A REFINEMENT OF WOLFF TYPE THEOREMS FOR THE MULTIPLIER ALGEBRAS OF DIRICHLET SPACES

by

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A DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the Graduate School of
The University of Alabama

TUSCALOOSA, ALABAMA

2016

ABSTRACT

In 2014, Banjade and Trent proved an analog of Wolff's theorem for the multiplier algebra of the Dirichlet space, which gives conditions for the cube of a function to be in a generated ideal.

This dissertation investigates conditions that guarantee the function itself is in the ideal. This result is then extended to the multiplier algebra of the weighted Dirichlet space, which is also seen as a refinement of a similar result by Banjade and Trent, but for the weighted space.

Following this, discussion takes place on how to generalize the results obtained in this dissertation.

LIST OF SYMBOLS

$A^2(\mathbb{D})$	Bergman space: analytic functions on the unit disk with square summable coefficients and weight $(n+1)^{-1}$
$A^\star$	Adjoint of the operator A
$B(H)$	Bounded linear operators from H to H
$\mathbb{C}$	Complex plane
$\mathbb{D}$	Open unit disk in the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert < 1\}$
$\mathcal{D}$	Dirichlet space: analytic functions on the unit disk with square summable coefficients and weight $(n+1)$
$\mathcal{D}_\alpha$	Weighted Dirichlet space: analytic functions on the unit disk with square summable coefficients and weight $(n+1)^\alpha$
Δ	Nonzero multiplicative linear functionals on $H^\infty(\mathbb{D})$
∂	Partial derivative with respect to z
$\bar{\partial}$	Partial derivative with respect to $\bar{z}$
$\partial\mathbb{D}$	Boundary of the unit disk, $\partial\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$H^2(\mathbb{D})$	Hardy space: analytic functions on the unit disk with square summable coefficients
$H^\infty(\mathbb{D})$	Bounded analytic functions on $\mathbb{D}$
$\mathcal{I}\left(\{f_j\}_{j=1}^n\right)$	Ideal generated by $\{f_j\}_{j=1}^n$
k_w	Reproducing Kernel (kernel)
M_ϕ	Multiplication operator associated to ϕ
$\mathcal{M}(\mathcal{H})$	Multiplier Algebra of the RKHS $\mathcal{H}$
RKHS	Reproducing Kernel Hilbert Space

ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor, Hyun Kwon, who not only provided guidance so that this dissertation became a reality, but also outlined the whole process with extreme clarity.

I would like to thank Dr. Trent for suggesting the problems that were ultimately solved in this dissertation.

I would like to thank the committee members, Dr. Ferguson, Dr. Moen, Dr. Moore, and Dr. Roskos for their support during this process, Dr. Moen, in particular, for the semester reading course that not only was a great learning experience about math, but laid the groundwork for me using Latex, which made typing this dissertation possible.

Various professors had a profound influence on the way I go about things. From my undergraduate university, I would like to specifically thank Dr. Abraham (who has since moved to Ohio University), Dr. Alexopoulos, and Dr. Kasturiarachi. From the University of Alabama, I would like to specifically thank Dr. Corson, Dr. Evans, and Dr. Knese (who has since moved to University of Washington).

I would like to thank my cat, Shynx, who never seemed to judge me when I was frustrated, unable to figure something out.

Finally, I would like to thank the shade of my heart, Alexandra.

CONTENTS

CHAPTER 1

PRELIMINARIES

1.1. Reproducing Kernel Hilbert Spaces (RKHS)

Before diving into the theorems to be proven, it is important to establish some preliminary notation and results to give appropriate context. The first of these topics are reproducing kernel Hilbert spaces. For a more in depth discussion on these topics, the reader may consult [**AM**] and [**S**].

1.1.1. Reproducing Kernel Hilbert Spaces. Suppose $H(X)$ is a Hilbert space of analytic functions on a set X satisfying

(i) for all $x \in X$, there exists $c_x < \infty$ so that $|h(x)| \leq c_x ||h||_H$, for all $h \in H$ and

(ii) $h(x) = 0$ for all $x \in X \Rightarrow h = 0$ in H,

then $H(X)$ is said to be a reproducing kernel Hilbert space on the set X.

1.1.2. Reproducing Kernel. Notice that the definition of a RKHS on a set X is simply a Hilbert space of functions on X, such that for every $x \in X$, point evaluation is a bounded linear functional on $H(X)$. Thus, by the Riesz representation theorem, for all $x \in X$, there exists a unique k_x, such that

$$h(x) = \langle h, k_x \rangle \text{ for all } h \in H(X).$$

The function k_x is called the reproducing kernel, or more briefly, the kernel for the RKHS $H(X)$.

Another fundamental property that RKHS can easily be seen to have from the definition is that convergence in norm implies pointwise convergence.

Some examples of RKHS are the following:

1.1.2.1. *Example 1.* The Hardy space $H^2(\mathbb{D})$, with inner product defined by

$$\langle f,\, g\rangle_{H^2} = \sum_{1}^{\infty} f_n \overline{g_n}, \text{ for all } f,\, g \in H^2(\mathbb{D}),$$

is a RKHS with kernel

$$k_w(z) = \frac{1}{1-\overline{w}z}, \text{ for all z, w} \in \mathbb{D}.$$

1.1.2.2. *Example 2.* The Dirichlet space $\mathcal{D}$, with inner product defined by

$$\langle f, g\rangle_{\mathcal{D}} = \sum_{1}^{\infty} (n+1)\, f_n \overline{g_n}, \text{ for all } f,\, g \in \mathcal{D},$$

is a RKHS with kernel

$$k_w(z) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1-z\overline{w}}\right), \text{ for all z, w} \in \mathbb{D}.$$

1.1.2.3. *Example 3.* The weighted Dirichlet space $\mathcal{D}_\alpha$, with $\alpha \in (0,1)$ and inner product defined by

$$\langle f, g\rangle_{\mathcal{D}_\alpha} = \sum_{n=0}^{\infty} (n+1)^\alpha\, f_n \overline{g_n}, \text{ for all } f,\, g \in \mathcal{D}_\alpha,$$

is a RKHS with kernel

$$k_w(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha} (z\overline{w})^n \text{ for } z,\, w \in \mathbb{D}.$$

1.1.2.4. *Example 4.* The Bergman space $A^2(\mathbb{D})$, with inner product defined by

$$\sum_{n=0}^{\infty} \frac{f_n \overline{g_n}}{(n+1)} \text{ for } z,\, w \in \mathbb{D},$$

is a RKHS with kernel

$$k_w(z) = \frac{1}{(1-\overline{w}z)^2}, \text{ for all } z, w \in \mathbb{D}.$$

1.1.3. Complete Nevanlinna-Pick Kernels. A kernel, k_w, is called a complete Nevanlinna-Pick kernel if it can be written as $\frac{1}{k_w(z)} = 1 - \sum_{n=1}^{\infty} a_n (z\overline{w})^n$, $a_n > 0$.

The Hardy space can easily be seen to have a complete Nevanlinna-Pick kernel, since

$$\frac{1}{k_w(z)} = 1 - z\overline{w}$$

It has been also been shown in [**AM**] and [**KT**] that both the Dirichlet space and weighted Dirichlet space possess complete Nevanlinna-Pick kernels. It is also worth noting that the Bergman space's reproducing kernel is not of this type.

1.2. Multiplier Algebras

1.2.1. Definitions. Let $H(\Omega)$ be a reproducing kernel Hilbert space of analytic functions on a domain, Ω. The multiplier algebra of $H(\Omega)$ is defined as the collection of all multipliers for $H(\Omega)$. That is,

$$\mathcal{M}(H(\Omega)) := \{\phi \text{ analytic on } \Omega \ : \ \phi f \in H(\Omega) \ \text{ for all } \ f \in H(\Omega)\}.$$

If the RKHS has the constant function 1 as a member, then $\mathcal{M}(H(\Omega)) \subset H(\Omega)$.

Similarly, M_ϕ is used to denote the multiplication operator defined by

$$M_\phi(f) = \phi f, \ \text{for all } f \in H(\Omega).$$

As mentioned previously, in a RKHS, convergence in norm implies pointwise convergence. A consequence of this is the following.

Lemma 1.2.1. *Let $H(\Omega)$ be a RKHS. Then ϕ is a multiplier if and only if M_ϕ is a bounded operator on $H(\Omega)$.*

Proof. If M_ϕ is a bounded operator on $H(\Omega)$, then it trivially follows that ϕ is a multiplier.

If ϕ is a multiplier, then suppose $\{f_n\}$ converges to $f \in H(\Omega)$ and that $\{M_\phi(f_n)\} = \{\phi f_n\}$ converges in norm to some $y \in H(\Omega)$. By the previous remark, $\{f_n\}$ converges pointwise to f and $\{\phi f_n\}$ converges pointwise to y, so that $y = \phi f = M_\phi(f)$. Thus, by the closed graph theorem, M_ϕ is bounded. $\square$

Since each multiplication operator M_ϕ is bounded, each of them have an adjoint, M_ϕ^*. However, much more can be said.

Lemma 1.2.2. *Let $H(\Omega)$ be a RKHS. If ϕ is a multiplier and k_w is the reproducing kernel for the RKHS $H(\Omega)$, then*

$$M_\phi^\star(k_w) = \overline{\phi(w)}k_w.$$

Moreover, ϕ is bounded on Ω, with

$$||\phi||_\infty \leq ||M_\phi||.$$

Proof. Let $f \in H(\Omega)$, so that

$$\langle f,\, M_\phi^\star\, k_w\rangle \;=\; \langle M_\phi f,\, k_w\rangle \;=\; \langle \phi f,\, k_w\rangle \;=\; \phi(w)f(w) = \langle f,\, \overline{\phi(w)}\, k_w\rangle.$$

Since this is true for all $f \in H(\Omega)$, it must be the case that $M_\phi^\star(k_w) = \overline{\phi(w)}k_w$.

Applying the norm to both sides, and using the facts that $||M_\phi^*|| = ||M_\phi||$ and M_ϕ is bounded gives

$$|\phi(w)| < ||M_\phi||, \text{ for all } w \in \Omega$$

Taking the supremum over all $w \in \Omega$ gives the result. □

The above lemma shows that if $H(\Omega)$ is a RKHS, then $\mathcal{M}(H(\Omega)) \subseteq H^\infty(\Omega)$. In the Hardy space, $H^2(\mathbb{D})$, the two are actually equal, since if $\phi \in H^\infty(\mathbb{D})$, then for all $f \in H^2(\mathbb{D})$,

$$\begin{aligned}
||\phi f||^2_{H^2(\mathbb{D})} &= \lim_{r\uparrow 1}\int_{-\pi}^{\pi} |(\phi f)(re^{it})|^2 \frac{d\sigma(t)}{2\pi} \\
&= \lim_{r\uparrow 1}\int_{-\pi}^{\pi} |\phi(re^{it})|^2 |f(re^{it})|^2 \frac{d\sigma(t)}{2\pi} \\
&\leq ||\phi||^2_{\infty,\mathbb{D}}\, \overline{\lim_{r\uparrow 1}}\int_{-\pi}^{\pi} |f(re^{it})|^2 \frac{d\sigma(t)}{2\pi} \\
&= ||\phi||^2_{\infty,\mathbb{D}}\; ||f||^2_{H^2(\mathbb{D})}.
\end{aligned}$$

This shows that every function in $H^\infty(\mathbb{D})$ is a member of $\mathcal{M}(H^2(\mathbb{D}))$. Using a similar argument, it can be shown that $\mathcal{M}(A^2(\mathbb{D})) = H^\infty(\mathbb{D})$.

1.2.2. Multipliers of $\mathcal{D}$ and $\mathcal{D}_\alpha$. The focus of this dissertation is on the Dirichlet space and weighted Dirichlet space, so a closer look into multipliers for these spaces is required. Before doing so, some equivalent norms must be mentioned.

Recall that the Dirichlet space is given by

$$\left\{ f : \mathbb{D} \to \mathbb{C} \ , \ f = \sum_{n=0}^{\infty} a_n z^n \ : \ ||f||_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)\, |a_n|^2 < \infty \right\}.$$

$||f||_{\mathcal{D}}$ is the norm of f, and of course $\mathcal{D} \subset H^2(\mathbb{D})$, as is seen by the inequality $||f||_{H^2(\mathbb{D})} \leq ||f||_{\mathcal{D}}$. An equivalent norm to $|| \cdot ||_{\mathcal{D}}$ is given by

$$||f||_{\mathcal{D}}^2 \equiv ||f||_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} |f'|^2 dA,$$

where dA is the area measure of $\mathbb{D}$.

As is shown in [**KT**], for functions in the Dirichlet space that are smooth on the boundary of $\mathbb{D}$, two other equivalent norms present themselves. They are

$$\|f\|_{\mathcal{D}}^2 \equiv \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{\mathbb{D}} |f\,'|^2 \, dA = ||f||_{\sigma}^2 + ||f\,'||_A^2 \quad \text{and}$$

$$\|f\|_{\mathcal{D}}^2 \equiv \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} \, d\sigma(t) d\sigma(\theta), \tag{1.2.1}$$

where $d\sigma$ is the Lebesgue measure on the interval $[-\pi, \pi]$.

It will also be useful to consider $\bigoplus_{1}^{\infty} \mathcal{D}$ as an l^2-valued Dirichlet space, where the norms are exactly the same as above but with the absolute values replaced by l^2-norms in the appropriate places. The harmonic Dirichlet space, restricted to the boundary of $\mathbb{D}$, is denoted by $\mathcal{HD}$ and it's square norm is given by (1.2.1).

For the weighted Dirichlet space, $\mathcal{D}_\alpha$, with $0 < \alpha < 1$, the situation is fairly similar. Recall that the weighted Dirichlet space, $\mathcal{D}_\alpha$ is given by

$$\left\{ f : \mathbb{D} \to \mathbb{C} \ , \ f = \sum_{n=0}^{\infty} a_n z^n \ : \ ||f||_{\mathcal{D}_\alpha}^2 = \sum_{n=0}^{\infty} (n+1)^\alpha \, |a_n|^2 < \infty \right\}.$$

Other equivalent norms are given by the following, with the second two being applicable to functions that are also smooth on the boundary.

$$\|f\|^2_{\mathcal{D}_\alpha} \equiv ||f||^2_{H^2(\mathbb{D})} + \int_{\mathbb{D}} |f'(z)|^2 \left(1-|z|^2\right)^{1-\alpha} dA \ ,$$

$$\|f\|^2_{\mathcal{D}_\alpha} \equiv \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{\mathbb{D}} |f'(z)|^2 \left(1-|z|^2\right)^{1-\alpha} dA \ , \text{ and}$$

$$\|f\|^2_{\mathcal{D}_\alpha} \equiv \int_{-\pi}^{\pi} |f|^2 d\sigma(t) + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it}-e^{i\theta}|^{1+\alpha}}\, d\sigma(t)\ d\sigma(\theta). \tag{1.2.2}$$

For ease of notation, the measure $(1-|z|^2)^{1-\alpha}dA$ will from now on be denoted by dA_α.

As in the regular Dirichlet space, it will be useful to consider $\bigoplus\limits_1^\infty \mathcal{D}_\alpha$ as an l^2-valued Dirichlet space, where the norms are exactly the same as above but with the absolute values replaced by l^2-norms in the appropriate places. The harmonic weighted Dirichlet space, restricted to the boundary of $\mathbb{D}$, is denoted by $\mathcal{HD}_\alpha$ and it's square norm is given by (1.2.2).

With that out of the way, notice that the constant function $1 \in \mathcal{D}$, from which it follows that $\mathcal{M}(\mathcal{D}) \subset \mathcal{D}$. From the discussion in the previous section, $\mathcal{M}(\mathcal{D}) \subset H^\infty(\mathbb{D})$. Using these two inclusions, an example of a function that is in $H^\infty(\mathbb{D})$, but not in $\mathcal{M}(\mathcal{D})$ can be obtained by finding a function that is a member of $H^\infty(\mathbb{D})$, but not a member of $\mathcal{D}$.

Consider $f(z) = \sum\limits_{n=1}^{\infty} \frac{z^{n^3}}{n^2}$, which is clearly in $H^\infty(\mathbb{D})$, however

$$\|f\|^2_{\mathcal{D}} = \sum_{n=1}^{\infty} (n+1)\,|a_n|^2 = \sum_{n=1}^{\infty} \frac{n^3+1}{n^4} \geq \sum_{n=1}^{\infty} \frac{1}{n},$$

which is a divergent series, showing that $f \notin \mathcal{D}$. Hence, $\mathcal{M}(\mathcal{D}) \subsetneq H^\infty(\mathbb{D})$.

For the weighted Dirichlet space the result is the same, as is seen by considering the function $f(z) = \sum\limits_{n=1}^{\infty} \frac{z^{n^{4m+1}}}{n^{2m\alpha}}$, $m = \left[\frac{1}{\alpha}\right] + 1$, $z \in \mathbb{D}$. This function is in $H^\infty(\mathbb{D})$, but not in $\mathcal{M}(\mathcal{D}_\alpha)$.

1.2.3. Carleson Measures. Perhaps the simplest classification of multipliers for the Dirichlet space can be given in terms of what are known as Carleson measures. A positive Borel measure μ on the unit disk $\mathbb{D}$ is a Carleson measure for $\mathcal{D}$ if there exists a constant

$C > 0$ such that

$$\int_{\mathbb{D}} |g|^2 d\mu \le C^2 \, ||g||_{\mathcal{D}}^2 \text{ for all } g \in \mathcal{D}.$$

LEMMA 1.2.3. *$\phi \in \mathcal{M}(\mathcal{D})$ if and only if $\phi \in H^\infty$ and $|\phi'|^2 dA$ is a Carleson measure for $\mathcal{D}$.*

PROOF. If $\phi \in \mathcal{M}(\mathcal{D})$, then $\phi \in H^\infty(\mathbb{D})$, and for any $f \in \mathcal{D}$,

$$\begin{aligned}
\int_{\mathbb{D}} |f|^2 |\phi'|^2 dA &= \int_{\mathbb{D}} |\,(\phi f)' - \phi f'|^2 dA \\
&\le 2 \int_{\mathbb{D}} |\,(\phi f)'\,|^2 dA + 2 \int_{\mathbb{D}} |\phi f'|^2 dA \\
&\le 2||M_\phi(f)||_{\mathcal{D}}^2 + 2||\phi||_\infty^2 ||f||_{\mathcal{D}}^2 \\
&\le 4||M_\phi||^2 ||f||_{\mathcal{D}}^2,
\end{aligned}$$

which shows $|\phi'|^2 dA$ is a Carleson measure for $\mathcal{D}$.

Conversely, if $\phi \in H^\infty(\mathbb{D})$ and $|\phi'|^2 dA$ is a Carleson measure for $\mathcal{D}$, with constant C, then ϕ is a multiplier of $H^2(\mathbb{D})$, so that for every $f \in \mathcal{D}$, ϕf is analytic on $\mathbb{D}$, with

$$\begin{aligned}
||\phi f||_{\mathcal{D}}^2 &= ||\phi f||_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} |(\phi f)'|^2 dA \\
&\le ||\phi||_\infty^2 ||f||_{H^2(\mathbb{D})}^2 + 2 \int_{\mathbb{D}} |\phi f'|^2 dA + 2 \int_{\mathbb{D}} |\phi' f|^2 dA \\
&\le 2||\phi||_\infty ||f||_{H^2(\mathbb{D})}^2 + 2||\phi||_\infty \int_{\mathbb{D}} |f'|^2 dA + 2 \int_{\mathbb{D}} |\phi' f|^2 dA \\
&\le (2||\phi||_\infty + 2C)||f||_{\mathcal{D}},
\end{aligned}$$

which shows $\phi f \in \mathcal{D}$ for all $f \in \mathcal{D}$. □

A similar result for the weighted Dirichlet space holds, by replacing every instance of dA with dA_α

1.3. Banach Algebra Techniques for $H^\infty(\mathbb{D})$

The purpose of this section is to give a brief overview of relevant material related to Banach algebras. This leads to a discussion about the space of nonzero multiplicative linear functionals of $H^\infty(\mathbb{D})$, culminating in the Carleson corona theorem, which proves that the open unit disk is dense in this space. This theorem is the first main influential factor for the subject of this dissertation.

1.3.1. Banach Algebras. A Banach algebra, is an associative complex algebra B, that is a Banach space, and satisfies

$$||xy|| \leq ||x|| \; ||y|| \text{ , for all } x, y \in B.$$

B is said to be commutative, whenvever the multiplication operation is commutative, and is said to be unital whenever there is a multiplicative identity, e. Such Banach algebras will be the focus of this discussion.

It is well known that in a commutative Banach algebra, if an element x satisfies $||x|| < 1$, then $e + x$ is invertible via multiplication. This is seen by observing that the formal power series of $(e+x)^{-1}$ converges to an element of the Banach algebra that is the inverse of $e+x$.

Since Banach algebras are in fact Banach spaces, one can consider the space of bounded linear functionals on them. Because of the additional multiplicative structure, attention can be focused on linear functionals that preserve the multiplication structure. It turns out that these are precisely the same as the ring homomorphisms from the Banach algebra to the complex plane.

LEMMA 1.3.1. *ϕ is a ring homomorphism from a commutative Banach algebra B to the complex plane, if and only if ϕ is a nonzero bounded multiplicative linear functional (mlf) satisfying $||\phi|| \leq 1$.*

PROOF. Of course any nonzero bounded multiplicative linear functional preserves the ring structure of B, by definition, so this direction is trivial.

To prove the other direction, let ϕ be a ring homomorphism from B to $\mathbb{C}$, so that the only thing to check is if ϕ is bounded. Suppose to the contrary that there exists an $x_0 \in B$, such that $|\phi(x_0)| > ||x_0||$, which implies that x_0 and $\phi(x_0)$ must be nonzero. Now, let $x = \frac{x_0}{\phi(x_0)}$, so that $||x|| < 1$, and $\phi(x) = 1$.

Since $||x|| < 1$, $e - x$ is invertible, with inverse y. However, due to the homomorphism preserving structure, this implies

$$1 = \phi(e) = \phi\left((e - x)y\right) = \phi(e - x)\phi(y) = (\phi(e) - \phi(x))\phi(y) = 0,$$

which is a contradiction. Thus, ϕ is a nonzero bounded multiplicative linear functional satisfying $||\phi|| \leq 1$. □

A subset J, of a a Banach algebra B, is said to be an ideal if it is closed under the addition operation, and $ab \in J$ for all $a \in J$, $b \in B$. In a commutative Banach algebra with identity, each ideal not containing an invertible element is contained in a maximal ideal, the maximal ideals are closed with respect to the norm, and every maximal ideal is the kernel of some multiplicative linear functional [**Ru**].

1.3.2. Maximal Ideal Space of $H^\infty(\mathbb{D})$, and the Carleson Corona Theorem. With pointwise addition and multiplication, along with the supremum norm,

$$||f||_\infty = \sup_{|z|<1} |f(z)|,$$

$H^\infty(\mathbb{D})$ is a commutative Banach algebra with identity 1. Thus, the commentary at the end of the previous section is relevant when discussing the ideals of $H^\infty(\mathbb{D})$.

Let Δ denote the set of all multiplicative linear functionals (mlf) on $H^\infty(\mathbb{D})$;

$$\Delta := \left\{\phi : \phi \quad \text{is a mlf on } H^\infty(\mathbb{D}),\, \phi \neq 0\right\}.$$

For each $\lambda \in \mathbb{D}$, the point evaluation functional, defined by

$$\phi_\lambda(f) = f(\lambda),$$

can be seen to be in Δ. Thus, under this identification, the unit disk can be embedded into Δ.

The corona is defined to be the set theoretic difference $\Delta \setminus \overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is understood to be the closure the point evaluation functionals in the weak*-topology, the topology of pointwise convergence for bounded linear functionals. In 1941, Kakutani asked whether the maximal ideal space $\mathcal{M}$ of $H^\infty(\mathbb{D})$ has a nontrivial corona.

It turns out that this topological question is equivalent to a result about bounded analytic functions on the disk. This is what is known as the Carleson corona theorem.

THEOREM 1 (Carleson Corona Theorem). *If $f_1,\ f_2, ..., f_n$ are bounded analytic functions in the open unit disk $\mathbb{D}$ such that*

$$|f_1(z)|^2 + ... + |f_n(z)|^2 \geq \delta^2 > 0, \ \ \textit{for all } z \in \mathbb{D}. \tag{1.3.1}$$

Then there exist bounded analytic functions $g_1,\ g_2,\ ..., g_n$ such that

$$f_1(z)g_1(z) + ... + f_n(z)g_n(z) = 1 \textit{ for all } z \in \mathbb{D}.$$

Before proving the equivalence of these two concepts, some remarks on the weak*-topology are prudent.

The topology of pointwise convergence for functions from a set X to a set Y coincides with the product topology on the Y^X. Indeed, each function from X to Y can be thought of as a point in this space by taking $\prod_{x\in X} f(x)$. The open sets in this topology are of the form $\prod_{x\in X} U_x$, where each U_x is open in Y, and all but finitely many satisfy $U_x = Y$.

Thus, the question being asked about the corona is whether or not the point evaluation functionals are dense in Δ as a subspace of $\mathbb{C}^{H^\infty(\mathbb{D})}$.

A basis $\mathcal{B}$, for $\mathbb{C}^{H^\infty(\mathbb{D})}$ is given by the collection of all possible open sets, where the finitley many $U_x \neq \mathbb{C}$ are open balls in the complex plane. Thus, a basis for Δ as a subspace of the product is given by $\mathcal{B}_\Delta = \{\Delta \cap B : B \in \mathcal{B}\}$.

With all of these notions established, the equivalence of the two concepts can be proven.

THEOREM 2. *The point evaluation functionals are a dense subset of Δ in weak*-topology if and only if the corona theorem holds.*

PROOF. Assume that the point evaluations are dense. Let $f_1, ...f_n \in H^\infty(\mathbb{D})$ such that $1 \notin \mathcal{I}$, the ideal generated by $f_1, ..., f_n$. Hence, $\mathcal{I}$ contains no invertible elements, and so is contained in some maximal ideal, which is the kernel of some $\phi \in \Delta$. Since point evaluations are dense, there exists $\phi_{\lambda_m} \to \phi$, as $m \to \infty$ in the weak*-topology. Hence, $\phi_{\lambda_m}(f_i) \to \phi(f_i) = 0$, for each $i = 1, ..., n$. This means that $\{\lambda_m\}$, is a sequence of points in $\mathbb{D}$, such that

$$|f_1(\lambda_m)|^2 + \cdots + |f_n(\lambda_m)|^2 \to 0,$$

and so the corona theorem holds.

Assume the point evaluations are not dense, so that there exists a point $\phi \in \Delta$, and a basic open set $O_\phi = \Delta \cap O$, where O is basic open in $\mathbb{C}^{H^\infty(\mathbb{D})}$, such that $\phi \in O_\phi$, and for each $\lambda \in \mathbb{D}$, $\phi_\lambda \notin O_\phi$. Since O is basic open, it is of the form

$$\prod_{h \in H^\infty(\mathbb{D})} O_h,$$

such that there exists $h_1, ..., h_n \in H^\infty(\mathbb{D})$, with $O_{h_1}, ..., O_{h_n}$, being balls in the complex plane, and all other $O_h = \mathbb{C}$.

It may be assumed that $\phi(h_1), ..., \phi(h_n)$ are the centers of these balls. Let δ be the smallest radius of all the balls. For each $\lambda \in \mathbb{D}$, $\phi_\lambda \in \Delta$, so it follows that there exists O_{h_i} such that $\phi_\lambda \notin O_{h_i}$. Thus,

$$|\phi_\lambda(h_1) - \phi(h_1)|^2 + \cdots + |\phi_\lambda(h_n) - \phi(h_n)|^2 \geq \delta^2 > 0.$$

Consider the functions $f_1 = h_1 - \phi(h_1), ..., f_n = h_n - \phi(h_n) \in H^\infty(\mathbb{D})$, and observe that

$$\phi(f_i) = \phi(h_i - \phi(h_i)) = \phi(h_i) - \phi(\phi(h_i)) = \phi(h_i) - \phi(h_i)\phi(1) = 0,$$

for each $i = 1, ..., n$, and for each $z \in \mathbb{D}$,

$$\begin{aligned}
|f_1(z)|^2 + \cdots + |f_n(z)|^2 &= |\phi_z(f_1)|^2 + \cdots + |\phi_z(f_n)|^2 \\
&= |\phi_z(h_1 - \phi(h_1))|^2 + \cdots + |\phi_z(h_1 - \phi(h_1))|^2 \\
&= |\phi_z(h_1) - \phi(h_1)|^2 + \cdots + |\phi_z(h_n) - \phi(h_n)|^2 \\
&\geq \delta^2.
\end{aligned}$$

However, if there exist $g_1, ..., g_n \in H^\infty(\mathbb{D})$ such that $f_1g_1 + \cdots + f_ng_n = 1$, it follows that

$$1 = \phi(1) = \phi(f_1g_1 + \cdots f_ng_n) = \phi(f_1)\phi(g_1) + \cdots + \phi(f_n)\phi(g_n) = 0,$$

and so the corona theorem does not hold. □

In 1962, Carleson proved the corona theorem, providing an answer to the question posed by Kakutani, namely that the corona is empty, and the point evaluations form a dense subset of Δ in the weak*-topology [**C**].

Carleson's theorem has since been generalized. Independently, Rosenblum [**R**], Tolokonnikov [**To**], and Uchiyama proved infinite versions of Carleson's work. Taking things a step further, Fuhrmann proved that the corona theorem holds for a matix of order $m \times n$, whenever $m, n < \infty$, and then Vasyunin extended this result to the case $m \times \infty$, $m < \infty$. In [**Tr3**], Trent obtained the best estimates for the matricial corona theorem up to this point. Trent and Zhang proved that the result of Vasyunin can be extended to any algebra where the corona theorem holds. Finally, in [**T2**], Treil showed that the corona theorem fails for a matrix of order $\infty \times \infty$.

These results prompted the question of whether or not the same results would hold in other multiplier algebras. For the Dirichlet space, the main focus of this dissertation, the analogue of the corona theorem was proven by Tolokonnikov [**To**], and the infinitely many generator version was proven by Trent [**Tr2**].

1.4. Wolff Ideal Problems

Given a RKHS, $H(\Omega)$, containg the constant function 1, for which the corona theorem holds, the result can be phrased in terms of ideals. Namely, if $f_1, f_2, ..., f_n \in \mathcal{M}(H(\Omega))$ satisfy

$$|f_1(z)|^2 + ... + |f_n(z)|^2 \geqq 1 \text{ for all } z \in \Omega,$$

then 1 belongs to the ideal generated by f_1, f_2, ..., f_n.

This leads to a more general question that can be asked, by replacing 1 with an arbitrary function $h \in \mathcal{M}(H(\Omega))$. That is, if $f_1,\ f_2, ..., f_n, h \in \mathcal{M}(H(\Omega))$, and

$$|f_1(z)|^2 + ... + |f_n(z)|^2 \geqq |h(z)|^2 \text{ for all } z \in \Omega, \tag{1.4.1}$$

can anything be said about whether or not h is in the ideal generated by those functions? These questions will be referred to as Wolff ideal problems, and positive results will be referred to as Wolff type theorems.

1.4.1. Wolff Type Theorems for $H^\infty(\mathbb{D})$. Given functions $f_1, ..., f_n, h \in H^\infty(\mathbb{D})$ that satisfy the boundedness condition (1.4.1), the Wolff ideal problem can be posed in this setting.

An example by Rao [**G**], shows that the condition need not imply that $h \in \mathcal{I}(\{f_j\}_{j=1}^n)$. However, it still makes sense to consider whether or not $h^p \in \mathcal{I}\left(\{f_j\}_{j=1}^n\right)$, for some power p, or in other terminology, that h is a member of the radical of $\mathcal{I}(\{f_j\}_{j=1}^n)$. The answer to this is affirmative, as is demonstrated by the following theorem due to Wolff [**G**].

THEOREM 1.4.1. *(Wolff's Ideal Theorem) If*

$$\{f_j\}_{j=1}^n \subset H^\infty(\mathbb{D}), H \in H^\infty(\mathbb{D}) \quad \textit{and}$$

$$|H(z)| \leq \left(\sum_{j=1}^n |f_j(z)|^2\right)^{\frac{1}{2}} \quad \textit{for all } z \in \mathbb{D},$$

then

$$H^3 \in \mathcal{I}(\{f_j\}_{j=1}^n),$$

the ideal generated by $\{f_j\}_{j=1}^n$ in $H^\infty(\mathbb{D})$.

This answered the big question, namely that the condition is sufficient to guarantee h is in the radical of the functions, but it was still unknown whether or not h^2 was in the ideal. This question remained open for almost 20 years, when Treil [**T1**] finally showed that the condition is not sufficient. Since this time, sufficient conditions have been given to guarantee h itself is in the ideal generated by the functions. These results have also been extended to the case where there are an infinite number of generators. The best result in this direction is the following result due to Treil [**T3**], but before presenting it, some relevant operators need to be defined formally for a RKHS, $H(\Omega)$.

Given $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(H(\Omega))$, let $F(z) = (f_1(z), f_2(z), \dots)$ for $z \in \mathbb{D}$. The row operator $M_F^R : \overset{\infty}{\underset{1}{\oplus}} H(\Omega) \to H(\Omega)$, is defined by by

$$M_F^R\left(\{h_j\}_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} f_j h_j \text{ for } \{h_j\}_{j=1}^{\infty} \in \overset{\infty}{\underset{1}{\oplus}} H(\Omega).$$

Similarly, the column operator $M_F^C : H(\Omega) \to \overset{\infty}{\underset{1}{\oplus}} H(\Omega)$ is defined by

$$M_F^C\,(h) = \{f_j h\}_{j=1}^{\infty} \text{ for } h \in \mathcal{M}(\Omega)$$

For $H^2(\mathbb{D})$, these are precisely the analytic Toeplitz operators T_F^R and T_F^C. The norm of these Toeplitz operators are given by

$$\left\|T_F^R\right\| = \left\|T_F^C\right\| = \sup_{z\in\mathbb{D}} \left(\sum_{j=1}^{\infty} |f_j(z)|^2\right)^{\frac{1}{2}}.$$

The theorem due to Treil can now be stated. Because of the equality given above, the assumption $F(z)F(z)^* \leq 1$ implies that the row and column operators are bounded.

Theorem 1.4.2. *Let $F = (f_1, f_2, ...)$ with $\{f_j\}_{j=1}^{\infty} \subset H^{\infty}(\mathbb{D})$ and $H \in H^{\infty}(\mathbb{D})$. Assume that*

$$(a)\ F(z)F(z)^* \leq 1$$

$$(b)\ |H(z)| \leq F(z)F(z)^* \alpha\left(F(z)F(z)^*\right)$$

for all $z \in \mathbb{D}$, where α is a positive, increasing, bounded function such that $\int_0^1 \frac{\alpha(t)}{t} dt < \infty$. Then there exists $G = (g_1, g_2, ...)$ with $\{g_j\}_{j=1}^{\infty} \subset H^{\infty}(\mathbb{D})$ satisfying

$$\|M_G^C\| < \infty$$

$$\text{and } FG^T = H.$$

1.4.2. Wolff's Theorem for $\mathcal{M}(\mathcal{D})$. As mentioned previously, analogs to both the finite generator and infinite generator versions of the corona theorem were established for $\mathcal{M}(\mathcal{D})$, the multiplier algebra of the Dirichlet space. With those settled, the focus shifted to investigating the Wolff ideal problem on $\mathcal{M}(\mathcal{D})$. The following analog of Wolff's theorem for $H^\infty(\mathbb{D})$ was proven by Banjade and Trent in 2014.

THEOREM 1.4.3. *Let $F = (f_1, f_2, ...)$ with $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$. Assume*

$$(a)\ \|M_F^C\| \leq 1$$

$$(b)\ |H(z)| \leq \left(\sum_{j=1}^{\infty} |f_j(z)|^2\right)^{\frac{1}{2}} \text{ for all } z \in \mathbb{D}.$$

Then there exists $G = (g_1, g_2, ...)$ with $\{g_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D})$ satisfying

$$\|M_G^C\| < \infty$$

$$\text{and } FG^T = H^3.$$

Notice that the first assumption differs from the assumption in the result due to Treil in the $H^\infty(\mathbb{D})$ case. This is because the pointwise hypothesis is not sufficient to conclude that M_F^R and M_F^C are bounded on the Dirichlet space. However, $\left\|M_F^R\right\| \leq \sqrt{18}\left\|M_F^C\right\|$ and $\|M_F^C\| \leq 1$ implies $\sum_{j=1}^\infty |f_j(z)|^2 \leq 1$ for all $z \in \mathbb{D}$ [**Tr2**].

This settled the question of whether or not a function, H, satisfying the boundedness condition would be in the radical of the ideal generated by the functions in the boundedness condition. All that remains is to attempt to come up with conditions to guarantee that a function itself is in the ideal, namely being able to replace the 3 with a 1 in the conclusion, without straying too far from the spirit of the original hypothesis. This is one of the main problems solved in this dissertation, the proof of which will be contained in the next chapter.

1.4.3. Wolff's Theorem for $\mathcal{M}(\mathcal{D}_\alpha)$. The analogue of the corona theorem for the algebra of multipliers on weighted Dirichlet spaces was established in Kidane-Trent [**KT**].

This was an infinite generator version. Banjade and Trent proved the following analog for Wolff's theorem in $\mathcal{M}(\mathcal{D}_\alpha)$.

Theorem 3. *Let* H, $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$. *Assume that*

$$\begin{aligned} &(a)\ \|M_F^C\| \leq 1 \\ \textit{and} \quad &(b)\ |H(z)| \leq \left(\sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{\frac{1}{2}} \quad \textit{for all } z \in \mathbb{D}. \end{aligned}$$

Then there exists $K(\alpha) < \infty$ *and* $\{g_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha)$ *with*

$$\begin{aligned} &\|M_G^C\| \leq K(\alpha) \\ \textit{and} \quad &F\,G^T = H^3. \end{aligned}$$

As in the regular Dirichlet space, the column operator was assumed to be bounded, which implies the row operator is bounded, since $\left\|M_F^R\right\| \leq \sqrt{10}\left\|M_F^C\right\|$ [**KT**].

Thus, the main focus turns to finding conditions to guarantee that H itself is in the ideal. This is another question that will be answered in this dissertation, the proof of which will be contained in the third chapter.

CHAPTER 2

REFINEMENT OF WOLFF'S THEOREM FOR $\mathcal{M}(\mathcal{D})$

As was discussed earlier, the next step in the progression concerning multipliers of the Dirichlet space, is to establish conditions that given $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$, and a function H satisfying these conditions, then H can be realized as a linear combination of the functions.

2.1. Statement of the Theorem

THEOREM 2.1.1. *Let $F = (f_1, f_2, ...)$ with $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$. Assume that*

(a) $\|M_F^C\| \leq 1$

(b) $|H(z)| \leq \left(\sum_{j=1}^{\infty} |f_j(z)|^2\right)^{\frac{3}{2}}$ *for all* $z \in \mathbb{D}$

(c) $|H'(z)| \leq C \left(\sum_{j=1}^{\infty} |f_j(z)|^2\right)^{\frac{1}{2}} \left(1 - |z|^2\right)^{\epsilon - \frac{1}{2}}$ *for all* $z \in \mathbb{D}$, *and fixed* C, $\epsilon > 0$.

Then there exists $G = (g_1, g_2, ...)$ with $\{g_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$ satisfying

$$\|M_G^C\| < \infty$$

$$\text{and } FG^T = H$$

Notice that the only conditions that are modified or added to the conditions given by Banjade and Trent is the change in an exponent bounding the modulus of H and a derivative condition. These are rather natural modifications given the fact that the change in the exponent corresponds to the $H^{\infty}(\mathbb{D})$ situation via Treils result, with $\alpha(t) = \sqrt{t}$ and the derivative condition appears because the proof of the theorem will travel through the functions in the Dirichlet space that are smooth along the boundary, for which there is a derivative element in the equivalent norms.

In the most broad sense, the proof for the theorem will involve proving an equivalent statement involving the row operator defined earlier. The equivalent statement turns out to be showing that the image of the row operator contains the image of the operator M_H, and the solutions to $M_F^R(x) = Hh$ satisfy a boundedness condition dependent on $||h||_{\mathcal{D}}$. This will involve writing down solutions, which then need to be checked that they are analytic solutions, and check that they satisfy the boundedness condition, which will imply that they are in $\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}$, proving that they are indeed solutions. This process will involve numerous lemmas.

2.2. Relevant Lemmas

The intention of this section is to compile a list of all the lemmas that will be needed to provide a cohesive proof for the theorem stated in the previous section. The hypotheses in the theorem are taken to be assumed for this section. Additionally, observe that it may be assumed that $H(0) \neq 0$, for H not identically zero.

Indeed, if $H(0) = 0$, but $H(a) \neq 0$, let $\beta(z) = \frac{a-z}{1-\bar{a}\,z}$ for $z \in \mathbb{D}$. Then since (b) and (c) hold for all $z \in \mathbb{D}$, they hold for $\beta(z)$. Thus, H and F can easily be replaced by $H \circ \beta$ and $F \circ \beta$ in condition (b).

To see that this can also be done in condition (c), observe that if $|\beta(z)| \leq |z|$, then $1 - |z|^2 \leq 1 - |\beta(z)|^2$, so that using the fact that $x^{\epsilon - \frac{1}{2}}$ is a decreasing function, and the fact that $\beta\,'$ is bounded gives that

$$\begin{aligned}|H'(b(z))b'(z)| &\leq C\left(\sum_{j=1}^{\infty}|f_j(\beta(z))|^2\right)^{\frac{1}{2}}\left(1 - |\beta(z)|^2\right)^{\epsilon-\frac{1}{2}}|\beta'(z)| \\ &\leq CC_1\left(\sum_{j=1}^{\infty}|f_j(\beta(z))|^2\right)^{\frac{1}{2}}\left(1 - |z|^2\right)^{\epsilon-\frac{1}{2}}.\end{aligned}$$

If $|\beta(z)| > |z|$, then $1-|z|^2 > 1-|\beta(z)|^2$, so that using the generalized Schwarz inequality on β, and the fact that $x^{\epsilon+\frac{1}{2}}$ is an increasing function gives that

$$\begin{aligned} |H'(b(z))b'(z)| &\leq C\left(\sum_{j=1}^{\infty}|f_j(\beta(z))|^2\right)^{\frac{1}{2}}\left(1-|\beta(z)|^2\right)^{\epsilon-\frac{1}{2}}|\beta'(z)| \\ &\leq C\left(\sum_{j=1}^{\infty}|f_j(\beta(z))|^2\right)^{\frac{1}{2}}\frac{\left(1-|\beta(z)|^2\right)^{\epsilon+\frac{1}{2}}}{1-|z|^2} \\ &\leq C\left(\sum_{j=1}^{\infty}|f_j(\beta(z))|^2\right)^{\frac{1}{2}}\frac{\left(1-|z|^2\right)^{\epsilon+\frac{1}{2}}}{1-|z|^2} \\ &= C\left(\sum_{j=1}^{\infty}|f_j(\beta(z))|^2\right)^{\frac{1}{2}}\left(1-|z|^2\right)^{\epsilon-\frac{1}{2}}. \end{aligned}$$

Taking the maximum of C and CC_1 allows for the full replacement of condition (c).

If the theorem can then be proven for $H\circ\beta$ and $F\circ\beta$, then there exists $G \in \mathcal{M}_{l^2}(\mathcal{D})$ so that $(F\circ\beta)\,G = H\circ\beta$. Hence $F(G\circ\beta^{-1}) = H$ and $G\circ\beta^{-1} \in \mathcal{M}_{l^2}(\mathcal{D})$, and so the result will hold even when $H(0) = 0$. Thus, it may be assumed that $H(0) \neq 0$, so that $\|F(0)\|_{l^2} \neq 0$. This normalization will allow the $H^2(\mathbb{D})$ portion of the norm to be bounded, as in [**Tr1**].

Most of the lemmas will be proven, but in the event that some are not, for any reason, a reference will be provided. The first of these lemmas will be one showing the existence of an operator that is essential to writing down the potential solutions. The proof of a more general version of the lemma can be found in [**Tr3**].

LEMMA 2.2.1. *Let* $\{c_j\}_{j=1}^{\infty} \in l^2$ *and* $C = (c_1, c_2, ...) \in B(l^2, \mathbb{C})$*,* $C \neq 0$*. Then there exists operators* Q*,*D *such that the entries of* Q *are either* 0 *or* $\pm c_j$ *for some* j*, the range of* Q *is equal to the kernel of* C*, the range of* D *is the kernel of* Q*,*

$$CC^*I - C^*C = QQ^*, \text{ and}$$

$$CC^*I - Q^*Q = DD^*.$$

This lemma will be used by applying it to $C = F(z)$, for each $z \in \mathbb{D}$. In the proof of the lemma, it is seen that Q is analytic on $\mathbb{D}$, and since the lemma holds pointwise on $\mathbb{D}$, differentiating with respect to z and $\overline{z}$ yields that

$$F'(z)F'(z)^*I - D'(z)^*D'(z) = Q'(z)^*Q'(z),$$

which implies

$$F'(z)F'(z)^*I \geq Q'(z)^*Q'(z)$$

The remaining lemmas will be used to show that the possible solution that is obtained by using Lemma 2.2.2 and the Cauchy transform, to be defined below, is indeed in $\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}$, by providing tools for bounding the norm of the solution. The first of these is a lemma relating weighted Bergman norms to weighted Dirichlet norms. The proof of this lemma can be found in [**D**].

Lemma 2.2.2. *For f holomorphic on the unit disk, $0 < p < \infty$, and $\beta > -1$,*

$$\int_{\mathbb{D}} |f|^p(1-|z|^2)^\beta dA \equiv |f(0)|^p + \int_{\mathbb{D}} |f'|^p(1-|z|^2)^{p+\beta} dA.$$

This lemma will be instrumental in bounding the norm of the solution, in particular the resulting term that contains H'. The next lemma involves the operator Q discussed in the lemma above.

Lemma 2.2.3. *If $\underline{w}$ is a vector valued harmonic function on $\overline{\mathbb{D}}$, then*

$$\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2 dA \leq 8\|\underline{w}\|_{\mathcal{HD}}^{\cdot}2$$

Proof. Let $\underline{w}$ be a vector-valued harmonic function on $\overline{\mathbb{D}}$. Write $\underline{w} = \underline{x} + \underline{\bar{y}}$, where $\underline{x}$ and $\underline{\bar{y}}$ are respectively the analytic and co-analytic parts of $\underline{w}$, so that

$$\begin{aligned}\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2\, dA &= \int_{\mathbb{D}} \|Q'\underline{x} + Q'\underline{\bar{y}}\|_{l^2}^2\, dA \\ &\leq 2\int_{\mathbb{D}} \|Q'\underline{x}\|_{l^2}^2\, dA + 2\int_{\mathbb{D}} \|Q'\underline{\bar{y}}\|_{l^2}^2\, dA.\end{aligned}$$

Now, using the fact that $F'(z)F'(z)^*I \geq Q'(z)^*Q'(z)$ and $||M_F^C|| \leq 1$,

$$\begin{aligned}
\int_{\mathbb{D}} \|Q'\underline{x}\|_{l^2}^2 \, dA &= \int_{\mathbb{D}} \langle Q'^{\star} Q'\underline{x}\,,\, \underline{x}\rangle_{l^2} dA \\
&\leq \int_{\mathbb{D}} \langle F'F'^{\star} I\underline{x},\, \underline{x}\rangle_{l^2} dA \\
&= \int_{\mathbb{D}} \sum_{k=1} |f_k' x_k|^2 \, dA \\
&\leq 2\int_{\mathbb{D}} \sum_{k=1}^{\infty} |(f_k x_k)'|^2 \, dA + 2\int_{\mathbb{D}} \sum_{k=1}^{\infty} |f_k x_k'|^2 \, dA \\
&\leq 2 \, \|M_F^C(\underline{x})\|^2 + 2 \, \|\underline{x}\|_{\mathcal{D}}^2 \\
&\leq 4 \, \|\underline{x}\|_{\mathcal{D}}^2
\end{aligned}$$

Similarly, it can be seen that $\int_{\mathbb{D}} \|Q'\underline{\bar{y}}\|_{l^2}^2 \, dA \leq 4 \, \|\underline{y}\|_{\mathcal{D}}^2$. Thus,

$$\begin{aligned}
\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2 \, dA &\leq 8 \, \|\underline{x}\|_{\mathcal{D}}^2 + 8 \, \|\underline{y}\|_{\mathcal{D}}^2 \\
&= 8 \, \|\underline{x} + \underline{\bar{y}}\|_{\mathcal{HD}}^2 \\
&= 8 \, \|\underline{w}\|_{\mathcal{HD}}^2 .
\end{aligned}$$

□

Lemma 2.2.4. *Let the operator T be defined on $L^2(\mathbb{D}, dA)$ by*

$$(Tf)(\lambda) = \int\limits_{\mathbb{D}} \frac{f(z)}{(z-\lambda)\left(1 - z\bar{\lambda}\right)} dA(z),$$

for $\lambda \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA)$. Then the operator T is bounded.

Proof. This lemma is proven using Schur's test. That is to say, that there exists a measurable function $p(z) > 0$ a.e. on $\mathbb{D}$ and a constant $C > 0$ such that

$$\int_{\mathbb{D}} \frac{p(z)}{|z-\lambda||1-z\bar{\lambda}|} \, dA(z) \leq C \, p(\lambda).$$

By Schur's test, the operator is then bounded. In order to proceed in this direction, consider the measurable function $p(z) = (1-|z|^2)^{-\frac{1}{2}}$. Then

$$\begin{aligned}
\int_{\mathbb{D}} \frac{(1-|z|^2)^{-\frac{1}{2}}}{|z-\lambda||1-z\bar{\lambda}|}\, dA(z) &= \int_{\mathbb{D}} \frac{(1-|z|^2)^{-\frac{1}{2}}}{|\varphi_\lambda(z)||1-z\bar{\lambda}|^2}\, dA(z), \quad \text{where } \varphi_\lambda(z) = \frac{\lambda - z}{1-z\bar{\lambda}} \\
&= \int_{\mathbb{D}} \frac{(1-|\varphi_\lambda(\zeta)|^2)^{-\frac{1}{2}}}{|\zeta||1-\varphi_\lambda(\zeta)\bar{\lambda}|^2}\, |\varphi_\lambda'(\zeta)|^2 dA(\zeta) \\
&= \left(1-|\lambda|^2\right)^{-\frac{1}{2}} \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta|\frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}\zeta|^2}}\, \frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}\zeta|^3} dA(\zeta) \\
&= \left(1-|\lambda|^2\right)^{-\frac{1}{2}} \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\, dA(\zeta),
\end{aligned}$$

so the proof will be complete if the remaining integral can be shown to be bounded independent of λ. Observe that

$$\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\, dA(\zeta) = \int_{\mathbb{D}_{\frac{1}{2}}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\, dA(\zeta) + \int_{\mathbb{D}-\left\{\mathbb{D}_{\frac{1}{2}}\right\}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\, dA(\zeta),$$

where $\mathbb{D}_{\frac{1}{2}}$ is the disk of radius $\frac{1}{2}$. For any $\zeta \in \mathbb{D}_{\frac{1}{2}}$ and $\lambda \in \mathbb{D}$,

$$\frac{1}{|1-\bar{\lambda}\zeta|} \le \frac{1}{1-\frac{1}{2}} = 2 \ \text{ and } \ \frac{1}{\sqrt{1-|\zeta|^2}} \le \frac{2}{\sqrt{3}}.$$

Therefore,

$$\int_{\mathbb{D}_{\frac{1}{2}}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\, dA(\zeta) \le \frac{4}{\sqrt{3}} \int_{\mathbb{D}_{\frac{1}{2}}} \frac{1}{|\zeta|}\, dA(\zeta) = \frac{4\pi}{\sqrt{3}}$$

and

$$\begin{aligned}
\int_{\mathbb{D}-\left\{\mathbb{D}_{\frac{1}{2}}\right\}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|\zeta||1-\bar{\lambda}\zeta|}\, dA(\zeta) &\le 2\int_{\mathbb{D}-\left\{\mathbb{D}_{\frac{1}{2}}\right\}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|}\, dA(\zeta) \\
&\le 2\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|}\, dA(\zeta).
\end{aligned}$$

Lemma 3.10 of [Z] implies that

$$\int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|}\, dA(\zeta) = \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|^{2-\frac{1}{2}-\frac{1}{2}}}\, dA(\zeta)$$

as a function of λ is uniformly bounded from above and bounded from below on $\mathbb{D}$.

Hence,

$$\sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{-\frac{1}{2}}}{|1-\bar{\lambda}\zeta|} \, dA(\zeta) \leq C_0 < \infty,$$

for some constant $C_0 > 0$, which shows T is a bounded linear operator on $L^2(\mathbb{D}, dA)$ with

$$\|T\| \leq C, \text{ where } C = \left(\frac{4\pi}{\sqrt{3}} + 2C_0\right).$$

□

LEMMA 2.2.5. *If Q is a multiplier of $\mathcal{D}$, then*

$$(1-|z|^2)|Q'(z)| \leq \|M_Q\|_{B(\mathcal{D})} \text{ for all } z \in \mathbb{D}$$

PROOF. Define $\varphi : \mathbb{D} \to \mathbb{D}$ as $\varphi(z) = \frac{Q(z)}{||M_Q||}$ for all $z \in \mathbb{D}$, so that φ is analytic, with $\|\varphi\|_\infty \leq 1$. A consequence of the generalized Schwarz lemma gives

$$|\varphi'(z)| \leq \frac{1}{1-|z|^2},$$

which yields the result. □

The next lemma involves the Beurling transform. It is formally defined by

$$\mathcal{B}(\phi) = \partial\left(\widehat{\phi}\right),$$

where $\phi \in C^1(\overline{\mathbb{D}})$, and $\hat{\phi}$ is the Cauchy transform of ϕ on $\mathbb{D}$. For background information on the Cauchy transform, see [**A**]. The most important properties of the Cauchy transform that are required for this dissertation are that it satisfies

$$\widehat{k}(z) = \frac{-1}{\pi} \int_{\mathbb{D}} \frac{k(w)}{w-z} dA(w) \quad \text{and} \quad \overline{\partial}\left(\widehat{k}\right) = k,$$

for all $k \in C^\infty(\overline{\mathbb{D}})$. The fact that the Buerling transform is bounded can be found in [**Ga**] and [**N**], as follows.

Lemma 2.2.6. *Let $\mathcal{B}$ be the Beurling transform. If $1 < p < \infty$, then there exists a constant $K(p)$ such that*

$$\int_{\mathbb{D}} |\mathcal{B}(\phi)|^p (1-|z|^2)^{p-2} dA(z) \leq K(p) \int_{\mathbb{D}} |\phi|^p (1-|z|^2)^{p-2} dA(z),$$

for all $\phi \in C^1(\overline{\mathbb{D}})$.

The next lemma that is needed to prove the result can be found in [**Tr2**].

Lemma 2.2.7. *If k is a smooth l^2-valued function on $\partial\mathbb{D}$, then*

$$\| \widetilde{\widehat{k}} \|^2_{\mathcal{HD}} \leq \|k\|^2_A + \|\widehat{k}\|^2_\sigma.$$

The final lemma that is needed to prove the result is the one that will guide the direction of the proof.

Lemma 2.2.8. *There exists a constant $K < \infty$ so that, for any $h \in \mathcal{D}$, there exists $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$ such that*

$$(i)\ \ M_F^R(\underline{u}_h) = Hh \quad \text{and}$$

$$(ii)\ \ \|\underline{u}_h\|_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}} \leq K\,\|h\|_{\mathcal{D}}$$

if and only if

$$M_H\, M_H^\star \leq K^2 M_F^R\, M_F^{\star R}.$$

Proof. Suppose there exists $K < \infty$ such that $M_H\, M_H^\star \leq K^2 M_F^R\, M_F^{\star R}$, so that by Douglas' lemma, there exists a $C \in B\left(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}}\mathcal{D}\right)$, such that $M_F^R\, C = M_H$ and $||C|| \leq K$.

For each $h \in \mathcal{D}$, let $\underline{u}_h = C(h)$, so that $\underline{u}_h \in \overset{\infty}{\underset{1}{\oplus}}\mathcal{D}$, $M_F^R\,(\underline{u}_h) = M_H h = Hh$, and $\|\underline{u}_h\|_{\mathcal{D}} \leq K\,\|h\|_{\mathcal{D}}$.

Conversely, suppose for each $h \in \mathcal{D}$, there exists a $u_h \in \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}$, such that (i) and (ii) hold. Let $\underline{v}_h = P_{\left(Ker M_F^R\right)^{\perp}}(\underline{u}_h)$. Since $M_F^R(\underline{v}_h) = Hh$ and $\|\underline{v}_h\|_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}} \leq \|\underline{u}_h\|_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}} \leq K\|h\|_{\mathcal{D}}$, (i) and (ii) hold for $\underline{v}_h$.

Define C, by $C(h) = \underline{v}_h$. By (i), if $\underline{v}_{h_1} \neq \underline{v}_{h_2}$, then

$$H(h_1 - h_2) = M_F^R\left(\underline{v}_{h_1} - \underline{v}_{h_1}\right) = M_F^R\left(P_{\left(Ker M_F^R\right)^{\perp}}\left(\underline{u}_{h_1} - \underline{u}_{h_2}\right)\right) \neq 0.$$

Thus, $h_1 \neq h_2$, and so C is well defined. A similar argument shows that C is linear, and by (ii), it follows that

$$||C(h)||_{\mathcal{D}} \leq ||\underline{v}_h||_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}} \leq K\,||h||_{\mathcal{D}},$$

so that C is an operator bounded by K, satisfying $M_F^R C = M_H$, by (i). Hence, by Douglas' lemma,

$$M_H M_H^\star = M_F^R C C^\star M_F^{\star R} \leq M_F^R K^2 M_F^{\star R} \leq K^2 M_F^R M_F^{\star R}.$$

□

The entire direction of the proof of the theorem, which is contained in the next section, is to show that conditions in the theorem imply (i) and (ii) hold in the smooth case, which gives rise to the conclusion of the lemma. This is done by proving it when h is a poynomial, and since the polynomials are dense in $\mathcal{D}$, they will hold for all h in $\mathcal{D}$. Following this, a compactness argument, which will incorporate a commutant lifting argument that holds for all RKHS with complete pick kernels, will be used to remove the smoothness condition, and give $G \in \mathcal{M}_{l^2}(\mathcal{D})$ satisfying the conclusion of the theorem.

2.3. Proof of the Theorem

Proof. Let $F = (f_1, f_2, ...)$ with $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$ all of which are analytic on $D_{1+\epsilon}(0)$, with $H(0) \neq 0$, and suppose h is a polynomial. Now, consider

$$u_h = F^*(FF^*)^{-1}Hh - Q\widehat{W},$$

defined pointwise for each $z \in \overline{\mathbb{D}}$, where $W = \frac{Q^* F'^* Hh}{(FF^*)^2}$.

Using the fact that $\overline{\partial}(\widehat{W}) = W$, that H, h, and Q are analytic, and the fact that $FF^*I - F^*F = QQ^*$ gives

$$\begin{aligned}
\overline{\partial}(u_h) &= \overline{\partial}\left(F^*(FF^*)^{-1}Hh\right) - \overline{\partial}\left(Q\widehat{W}\right) \\
&= F'^*(FF^*)^{-1}Hh - \frac{F^*FF'^*Hh}{(FF^*)^2} - QW \\
&= F'^*(FF^*)^{-1}Hh - \frac{F^*FF'^*Hh}{(FF^*)^2} - \frac{QQ^*F'^*Hh}{(FF^*)^2} \\
&= F'^*(FF^*)^{-1}Hh - \frac{F^*FF'^*Hh}{(FF^*)^2} - \frac{(FF^*I - F^*F)F'^*Hh}{(FF^*)^2} \\
&= 0,
\end{aligned}$$

so that u_h is analytic.

Using the fact that the range of Q is equal to the kernel of F, it is immediate that

$$M_F^R(u_h) = Hh$$

Next it is shown that there exists a constant K, independent of h and ϵ, such that $\|u_h\|_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}} \leq K\|h\|_{\mathcal{D}}$.

$$\begin{aligned}
\|u_h\|^2_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}} &= \int_{-\pi}^{\pi} \|u_h(e^{it})\|^2_{l^2} d\sigma(t) + \int_{\mathbb{D}} \|u_h'(z)\|^2_{l^2} dA(z) \\
&= \underbrace{\int_{-\pi}^{\pi} \|u_h(e^{it})\|^2_{l^2} d\sigma(t)}_{(i)} + \underbrace{\int_{\mathbb{D}} \|\partial u_h(z)\|^2_{l^2} dA(z)}_{(ii)}.
\end{aligned}$$

It can be shown using condition (b) of the theorem and a Carleson measure technique as seen in [**Tr1**], by taking $\alpha(t) = \sqrt{t}$, that $(i) \leq K_1^2\|h\|^2_\sigma \leq K_1^2\|h\|^2_{\mathcal{D}}$. This is where the assumption $H(0) \neq 0$ is used. For (ii) the integral will be split into five pieces to obtain a similar bound.

$$(ii) \leq 2\int_{\mathbb{D}} \left\| \partial \left(\frac{F^* H h}{FF^*} \right) \right\|_{l^2}^2 dA(z) + 2\int_{\mathbb{D}} \left\| \partial \left(Q\widehat{W} \right) \right\|_{l^2}^2 dA(z)$$

$$\leq \underbrace{4\int_{\mathbb{D}} \left\| \frac{F^* H' h}{FF^*} \right\|_{l^2}^2 dA(z)}_{(a')} + \underbrace{8\int_{\mathbb{D}} \left\| \frac{F^* H h'}{FF^*} \right\|_{l^2}^2 dA(z)}_{(b')} + \underbrace{8\int_{\mathbb{D}} \left\| \frac{F^* H h F' F^*}{(FF^*)^2} \right\|_{l^2}^2 dA(z)}_{(c')}$$

$$+ \underbrace{4\int_{\mathbb{D}} \left\| Q'\widehat{W} \right\|_{l^2}^2 dA(z)}_{(d')} + \underbrace{4\int_{\mathbb{D}} \left\| Q\partial\widehat{W} \right\|_{l^2}^2 dA(z)}_{(e')}.$$

Condition (c) of the theorem and Lemma 2.2.2 give

$$(a') = \int_{\mathbb{D}} \left\langle \frac{F^* H' h}{FF^*}, \frac{F^* H' h}{FF^*} \right\rangle_{l^2} dA(z) = \int_{\mathbb{D}} \left\langle H'h, \frac{H'h}{FF^*} \right\rangle dA(z)$$

$$= \int_{\mathbb{D}} \frac{|H'h|^2}{FF^*} dA(z) \leq C \int_{\mathbb{D}} |h|^2 (1 - |z|^2)^{2\epsilon - 1} dA(z)$$

$$\leq CK \left(|h(0)|^2 + \int_{\mathbb{D}} |h'|^2 (1 - |z|^2)^{2\epsilon + 1} dA \right)$$

$$\leq CK \left(\|h\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} |h'|^2 dA \right)$$

$$= CK \|h\|_{\mathcal{D}}^2 .$$

An application of condition (b) of the theorem gives that

$$(b') = \int_{\mathbb{D}} \left\langle \frac{F^* H h'}{FF^*}, \frac{F^* H h'}{FF^*} \right\rangle_{l^2} dA(z) = \int_{\mathbb{D}} \left\langle H'h, \frac{H'h}{FF^*} \right\rangle dA(z)$$

$$= \int_{\mathbb{D}} \frac{|Hh'|^2}{FF^*} dA(z) \leq \int_{\mathbb{D}} |h'|^2 dA(z) \leq \|h\|_{\mathcal{D}}^2,$$

and also along with condition (a) of the theorem that

$$(c') = \int\limits_{\mathbb{D}} \left\langle \frac{F^*F'F^*Hh}{(FF^*)^2}, \frac{F^*F'F^*Hh}{(FF^*)^2} \right\rangle_{l^2} dA(z) = \int\limits_{\mathbb{D}} \frac{|F'F^*Hh|^2}{(FF^*)^3} dA(z)$$

$$\leq \int\limits_{\mathbb{D}} |F'F^*h|^2 dA(z) = \int_{\mathbb{D}} \Big| \sum_{j=1}^{\infty} f_j' \overline{f} h \Big|^2 dA(z) \leq \int_{\mathbb{D}} \sum_{j=1}^{\infty} |f_j' h|^2 dA(z)$$

$$\leq 2 \int_{\mathbb{D}} \sum_{j=1}^{\infty} |(f_j h)'|^2 \, dA + 2 \int_{\mathbb{D}} \sum_{j=1}^{\infty} |f_j h'|^2 \, dA$$

$$\leq 2 \left\| M_F^C(h) \right\|_{\mathcal{D}}^2 + 2 \|h\|_{\mathcal{D}}^2 \leq 4 \|h\|_{\mathcal{D}}^2 .$$

Using the fact that $\|Q(z)\|_{B(l^2)} \leq 1$, Lemma 2.2.6, condition (b), and $QQ^* \leq FF^*$ gives

$$(e') = \int\limits_{\mathbb{D}} \|Q\mathcal{B}(W)\|_{l^2}^2 \, dA(z) \leq K_2 \int\limits_{\mathbb{D}} \|W\|_{l^2}^2 \, dA(z)$$

$$= K_2 \int\limits_{\mathbb{D}} \left\langle \frac{Q^*F'^*Hh}{(FF^*)^2}, \frac{Q^*F'^*Hh}{(FF^*)^2} \right\rangle_{l^2} dA(z) \leq K_2 \int_{\mathbb{D}} \sum_{j=1}^{\infty} |f_j' h|^2 dA(z)$$

$$\leq 4K_2 \|h\|_{\mathcal{D}}^2 .$$

To bound (d'), it is split into two integrals as follows:

$$(d') \leq \underbrace{2 \int\limits_{\mathbb{D}} \left\| Q' \left(\widehat{W} - \widetilde{\widehat{W}} \right) \right\|_{l^2}^2 dA(z)}_{(i)} + \underbrace{2 \int\limits_{\mathbb{D}} \left\| Q' \widetilde{\widehat{W}} \right\|_{l^2}^2 dA(z)}_{(ii)},$$

where $\widetilde{\widehat{W}} = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}\left(e^{it}\right) d\sigma(t)$, is the harmonic extension of $\widehat{W}$ from $\partial\mathbb{D}$ to $\mathbb{D}$.

Using the definition of the Cauchy transform for smooth functions, the definition of the harmonic extension, Lemma 2.2.4 and Lemma 2.2.5, it follows that

$$
\begin{aligned}
(i) &= \int\limits_{\mathbb{D}} \left\| Q' \left(\frac{-1}{\pi} \int\limits_{\mathbb{D}} \frac{W(w)}{w-z} dA(w) - \int\limits_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}\left(e^{it}\right) d\sigma(t) \right) \right\|_{l^2}^2 dA(z) \\
&= \frac{1}{\pi^2} \int\limits_{\mathbb{D}} \left\| Q' \left(\int\limits_{\mathbb{D}} \frac{W(w)}{w-z} dA(w) - \int\limits_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \int\limits_{\mathbb{D}} \frac{W(w)}{w-e^{it}} dA(w) d\sigma(t) \right) \right\|_{l^2}^2 dA(z) \\
&= \frac{1}{\pi^2} \int\limits_{\mathbb{D}} \left\| Q' \int\limits_{\mathbb{D}} W(w) \left(\frac{1}{w-z} + \int\limits_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \frac{e^{-it}}{1-we^{-it}} d\sigma(t) \right) dA(w) \right\|_{l^2}^2 dA(z) \\
&= \frac{1}{\pi^2} \int\limits_{\mathbb{D}} \left\| Q' \int\limits_{\mathbb{D}} W(w) \left(\frac{1}{w-z} + \int\limits_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \frac{e^{-it}}{1-we^{-it}} d\sigma(t) \right) dA(w) \right\|_{l^2}^2 dA(z) \\
&= \frac{1}{\pi^2} \int\limits_{\mathbb{D}} \left\| Q' \int\limits_{\mathbb{D}} W(w) \left(\frac{1-|z|^2}{(w-z)(1-w\bar{z})} \right) dA(w) \right\|_{l^2}^2 dA(z) \\
&= \frac{1}{\pi^2} \int\limits_{\mathbb{D}} \left\| Q' \left(1-|z|^2\right) T\left(W\right) \right\|_{l^2}^2 dA(z) \\
&\leq \frac{1}{\pi^2} \left\| M_Q \right\|^2 \left\| T\left(W\right) \right\|^2 \\
&\leq \frac{C^2}{\pi^2} \left\| M_Q \right\|^2 \int\limits_{\mathbb{D}} \left\| W \right\|^2 dA(z) \\
&\leq \frac{4C^2}{\pi^2} \left\| M_Q \right\|^2 \left\| h \right\|_{\mathcal{D}}^2,
\end{aligned}
$$

where the second to last equality comes from observing that $\int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \frac{e^{-it}}{1-we^{-it}} d\sigma(t)$ is the harmonic extension of $\frac{\bar{z}}{1-w\bar{z}}$, which is harmonic in it's own right, and since harmonic extensions are unique, they are equal.

Finally, the use Lemmas 2.2.3 and 2.2.7, and the fact that $\|\widehat{W}\|_\sigma^2 \leq 15\|h\|_\sigma^2$ [**Tr2**], show that

$$
(ii) \leq 8\|\widetilde{\widehat{W}}\|_{\mathcal{HD}}^2 \leq 8\left(\|W\|_A^2 + \|\widehat{W}\|_\sigma^2\right) \leq 8\left(4\|h\|_{\mathcal{D}}^2 + 15\|h\|_\sigma^2\right).
$$

Thus, for every polynomial h, when all the functions are smooth on $\partial\mathbb{D}$, there exists $u_h \in \overset{\infty}{\underset{1}{\oplus}}\mathcal{D}$ and a constant K, independent of h, such that:

$$M_F^R(u_h) = Hh$$

$$\|u_h\|_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}} \leq K\|h\|_{\mathcal{D}}$$

Since polynomials are dense in $\mathcal{D}$ and M_F^R is continuous, this holds for all $h \in \mathcal{D}$. By lemma 2.2.8, this is equivalent to saying that when all the functions in question are smooth, there existing a constant K such that

$$M_H M_H^* \leq K^2 M_F M_F^*$$

Hence, for every $0 \leq r < 1$, if F_r is defined by $F_r(z) = F(rz)$ and H_r is similarly defined, then

$$M_{H_r} M_{H_r}^* \leq K^2 M_{F_r} M_{F_r}^*$$

with $M_{F_r}^R \to M_F^R$ and $M_{H_r} \to M_H$ as $r \uparrow 1$, where convergence of the operators is in the strong$-$topology. Using a commutant lifting argument similar to that seen in [**Tr2**], there exist $G_r \in \mathcal{M}(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}}\mathcal{D})$ such that $M_{F_r}^R M_{G_r}^C = H_r$.

By compactness of the closed ball of radius K in the weak operator topology, there exists a net with $G_{r_\alpha}^* \to G^*$ as $r_\alpha \uparrow 1$. Since the multiplier algebra, as operators, is closed in the weak operator topology, $G \in \mathcal{M}(\mathcal{D}, \overset{\infty}{\underset{1}{\oplus}}\mathcal{D})$. Also, it follows that $M_{G_{r_\alpha}}^{C*} M_{F_{r_\alpha}}^{R*}$ converges to $M_G^{C*} M_F^{R*}$ in the weak operator topology, since $M_{F_{r_\alpha}}^{R*}$ converges strongly to M_F^{R*}. Hence, $M_F^R M_G^C = M_H$ with entries of G in $\mathcal{M}(\mathcal{D})$ and $\|M_G^C\| \leq K$. □

CHAPTER 3

REFINEMENT OF WOLFF'S THEOREM FOR $\mathcal{M}(\mathcal{D}_\alpha)$

In this chapter, the result from the previous chapter will be extended to the multiplier algebra of weighted Dirichlet spaces, $\mathcal{D}_\alpha$, for $\alpha \in (0,1)$. The result is known for $H^\infty(\mathbb{D}) = \mathcal{M}(H^2(\mathbb{D}))$, which corresponds to $\alpha = 0$, and was proven in the previous chapter for the Dirichlet space, which corresponds to $\alpha = 1$, so this theorem gives everything in between. The technique for proving the theorem is similar to that of the previous chapter, and the process will be followed as closely as possible to illustrate just how similar things can be when working on these spaces.

3.1. Statement of the Theorem

Theorem 3.1.1. *Let $F = (f_1, f_2, ...)$ with $\{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D}_\alpha)$ and $H \in \mathcal{M}(\mathcal{D}_\alpha)$. Assume that*

(a) $\|M_F^C\| \leq 1$

(b) $|H(z)| \leq \left(\sum_{j=1}^\infty |f_j(z)|^2\right)^{\frac{3}{2}}$ *for all $z \in \mathbb{D}$*

(c) $|H'(z)| \leq \left(\sum_{j=1}^\infty |f_j(z)|^2\right)^{\frac{1}{2}} \left(1-|z|^2\right)^{\epsilon - \frac{2-\alpha}{2}}$ *for all $z \in \mathbb{D}$, and fixed $\epsilon > 0$.*

Then there exists $G = (g_1, g_2, ...)$ with $\{g_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D}_\alpha)$ satisfying

$$\|M_G^C\| < \infty \text{ and } FG^T = H.$$

Just like the case for the regular Dirichlet space, the only conditions that are different from the traditional Wolff's Theorem on $\mathcal{M}(\mathcal{D}_\alpha)$, that gives H^3, are the replacement of the exponent $\frac{1}{2}$ with the exponent $\frac{3}{2}$, in condition (b), and the addition of a derivative condition. This derivative condition can be seen as the analog of the derivative condition in the previously proven theorem, since putting $\alpha = 1$, which corresponds to the Dirichlet space, gives that condition.

3.2. Relevant Lemmas

Following the example set by the previous chapter, this section will be a compilation of all the relevant facts needed to provide a complete proof of the theorem in the next section. Some are repeated, but are included here, for completeness of the chapter and because some of the sources change for the weighted case. Like before, the first observation is that it may be assumed that $H(0) \neq 0$.

Unlike the previous chapter, most of the lemmas will not be proven, due to the fact that proofs are extremely similar, but if one is omitted for any other reason, a reference will be provided. The first two lemmas don't change at all from the previous chapter, due to the generality with which they were presented. Again, a proof for a more general version of the first lemma can be found in [**Tr3**].

LEMMA 3.2.1. *Let* $\{c_j\}_{j=1}^{\infty} \in l^2$ *and* $C = (c_1, c_2, ...) \in B(l^2, \mathbb{C})$, $C \neq 0$. *Then there exists operators* Q,D *such that the entries of* Q *are either* 0 *or* $\pm c_j$ *for some* j, *the range of* Q *is equal to the kernel of* C, *the range of* D *is the kernel of* Q,

$$CC^*I - C^*C = QQ^*, \text{ and}$$

$$CC^*I - Q^*Q = DD^*.$$

Just like the regular Dirichlet space, this will be applied to $C = F(z)$, for each $z \in \mathbb{D}$, and the fact that

$$F'(z)F'(z)^*I \geq Q'(z)^*Q'(z),$$

will be needed. A reference for the proof of the second lemma is [**D**].

Lemma 3.2.2. *For f holomorphic on the unit disk, $0 < p < \infty$, and $\beta > -1$,*

$$\int_{\mathbb{D}} |f|^p (1-|z|^2)^\beta dA \equiv |f(0)|^p + \int_{\mathbb{D}} |f'|^p (1-|z|^2)^{p+\beta} dA.$$

This lemma will again be instrumental bounding the norm of the term that contains H'. The next lemma is the first one that at least appears to be different, however the proof follows by replacing every instance of dA in the previous chapter's proof with dA_α.

Lemma 3.2.3. *If $\underline{w}$ is a vector valued harmonic function on $\overline{\mathbb{D}}$, then*

$$\int_{\mathbb{D}} \|Q'\underline{w}\|_{l^2}^2 dA_\alpha \leq 8\|\underline{w}\|_{\mathcal{HD}_\alpha}^2$$

Lemma 3.2.4. *Let the operator T be defined on $L^2(\mathbb{D}, dA_\alpha)$ by*

$$(Tf)(\lambda) = \int_{\mathbb{D}} \frac{f(z)}{(z-\lambda)\left(1-z\bar{\lambda}\right)} dA_\alpha(z),$$

for $\lambda \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA_\alpha)$. Then the operator T is bounded.

Proof. This is proven again using Schur's Test, but with $p(z) = (1-|z|^2)^{\alpha-\frac{3}{2}}$. By the previous result involving this integral operator, for any $\lambda \in \mathbb{D}$,

$$\begin{aligned}
(Tp)(\lambda) &= \int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-\frac{3}{2}}}{(z-\lambda)\left(1-z\bar{\lambda}\right)} dA_\alpha \\
&= \int_{\mathbb{D}} \frac{(1-|z|^2)^{-\frac{1}{2}}}{(z-\lambda)\left(1-z\bar{\lambda}\right)} dA \\
&\leq C\left(1-|\lambda|^2\right)^{-\frac{1}{2}} \\
&\leq C\left(1-|\lambda|\right)^{\alpha-\frac{3}{2}},
\end{aligned}$$

and so the operator is bounded by Schur's test. □

The next lemma is another whose proof only changes by replacing with the appropriate norms.

LEMMA 3.2.5. *If Q is a multiplier of $\mathcal{D}_\alpha$, then*

$$(1-|z|^2)|Q'(z)| \leq \|M_Q\|_{B(\mathcal{D}_\alpha)} \quad \textit{for all } z \in \mathbb{D}$$

The next lemma involves the Beurling transform. Recall that it is formally defined by

$$\mathcal{B}(\phi) = \partial\left(\widehat{\phi}\right),$$

where $\phi \in C^1(\overline{\mathbb{D}})$, and $\hat{\phi}$ is the Cauchy transform of ϕ on $\mathbb{D}$. For background information on the Cauchy transform, see [**A**]. Like before, the most important properties of the Cauchy transform that are required for this dissertation are that it satisfies

$$\widehat{k}(z) = \frac{-1}{\pi}\int\limits_{\mathbb{D}} \frac{k(w)}{w-z}dA(w) \quad \text{and} \quad \overline{\partial}\left(\widehat{k}\right) = k,$$

for all $k \in C^\infty(\overline{\mathbb{D}})$. The proof of the fact that the Buerling transform is bounded in this case is a rather tedious application of Schur's test, and so it is omitted. A complete proof can be found in [**BT2**].

LEMMA 3.2.6. *Let $\mathcal{B}$ be the Beurling transform. Then*

$$\int\limits_{\mathbb{D}} |\mathcal{B}(\phi)|^2 dA_\alpha \leq \frac{23}{\alpha}\int\limits_{\mathbb{D}} |\phi|^2 dA_\alpha,$$

for all $\phi \in C^1(\overline{\mathbb{D}})$.

The next lemma that is needed to prove the result can be found in [**KT**].

LEMMA 3.2.7. *If k is a smooth l^2-valued function on $\partial\mathbb{D}$, then there exists a constant, C_1, independent of α, such that*

$$\| \,\widetilde{\hat{k}}\, \|^2_{\mathcal{HD}_\alpha} \leq C_1\|k\|^2_{A_\alpha}.$$

The final lemma that is needed to prove the result is the one that will guide the direction of the proof. The proof of this lemma follows analogously to that of the proof in the previous chapter.

LEMMA 3.2.8. *There exists a constant* $K < \infty$ *so that, for any* $h \in \mathcal{D}_\alpha$*, there exists* $\underline{u}_h \in \bigoplus_1^\infty \mathcal{D}_\alpha$ *such that*

$$\text{(i)}\ \ M_F^R(\underline{u}_h) = Hh \quad \textit{and}$$

$$\text{(ii)}\ \ \|\underline{u}_h\|_{\bigoplus_1^\infty \mathcal{D}_\alpha} \leq K\,\|h\|_{\mathcal{D}_\alpha}$$

if and only if

$$M_H\, M_H^\star \leq K^2 M_F^R\, M_F^{\star R}.$$

The entire direction of the proof of the theorem, which is contained in the next section, is to show that conditions in the theorem imply that (i) and (ii) in this lemma hold in the smooth case. This is done by proving that (i) and (ii) hold when all of the functions in question are smooth on the boundary of $\mathbb{D}$, and h is a poynomial. Since the polynomials are dense in $\mathcal{D}_\alpha$, the hypotheses will hold for all h in $\mathcal{D}_\alpha$. Following this, a compactness argument, which will incorporate a commutant lifting argument that holds for all RKHS with complete pick kernels, will be used to remove the smoothness condition, and complete the proof.

3.3. Proof of the Theorem

PROOF. Let $F = (f_1, f_2, ...)\ \{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathcal{D}_\alpha)$ and $H \in \mathcal{M}(\mathcal{D}_\alpha)$ all of which are analytic on $D_{1+\epsilon}(0)$, with $H(0) \neq 0$, and suppose h is a polynomial. Now consider the following;

$$u_h = F^*(FF^*)^{-1}Hh - Q\widehat{W},$$

defined pointwise for each $z \in \overline{\mathbb{D}}$, where

$$W = \frac{Q^* F'^* H h}{(FF^*)^2}.$$

Direct computation, using the fact that that $\overline{\partial}(\widehat{W}) = W$, gives that $\overline{\partial}(u_h) = 0$, so that u_h is analytic. Using the fact that the range of Q is equal to the kernel of F, it is immediate that

$$M_F^R(u_h) = Hh$$

Next it is shown that there exists a constant K, independent of h and ϵ, such that $\|u_h\|_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}_\alpha} \leq K\|h\|_{\mathcal{D}_\alpha}$.

$$\begin{aligned}
\|u_h\|^2_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}_\alpha} &= \int_{-\pi}^{\pi} \|u_h(e^{it})\|^2_{l^2} d\sigma(t) + \int_{\mathbb{D}} \|u_h'(z)\|^2_{l^2} dA_\alpha(z) \\
&= \underbrace{\int_{-\pi}^{\pi} \|u_h(e^{it})\|^2_{l^2} d\sigma(t)}_{(i)} + \underbrace{\int_{\mathbb{D}} \|\partial u_h(z)\|^2_{l^2} dA_\alpha(z).}_{(ii)}
\end{aligned}$$

It can be shown using condition (b) of the theorem and a Carleson measure technique as seen in [**Tr1**], by taking $\alpha(t) = \sqrt{t}$, that $(i) \leq K_1^2\|h\|^2_\sigma \leq K_1^2\|h\|^2_{\mathcal{D}_\alpha}$. This is where the assumption $H(0) \neq 0$ is used. For (ii) the integral will be split into five pieces to obtain a similar bound.

$$
\begin{aligned}
(ii) &\leq 2\int_{\mathbb{D}} \left\| \partial\left(\frac{F^*Hh}{FF^*}\right)\right\|_{l^2}^2 dA_\alpha(z) + 2\int_{\mathbb{D}} \left\|\partial\left(Q\widehat{W}\right)\right\|_{l^2}^2 dA_\alpha(z) \\
&\leq \underbrace{4\int_{\mathbb{D}} \left\|\frac{F^*H'h}{FF^*}\right\|_{l^2}^2 dA_\alpha(z)}_{(a')} + \underbrace{8\int_{\mathbb{D}} \left\|\frac{F^*Hh'}{FF^*}\right\|_{l^2}^2 dA_\alpha(z)}_{(b')} + \underbrace{8\int_{\mathbb{D}} \left\|\frac{F^*HhF'F^*}{(FF^*)^2}\right\|_{l^2}^2 dA_\alpha(z)}_{(c')} \\
&+ \underbrace{4\int_{\mathbb{D}} \left\|Q'\widehat{W}\right\|_{l^2}^2 dA_\alpha(z)}_{(d')} + \underbrace{4\int_{\mathbb{D}} \left\|Q\partial\widehat{W}\right\|_{l^2}^2 dA_\alpha(z)}_{(e')}.
\end{aligned}
$$

Condition (c) of the theorem and Lemma 3.2.2 give

$$
\begin{aligned}
(a') &= \int_{\mathbb{D}} \left\|\frac{F^*}{\sqrt{FF^*}}\frac{H'}{\sqrt{FF^*}}h\right\|_{l^2}^2 dA_\alpha(z) \leq C\int_{\mathbb{D}} |h|^2 (1-|z|^2)^{2\epsilon-2+\alpha} dA_\alpha(z) \\
&= C\int_{\mathbb{D}} |h|^2 (1-|z|^2)^{2\epsilon-1} dA(z) \\
&\leq CK\left(|h(0)|^2 + \int_{\mathbb{D}} |h'|^2(1-|z|^2)^{2\epsilon+1} dA\right) \\
&\leq CK\left(\|h\|_{H^2(\mathbb{D})}^2 + \int_{\mathbb{D}} |h'|^2 dA_\alpha\right) \\
&= C\|h\|_{\mathcal{D}_\alpha}^2.
\end{aligned}
$$

An application of condition (b) of the theorem gives that

$$
(b') = \int_{\mathbb{D}} \left\|\frac{F^*}{\sqrt{FF^*}}\frac{H}{\sqrt{FF^*}}h'\right\|_{l^2}^2 dA_\alpha(z) \leq \int_{\mathbb{D}} |h'|^2 dA_\alpha(z) \leq \|h\|_{\mathcal{D}_\alpha}^2,
$$

and also along with condition (a) of the theorem that

$$
\begin{aligned}
(c') &= \int\limits_{\mathbb{D}} \left\| \frac{F^*F'F^*}{\sqrt{FF^*}} \frac{H}{(FF^*)^{\frac{3}{2}}} h \right\|_{l^2}^2 dA_\alpha(z) \le \int\limits_{\mathbb{D}} \|F'^*h\|_{l^2}^2 \, dA_\alpha(z) \\
&\le 2 \int\limits_{\mathbb{D}} \sum_{j=1}^{\infty} |(f_j h)'|^2 \, dA_\alpha(z) + 2 \int \mathbb{D} \sum_{j=1}^{\infty} |f_j h'|^2 dA_\alpha(z) \\
&\le 2 \left\| M_F^C(h) \right\|^2 + 2 \|h\|_{\mathcal{D}_\alpha}^2 \le 4 \|h\|_{\mathcal{D}_\alpha}^2 .
\end{aligned}
$$

Using the fact that $\|Q(z)\|_{B(l^2)} \le 1$, Lemma 3.2.6, and that $QQ^* \le FF^*$ gives

$$
\begin{aligned}
(e') &= \int\limits_{\mathbb{D}} \|Q\mathcal{B}(W)\|_{l^2}^2 \, dA_\alpha(z) \le K_2 \int\limits_{\mathbb{D}} \|W\|_{l^2}^2 \, dA_\alpha(z) \\
&= K_2 \int\limits_{\mathbb{D}} \left\| \frac{Q^*F'^*Hh}{(FF^*)^2} \right\|_{l^2}^2 dA_\alpha(z) = K_2 \int\limits_{\mathbb{D}} \left\| \frac{Q^*}{\sqrt{FF^*}} F'^* \frac{H}{(FF^*)^{\frac{3}{2}}} h \right\|_{l^2}^2 dA_\alpha(z) \\
&\le K_2 \int\limits_{\mathbb{D}} \|F'^*h\|_{l^2}^2 \, dA_\alpha(z) \le 4K_2 \|h\|_{\mathcal{D}_\alpha}^2
\end{aligned}
$$

To bound (d'), it is split into two integrals as follows:

$$
(d') \le \underbrace{2\int\limits_{\mathbb{D}} \left\| Q' \left(\widehat{W} - \widetilde{\widehat{W}} \right) \right\|_{l^2}^2 dA_\alpha(z)}_{(i)} + \underbrace{2\int\limits_{\mathbb{D}} \left\| Q' \widetilde{\widehat{W}} \right\|_{l^2}^2 dA_\alpha(z)}_{(ii)},
$$

where $\widetilde{\widehat{W}} = \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}\left(e^{it}\right) d\sigma(t)$, the harmonic extension of $\widehat{W}$ from $\partial\mathbb{D}$ to $\mathbb{D}$. Using the definition of the Cauchy transform for smooth functions, the definition of the harmonic extension, Lemma 3.2.4 and Lemma 3.2.5, it follows that

$$
\begin{aligned}
(i) &= \int_{\mathbb{D}} \left\| Q' \left(\frac{-1}{\pi} \int_{\mathbb{D}} \frac{W(w)}{w-z} dA(w) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \widehat{W}\left(e^{it}\right) d\sigma(t) \right) \right\|_{l^2}^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \left(\int_{\mathbb{D}} \frac{W(w)}{w-z} dA(w) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \int_{\mathbb{D}} \frac{W(w)}{w-e^{it}} dA(W) d\sigma(t) \right) \right\|_{l^2}^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \int_{\mathbb{D}} W(w) \left(\frac{1}{w-z} + \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \frac{e^{-it}}{1-we^{it}} d\sigma(t) \right) dA(w) \right\|_{l^2}^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \int_{\mathbb{D}} W(w) \left(\frac{1}{w-z} + \int_{-\pi}^{\pi} \frac{1-|z|^2}{|1-e^{-it}z|} \frac{e^{-it}}{1-we^{it}} d\sigma(t) \right) dA(w) \right\|_{l^2}^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \int_{\mathbb{D}} W(w) \left(\frac{1-|z|^2}{(w-z)(1-w\bar{z})} \right) dA(w) \right\|_{l^2}^2 dA_\alpha(z) \\
&= \frac{1}{\pi^2} \int_{\mathbb{D}} \left\| Q' \left(1-|z|^2\right) T\left(W\right) \right\|_{l^2}^2 dA_\alpha(z) \\
&\leq \frac{1}{\pi^2} \left\| M_Q \right\|^2 \left\| T\left(W\right) \right\|^2 \\
&\leq \frac{C^2}{\pi^2} \left\| M_Q \right\|^2 \int_{\mathbb{D}} \left\| W \right\|^2 dA_\alpha(z) \\
&\leq \frac{4C^2}{\pi^2} \left\| M_Q \right\|^2 \left\| h \right\|_{\mathcal{D}_\alpha}^2
\end{aligned}
$$

Finally, the use Lemmas 3.2.3 and 3.2.7, show that

$$
(ii) \leq 8\|\widetilde{\widehat{W}}\|_{\mathcal{HD}}^2 \leq 8C_1\|W\|_{A_\alpha} \leq 32C_1\|h\|_{\mathcal{D}_\alpha}^2.
$$

Thus, for every polynomial h, when all the functions are smooth on $\partial\mathbb{D}$, there exists $u_h \in \overset{\infty}{\underset{1}{\oplus}}\mathcal{D}_\alpha$ and a constant K, independent of h, such that:

$$
M_F^R(u_h) = Hh
$$

$$
\|u_h\|_{\overset{\infty}{\underset{1}{\oplus}}\mathcal{D}_\alpha} \leq K\|h\|_{\mathcal{D}_\alpha}.
$$

Since polynomials are dense in $\mathcal{D}_\alpha$ and M_F^R is continuous, this holds for all $h \in \mathcal{D}_\alpha$. By lemma 2.2.8, this is equivalent to saying that when all the functions in question are smooth, there existing a constant K such that

$$M_H M_H^* \leq K^2 M_F M_F^*$$

Hence, for every $0 \leq r < 1$, if F_r is defined by $F_r(z) = F(rz)$ and H_r is similarly defined, then

$$M_{H_r} M_{H_r}^* \leq K^2 M_{F_r} M_{F_r}^*,$$

with $M_{F_r}^R \to M_F^R$ and $M_{H_r} \to M_H$ as $r \uparrow 1$, where convergence of the operators is in the strong$-$topology. Using a commutant lifting argument similar to that seen in [**Tr2**], there exist $G_r \in \mathcal{M}(\mathcal{D}_\alpha, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha)$ such that $M_{F_r}^R M_{G_r}^C = H_r$.

By compactness, there exists a net with $G_{r_\beta}^* \to G^*$ as $r_\beta \to 1^-$. Since the multiplier algebra, as operators, is closed in the weak operator topology, $G \in \mathcal{M}(\mathcal{D}_\alpha, \overset{\infty}{\underset{1}{\oplus}} \mathcal{D}_\alpha)$, and since F_{r_β} converges strongly to F^*, it follows that $M_{G_r}^{C*} M_{F_r}^{R*}$ converges to $M_G^{C*} M_f^{R*}$ in the weak operator topology. Hence, $M_F^R M_G^C = M_H$ with entries of G in $\mathcal{M}(\mathcal{D}_\alpha)$ and $\|M_G^C\| \leq K$. $\square$

CHAPTER 4

FUTURE RESEARCH

Having found conditions that allow the replacement of H^3 with H itself in the Wolff theorems for both the Dirichlet space and the weighted Dirichlet space, one immediate possibility for further research presents itself, namely improving the conditions.

One course of action in this direction is to minimize the derivative condition. The derivative condition used is the best possible exponent for $(1-|z|^2)$ that uses the lemma due to Hardy and Littlewood. A result due to E.G Kwon [**Kw**] generalizes this result from $(1-|z|^2)$ to what are known in the literature as "secure weights". Strengthening the derivative condition in this direction seems to be the most likely to succeed, although it may end in results that are less elegant.

Looking back to the $H^\infty(\mathbb{D})$ case, the progression was the corona theorem, Wolff's Theorem, then finding conditions to replace H^3 with H, then improving the conditions. When improving the conditions, various results were obtained, but two particular ones stand out. The first was a result due to Trent [**Tr1**], which is the following.

Theorem 4.3.1. *Let $F = (f_1, f_2, ...)$ with $\{f_j\}_{j=1}^\infty \subset H^\infty(\mathbb{D})$ and $H \in H^\infty(\mathbb{D})$, with*

$$(a)\ F(z)F(z)^* \leq 1$$

$$(b)\ |H(z)| \leq F(z)F(z)^* \alpha\left(F(z)F(z)^*\right)$$

for all $z \in \mathbb{D}$, where α is a positive, increasing, bounded function such that $\int_0^1 \frac{\alpha(t)}{t} dt$ and $\int_0^1 \frac{1}{t} \int_0^1 \frac{\alpha(u)}{u} du\, dt < \infty$. Then there exists $G = (g_1, g_2, ...)$ with $\{g_j\}_{j=1}^\infty \subset H^\infty(\mathbb{D})$ satisfying

$$\|M_G^C\| < \infty$$

$$\text{and } FG^T = H.$$

The second of which is the result due to Treil that was mentioned earlier as the best known conditions for the $H^\infty(\mathbb{D})$ case, which removes the second integral condition.

Since the proof of the theorem in this dissertation and Wolff's theorem for the Dirichlet space uses similar tools and processes as the theorem due to Trent, the first question that seems reasonable to ask, is whether or not it holds for the multipliers of the Dirichlet space. At this point it isn't clear that it does, and it remains unclear if the derivative condition given in this dissertation is added to the conditions. Regardless, adding it, or even replacing it with $|H'(z)| \leq \alpha\,(F(z)F(z)^*)$, might lead to positive result.

The second question that can be asked, which bypasses the above question for a stronger result, is whether or not the result due to Treil can be proven for the multipliers of the Dirichlet space. The techniques used to prove Treil's result differ from those used here, so an analysis of how to apply them to the multipliers of the Dirichlet space might prove to be a fruitful endeavor.

Also, due to the similarity in the proof techniques used in solving corona problems, it makes sense to attempt to prove corona theorems for any reproducing kernel Hilbert space that lack such a theorem. Many of the tools used in this dissertation can be used in more general settings and would still apply. Following that, the progression that has been seen in both $H^\infty(\mathbb{D})$ and $\mathcal{M}(\mathcal{D})$ can be followed. That is, investigate Wolff ideal problems on these space, and attempt to prove Wolff's theorem for such spaces. For RKHS that already have a corona theorem, it makes sense to continue the progression and start thinking about the Wolff ideal problem.

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