 \sum_{k=1}^{\infty} |u_k|^2\, d\sigma \\ &\leq \int_{\partial\mathbb{D}} \sum_{k=1}^{\infty} |u_k|^2\, d\sigma = \| \underline{U} \|_\sigma^2\end{aligned}$$

second if we set $\omega(z) = (1 - |z|^2)^{1-\alpha}$, then

$$\begin{aligned}\int_{\mathbb{D}} |(\sum_{k=1}^{\infty} f_k u_k)'|^2 (1 - |z|^2)^{1-\alpha}\, dA &= \int_{\mathbb{D}} |(\sum_{k=1}^{\infty} f_k' u_k + f_k u_k')|^2 \omega(z)\, dA \\ &\leq \int_{\mathbb{D}} (2 |\sum_{k=1}^{\infty} f_k' u_k|^2 + 2 |\sum_{k=1}^{\infty} f_k u_k'|^2)\, \omega(z)\, dA \\ &\leq \int_{\mathbb{D}} (2 |\sum_{k=1}^{\infty} f_k' u_k|^2 + 2 \sum_{k=1}^{\infty} |u_k'|^2)\, \omega(z)\, dA\end{aligned}$$

Therefore,

$$\begin{aligned}\left\| M_F^R(\{u_k\}_{k=1}^{\infty}) \right\|^2 &\leq \| \underline{U} \|_\sigma^2 + \int_{\mathbb{D}} (2 |\sum_{k=1}^{\infty} f_k' u_k|^2 + 2 \sum_{k=1}^{\infty} |u_k'|^2)\, \omega(z)\, dA \\ &\leq 2\, \| \underline{U} \|_{\mathcal{D}_\alpha}^2 + 2 \int_{\mathbb{D}} (| \sum_{k=1}^{\infty} f_k' u_k|^2)\, \omega(z)\, dA\end{aligned}$$

But,

$$\begin{aligned}|\sum_{k=1}^{\infty} f_k' u_k|^2 &= |\sum_{k=1}^{\infty} f_k' u_k \sum_{k=1}^{\infty} \overline{f_k'}\,\overline{u_k}| = |\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} f_k' u_k \overline{f_j'}\,\overline{u_j}| \\ &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |f_k' u_k f_j' u_j|\end{aligned}$$

and since

$$\begin{aligned}|f_k' u_k f_j' u_j| = |f_k' u_j f_j' u_k| &\leq \frac{1}{2}|f_k' u_j|^2 + \frac{1}{2}|f_j' u_k|^2 \\ \Rightarrow 2|f_k' u_k f_j' u_j| &\leq |f_k' u_j|^2 + |f_j' u_k|^2\end{aligned}$$

we have

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |f_k' u_k f_j' u_j| &\leq \frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(|f_k' u_j|^2 + |f_j' u_k|^2) \\ &\leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |f_k' u_j|^2\end{aligned}$$

Hence

$$\begin{aligned}\left\| M_F^R(\{u_k\}_{k=1}^{\infty})\right\|^2 &\leq 2\,\|\underline{U}\|_{\mathcal{D}_\alpha}^2 + 2\int_{\mathbb{D}} (|\sum_{k=1}^{\infty} f_k' u_k|^2)\,\omega(z)\,dA \\ &\leq 2\,\|\underline{U}\|_{\mathcal{D}_\alpha}^2 + 2\int_{\mathbb{D}} (\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |f_k' u_j|^2)\,\omega(z)\,dA \\ &\leq 2\,\|\underline{U}\|_{\mathcal{D}_\alpha}^2 + 2\sum_{j=1}^{\infty}\int_{\mathbb{D}}\sum_{k=1}^{\infty} |f_k' u_j|^2\,\omega(z)\,dA\end{aligned}$$

Next we consider

$$\begin{aligned}\int_{\mathbb{D}}\sum_{k=1}^{\infty}|\,f_k'u_j\,|^2\,\omega(z)\,dA &= \int_{\mathbb{D}}\sum_{k=1}^{\infty}|\,f_k'u_j+f_ku_j'-f_ku_j'\,|^2\,\omega(z)\,dA\\ &\leq 2\int_{\mathbb{D}}\sum_{k=1}^{\infty}(\,|\,f_k'u_j+f_ku_j'\,|^2+|\,f_ku_j'\,|^2)\,\omega(z)\,dA\end{aligned}$$

Since

$$\begin{aligned}\left\|M_F^C(u_j)\right\|^2_{\bigoplus_1^\infty\mathcal{D}_\alpha} &= \|(f_1u_j,\ f_2u_j,\ \ldots\ )\|^2_{\bigoplus_1^\infty\mathcal{D}_\alpha}=\sum_{k=1}^{\infty}(\,\|\,f_ku_j\|^2_{\mathcal{D}_\alpha}\\ &= \int_{\partial\mathbb{D}}\sum_{k=1}^{\infty}|\,f_ku_j\,|^2d\sigma+\int_{\mathbb{D}}\sum_{k=1}^{\infty}|\,(f_ku_j)'\,|^2\omega(z)\,dA,\end{aligned}$$

we have

$$\begin{aligned}\int_{\mathbb{D}}\sum_{k=1}^{\infty}|\,f_k'u_j\,|^2\,\omega(z)\,dA &\leq 2\left\|M_F^C(u_j)\right\|^2_{\bigoplus_1^\infty\mathcal{D}_\alpha}+2\,\|u_j\|_{\mathcal{D}_\alpha}\\ &\leq 2\,\|u_j\|_{\mathcal{D}_\alpha}+2\,\|u_j\|_{\mathcal{D}_\alpha}=4\,\|u_j\|_{\mathcal{D}_\alpha}\end{aligned}$$

Therefore

$$\begin{aligned}\left\|M_F^R(\{u_k\}_{k=1}^{\infty})\right\|^2 &\leq 2\,\|\,\underline{U}\,\|^2_{\mathcal{D}_\alpha}+2\,\sum_{j=1}^{\infty}\int_{\mathbb{D}}\sum_{k=1}^{\infty}|\,f_k'u_j\,|^2\,\omega(z)\,dA\\ &\leq 2\,\|\,\underline{U}\,\|^2_{\mathcal{D}_\alpha}+2\sum_{j=1}^{\infty}(4\,\|u_j\|^2_{\mathcal{D}_\alpha})\\ &\leq 2\,\|\,\underline{U}\,\|^2_{\mathcal{D}_\alpha}+8\sum_{j=1}^{\infty}\|u_j\|^2_{\mathcal{D}_\alpha}\\ &\leq 10\,\|\,\underline{U}\,\|^2_{\mathcal{D}_\alpha}\end{aligned}$$

which implies, $\left\|M_F^R\right\|\leq\sqrt{10}\left\|M_F^C\right\|$

□

Next we state the corona theorem for the Weighted Dirichlet Space ; we refer to this theorem as the $\mathcal{D}_\alpha$ **Corona Theorem**.

Theorem 5.5.2 (**The $\mathcal{D}_\alpha$ Corona Theorem**)

Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$. Assume that , $\| M_F^C \| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^{\infty} | f_j(z) |^2$ for all $z \in \mathbb{D}$. Then there exists a positive number $C(\epsilon, \alpha)$ and $\{g_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$ such that

$$(i) \sum_{j=1}^{\infty} f_j g_j = 1$$

$$(ii) \| M_G^C \| \leq C(\epsilon, \alpha)$$

We prove the $\mathcal{D}_\alpha$ Corona Theorem by establishing the following two theorems: **Theorem A** and **Theorem B**

Theorem 5.5.3. (**Theorem A**) Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$. Assume that $\| M_F^C \| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^{\infty} | f_j(z) |^2$ for all $z \in \mathbb{D}$. Then there exists a $k(\epsilon, \alpha) > 0$, such that $k(\epsilon, \alpha)^{-2} I \leq M_F^R (M_F^R)^* \leq I$

Theorem 5.5.4. (**Theorem B**) Assume that $\delta^2 I \leq M_F^R (M_F^R)^* \leq I$. Then there exists $\{g_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$, such that

$$(i) \sum_{j=1}^{\infty} f_j g_j = 1$$

$$(ii) \| M_G^C \| \leq 1/\delta$$

Theorem A & B with $\delta = [k(\epsilon, \alpha)]^{-1}$ complete the $\mathcal{D}_\alpha$ Corona Theorem.

First we prove **Theorem B**; however we first state some facts about $\mathcal{D}_\alpha$.

$\mathcal{D}_\alpha$ is a RKHS with reproducing kernel given by

$$k_\omega(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\, \overline{\omega}^n z^n,\ \forall z, \omega \in \mathbb{D}.$$

Furthermore the reproducing kernel $k_\omega(z)$ has one-positive square, i.e.:

$$1 - \frac{1}{k_\omega(z)} = \sum_{n=1}^{\infty} C_n\, \overline{\omega}^n z^n, \quad \text{where } C_n \ \geq\ 0\ \ \forall n.$$

We would like to say a little more about this property

Definition. We say a reproducing kernel $k_w(z)\ \in\ \mathcal{D}_\alpha$ has " **one positive square** "

if there exists a b_n in $\mathcal{D}_\alpha$ such that:

$$\frac{1}{k_w(z)} = 1 - \sum_{1}^{\infty} b_n(z)\overline{b_n(w)} \quad \text{for all } \ z,\ w\ \in \mathbb{D}.$$

In our case we claim that there are $C_n \geq 0$ for all n such that:

$$1 - \frac{1}{k_\omega(z)} = \sum_{n=1}^{\infty} C_n\, \overline{\omega}^n z^n,$$

Computing:

$$1 - \frac{1}{k_\omega(z)} = 1 - \frac{1}{\sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\, \overline{\omega}^n z^n} = \sum_{n=1}^{\infty} C_n\, \overline{\omega}^n z^n$$

we want to show that $C_n \geq 0\ \forall n$

$$1 - \frac{1}{\sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\, \overline{\omega}^n z^n} = \sum_{n=1}^{\infty} C_n\, \overline{\omega}^n z^n$$

which implies

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\,\overline{\omega}^n z^n (1 - \sum_{n=1}^{\infty} C_n\,\overline{\omega}^n z^n) = 1$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\,\overline{\omega}^n z^n (-1 + \sum_{n=1}^{\infty} C_n\,\overline{\omega}^n z^n) = -1$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha}\,\overline{\omega}^n z^n (\sum_{n=0}^{\infty} C_n\,\overline{\omega}^n z^n) = -1, \quad set\ C_0 = -1$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{C_k}{(n+1)^\alpha}\,\overline{\omega}^n z^n\,\overline{\omega}^k z^k = -1$$

Thus for $\alpha \in (0,1)$ we have:

$$C_0 = -1, \quad C_1 = 1/2^\alpha > 0, \quad C_2 = 1/3^\alpha - 1/4^\alpha > 0$$

Assume for some $n > 1$, $C_{n-1} \geq 0$; we will show $C_n \geq 0$.

To this end we consider

$$\frac{C_0}{n^\alpha} + \frac{C_1}{(n-1)^\alpha} + \frac{C_2}{(n-2)^\alpha} + \ \ldots + \frac{C_{n-2}}{2^\alpha} + C_{n-1} = 0 \tag{5.5.4.1}$$

$$\frac{C_0}{(n+1)^\alpha} + \frac{C_1}{n^\alpha} + \frac{C_2}{(n-1)^\alpha} + \ \ldots + \frac{C_{n-2}}{3^\alpha} + \frac{C_{n-1}}{2^\alpha} + C_n = 0 \tag{5.5.4.2}$$

Multiply (5.5.4.1) by n^α and (5.5.4.2) by $(n+1)^\alpha$ to obtain

$$C_0 + \frac{n^\alpha C_1}{(n-1)^\alpha} + \frac{n^\alpha C_2}{(n-2)^\alpha} + \ \ldots + \frac{n^\alpha C_{n-2}}{2^\alpha} + n^\alpha C_{n-1} = 0 \tag{5.5.4.3}$$

$$C_0 + \frac{(n+1)^\alpha C_1}{n^\alpha} + \frac{(n+1)^\alpha C_2}{(n-1)^\alpha} + \ \ldots + \frac{(n+1)^\alpha C_{n-1}}{2^\alpha} + (n+1)^\alpha C_n = 0 \tag{5.5.4.4}$$

Subtracting (5.5.4.3) from (5.5.4.4) we get

$$C_1\left(\frac{(n+1)^\alpha}{n^\alpha} - \frac{n^\alpha}{(n-1)^\alpha}\right) + C_2\left(\frac{(n+1)^\alpha}{(n-1)^\alpha} - \frac{n^\alpha}{(n-2)^\alpha}\right) + \ \ldots + C_n(n+1)^\alpha = 0$$

Now for each $k = 1, \ldots n-1$ consider

$$\frac{(n+1)^\alpha}{(n+1-k)^\alpha} - \frac{n^\alpha}{(n-k)^\alpha} = \frac{[(n+1)(n-k)]^\alpha - [n(n+1-k)]^\alpha}{(n+1-k)^\alpha (n-k)^\alpha}$$
$$= \frac{(n^2 - nk + n - k)^\alpha - (n^2 + n - nk)^\alpha}{(n+1-k)^\alpha (n-k)^\alpha} \tag{5.5.4.5}$$

Using the fact that $1 < b < a$ and $\alpha > 0$ if and only if $a^\alpha > b^\alpha$.

We see that (5.5.4.5) > 0 if and only if $(n^2 - nk + n - k) - (n^2 + n - nk) > 0$.

But $(n^2 - nk + n - k) - (n^2 + n - nk) = -k < 0$ for all $k = 1, \ldots n-1$

implies that

$$\frac{(n^2 - nk + n - k)^\alpha - (n^2 + n - nk)^\alpha}{(n+1-k)^\alpha (n-k)^\alpha} < 0, \quad \text{for all } k = 1, \ldots n-1$$

By assumption, $C_k \geq 0$ for $k = 1, \ldots n-1$ and $\left[\frac{(n+1)^\alpha}{(n+1-k)^\alpha} - \frac{n^\alpha}{(n-k)^\alpha}\right] < 0$; so we have that $C_k \left[\frac{(n+1)^\alpha}{(n+1-k)^\alpha} - \frac{n^\alpha}{(n-k)^\alpha}\right] < 0$ for all $k = 1, \ldots n-1$. This implies

$$C_1 \left(\frac{(n+1)^\alpha}{n^\alpha} - \frac{n^\alpha}{(n-1)^\alpha}\right) + C_2 \left(\frac{(n+1)^\alpha}{(n-1)^\alpha} - \frac{n^\alpha}{(n-2)^\alpha}\right) + \ldots + C_{n-1} \left(\frac{(n+1)^\alpha}{2^\alpha} - n^\alpha\right) < 0,$$

Thus, since $(n+1)^\alpha > 0$ and

$$C_1 \left(\frac{(n+1)^\alpha}{n^\alpha} - \frac{n^\alpha}{(n-1)^\alpha}\right) + C_2 \left(\frac{(n+1)^\alpha}{(n-1)^\alpha} - \frac{n^\alpha}{(n-2)^\alpha}\right) + \ldots + C_n (n+1)^\alpha = 0$$

It must hold that $C_n \geq 0$ for all $n = 1, 2, \ldots$ This prove our claim.

The fact that $\mathcal{D}_\alpha$ is a RKHS with one positive square makes it possible to use a corresponding Commutant Lifting Theorem (CLT) to prove Theorem B.

So, next we state the commutant lifting theorem for $\mathcal{D}_\alpha$, and then we prove Theorem B.For more detail refer to Ball-Trent-Vinnikov [BTV].

Theorem 5.5.5. (CLT) Let M_* and N_* be invariant subspaces for M_z^* on $\bigoplus_1^M \mathcal{D}_\alpha$ and $\bigoplus_1^N \mathcal{D}_\alpha$ respectively, where $1 \le M,\ N\ \le \infty$. Assume that $X^* \in B(M_*,\ N_*)$ satisfies $X^* M_z^*|_{M_*} = M_z^*|_{N_*} X^*$, then there exists $Y^* \in B(\bigoplus_1^M \mathcal{D}_\alpha,\ \bigoplus_1^N \mathcal{D}_\alpha)$ so that: $(i)\quad Y^*|_{M_*}\ =\ X^*\ \ (X^*\ lifts\ to\ Y^*)$

$(ii)\ \ Y^* M_z^* = M_z^* Y^*\ \ (Y\ is\ in\ the\ commutant\ of\ M_z)$

$(iii)\ \ \|Y^*\|\ =\ \|X^*\|\quad (Norms\ are\ preserved)$

Theorem B Assume that $\delta^2 I\ \le\ M_F^R (M_F^R)^* \le\ I$, then there exists a $\{g_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D}_\alpha)$, such that (i) $\sum_{j=1}^\infty f_j g_j = 1$

ii) $\left\|\ M_G^C\ \right\| \le\ 1/\delta$

Proof. Define $M = \infty$, $N = 1$, $M_*\ =\ range\ (M_F^R)^*$, $\ N_*\ =\ \mathcal{D}_\alpha$

Let $X^*\ =\ [M_F^R\,(M_F^R)^*]^{-1}(M_F^R)$, $\qquad$ Note $\quad X^*\ :\ \bigoplus_1^M \mathcal{D}_\alpha\ \to\ \mathcal{D}_\alpha$

$$
\begin{aligned}
\|X^*\|^2 = \|X^* X\|\ &=\ \left\|[M_F^R\,(M_F^R)^*]^{-1} M_F^R [\,[M_F^R\,(M_F^R)^*]^{-1} M_F^R\,]^*\right\| \\
&= \left\|[M_F^R\,(M_F^R)^*]^{-1} M_F^R (M_F^R)^* [\,[M_F^R\,(M_F^R)^*]^{-1}\,]^*\right\| \\
&= \left\|[\,[M_F^R\,(M_F^R)^*]^{-1}\,]^*\right\| \\
&= \left\|[\,[M_F^R\,(M_F^R)^*]^{-1}\,]\right\|
\end{aligned}
$$

From $\delta^2 I\ \le\ M_F^R (M_F^R)^* \le\ I$ it follows that

$$I\ \le\ [M_F^R (M_F^R)^*]^{-1} \le\ 1/\delta^2\, I$$

which implies that $\left\| [[M_F^R (M_F^R)^*]^{-1}] \right\| \leq 1/\delta^2$

Thus, $\|X^*\|^2 \leq 1/\delta^2$, and so $\|X^*\| \leq 1/\delta$

Now consider, $M_z M_F^R(\{u_j\}) = M_z \sum_{j=1}^{\infty} f_j u_j = z \sum_{j=1}^{\infty} f_j u_j$

$$= \sum_{j=1}^{\infty} f_j(zu_j) = M_F^R(\{zu_j\}) = M_F^R(M_z(\{u_j\})).$$

This justifies $M_z M_F^R = M_F^R M_z$, and from this it follows that

$(M_z)^*(M_F^R)^* = (M_F^R)^*(M_z)^*.$

Let now $u \in \mathcal{D}_\alpha$ be any, then

$$X^* {M_z}^*((M_F^R)^* u) = X^* (M_F^R)^* {M_z}^* u = {M_z}^* u$$

$$= M^* u$$

$$X^* {M_z}^*((M_F^R)^* = {M_z}^* (X^* (M_F^R)^*)$$

So we have $X^* {M_z}^*|_{M_*} = {M_z}^*|_{N_*} X^*$

Thus, by the CLT there exist $Y^* \in B(\bigoplus_1^M \mathcal{D}_\alpha, \mathcal{D}_\alpha)$ satisfying (i), (ii), *and* (iii).

By (ii) Y has entries in $\mathcal{M}(D_\alpha)$ say $g_i \in \mathcal{M}(D_\alpha)$. So, $Y = M_G^C$.

By (i), $Y^*(M_F^R)^* = I$, which implies $M_F^R M_G^C = I$.

Finally (iii) implies $\left\|M_G^C\right\| = \|Y\| = \|X\| \leq 1/\delta$

□

Next we prove **Theorem A** which is done by proving **Theorem A'**, an operator theory equivalence of Theorem A. To prove the equivalence we use Douglas's Lemma, which is stated below.

Lemma 5.5.6 (**Douglas Lemma**)

Suppose $A \in \mathcal{B}(H,\ K_1)$ *and* $B \in \mathcal{B}(H,\ K_2)$, then there exists $C \in \mathcal{B}(K_1,\ K_2)$ with $\|C\| \leq \delta$ *and* $CA\ =\ B$ if and only if $A^*A \geq 1/\delta^2\ B^*B$.

Theorem A' Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$. Assume that $\left\| M_F^C \right\| \leq\ 1$, and $0\ < \epsilon^2 \leq \sum_{j=1}^{\infty} \mid f_j(z) \mid^2$ for all $z \in \mathbb{D}$. Then there exists a positive number $C(\epsilon, \alpha) < \infty$, and for all $h \in \mathcal{D}_\alpha$ there exists $\mathbf{U}_h \in \bigoplus_1^{\infty} \mathcal{D}_\alpha$ such that

$(i)\quad M_F^R(\mathbf{U}_h) = h$

$(ii)\quad \| \mathbf{U}_h \|_{\bigoplus_1^{\infty} \mathcal{D}_\alpha}\ \leq\ c(\epsilon, \alpha) \| h \|_{\mathcal{D}_\alpha}.$

Next we give proof of the equivalence:

Proof. (Equivalence between Theorem A and A')

Note the conclusion of Theorem A is

$k(\epsilon)^{-2} I\ \leq\ M_F^R (M_F^R)^* \leq\ I\ \equiv\ I\ \geq\ M_F^R (M_F^R)^* \geq\ k(\epsilon)^{-2} I^* I$

To apply Douglas's Lemma Consider

$M_F^R (M_F^R)^* \geq\ k(\epsilon)^{-2} I^* I\ \equiv\ (M_F^{R^*})^* (M_F^R)^* \geq\ k(\epsilon)^{-2} I^* I$, where

$I\ : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ and $M_F^{R^*}\ : \mathcal{D}_\alpha \to \bigoplus_1^{\infty} \mathcal{D}_\alpha$

Thus, by Douglas's Lemma there exists a $C\ : \bigoplus_1^{\infty} \mathcal{D}_\alpha\ \to \mathcal{D}_\alpha$, such that

$(i)\quad C M_F^{R^*} = I\ \equiv\ M_F^{R^*} C = I$ and $(ii)\quad \|C\| \leq k(\epsilon)$

So now $M_F^{R^*} C = I$ implies that $M_F^R C^*(h) = h,\ \forall\ h \in \mathcal{D}_\alpha$.

And for $C^*\ : \mathcal{D}_\alpha \to \bigoplus_1^{\infty} \mathcal{D}_\alpha$, if we set $C^*(h) = U_h$,

then we get $M_F^R(U_h)\ =\ h,$ for all $h \in \mathcal{D}_\alpha$

And $\left\| M_F^R C^*(h) \right\|_{\mathcal{D}_\alpha}^2 \le \|C^* h\|_{\mathcal{D}_\alpha}^2 \le k(\epsilon)^2 \|h\|_{\mathcal{D}_\alpha}^2$, implies that

$\|U_h\|_{\mathcal{D}_\alpha} \le k(\epsilon) \|h\|_{\mathcal{D}_\alpha}$ proving the equivalence.

□

Now we are ready to prove Theorem A' which is much harder to prove than Theorem B. To prove Theorem A' we use a sequence of lemmas. Let's start by making the following observation.

It suffices to prove **Theorem A'** for any dense set of functions in $\mathcal{D}_\alpha$. Then a compactness argument establishes **Theorem A'**, and by the equivalence **Theorem A** follows.

We consider sets of functions which are smooth across the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. For our case, we consider trigonometric polynomials which are dense in $\mathcal{D}_\alpha$ and smooth on $\partial\mathbb{D}$.

The General plan is as follows. Assume F is analytic on $\mathbb{D}_{1+\epsilon}(0)$, given $h \in \mathcal{D}_\alpha$ and h analytic on $\mathbb{D}_{1+\epsilon}(0)$; write the most general solution of the point wise problem

$$\underline{V}_h(z) = F(z)^*[\,F(z)F(z)^*]^{-1}h(z) \; - \; Q(z)\underline{K}(z) \quad \text{on } \mathbb{D},$$

where Range $Q(z) =$ Kernel $F(z)$, $Q(z)$ is analytic, and $\underline{K}(z) \in l^2 \; for \; z \in \overline{\mathbb{D}}$. In fact it holds that:

$$F(z)F(z)^* \, I \; - \; F(z)^*F(z) = Q(z)Q(z)^*.$$

See [Tr3] for more detail.

Note we want to find a $\underline{K}(z)$ and a positive number $C < \infty$, so that

$$\underline{V}_h \in \bigoplus_1^{\infty} \mathcal{D}_\alpha, \text{ satisfies}$$

$$\|\underline{V}_h\| \le C\,\|h\|$$

$$\text{with} \quad \overline{\partial}\underline{V}_h = 0 \; in \; \mathbb{D}.$$

The following sections will do that for us.

5.6. The Cauchy Transform

Definition: For $\underline{\mathcal{K}}$ smooth on $\overline{\mathbb{D}}$ *and* $z \in \mathbb{D}$ the Cauchy Transform $\widehat{\underline{\mathcal{K}}}$ is defined by

$$\widehat{\underline{\mathcal{K}}}(z) = \int_{\mathbb{D}} \frac{\underline{k}(\omega)}{z-\omega}\, dA(\omega), \quad \text{where } dA(\omega) = \frac{1}{\pi} dm$$

Note it holds that $\overline{\partial_z}\,\widehat{\underline{\mathcal{K}}} = \underline{k} \; in \; \mathbb{D}$ for the proof see [AM].

So now, if we set

$$\underline{U}_h = F^*(FF^*)^{-1}h - Q\,\widehat{\frac{Q^*\,F'^*\,h}{(FF^*)^2}},$$

then

$$F\underline{U}_h = FF^*(FF^*)^{-1}h - FQ\,\widehat{\frac{Q^*\,F'^*\,h}{(FF^*)^2}}$$

$$= h - FQ\,\widehat{\frac{Q^*\,F'^*\,h}{(FF^*)^2}}$$

and since $Range\; Q(z) = \; Kernel\; F(z)$ we have

$$FQ\,\widehat{\frac{Q^*\,F'^*\,h}{(FF^*)^2}} = 0, \text{ thus}$$

$$F\underline{U}_h = h$$

Also it is not hard to show that $\overline{\partial}_z\,\underline{U}_h = 0$ in $\mathbb{D}$.

So, we have $F\underline{U}_h = h$, and $\underline{U}_h$ is analytic, Thus we are done in the smooth case of F if we show that:

$$\|\underline{U}_h\|_{\bigoplus_1^\infty \mathcal{D}_\alpha} \leq N(\epsilon) \|h\|_{\mathcal{D}_\alpha}$$

To this end, let $\underline{K} = \frac{Q^* F'^* h}{(FF^*)^2}$. Our procedure is to show

$$(i) \quad \|F^*(FF^*)^{-1}h\|_{\bigoplus_1^\infty \mathcal{HD}_\alpha} \leq C_1 \|h\|_{\mathcal{D}_\alpha},$$

$$(ii) \quad \left\|Q\underline{\widehat{K}}\right\|_{\bigoplus_1^\infty \mathcal{HD}_\alpha} \leq C_2 \left\|\underline{\widehat{K}}\right\|_{\bigoplus_1^\infty \mathcal{D}_\alpha}, \text{ and the main estimate}$$

$$(iii) \quad \left\|\widehat{K}\right\|_{\bigoplus_1^\infty \mathcal{HD}_\alpha} \leq C_2 \|h\|_{\mathcal{D}_\alpha}$$

Once we identify clearly what we need to do for the smooth case we proceed to prove (i), (ii), and (iii). Lemma 5.6.1. - Lemma 5.6.6. proves (i) and (ii), while Cauchy Transform and Schur's Theorem helps us to establish (iii). Note that, Lemma 5.6.1. - Lemma 5.6.3 extend multipliers on $\mathcal{D}_\alpha$ to multipliers on $\mathcal{HD}_\alpha$.

Lemma 5.6.1.

(a) Let $M_\varphi \in M(\mathcal{D}_\alpha)$, then $M_\varphi \in M(\mathcal{HD}_\alpha)$ and $\|M_\varphi\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq \sqrt{20}\, \|M_\varphi\|_{\mathcal{B}(\mathcal{D}_\alpha)}$.

(b) Let $\{f_i\}_{i=1}^\infty \subseteq \mathcal{M}(\mathcal{D}_\alpha)$. Then $\|M_F^C\|_{\mathcal{B}(\mathcal{HD}_\alpha, \bigoplus_1^\infty \mathcal{HD}_\alpha)} \leq \sqrt{20}\, \|M_F^C\|_{\mathcal{B}(\mathcal{D}_\alpha, \bigoplus_1^\infty \mathcal{D}_\alpha)}$.

Proof. : We need only prove (a), since (b) follows by summing the result of (a). Let $M_\varphi \in M(\mathcal{D}_\alpha)$, and $\| M_\varphi \| = C$. We prove the result for trigonometric polynomials i.e. polynomials of the type $f(t) = \sum_{n=-N}^{N} C_n e^{int}$. To this end, we consider analytic polynomials p and q_0 with $q_0(0) = 0$, and we need only estimate $\|M_\varphi(p + \overline{q}_0)\|_{\mathcal{HD}_\alpha}$, since $\{p + \overline{q}_0\}$ is dense in $\mathcal{HD}_\alpha \subseteq L^2(\mathbb{D})$.

Note $p=\sum_{k=0}^{N} a_k e^{ikt}$, and $q_0=\sum_{k=1}^{N} b_k e^{ikt}$ thus $\overline{q}_0=\sum_{k=1}^{N} \overline{b}_k e^{-ikt}$.

Recall $f \in \mathcal{M}(\mathcal{HD}_\alpha)$ implies that

$$\|f\|^2_{\mathcal{HD}_\alpha} = \int_{\partial\mathbb{D}} |f|^2 \, d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|f(e^{it})-f(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} d\sigma d\sigma,$$

and so

$$\|\varphi(p+\overline{q}_0)\|^2_{\mathcal{HD}_\alpha} = \int_{-\pi}^{\pi} |\varphi(p+\overline{q}_0)|^2 \, d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi(p+\overline{q}_0))(e^{it})-(\varphi(p+\overline{q}_0))(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} d\sigma d\sigma$$

$$\begin{aligned}
\|\varphi(p+\overline{q}_0)\|^2_{\mathcal{HD}_\alpha} &\le 2\int_{-\pi}^{\pi} |\varphi p|^2 \, d\sigma + 2\int_{-\pi}^{\pi} |\varphi\overline{q}_0|^2 \, d\sigma + 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi p)(e^{it})-(\varphi p)(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} d\sigma d\sigma \\
&\quad +2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi\overline{q}_0)(e^{it})-(\varphi\overline{q}_0)(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} d\sigma d\sigma \\
&\le 2\left\|\varphi p\right\|_{\mathcal{D}_\alpha} + 2c^2\int_{-\pi}^{\pi} |\overline{q}_0|^2 \, d\sigma + 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi\overline{q}_0)(e^{it})-(\varphi\overline{q}_0)(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} d\sigma d\sigma \\
&\le 2c_0^2\left\|p\right\|_{\mathcal{D}_\alpha} + 2c^2\int_{-\pi}^{\pi} |\overline{q}_0|^2 \, d\sigma \\
&\quad +2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(\varphi\overline{q}_0)(e^{it})-\varphi(e^{it})\overline{q}_0(e^{i\theta})+\varphi(e^{it})\overline{q}_0(e^{i\theta})-(\varphi\overline{q}_0)(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} d\sigma d\sigma \\
&\le 2c^2\left\|p\right\|_{\mathcal{D}_\alpha} + 2c^2\int_{-\pi}^{\pi} |\overline{q}_0|^2 \, d\sigma + 4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|\overline{q}_0(e^{it})-\overline{q}_0(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} |\,\varphi(e^{it})\,|^2 d\sigma d\sigma \\
&\quad + \quad 4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|\varphi(e^{it})-\varphi(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} |\,\overline{q}_0(e^{i\theta})\,|^2 d\sigma d\sigma
\end{aligned}$$

$$\|\varphi(p+\overline{q}_0)\|^2_{\mathcal{HD}_\alpha} \leq 2c^2\,\|p\|^2_{\mathcal{D}_\alpha} + 4c^2\,\|\overline{q}_0\|_{\mathcal{HD}_\alpha} + 4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|\varphi(e^{it})-\varphi(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}|\,q_0(e^{i\theta})\,|^2 d\sigma d\sigma$$

But, $\|\overline{q}_0\|_{\mathcal{HD}_\alpha} = \|q_0\|_{\mathcal{D}_\alpha}$ and

$$\begin{aligned}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|\varphi(e^{it})-\varphi(e^{i\theta})\right|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}}|\,q_0(e^{i\theta})\,|^2 d\sigma d\sigma &\leq 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\,\varphi(e^{it})q_0(e^{it})-\varphi(e^{i\theta})q_0(e^{i\theta})\,|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma \\ &\quad +2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|\,\varphi(e^{it})q_0(e^{i\theta})-\varphi(e^{it})q_0(e^{it})\,|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma \\ &\leq 2c^2\,\|\,q_0\,\|^2_{\mathcal{D}_\alpha} + 2c^2\,\|q_0\,\|^2_{\mathcal{D}_\alpha}\end{aligned}$$

So, we have

$$\begin{aligned}\|\varphi(p+\overline{q}_0)\|^2_{\mathcal{HD}_\alpha} &\leq 2c^2\,\|p\|^2_{\mathcal{D}_\alpha} + 4c^2\,\|q_0\|^2_{\mathcal{D}_\alpha} + 16\,c^2\,\|\,q_0\,\|^2_{\mathcal{D}_\alpha} \\ &\leq 20\,c^2\,(\|\,p\,\|^2_{\mathcal{D}_\alpha} + \|\,q_0\,\|^2_{\mathcal{D}_\alpha}) \\ &\leq 20\,c^2\,(\|\,p\,+\,\overline{q}_0\,\|^2_{\mathcal{HD}_\alpha}\end{aligned}$$

Thus, $\|M_\varphi\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq \sqrt{20}\,\|M_\varphi\|_{\mathcal{B}(\mathcal{D}_\alpha)}$.

□

Lemma 5.6.2. Assume $\left\|M_F^C\right\| \leq 1$, then $M_{FF^*} \in \mathcal{B}(\mathcal{HD}_\alpha)$ and $\|M_{FF^*}\| \leq 86$.

PROOF. : First we note that for a positive operator P, $M_P = M_{\sqrt{P}}M_{\sqrt{P}}$.

and since $FF^* \geq \epsilon > 0$ we have $M_{\sqrt{FF^*}}\,M_{\sqrt{FF^*}} = M_{(FF^*)}$.

This implies that $\left\|\,M_{\sqrt{FF^*}}\,\right\|^2 = \|M_{FF^*}\|$. Hence, to prove the lemma we need only show that $M_{\sqrt{FF^*}} \in \mathcal{B}(\mathcal{HD}_\alpha)$ with $\left\|\,M_{\sqrt{FF^*}}\right\| \leq \sqrt{86}$.

Let p and q_0 be analytic polynomials with $q_0(0) = 0$, and $u = p + \overline{q}_0$, then

$$\begin{aligned}
\| M_{(FF^*)^{1/2}}(u) \|^2_{\mathcal{HD}_\alpha} &= \int_{\partial\mathbb{D}} |FF^*|\,|u|^2\, d\sigma \;+ \\
&\qquad \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(FF^*)^{1/2}(e^{it})u(e^{it}) - (FF^*)^{1/2}(e^{i\theta})u(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} d\sigma d\sigma \\
&\le 2\,\|u\|_{\mathcal{HD}_\alpha} + \underbrace{2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|(FF^*)^{1/2}(e^{it}) - (FF^*)^{1/2}(e^{i\theta})\right|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} |\, u(e^{i\theta})\,|^2 d\sigma d\sigma}_{(a)}
\end{aligned}$$

Now first note that $FF^*(z) = \sum_{j=1}^{\infty} |\, f_j(z)\,|^2$, and from Cauchy Schwartz Inequality it follows that:

$$\left|(FF^*)^{1/2}(e^{it}) - (FF^*)^{1/2}(e^{i\theta})\right|^2 \le \sum_{j=1}^{\infty} |\, f_j(e^{it}) - f_j(e^{i\theta})\,|^2$$

which implies that

$$\begin{aligned}
(a) &\le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sum_{j=1}^{\infty} \frac{|\, f_j(e^{it}) - f_j(e^{i\theta})\,|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} |\, u(e^{i\theta})\,|^2 d\sigma d\sigma \\
&\le \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sum_{j=1}^{\infty} \frac{|\, f_j(e^{it})(u(e^{it}) - u(e^{i\theta})) - f_j(e^{i\theta})(u(e^{it}) - u(e^{i\theta}))\,|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} |\, u(e^{i\theta})\,|^2 d\sigma d\sigma \\
&\le 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sum_{j=1}^{\infty} \frac{|\, u(e^{it}) - u(e^{i\theta})\,|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} |\, f_j(e^{it})\,|^2 d\sigma d\sigma \;+ \\
&\qquad 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sum_{j=1}^{\infty} \frac{|\, f_j(e^{it})u(e^{it}) - f_j(e^{i\theta})u(e^{i\theta})\,|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}} | d\sigma d\sigma
\end{aligned}$$

$$\leq \ 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\big|\,u(e^{it})-u(e^{i\theta})\,\big|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}|\,d\sigma d\sigma \ \ +2\int_{\partial\mathbb{D}}\sum_{j=1}^{\infty}|\,f_j u\,|^2 d\sigma \ +$$

$$2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{j=1}^{\infty}\frac{\big|\,f_j(e^{it})u(e^{it})-f_j(e^{i\theta})u(e^{i\theta})\,\big|^2}{|e^{it}-e^{i\theta}|^{\alpha+1}}|d\sigma d\sigma$$

$$\leq \ 2\,\|u\|^2_{\mathcal{HD}_\alpha}+2\sum_{j=1}^{\infty}\|f_j u\|^2_{\mathcal{HD}_\alpha}=2\,\|u\|^2_{\mathcal{HD}_\alpha}+2\,\left\|M_F^C(u)\right\|^2_{\mathcal{HD}_\alpha}$$

Thus we have

$$\begin{aligned}\left\|\,M_{(FF^*)^{1/2}}(u)\,\right\|^2_{\mathcal{HD}_\alpha} &\leq \ 2\,\|u\|^2_{\mathcal{HD}_\alpha}+4\,\|u\|^2_{\mathcal{HD}_\alpha}+4\,\left\|M_F^C(u)\right\|^2_{\mathcal{HD}_\alpha}\\ &\leq \ 6\,\|u\|^2_{\mathcal{HD}_\alpha}+4\,\left\|M_F^C(u)\right\|^2_{\mathcal{HD}_\alpha}\end{aligned}$$

$$\begin{aligned}\left\|\,M_{(FF^*)^{1/2}}(u)\,\right\|^2_{\mathcal{HD}_\alpha} &\leq \ 6\,\|u\|^2_{\mathcal{HD}_\alpha}+4\,(20\,\left\|M_F^C\right\|^2_{\mathcal{HD}_\alpha}\,\|u\|^2_{\mathcal{HD}_\alpha})\\ &\leq \ 86\,\|u\|^2_{\mathcal{HD}_\alpha}\end{aligned}$$

which implies

$$\left\|\,M_{(FF^*)^{1/2}}\,\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)}\leq 86,\quad \text{and hence } \|\,M_{FF^*}\,\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)}\leq 86$$

□

The next lemma proves that $M_{1/H}$ is in $M(\mathcal{D}_\alpha)$, provided $M_H\in M(\mathcal{D}_\alpha)$ and H is bounded form below on $\mathbb{D}$.

Lemma 5.6.3. Let $H\in\mathcal{M}(\mathcal{HD}_\alpha)$ with $1\geq|\,H(e^{it}\,|\geq\epsilon>0$, for σ a.e. $t\in[-\pi,\,\pi]$. Then $M_{1/H}\in M(\mathcal{HD}_\alpha)$ and $\left\|M_{1/H}\right\|_{\mathcal{B}(\mathcal{HD}_\alpha)}\leq(\sqrt{10}/\epsilon^2)\,\|M_H\|_{\mathcal{B}(\mathcal{D}_\alpha)}$.

Proof. : Let $r\in\mathcal{HD}_\alpha$ be a trigonometric polynomial on $\partial\mathbb{D}$

and let $\|M_H\|_{\mathcal{B}(\mathcal{HD}_\alpha)}=C_0$. Then,

$$\left\| M_{\frac{1}{H}} r \right\|^2_{\mathcal{HD}_\alpha} = \int_{-\pi}^{\pi} |r/H|^2 \, d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|\frac{r}{H}(e^{it}) - \frac{r}{H}(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma$$

$$\leq \frac{1}{\epsilon^2}\int_{\partial\mathbb{D}} | r^2 | d\sigma + \frac{1}{\epsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|H(e^{i\theta})r(e^{it}) - H(e^{it})r(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma$$

Using the fact that $|\, a + b\,|^2 \leq 2\,|a|^2 + 2\,|b|^2$ we get

$$\left\| M_{\frac{1}{H}} r \right\|^2_{\mathcal{HD}_\alpha} \leq \frac{1}{\epsilon^2}\int_{\partial\mathbb{D}} | r^2 | d\sigma + \frac{2}{\epsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | H(e^{i\theta}) |^2 \frac{\left|r(e^{it}) - r(e^{i\theta})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma +$$

$$\frac{2}{\epsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|H(e^{i\theta}) - H(e^{it})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} | r(e^{i\theta}) |^2 d\sigma d\sigma$$

$$\left\| M_{\frac{1}{H}} r \right\|^2_{\mathcal{HD}_\alpha} \leq \frac{2\,C_0^2}{\epsilon^4} \|r\|^2_{\mathcal{HD}_\alpha} + \frac{2}{\epsilon^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left|H(e^{i\theta}) - H(e^{it})\right|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} | r(e^{i\theta}) |^2 d\sigma d\sigma$$

$$\leq \frac{2\,C_0^2}{\epsilon^4} \|r\|^2_{\mathcal{HD}_\alpha} + \frac{2}{\epsilon^4}(4\,C_0^2 \,\|r\|^2_{\mathcal{HD}_\alpha}) = \frac{10\,C_0^2}{\epsilon^4} \|r\|^2_{\mathcal{HD}_\alpha}$$

This implies that

$$\left\| M_{\frac{1}{H}} \right\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq \frac{\sqrt{10}}{\epsilon^2} \|M_H\|_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq \frac{\sqrt{10}}{\epsilon^2} \|M_H\|_{\mathcal{B}(\mathcal{D}_\alpha)}$$

□

Lemma 5.6.4. Let $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$. Assume that $\left\|M_F^C\right\| \leq 1$ and $0 < \epsilon^2 \leq F(z)F(z)^*$ for all $z \in \mathbb{D}$. Then for $h \in \mathcal{D}_\alpha$, we have

$$\left\| \frac{F^*}{FF^*} h \right\|^2_{\mathcal{H}D_\alpha} \leq \frac{10.86^2.20}{\epsilon^4} \|h\|^2_{\mathcal{D}_\alpha}$$

PROOF. Let r be a trigonometric polynomial in $\partial\mathbb{D}$. Then by Lemma 5.6.1.

$$\begin{aligned}
\|F^* r\|^2_{\mathcal{HD}_\alpha} &= \left\|(\overline{f}_1 r\,,\ \overline{f}_2 r\,,\ \ldots)^T\right\| \\
&= \sum_{j=1}^{\infty} \|f_j \overline{r}\|^2_{\mathcal{H}D_\alpha} \\
&= \left\|M_F^C\,\overline{r}\,\right\|^2_{\mathcal{HD}_\alpha} \leq \left\|M_F^C\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)} \|\overline{r}\|^2_{\mathcal{H}D_\alpha} \\
&\leq 20\,\left\|M_F^C\right\|^2_{\mathcal{B}(\mathcal{D}_\alpha)} \|\overline{r}\|^2_{\mathcal{HD}_\alpha}
\end{aligned}$$

By Lemma 5.6.2. and 5.6.3. we have

$$\begin{aligned}
\left\|\,M_{(FF^*)^{-1}}\,\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)} &\leq \frac{10}{\epsilon^4}\,\|FF^*\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)} \leq \frac{10}{\epsilon^4}\,\|\,M_{FF^*}\,\|^2_{\mathcal{B}(\mathcal{HD}_\alpha)} \\
&\leq \frac{86^2\,10}{\epsilon^4}
\end{aligned}$$

Now for h in $\mathcal{D}_\alpha$

$$\begin{aligned}
\left\|\,(FF^*)^{-1}\,F^*h\,\right\|^2 &= \left\|\,M_{(FF^*)^{-1}}\,(F^*h)\,\right\|^2 \\
&\leq \left(\frac{86\sqrt{10}}{\epsilon^2}\right)^2 \|F^*h\|^2 \\
&\leq \frac{86^2.10.20}{\epsilon^4}\,\|\,h\,\|^2_{\mathcal{HD}_\alpha} = \frac{86^2.10.20}{\epsilon^4}\,\|\,h\,\|^2_{\mathcal{D}_\alpha}
\end{aligned}$$

□

The next Lemma will enable us to write down the most general solution of $F\underline{U}_h(z) = h(z)$. A more general version of this Lemma can be found in Trent [Tr3]. For completeness, we include a proof

Lemma 5.6.5. Let $\{ c_j \}_{j=1}^{\infty} \in l^2$ and $C = (c_1 \, , \; c_2 \, , \; \ldots \,) \in \mathcal{B}(l^2, \; \mathbb{C})$. Then there exists a Q such that entries of Q are either 0 or $+c_j$ or $-c_j$ for some j and $\quad CC^*I - C^*C = QQ^*$.

Proof. For $k = 1, 2, \ldots,$ let

$$A_k = \begin{bmatrix} 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ c_{k+1} & c_{k+2} & c_{k+3} & \ldots \\ -c_k & 0 & 0 & \ldots \\ 0 & -c_k & 0 & \ldots \\ 0 & 0 & -c_k & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where the first k rows of A_k have only 0 entries.

Then

$$A_k A_k^* = \begin{bmatrix} 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\ \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \ldots \\ 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\ 0 & \ldots & 0 & \sum\limits_{j=k+1}^{\infty} |c_j|^2 & -\overline{c}_k c_{k+2} & -\overline{c}_k c_{k+3} & \ldots \\ 0 & \ldots & 0 & -c_k \overline{c}_{k+2} & |c_k|^2 & 0 & \ldots \\ 0 & \ldots & 0 & -c_k \overline{c}_{k+3} & 0 & |c_k|^2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Thus

$$\sum_{k=1}^{\infty} A_k A_k^* = \begin{bmatrix} \sum\limits_{k\neq 1}^{\infty} |c_k|^2 & -\overline{c}_1 c_2 & -\overline{c}_1 c_3 & \dots \\ -\overline{c}_2 c_1 & \sum\limits_{k\neq 2}^{\infty} |c_k|^2 & -\overline{c}_2 c_3 & \dots \\ -\overline{c}_3 c_1 & -\overline{c}_3 c_2 & \sum\limits_{k\neq 3}^{\infty} |c_k|^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= CC^* I - C^* C.$$

This completes the prof if we let

$$Q = [A_1, A_2, \dots] \in B(\bigoplus_1^{\infty} l^2, l^2).$$

□

Lemma 5.6.6. Let $\{ f_j \}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$. Assume that for each j, f_j is analytic on $\mathbb{D}_{1+\epsilon}(0)$ and $\left\| M_F^C \right\|_{\mathcal{B}(\mathcal{D}_\alpha)} \leq 1$. Associate $Q(z)$ *to* $F(z)$ for each $|\, z \,| = 1$. Then, $\|\, M_Q \,\|_{\mathcal{B}(\bigoplus_1^{\infty} \mathcal{H}\mathcal{D}_\alpha)} \leq \sqrt{86}$.

Proof. : $F(z) = (f_1(z),\ f_2(z),\ \dots\ ), \quad z \in \mathbb{D}_{1+\epsilon}(0)$

Recall, $\left\| M_F^C \right\|_{\mathcal{B}(\mathcal{D}_\alpha)} \leq 1$ implies $\|F(z)\|_{l^2}^2 \leq 1$.

By lemma 5.6.5. for each $z \in \overline{\mathbb{D}}$ there exists a $Q(z)$ such that

$F(z)F(z)^* I - F(z)^* F(z) = Q(z)Q(z)^*$.

Since $F(z)^* F(z) \geq 0$ we have

$Q(z)Q(z)^* \leq (\ F(z)F(z)^*) I_{l^2}$, for all $z \in \overline{\mathbb{D}}$ and this implies

$\|Q(z)\|_{\mathcal{B}(l^2)} \leq 1$

Now let $\underline{r} \in \bigoplus_{1}^{\infty} \mathcal{HD}_\alpha$ be a trigonometric polynomial in z, then

$$\|Q\underline{r}\|_{\bigoplus_1^\infty \mathcal{HD}_\alpha} = \int_{\partial\mathbb{D}} \|\,(Q\underline{r})(e^{it})\,\|^2\, d\sigma \;+\; \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|(Q\underline{r})(e^{it}) - (Q\underline{r})(e^{i\theta})\right\|^2}{\left|e^{it}-e^{i\theta}\right|^{\alpha+1}} d\sigma d\sigma$$

First let's consider

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|\, F(e^{it}) \,-\, F(e^{i\theta})\,\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \,||\,\underline{r}(e^{it})\,||^2 d\sigma d\sigma$$

$$\begin{aligned}
\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|\, F(e^{it}) \,-\, F(e^{i\theta})\,\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \,||\,\underline{r}(e^{it})\,||^2 d\sigma d\sigma \;&=\; \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|\,(F\underline{r})(e^{it}) \,-\, F(e^{i\theta})\underline{r}(e^{it})\,\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma d\sigma \\
&\leq\; 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\|\,F(e^{it})\,\|^2\,||\,\underline{r}(e^{it}) \,-\, \underline{r}(e^{i\theta})\,||^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma d\sigma \,+ \\
&\qquad 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|\,(F\underline{r})(e^{it}) \,-\, (F\underline{r})(e^{i\theta})\,\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma d\sigma \\
&\leq\; 2\,\|\underline{r}\|^2_{\mathcal{HD}_\alpha} + 2\,\|F\underline{r}\|^2_{\mathcal{B}(\mathcal{HD}_\alpha,\bigoplus\mathcal{HD}_\alpha)} \\
&\leq\; 2\,\|\underline{r}\|^2_{\mathcal{HD}_\alpha} + 2\,\left\|M_F^C\right\|^2_{\mathcal{B}(\mathcal{HD}_\alpha,\bigoplus\mathcal{HD}_\alpha)}\,\|\underline{r}\|^2_{\mathcal{HD}_\alpha} \\
&\leq\; 2\,\|\underline{r}\|^2_{\mathcal{HD}_\alpha} + 2.20\;\|\underline{r}\|^2_{\mathcal{HD}_\alpha} = 42\;\|\underline{r}\|^2_{\mathcal{HD}_\alpha}
\end{aligned}$$

Now let $\underline{r} \in \bigoplus_{1}^{\infty} \mathcal{HD}_\alpha$, then

$$\|Q\underline{r}\|^2_{\mathcal{HD}_\alpha} \;=\; \int_{-\pi}^{\pi} \left\|(Q\underline{r})(e^{it})\right\|^2_{l^2} d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|\,(Q\underline{r})(e^{it}) \,-\, (Q\underline{r})(e^{i\theta})\,\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}}\, d\sigma d\sigma$$

Then using he fact that $|\,a \,+\, b\,|^2 \leq 2\,|\,a\,|^2 +\, 2\,|\,b\,|^2$ we get:

$$\|Q\underline{r}\|^2_{\mathcal{HD}_\alpha} \le \int_{-\pi}^{\pi} \left\|\underline{r}(e^{it})\right\|^2_{l^2} d\sigma + 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\| Q(e^{it}) - Q(e^{i\theta}) \right\|^2_{\mathcal{B}(l^2)}}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \left\|\underline{r}(e^{it})\right\|^2_{l^2} d\sigma d\sigma +$$

$$2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\| \underline{r}(e^{it}) - \underline{r}(e^{i\theta}) \right\|^2_{l^2}}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \left\|Q(e^{i\theta})\right\|^2_{\mathcal{B}(l^2)} d\sigma d\sigma$$

Now by Lemma 5.6.5., if we set $C_j = f_j(e^{it}) - f_j(e^{i\theta})$, then we have

$\{ f_j(e^{it}) - f_j(e^{i\theta}) \}_{j=1}^{\infty}$ is in l^2.

Let $\quad F(e^{it}) - F(e^{i\theta}) = (f_1(e^{it}) - f_1(e^{i\theta}),\ f_2(e^{it}) - f_2(e^{i\theta}),\ \ldots\ )$

And it holds that

$$(Q(e^{it}) - Q(e^{i\theta}))(Q(e^{it}) - Q(e^{i\theta}))^* \le (F(e^{it}) - F(e^{i\theta}))(F(e^{it}) - F(e^{i\theta}))^* I_{l^2},$$

which implies that

$$\left\|Q(e^{it}) - Q(e^{i\theta})\right\|^2 \le \left\|F(e^{it}) - F(e^{i\theta})\right\|^2$$

Thus now we have

$$\begin{aligned}
\| Q\underline{r} \|^2_{\bigoplus_1^\infty \mathcal{HD}_\alpha} &\le \int_{\partial\mathbb{D}} \left\| \underline{r}(e^{it}) \right\|^2 d\sigma + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|\underline{r}(e^{it}) - \underline{r}(e^{i\theta})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \left\|Q(e^{i\theta})\right\|^2 d\sigma d\sigma + \\
&\quad 2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|Q(e^{it}) - Q(e^{i\theta})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \left\|\underline{r}(e^{it})\right\|^2 d\sigma d\sigma \\
&\le 2 \, \| \underline{r} \|^2_{\mathcal{HD}_\alpha} + 2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\left\|F(e^{it}) - F(e^{i\theta})\right\|^2}{\left|e^{it} - e^{i\theta}\right|^{\alpha+1}} \left\|\underline{r}(e^{it})\right\| d\sigma d\sigma \\
&\le 2 \, \| \underline{r} \|^2_{\mathcal{HD}_\alpha} + 2(42) \, \| \underline{r} \|^2_{\mathcal{HD}_\alpha}.
\end{aligned}$$

Thus, $\| M_Q \|_{\mathcal{B}(\bigoplus_1^\infty \mathcal{HD}_\alpha, \bigoplus_1^\infty \mathcal{D}_\alpha)} \le \sqrt{86}$

□

Let's summarize the results we obtained so far.

First set $\underline{\widehat{\mathcal{K}}} = \widehat{\frac{Q^*\,F'^*\,h}{(FF^*)^2}}$, then by Lemma 5.6.6. we have

$$\left\| M_Q \underline{\widehat{\mathcal{K}}} \right\|_{\mathcal{B}(\bigoplus_1^\infty \mathcal{HD}_\alpha)} \leq \sqrt{86} \left\| \underline{\widehat{\mathcal{K}}} \right\|_{\bigoplus_1^\infty \mathcal{HD}_\alpha}$$

So we are left to show that

$$\left\| \underline{\widehat{\mathcal{K}}} \right\|_{\bigoplus_1^\infty \mathcal{HD}_\alpha} \leq C \,\| h \|_{\mathcal{D}_\alpha}$$

To this end we will show that

$$\left\| \underline{\widehat{\mathcal{K}}} \right\|^2_{\bigoplus_1^\infty \mathcal{HD}_\alpha} \leq C_2 \,\| \mathcal{K} \|^2_{\mathcal{A}_\alpha} = C_2 \int_{\mathbb{D}} \| \mathcal{K}(z) \|^2 \, (1-|z|^2)^{1-\alpha} \, dA(z)$$

If we showed this then:

$$\begin{aligned}
\| \mathcal{K} \|^2_{\mathcal{A}_\alpha} &= \left\| \frac{Q^* F' h}{(FF^*)^2} \right\|^2_{\mathcal{A}_\alpha} = \int_{\mathbb{D}} \left\| \frac{Q^* F' h}{(FF^*)^2}(z) \right\|^2 (1-|z|^2)^{1-\alpha} dA(z) \\
&= \int_{\mathbb{D}} \left\| \frac{Q^* F' h}{(FF^*)^{3/2}\sqrt{FF^*}}(z) \right\|^2 (1-|z|^2)^{1-\alpha} dA(z) \\
&\leq \frac{1}{\epsilon^6} \int_{\mathbb{D}} \left\| \frac{Q^* F' h}{\sqrt{FF^*}}(z) \right\|^2 (1-|z|^2)^{1-\alpha} dA(z), \quad \text{by Lemma 5.6.4}
\end{aligned}$$

by Lemma 5.6.5 $\|Q(z)\| \leq \|F(z)\|$ thus,

$$\begin{aligned}
\| \mathcal{K} \|^2_{\mathcal{A}_\alpha} &\leq \frac{1}{\epsilon^6} \int_{\mathbb{D}} \|F' h(z)\|^2 \, (1-|z|^2)^{1-\alpha} dA(z) \\
&\leq \frac{1}{\epsilon^6} \left\| M_F^C(h) \right\|^2_{\bigoplus_1^\infty \mathcal{HD}_\alpha} \\
&\leq \frac{20}{\epsilon^6} \, \|h\|^2_{\mathcal{D}_\alpha}
\end{aligned}$$

5.7. Estimates for $\widehat{K}$

Suppose (X, μ) is a measure space and K is a function on $X \times X$. Let T be the integral operator with kernel K, given by.

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y)$$

For more detail and the proof of the following results see Theory of Bergman Spaces by H. Hedenmalm, B. Korenblum, and K. Zhu [HKZ].

Theorem 5.7.1.(Schur's Theorem)

Suppose K is a non negative measurable function on $X \times X$. Let T be the integral operator with kernel K and $1 < p < \infty$ with $1/p + 1/q = 1$. If there are positive constants $C_1,\ C_2$ and positive measurable functions h and l on X such that

$$\int_X K(x,y)[\,h(y)\,]^q d\mu(y) \ \le C_1[\,l(x)\,]^q \text{ for a.e } x \in X$$

and

$$\int_X K(x,y)[\,l(x)\,]^p d\mu(x) \ \le C_2[\,h(y)\,]^p \text{ for a.e } y \in X,$$

then T is a bounded linear operator on $L^p(X, d\mu)$ with norm $\le\ C_1^{1/q}C_2^{1/p}$.

Lemma 5.7.2. Suppose $z \in \mathbb{D},\ c \in \Re,\ t > -1$, and

$I_{c,t}(z) \ = \ \int_{\mathbb{D}} \frac{(1-|\,w\,|^2)^t}{|\,1-z\overline{w}\,|^{2+t+c}}\, dA(w)$, then the following hold

a) If $c < 0$, then as a function of z $I_{c,t}(z)$ is bounded from above and bounded from below *on* $\mathbb{D}$

b) If $c > 0$, then $I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c}, |z| \rightarrow 1^-$

c) If $c = 0$, then, $I_{c,t}(z) \sim log\frac{1}{1-|z|^2}, |z| \rightarrow 1^-$

In the following we will try to find an estimate for $\left\| \widehat{k} \right\|_{\mathcal{D}_\alpha} = \left\| \widehat{k} \right\|_\alpha$, where $\widehat{k}$ is the Cauchy transform. To this end, first we will state Pompeiu's formula.

Pompeiu's Formula. Let Ω be a bounded domain. If $f \in C^1(\overline{\Omega})$ and $w \in \Omega$, then

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}\,dz - \frac{1}{\pi}\int_{\Omega} \frac{\partial f}{\partial \overline{u}}(u)\,\frac{dA(u)}{u-w}.$$

Proof. See Anderson [A], "*Topics in Complex Analysis.*"

Lemma 5.7.3. Let $\varphi \in C^2(\mathbb{D}_{1+\epsilon})$, then

$$\varphi(z) = \underbrace{\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\varphi(u)}{u-z}\,dz}_{\varphi_1(z)} - \underbrace{\frac{1}{\pi}\int_{\mathbb{D}} \frac{\partial\varphi/\partial\overline{u}}{u-z}\,dA(u)}_{\varphi_2(z)}, \text{ with}$$

$\varphi_1(e^{it}) = \sum_{n=0}^{\infty} \varphi_n\, e^{int}$ and $\varphi_2(e^{it}) = \sum_{n=1}^{\infty} \varphi_{-n}\, e^{-int}$,

where $\varphi_n = \int\limits_{-\pi}^{\pi} \varphi(e^{it})e^{-int}d\sigma(t) = \langle \varphi, \ e^{int} \rangle$

PROOF. : That, $\varphi(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\varphi(u)}{u-z}\,dz - \frac{1}{\pi}\int_{\mathbb{D}} \frac{\partial\varphi/\partial\overline{u}}{u-z}\,dA(u)$

is a direct consequence of Pompeiu's Formula. To prove the rest consider

$$\varphi_1(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\varphi(u)}{u-z}\,du,$$

Set $u = e^{it}$, then $du = ie^{it}dt$ and

$$\begin{aligned}
\varphi_1(z) &= \int_{-\pi}^{\pi} \frac{\varphi(e^{it})}{1- \ ze^{-it}}\, d\sigma, \quad \text{where } d\sigma = \frac{dt}{2\pi} \\
&= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int} z^n d\sigma(t) \\
&= \sum_{n=0}^{\infty} \left(\int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int} d\sigma(t) \right) z^n \\
&= \sum_{n=0}^{\infty} \varphi_n z^n
\end{aligned}$$

where $\varphi_n = \int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int} d\sigma(t)$, i.e. $\varphi_n = \langle \varphi, \ e^{int} \rangle$

On the other hand $\varphi \in C^2(\mathbb{D}_{1+\epsilon})$, implies that φ has a Fourier series representation, which is:

$\varphi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \varphi_n e^{in\theta} = \sum_{n=0}^{\infty} \varphi_n \, e^{in\theta} + \sum_{n=1}^{\infty} \varphi_{-n} \, e^{-in\theta}$.

This combied with the above result gives

$$\sum_{n=0}^{\infty} \varphi_n \, e^{in\theta} = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\varphi(u)}{u - e^{i\theta}} \, du = \varphi_1(e^{i\theta})$$

and so

$$\begin{aligned}
\sum_{n=1}^{\infty} \varphi_{-n} \, e^{-in\theta} = \sum_{n=1}^{\infty} \varphi_{-n} \, e^{-int} &= \varphi_2(e^{i\theta}) \\
&= -\frac{1}{\pi} \int_{\mathbb{D}} \frac{\partial \varphi / \partial \, \overline{u}}{u - e^{-i\theta}} \, dA(u)
\end{aligned}$$

Thus we have $\varphi(z) = \varphi_1(z) + \varphi_2(z)$ giving the desired result.

□

Note that

$$\sum_{n=1}^{\infty} \varphi_{-n}\, e^{-in\theta} = \widehat{\frac{\overline{\partial}}{\partial u}\, \varphi}\, (e^{i\theta})$$

For the lemma below we need the following approximation formula for the details see K. Zhu [Zh].

$$\sum_{n=1}^{\infty} n^{c-1} |\, z\,|^{2n} \sim \frac{1}{(1-|z|^2)^c}$$

Lemma 5.7.4. Let $k \in C^2\,(\mathbb{D}_{1+\epsilon})$ and $\widehat{k}(z) = \int\limits_{\mathbb{D}} \frac{k(w)}{z-w}\, dA(w)$, where $z \in \mathbb{D}$ and $dA(w) = \frac{1}{\pi} dm$. Then $\left\| \,\widehat{k}\, \right\|_{\alpha}^{2} \sim \int\limits_{\mathbb{D}} \int\limits_{\mathbb{D}} k(w)\, \overline{k(z)} \frac{1}{(1-w\bar{z})^{1+\alpha}}\, dA(w) dA(z).$

Proof. :

$\widehat{k}(e^{i\theta}) = \sum\limits_{n=1}^{\infty} \widehat{k}_{-n}\, e^{-in\theta}$, *where* $\widehat{k}_{-n} = \left\langle \widehat{k},\ \ e^{-int} \right\rangle = \int\limits_{-\pi}^{\pi} \widehat{k}(e^{it}) e^{int} d\sigma(t)$

$$\begin{aligned}
\int_{-\pi}^{\pi} \widehat{k}(e^{it}) e^{int} d\sigma(t) &= \int_{-\pi}^{\pi} \left(- \int_{\mathbb{D}} \frac{k(w)}{w - e^{it}}\, dA(w) \right) e^{int} d\sigma(t) \\
&= \int_{\mathbb{D}} \left(\int_{-\pi}^{\pi} \frac{e^{i(n-1)t}}{1 - we^{-it}} d\sigma(t) \right) k(w) dA(w) \\
&= \int_{\mathbb{D}} \left(\int_{-\pi}^{\pi} \sum_{k=0}^{\infty} e^{-ikt} w^k e^{i(n-1)t} d\sigma(t) \right) k(w) dA(w) \\
&= \int_{\mathbb{D}} \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} e^{it(n-1-k)} d\sigma(t) w^k k(w) dA(w) \\
&= \int_{\mathbb{D}} k(w) w^{n-1} dA(w) \qquad (5.7.4.1)
\end{aligned}$$

The last equality follows from the fact that

$$\int_{-\pi}^{\pi} e^{it(n-1-k)} d\sigma(t) = \begin{cases} 0, & if \;\; k \neq n-1 \\ 1, & if \;\; k = n-1 \end{cases}$$

On the other hand since $\widehat{k}(z) = \sum_{n=1}^{\infty} \widehat{k}_{-n}\overline{z}^n \;\; for \;\; z \in \partial\mathbb{D}$, the weighted Dirichlet norm for $\widehat{k}$ is given by

$$\begin{aligned} \left\| \widehat{k} \right\|_{\alpha}^{2} &= \sum_{n=1}^{\infty} (|-n|+1)^{\alpha} \left| \widehat{k}_{-n} \right|^{2} \\ &= \sum_{n=1}^{\infty} (n+1)^{\alpha} \left| \widehat{k}_{-n} \right|^{2} \\ &= \sum_{n=1}^{\infty} (n+1)^{\alpha} \left| \int_{\mathbb{D}} k(w) w^{n-1} dA(w) \right|^{2}, \qquad \text{by (5.7.4.1)} \end{aligned}$$

By Stirling's formula

$$\sum_{n=1}^{\infty} (n+1)^{\alpha} \sim \sum_{n=1}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)\Gamma(n+1)}$$

Thus

$$\left\| \widehat{k} \right\|_{\alpha}^{2} \sim \sum_{n=1}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)\Gamma(n+1)} \int_{\mathbb{D}} \int_{\mathbb{D}} k(w)\, \overline{k(z)}\, w^{n-1} \overline{z}^{n-1} dA(w) dA(z)$$

and

$$\int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\,\overline{k(z)}\sum_{n=1}^{\infty}\frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)\Gamma(n+1)}(w\,\overline{z})^n dA(w)dA(z)$$

$$= \int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\,\overline{k(z)}\frac{1}{(1-w\overline{z})^{1+\alpha}}dA(w)dA(z), \qquad (5.7.4.2)$$

This proves Lemma 5.7.4

□

Note that by Lemma 5.7.4 we can redefine the norm $\left\| \widehat{k} \right\|_{\alpha}^{2}$ by:

$$\left\| \widehat{k} \right\|_{\alpha}^{2} = \int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\,\overline{k(z)}\frac{1}{(1-w\overline{z})^{1+\alpha}}\;dA(w)dA(z) \qquad (5.7.4.3)$$

Lemma 5.7.5. Let $\widehat{k}$ be as in Lemma 5.7.4. Then for some constant C_0

$$\left\| \widehat{k} \right\|_{\alpha}^{2} \;\leq\; C_0 \int_{\mathbb{D}} \left| k(z)(1-|z|^2)^{(1+\alpha)/2} \right|^2 dA(z), \qquad (5.7.5.1)$$

Proof. : We know that

$$\begin{aligned}\left\| \widehat{k} \right\|_{\alpha}^{2} &= \int_{\mathbb{D}}\int_{\mathbb{D}} k(w)\,\overline{k(z)}\frac{1}{(1-w\overline{z})^{1+\alpha}}dA(w)dA(z)\\ &= \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{k(w)\,(1-|w|^2)^{\beta}\;\;\overline{k(z)}\,(1-|z|^2)^{\beta}}{(1-|w|^2)^{\beta}\,(1-|z|^2)^{\beta}\,(1-w\overline{z})^{1+\alpha}}\;dA(w)dA(z)\end{aligned}$$

Set

$$\begin{aligned}K(z,\;w) &= \frac{1}{(1-|w|^2)^{\beta/2}\,(1-|z|^2)^{\beta/2}\,|\,1-w\overline{z}\,|^{\alpha+1}}\\ L(z) &= k(z)(1-|z|^2)^{\beta/2},\end{aligned}$$

then $\left\| \widehat{k} \right\|_{\alpha}^{2} = \int\limits_{\mathbb{D}} \int\limits_{\mathbb{D}} L(w)\overline{L(z)}K(z,\ w)\ dA(z)dA(w)$ which implies

$$\left\| \widehat{k} \right\|_{\alpha}^{2} \quad \leq \quad \int\limits_{\mathbb{D}} |L(w)| \int\limits_{\mathbb{D}} |\ L(z)\ | K(z,\ w)\ dA(z)dA(w)$$

giving (5.7.5.1) which is equivalent to:

$$\int\limits_{\mathbb{D}} |L(w)| \int\limits_{\mathbb{D}} |\ L(z)\ | K(z,\ w)\ dA(z)dA(w) \ \leq\ C_0 \int\limits_{\mathbb{D}} |L(z)|^2\ dA(z) \tag{5.7.5.2}$$

To prove (5.7.5.2) let's define:

$$Tf(w) \quad = \quad \int\limits_{\mathbb{D}} f(z)\ K(z,\ w)\ dA(z),$$

Note that if T is a bounded integral operator, then we have

$$\begin{aligned}
\left\| \widehat{k} \right\|_{\alpha}^{2} \quad &\leq \quad \int\limits_{\mathbb{D}} |\ L(w)\ | \int\limits_{\mathbb{D}} |\ \overline{L(z)}\ | K(z,\ w)\ dA(z)dA(w) \\
&= \quad \int\limits_{\mathbb{D}} |L(w)|\, T(|L(w)|)dA(w) \\
&= \quad \langle |L|,\ T(|L|) \rangle_2 \\
&\leq \quad \|\ L\ \|\ \|\ T\ \|\ \|\ L\ \| = \|\ T\ \|\ \|\ L\ \|_2^2 \\
&\leq \quad \|\ T\ \| \int\limits_{\mathbb{D}} |\ L(z)\ |^{\ 2}\ dA(z)
\end{aligned}$$

which gives (5.7.5.2) in case $\|T\| = C_0$ and so proving Lemma 5.7.5.

$\square$

It is left to justify, T is a bounded integral operator. The following Lemma will do that.

Lemma 5.7.6. Let $k(z)$ be smooth on $\mathbb{D}(1+\epsilon)$, $L(z) = k(z)(1-|z|^2)^{\beta/2}$ where $\beta = 1-\alpha$. Let $K(z, w) = \frac{1}{(1-|w|^2)^{\beta/2}\,(1-|z|^2)^{\beta/2}\,|1-w\overline{z}|^{\alpha+1}}$, and $TL(w) = \int_{\mathbb{D}} \left| \overline{L(z)} \right| K(z, w)\, dA(z)$. If there is a positive function P such that $\int_{\mathbb{D}} P(z)\, K(z, w)\, dA(z) \leq C_0\, P(w)$ for some constant C_0, then the integral operator T is bounded with $\| T \| = C_0$.

Proof. : An immediate consequence of Schur's Theorem.

□

To use Lemma 5.7.6. for our purpose we must find a $P(z) > 0$ such that for some constant C_0

$$\int_{\mathbb{D}} P(z)\, K(z, w)\, dA(z) = \int_{\mathbb{D}} \frac{P(z)}{(1-|w|^2)^{\beta/2}\,(1-|z|^2)^{\beta/2}\,|1-w\overline{z}|^{1+\alpha}}\, dA(z)$$

$$\leq C_0 P(w) \qquad (5.7.5.3)$$

which implies that

$$\int_{\mathbb{D}} \frac{P(z)}{(1-|z|^2)^{\beta/2}\,|1-w\overline{z}|^{1+\alpha}}\, dA(z) \leq C_0 P(w)\,(1-|w|^2)^{\beta/2}$$

Set $P(w) = 1/(1-|w|^2)^{\gamma}$ for some $\gamma > \beta/2$ and $t = -(\beta/2 + \gamma)$, then

$$\int_{\mathbb{D}} \frac{P(z)}{(1-|z|^2)^{\beta/2}\,(1-w\overline{z})^{\alpha+1}}\,dA(z) = \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{\beta/2+\gamma}\,|\,1-w\overline{z}\,|^{\alpha+1}}\,dA(z)$$

$$= \int_{\mathbb{D}} \frac{(1-|z|^2)^{-(\beta/2+\gamma)}}{|\,1-w\overline{z}\,|^{\alpha+1}}\,dA(z)$$

$$= \int_{\mathbb{D}} \frac{(1-|z|^2)^{t}}{|\,1-w\overline{z}\,|^{\,2+t+(-1-t+\alpha)}}\,dA(z)$$

By Lemma 5.7.2. we require

(i) $t > -1$ that is $-(\beta/2 + \gamma) > -1$ which is equivalent to $\beta/2 + \gamma < 1$, and

(ii) $\gamma > \beta/2$.

Question is, does such γ exists for $\beta \in (0, 1) \equiv \beta/2 \in (0, 1/2)$?

The answer is **yes**.

For example a $\gamma \in (1/2,\ 1-\beta/2)$ given by $\gamma = (\beta/2 + 1/2)/2 = \frac{\beta+1}{4}$ works.

So, such a γ always exists.

Now to justify (5.7.5.3) let

$$I_{c,t}(w) = \int_{\mathbb{D}} \frac{P(z)}{(1-|z|^2)^{\beta/2}\,|\,1-w\overline{z}\,|^{\alpha+1}}\,dA(z)$$

$$= \int_{\mathbb{D}} \frac{(1-|z|^2)^{t}}{|\,1-w\overline{z}\,|^{\,2+t+(-1-t+\alpha)}}\,dA(z),$$

where $c = -1 - t + \alpha$

Note that

(i) $\quad t = -(\beta/2 \; + \; \gamma) > -1$, and

(ii) $\quad c = -1 - t + \alpha \; = \; -1 + \frac{1-\alpha}{2} \; + \gamma \; + \; \alpha$

$$= \; -1/2 \; + \; \alpha/2 \; + \; \gamma \; = \; -\tfrac{1-\alpha}{2} \; + \; \gamma$$

$$= \; \tfrac{-\beta}{2} \; + \; \gamma \; > \; 0.$$

Thus applying Lemma 5.7.2, we get:

$$I_{c,\,t}(w) \; \sim \; 1/(1 - \mid w \mid^2)^{\gamma-\beta/2} \; = \; P(w)(1 - \mid w \mid^{\beta/2})$$

that is

$$I_{c,t}(w) \; \sim \; 1/(1 - \mid w \mid^2)^{\gamma-\beta/2}$$

which implies that for some C_0

$$I_{c,\,t}(w) \;\; \leq \; C_0 \; P(w)(1 - \mid w \mid^2)^{\beta/2}$$

Hence,

$$\int\limits_{\mathbb{D}} \; P(z) \; K(z, \, w) \; dA(z) \; \leq \; C_0 \; P(w)$$

This justifies (5.7.5.3)

Next we would give approximation for $\| \, T \, \| \; = \; C_0$. This will require several lemmas, but first we state a lemma for the proof refer P. Duren & A. Schuster [DS].

Lemma 5.7.7. Let s and t be real numbers satisfying $1 < t < s$. Then there is a constant C, depending only on s and t, such that

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{t-2}}{|1-z\overline{w}|^s}\, dA(z) \;\le\; C(1-|w|^2)^{t-s} \qquad \text{for all } w \in \mathbb{D}. \tag{5.7.7.1}$$

To apply Lemma 5.7.7. for our case we need to consider $I_{c,t}(w)$ of Lemma 5.7.2.

$$I_{c,t}(w) \;=\; \int_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-w\overline{z}|^{1+\alpha}}\, dA(z)$$

where $t = -(\beta/2 + \gamma) > -1$, and $\beta/2 = (1-\alpha)/2$.

In terms of (5.7.7.1)

$$I_{c,t}(w) \;=\; \int_{\mathbb{D}} \frac{(1-|z|^2)^{t_0-2}}{|1-w\overline{z}|^s}\, dA(z)$$

where $t_0 = t+2$ and $s = 1+\alpha$.

Lemma 5.7.8. Let $t_0 = t+2$, $s = 1+\alpha$, $I_{c,t}(w) \;= \int_{\mathbb{D}} \frac{(1-|z|^2)^{t_0-2}}{|1-w\overline{z}|^s}\, dA(z)$, $t > -1$, and $\beta/2 = (1-\alpha)/2$. Then there is a constant C_0 depending only on α such that

$$I_{c,t}(w) \;\le\; C_0\,(1-|w|^2)^{t_0-s}.$$

PROOF. : By Lemma 5.7.7 we need only verify $1 < t_0 < s$.

To show that we will prove $(a)\quad t_0 - 1 \,>\, 0$ *and* $(b)\quad s - t_0 \,>\, 0$

$$(a)\quad t_0 - 1 = t + 2 - 1 = -(\beta/2 + \gamma) + 1 \,>\, 0 \text{ since in } I_{c,\,t}(w)$$

$$-(\beta/2 + \gamma) \,>\, -1$$

$$(b)\quad s - t_0 = 1 + \alpha - t - 2 = \alpha + \beta/2 + \gamma - 1$$

$$= \alpha + (1 - \alpha)/2 + \gamma - 1 \;=\; \tfrac{1}{2}\alpha + \gamma - 1/2 \,>\, 0$$

since γ can be chosen to be greater than $\frac{1}{2}\,(1 - \alpha)$.

This proves $1 \,<\, t_0 \,<\, s$.

Thus by Lemma 5.7.7. there is a C_0 depending on t_0 and s satisfying

$$I_{c,\,t}(w) \;\leq\; C_0\,(1 - |\,w\,|^2)^{t_0 - s}.$$

Left to show C_0 depends only on α; but then, since $s = \,1 + \alpha$, $t = -(\beta/2\, + \gamma)$ and $t_0 =\, t + 2$ where γ can be made to be dependent on $\beta = 1 - \alpha$, the result follows.

□

Given $I_{c,\,t}(w) \;\leq\; C_0\,(1 - |\,w\,|^2)^{t_0 - s}$, to give an approximation to C_0 in terms of α; it suffice to consider the case for $w \,=\, \rho \,>\, 0$. There are two cases:

$(i)\quad \rho \,\leq\, 1/2$ and

$(ii)\quad \rho \,>\, 1/2$.

Next we give estimates for (i) and (ii)

$(i)\quad \rho \,\leq\, 1/2$

By Lemma 5.7.7 $\quad I_{c,\,t}(w) \;\leq\; C_1\,(1 - |\,w\,|^2)^{t_0 - s} = C_1\,(1 - \rho^2)^{t + 2 - s}$, where C_1 is given by

$$\begin{aligned} C_1 &= 2^{t+2} \int_{\mathbb{D}} (1 - |\, z\,|^2)^t dA(z) \\ &= \frac{2^{t+2}}{\pi} \int_0^{2\pi} \int_0^1 (1-r^2)^t r dr d\theta \\ &= \frac{2^{t+2}}{t+1} \end{aligned}$$

(ii) $\quad \rho > 1/2$ equivalent to $\frac{1}{2\rho} < 1$, two sub-cases $|z| \leq 1/2\rho$ and $|z| > 1/2\rho$

First $|z| \leq 1/2\rho$; $\frac{1}{2^s} = |\, 1 - \rho \frac{1}{2\rho}\,|^s \leq |\, 1 - \rho z\,|^s$.

By (b) of Lemma 5.7.8. $t_0 - s < 0$, hence

$$\begin{aligned} \int\limits_{|z| \leq 1/2\rho} \frac{(1-|z\,|^2)^t}{|1-\rho z\,|^s} dA(z) &\leq 2^s \int\limits_{\mathbb{D}} (1-|z\,|^2)^t \, dA(z) \\ &\leq C_2 (1-\rho)^{t_0 - s} \end{aligned}$$

where $C_2 = 2^s \int\limits_{\mathbb{D}} (1 - |z\,|^2)^t dA(z) = 2^s/(t+1)$

Left to consider the case

$$\int\limits_{|z|>1/2\rho} \frac{(1-|z\,|^2)^t}{|1-\rho z\,|^s} dA(z) = \frac{2}{\pi} \int\limits_{1/2\rho}^{1} \int_0^\pi \frac{(1-r^2)^t}{(1-2\rho\, r\, cos\theta + \rho^2 r^2)^{s/2}} r dr d\theta$$

Since, $0 \leq x \leq \pi/2$ implies $sinx \geq 2x/\pi$, we have

$$1 - 2\rho\, r\, cos\theta + \rho^2\, r^2 = (1-\rho\, r)^2 + 4\rho\, r\, sin^2(\theta/2) \geq (1-\rho\, r)^2 + 4\rho\, r\, \theta^2/\pi^2.$$

This implies

$$\int_0^\pi \frac{(1-r^2)^t}{(1-2\rho\, r\, cos\theta + \rho^2 r^2)^{s/2}} r dr d\theta \leq \frac{1}{(1-\rho\, r)^s} \int\limits_0^\pi \frac{d\theta}{[1 + \frac{2}{\pi^2}(\frac{\theta}{1-\rho\, r})^2]^{s/2}}.$$

Next we will prove a lemma which basically gives an upper bound for the integral on the right hand side of the above inequality.

Lemma 5.7.9. Let $s = 1+\alpha$, $\rho > 1/2$, and $r > 1/2\rho$, then

$$\int_0^{\pi} \frac{d\theta}{[1+\frac{2}{\pi^2}(\frac{\theta}{1-\rho r})^2]^{s/2}} \leq \frac{\pi^2}{\alpha}(1-\rho r).$$

Proof. Let $u = \frac{\theta}{1-\rho r}$, then

$$\int_0^{\pi} \frac{d\theta}{[1+\frac{2}{\pi^2}(\frac{\theta}{1-\rho r})^2]^{s/2}} = (1-\rho r) \int_0^{\pi/1-\rho r} \frac{du}{[1+\frac{2}{\pi^2}(u)^2]^{s/2}}$$

So, we are done if we show

$$\int_0^{\pi/1-\rho r} \frac{du}{[1+\frac{2}{\pi^2}(u)^2]^{s/2}} \leq \pi^2/\alpha$$

$$\int_0^{\pi/1-\rho r} \frac{du}{[1+\frac{2}{\pi^2}(u)^2]^{s/2}} \leq \int_0^{\infty} \frac{du}{[1+\frac{2}{\pi^2}(u)^2]^{s/2}}$$

$$\int_0^{\infty} \frac{du}{[1+\frac{2}{\pi^2}(u)^2]^{s/2}} = \frac{\pi}{\sqrt{2}} \int_0^{\infty} \frac{dv}{[1+v^2]^{s/2}}$$

$$
\begin{aligned}
\frac{\pi}{\sqrt{2}}\int_0^\infty \frac{dv}{[1+v^2]^{s/2}} &= \frac{\pi}{\sqrt{2}(s-1)}\left[s\int_0^\infty \frac{dv}{[1+v^2]^{s/2+1}} - \left\lfloor \frac{v}{(1+\ v^2)^{s/2}}\right\rfloor_0^\infty\right] \\
&\leq \frac{\pi}{\sqrt{2}(s-1)}\left[s\int_0^\infty \frac{dv}{[1+v^2]^{1}} - \left\lfloor \frac{v}{v^s(1+1/v^2)^{s/2}}\right\rfloor_0^\infty\right] \\
&\leq \frac{\pi\, s}{\sqrt{2}\,(s-1)}\,(\pi/2)\,-\,0 \\
&\leq \frac{\pi^2\,(\alpha+1)}{2\sqrt{2}\,(\alpha+1-1)} \leq \frac{\pi^2}{\sqrt{2}\ \alpha} \leq \frac{\pi^2}{\alpha}
\end{aligned}
$$

□

Lemma 5.7.9. together with

$$
\int_0^\pi \frac{(1-r^2)^t}{(1-2\rho\, r\, cos\theta+\rho^2 r^2)^{s/2}}\, rdrd\theta \leq \frac{1}{(1-\rho\, r)^s}\int_0^\pi \frac{d\theta}{[1+\frac{2}{\pi^2}(\frac{\theta}{1-\rho\, r})^2]^{s/2}}
$$

implies

$$
\int_0^\pi \frac{(1-r^2)^t}{(1-2\rho\, r\, cos\theta+\rho^2 r^2)^{s/2}}\, rdrd\theta \leq \frac{1}{(1-\rho\, r)^s}\,\frac{\pi^2}{\alpha}(1-\rho\, r)\;=\;\frac{\pi^2}{\alpha}\,(1-\rho\, r)^{1-s}.
$$

Hence,

$$
\begin{aligned}
\int_{|z|\,>\,1/2\rho} \frac{(1-|z\,|^2)^t}{|1-\rho z\,|^s}\, dA(z) &= \frac{2}{\pi}\int_{1/2\rho}^1 \int_0^\pi \frac{(1-r^2)^t}{(1-2\rho\, r\, cos\theta+\rho^2 r^2)^{s/2}}\, rdrd\theta \\
&\leq \frac{2}{\pi}\,\frac{\pi^2}{\alpha}\int_{1/2\rho}^1 \frac{(1-r^2)^t}{(1-\rho\, r\,)^{s-1}}\, rdr \\
&\leq \frac{2\pi}{\alpha}\int_{1/2\rho}^1 \frac{(1-r^2)^t}{(1-\rho\, r\,)^{s-1}}\, rdr
\end{aligned}
$$

Our next task is to approximate the integral $\int_{1/2\rho}^{1} \frac{(1-r^2)^t}{(1-\rho r)^{s-1}}\, rdr$ from above.

First recall $t = -(\beta/2 + \gamma)$, $s = 1+\alpha \Leftrightarrow t+2-s < 0$. And $\rho > 1/2$ it holds

$$\int_{1/2\rho}^{1} \frac{(1-r^2)^t}{(1-\rho r)^{s-1}}\, rdr \le \underbrace{\int_{0}^{\rho} \frac{(1-r^2)^t}{(1-\rho r)^{s-1}}\, rdr}_{(I_1)} + \underbrace{\int_{\rho}^{1} \frac{(1-r^2)^t}{(1-\rho r)^{s-1}}\, rdr}_{(I_2)}.$$

It can be easily shown that $0 < r < \rho \Rightarrow \rho r > r^2 \Rightarrow 1-\rho r < 1-r^2$, thus

$$(I_1) \le \frac{2}{s-(t+2)} (1-\rho)^{t+2-s},$$

and $\rho < r < 1 \Rightarrow \rho^2 < r^2 \Rightarrow 1-\rho^2 > 1-r^2$, thus

$$(I_2) \le \frac{1}{2(t+1)} (1-\rho)^{t+2-s}$$

So, we have

$$\begin{aligned}\int_{1/2\rho}^{1} \frac{(1-r^2)^t}{(1-\rho r)^{s-1}}\, rdr &\le \frac{2}{s-(t+2)} (1-\rho)^{t+2-s} + \frac{1}{2(t+1)} (1-\rho)^{t+2-s} \\ &\le \left(\frac{2}{s-(t+2)} + \frac{1}{2(t+1)}\right) (1-\rho)^{t+2-s}\end{aligned}$$

Thus, for case (ii) $\quad \rho > 1/2$

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-\rho z|^s}\, dA(z) &= \int_{|z|\le 1/2\rho} \frac{(1-|z|^2)^t}{|1-\rho z|^s}\, dA(z) + \int_{|z|>1/2\rho} \frac{(1-|z|^2)^t}{|1-\rho z|^s}\, dA(z) \\ &\le \left[\frac{2^s}{t+1} + \frac{2\pi}{\alpha}\left(\frac{2}{s-(t+2)} + \frac{1}{2(t+1)}\right)\right] (1-\rho)^{t+2-s}\end{aligned}$$

Notice that: $\alpha \in (0, 1)$, $s = 1+\alpha$, $\beta/2 = (1-\alpha)/2$, $t = -(\beta/2+\gamma)$, where γ is chosen to be in $(\beta/2, 1-\beta/2) = ((1-\alpha)/2, 1-(1-\alpha)/2)$. In particular if we take $\gamma \in \left(\frac{1}{2}(1-\alpha), \frac{1}{2}(1+\alpha)\right)$ we get,

$$\frac{2^{t+2}}{t+1} = \frac{2^{5/2+\alpha/2-\gamma}}{1+\alpha-2\gamma} \quad \textit{and}$$

$$\frac{2^s}{t+1} + \frac{2\pi}{\alpha}\left(\frac{2}{s-(t+2)} + \frac{1}{2(t+1)}\right) = \frac{2^{2+\alpha}}{1+\alpha-2\gamma} + \frac{2\pi}{\alpha}\left(\frac{4}{\alpha+2\gamma-1} + \frac{1}{1+\alpha-2\gamma}\right)$$

Also for such a choice of γ it holds $\alpha+2\gamma-1 > 0$, $1+\alpha-2\gamma > 0$, and it is readily seen that $2^{2+\alpha} > 2^{\frac{1}{2}(5+\alpha)-\gamma}$, which implies $\frac{2^{t+2}}{t+1} < \frac{2^s}{t+1}$. Thus the

$$\max\left\{\frac{2^{t+2}}{t+1}, \quad \frac{2^s}{t+1} + \frac{2\pi}{\alpha}\left(\frac{2}{s-(t+2)} + \frac{1}{2(t+1)}\right)\right\} = \frac{2^s}{t+1} + \frac{2\pi}{\alpha}\left(\frac{2}{s-(t+2)} + \frac{1}{2(t+1)}\right)$$
$$= \frac{2^{2+\alpha}}{1+\alpha-2\gamma} + \frac{2\pi}{\alpha}\left(\frac{4}{1\alpha+2\gamma-1} + \frac{1}{1+\alpha-2\gamma}\right)$$

And so we have the following theorem proved.

Theorem 5.7.10. Let $\alpha \in (0, 1)$, $s = 1+\alpha$, $t = -[(1-\alpha)/2+\gamma]$ where γ is chosen to be in $\left(\frac{1}{2}(1-\alpha), \frac{1}{2}(1+\alpha)\right)$, and $I_{c,t}(w) = \int_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-w\bar{z}|^s}\, dA(z)$, then $I_{c,t}(w) = C_0(1-|w|^2)^{t+2-s}$, where

$$C_0 = \ max\left\{\frac{2^{t+2}}{t+1}, \quad \frac{2^s}{t+1} + \frac{2\pi}{\alpha}\left(\frac{2}{s-(t+2)} + \frac{1}{2(t+1)}\right)\right\}$$
$$= \frac{2^{1+\alpha}}{1+\alpha-2\alpha} + \frac{2\pi}{\alpha}\left(\frac{4}{\alpha+2\gamma-1} + \frac{1}{1+\alpha-2\gamma}\right).$$

We are now ready to give the proof of Theorem A' in the smooth case.

Theorem A' Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$, $(0 < \alpha < 1)$. Assume that $\left\| M_F^C \right\| \leq 1$, and $0 < \epsilon^2 \leq \sum_{j=1}^{\infty} | f_j(z) |^2$ *for all* $z \in \mathbb{D}$. Then there is a positive number $C(\epsilon, \alpha)$ and for all $h \in \mathcal{D}_\alpha$ there exists $\mathbf{U}_h \in \bigoplus_1^{\infty} \mathcal{D}_\alpha$ such that

$$(i)\ \ M_F^R(\mathbf{U}_h) = h$$

$$(ii)\ \ \| \mathbf{U}_h \|_{\bigoplus_1^{\infty} \mathcal{D}_\alpha} \leq C(\epsilon, \alpha) \| h \|_{\mathcal{D}_\alpha}$$

Proof. Assume that $\{f_j\}_{j=1}^{\infty}$ are analytic in $| z | < 1 + \delta$, $\forall j$, $\left\| M_F^C \right\| \leq 1$,

and $0 < \epsilon^2 \leq F(z)F(z)^* \;\; \forall z \in \mathbb{D}$, where $F(z) = (f_1(z),\ f_2(z),\ \ldots\ )$.

Let h be analytic in $| z | < 1 + \delta$. Define

$$\underline{U}_h = F^*(FF^*)^{-1}h - Q\left(\frac{\widehat{Q^*\, F'^*}\, h}{(FF^*)^2} \right) \qquad (*)$$

point wise on $\overline{\mathbb{D}}$.

By Lemma 5.6.5, the entries of $Q(z)$ are either $-f_j$, 0, or f_j. So, $\overline{\partial}\underline{U}_h(z) = \underline{0}$ *for* $z \in D$. Thus we need only show that

$$\| \mathbf{U}_h \|_{\bigoplus_1^{\infty} \mathcal{D}_\alpha} \leq C(\epsilon, \alpha) \| h \|_{\mathcal{D}_\alpha}$$

Taking norms in (*) above we get

$$\|\underline{U}_h\|_{\bigoplus_1^{\infty} \mathcal{D}_\alpha} = \left\| F^*(FF^*)^{-1}h + Q\,\frac{\widehat{Q^*\, F'^*}\, h}{(FF^*)^2} \right\|_{\bigoplus_1^{\infty} \mathcal{H}\mathcal{D}_\alpha}$$

$$\|\underline{U}_h\|_{\bigoplus_1^{\infty} \mathcal{D}_\alpha} \leq \left\| F^*(FF^*)^{-1}h \right\| + \left\| Q\,\frac{\widehat{Q^*\, F'^*}\, h}{(FF^*)^2} \right\|_{\bigoplus_1^{\infty} \mathcal{H}\mathcal{D}_\alpha}$$

$$
\begin{aligned}
&\le \frac{86\sqrt{200}}{\epsilon^2}\,\|\,h\,\|_{\mathcal{D}_\alpha} + \sqrt{86}\left\|\frac{\widehat{Q * F' * h}}{(FF*)^2}\right\|_{\bigoplus_1^\infty \mathcal{H}\mathcal{D}_\alpha} \\
&\le \frac{86\sqrt{200}}{\epsilon^2}\,\|\,h\,\|_{\mathcal{D}_\alpha} + \sqrt{86}\left\|\widehat{k}\right\|_{\bigoplus_1^\infty \mathcal{H}\mathcal{D}_\alpha} \\
&\le \frac{86\sqrt{200}}{\epsilon^2}\,\|\,h\,\|_{\mathcal{D}_\alpha} + \sqrt{86}\,\|T\|\,\|\,k\,\|_{A_\alpha} \\
&\le \frac{86\sqrt{200}}{\epsilon^2}\,\|\,h\,\|_{\mathcal{D}_\alpha} + \sqrt{86}\,\|T\|\,\frac{\sqrt{20}}{\epsilon^3}\,\|h\|_{\mathcal{D}_\alpha} \\
&\le \left(\frac{86\sqrt{200}}{\epsilon^2} + \sqrt{86}\,C_0\frac{\sqrt{20}}{\epsilon^3}\right)\|\,h\,\|_{\mathcal{D}_\alpha}
\end{aligned}
$$

where $C(\epsilon,\alpha) = \left(\frac{86\sqrt{200}}{\epsilon^2} + \sqrt{86}\frac{\sqrt{20}}{\epsilon^3}\,C_0\right)$. □

We will show that the same estimates hold for the general case. To this end, first we will state two lemmas: Lemma 5.7.11 and Lemma 5.7.12. for their proof we will refer Trent [Tr2] with the following observation. We found a slight error in the proof of the second lemma in Trent [Tr2] with that fixed both lemmas holds true for any reproducing kernel with one positive square on the unit disk.

Lemma 5.7.11. Let $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$ with $\left\| M_F^C \right\|_\alpha \leq 1$. For $0 \leq r \leq 1$. Let $F_r(z) = F(rz)$. Then $\left\| M_{F_r}^C \right\|_\alpha \leq \left\| M_F^C \right\|_\alpha$ and thus $F_r \in M(\mathcal{D}_\alpha, \bigoplus_1^\infty \mathcal{D}_\alpha)$.

Lemma 5.7.12. Let $\mathcal{F} \in M(\bigoplus_1^\infty \mathcal{D}_\alpha)$. Then $s - \lim_{r \to 1^-} M_{\mathcal{F}_r}^* = M_{\mathcal{F}}^*$.

That is $M_{\mathcal{F}_r}^*$ converges strongly to $M_{\mathcal{F}}^*$ as $r \to 1^-$.

Now we are in a position to prove Theorem A' and hence the $\mathcal{D}_\alpha$ Corona theorem for the general case.

Proof. Let $\{f_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$, $\left\| M_F^C \right\| \leq 1$ and $0 < \epsilon^2 \leq F(z)F(z)^*$

for all $z < 1$

By Lemma 5.7.11. for $0 \leq r < 1$, we have $\left\| M_{F_r}^C \right\| \leq 1$ and

$0 < \epsilon^2 \leq F_r(z)F_r(z)^*$ for all $z < 1$.

By the proof of Theorem A' in the smooth case we have

$(C(\epsilon))^{-2} I \leq M_{F_r}^R (M_{F_r}^R)^* \leq I$.

By Theorem B, $\exists\, G_r \in M(\mathcal{D}_\alpha, \bigoplus_1^\infty \mathcal{D}_\alpha)$ so that

$M_{F_r}^R (M_{G_r}^C)^* = I$ and $\left\| (M_{G_r}^C) \right\| \leq C(\epsilon)$

By compactness we may choose a net with $G_{r_x}^* \overset{WOT}{\rightarrow} G^*$ as $r_x \to 1^-$.

Note, $G \in M(\mathcal{D}_\alpha, \bigoplus_1^\infty \mathcal{D}_\alpha)$, since the multiplier algebra (as operators) is weak operator closed (WOT).

Now by Lemma 5.8.12. $F_{r_x}^* \overset{s}{\rightarrow} F^*$. thus we get

$I = G_{r_x}^* F_{r_x}^* \overset{WOT}{\rightarrow} G^* F^*$ which implies $FG = I$.

with entries of G in $\mathcal{M}(\mathcal{D}_\alpha)$ and $\left\| M_G^C \right\| \leq C(\epsilon)$

This ends the proof of the $\mathcal{D}_\alpha$ Theorem . □

In Summary; we proved the $\mathcal{D}_\alpha$ **Corona Theorem**

"If $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$, $\left\| M_F^C \right\| \leq 1$ and $0 < \epsilon^2 \leq \sum_{j=1}^{\infty} | f_j(z) |^2$ for all $z \in \mathbb{D}$, then there exists a positive number $C(\epsilon, \alpha)$, depending on ϵ and α, and a $\{g_j\}_{j=1}^{\infty} \subseteq \mathcal{M}(\mathcal{D}_\alpha)$ such that

$$(i) \quad \sum_{j=1}^{\infty} f_j g_j = 1 \quad \text{and} \quad (ii) \ \left\| M_G^C \right\| \leq C(\epsilon, \alpha) \text{''}$$

by establishing two theorems, **Theorem A** and **Theorem B**.

We showed that Theorem A is equivalent to Theorem A'. That is, given the hypothesis of Theorem A, we proved that for every $h \in \mathcal{D}_\alpha$ there is a $\underline{\mathbf{U}}_h \in \bigoplus_1^\infty \mathcal{D}_\alpha$ such that

$$F \underline{\mathbf{U}}_h = h \text{ and } \| \mathbf{U}_h \|_{\bigoplus_1^\infty \mathcal{D}_\alpha} \leq C(\epsilon, \alpha) \| h \|_{\mathcal{D}_\alpha}$$

where the estimates

$$C(\epsilon, \alpha) \leq \left(\frac{86\sqrt{200}}{\epsilon^2} + \sqrt{86} \, \frac{\sqrt{20}}{\epsilon^3} \, C_0 \right)$$

and C_0 is as in Theorem 5.7.10.

And the CLT, by establishing Theorem B, finishes the proof of the Corona Theorem for the space $\mathcal{D}_\alpha$.

6. FUTURE RESEARCH

We close this dissertation paper by posing some immediate problems for future research.

An immediate future problems include: the Matrix-Valued corona Problem for the $\mathcal{D}_\alpha$ spaces with appropriate weights.

In this thesis, we proved the weighted Corona theorem for infinitely many functions. We can think of the infinitely many functions just as a row matrix. Question, can we extend this to a matrix case? The answer is yes.

In fact, by results of Trent Zhang [TZ], we can immediately get a matrix corona theorem. However, the general result of Trent Zhang [TZ] lead to bad estimates in our case. We believe that we can get better estimates on solutions for the matrix corona problem on weighted Dirichlet space.

Another possible future research problem is, "*Corona Theorems*" on the space $H_\alpha(\mathbb{D})$ for a general choices of the weight α.

$$H_\alpha(\mathbb{D}) \triangleq \{f \ : f \in Hol(\mathbb{D}),\ f(z) = \sum_{n=0}^{\infty} f_n z^n,\ \sum_{n=0}^{\infty} \alpha_n | \, f_n \, |^2 < \infty\}$$

Final remark:

The several variable version of this dissertation has recently been solved by Arocena, Sawyer and Wick. These paper should appear in Acta Mathematica.

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